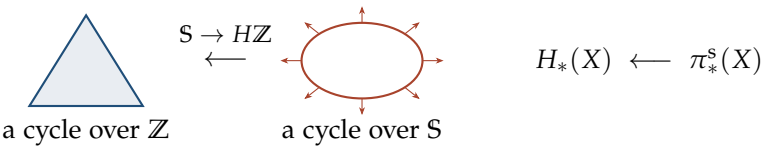
Higher Categorical Semantics

Algebraic and Differential Topology in Higher Categories

Chains and framed manifolds as the linear algebra of the integers and of the sphere



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This volume is a treatment, in categorical language, of the material of L. S. Pontryagin, *Selected Works, Volume 3: Algebraic and Differential Topology* (Gordon and Breach, 1986), which contains two monographs: *Foundations of Algebraic Topology* and *Smooth Manifolds and Their Applications in Homotopy Theory*. It reproduces none of that book's text or figures; where an example of the source is recomputed, this is said. An appendix gives a section-level concordance.

Every numbered statement carries a status tag: [classical] a classical result; [literature; source] a result of the cited literature; [reformulation; cf. source] a known result restated in the language of this volume; [observation] an observation of this volume, proved here; [programme] a proposal. Tags of the form [L1] name a check in the companion script `verify-topology.py`; tags of the form [source, I, §10] point to the source, the Roman numeral giving the monograph.

This book was drafted with the assistance of an AI system (Anthropic's Claude), which also wrote and ran the companion script and drew the figures. The bibliography was assembled from memory, except for the entries marked as checked, and should be verified before it is relied on.

Preface

The third volume of Pontryagin's *Selected Works* [Pon86] contains two short monographs. The first, *Foundations of Algebraic Topology*, defines the homology groups of a polyhedron, proves that they do not depend on the triangulation, and ends with the fixed point theorem of Lefschetz and Hopf. The second, *Smooth Manifolds and Their Applications in Homotopy Theory*, computes the first homotopy groups of spheres that are not given by homology, by representing a map of spheres as a manifold with a field of normal frames. They have no notation in common.

This volume reads them as one theory with two coefficient rings.

- *Chains are a free module.* The chain complex of a space is the free module over the integers on its homotopy type, and homology is forced by that (Theorem 2.4). The constructions of the first monograph, the polyhedron, subdivision, the cone, the cylinder, are a colimit, a functor, a contraction and a product (Chapters 1, 3 and 5).
- *A fixed point theorem is a statement about traces.* The Lefschetz number is the trace of a dualizable object, and Sperner's lemma is a degree computed by Stokes' theorem (Theorems 4.1 and 6.5).
- *Framed manifolds are a free module too.* Pontryagin's construction identifies framed manifolds up to cobordism with the homotopy of the sphere spectrum, which is the free commutative group, in the homotopical sense, on one point (Theorems 8.2 and 8.5).
- *The first three groups.* A signed count of points, the parity of the turns of a frame along a circle, and the Arf invariant of a quadratic form on a surface (Theorems 9.1, 10.3 and 11.5).
- *The comparison.* The first monograph is the second tensored with the integers; what is lost is everything in positive degrees (Theorem 12.1).

Each qualitative statement has a computed companion: homology with torsion by Smith normal form, signed counts of Sperner cells, Lefschetz numbers of all automorphisms of a small torus, linking numbers of Hopf fibres by Gauss's integral, lifts of loops of rotations to quaternions, counts of quadratic forms by Arf invariant. The companion script performs 63 checks. There are thirty-one figures, about half of them computed.

Nothing here is a new theorem, and the long proofs are indicated, not given. What is higher, and how much, is assessed in Chapter 13: of the volumes of this series, this is the one whose central object, the sphere spectrum, cannot be stated without higher categories. A companion paper [ADTML] applies the material to machine learning.

The volume belongs to a series with [GIHC], [ESCHC], [DTHC], [EGHC] and [PDEHC]. It overlaps with [DTHC], which treats the Pontryagin–Thom construction for general tangential structures; here the framed case is taken slowly and all the way to the first computations.

Contents

I Complexes 1

1 Complexes as Presheaves 2
- 1.1 Simplices, complexes, polyhedra . . . 2
- 1.2 What is wrong with complexes: products . . . 3
- 1.3 Nerves . . . 4
- 1.4 Six complexes . . . 4

2 Homology as a Functor 7
- 2.1 Chains . . . 7
- 2.2 Why chains: the free abelian group on a space . . . 8
- 2.3 Euler and Poincaré . . . 9
- 2.4 Cones, suspensions, products . . . 9

3 Invariance 11
- 3.1 Simplicial approximation . . . 11
- 3.2 Subdivision is a functor . . . 12
- 3.3 The cone is a contraction . . . 13
- 3.4 The theorem . . . 13

II Maps 15

4 Sperner, Brouwer, Dimension 16
- 4.1 The lemma . . . 16
- 4.2 No retraction, and fixed points . . . 18
- 4.3 Dimension . . . 19

5 Homotopy and the Cylinder 21
- 5.1 The cylindric construction . . . 21
- 5.2 What a chain homotopy is . . . 22
- 5.3 Degree . . . 22
- 5.4 What homology cannot see . . . 23

6 Traces and Fixed Points **25**
6.1 The Hopf trace formula . . . 25
6.2 Examples . . . 26
6.3 The trace in a symmetric monoidal category . . . 27

III Framed Manifolds **29**

7 General Position **30**
7.1 The count . . . 30
7.2 Sard's theorem . . . 31
7.3 Why general position is categorical . . . 32

8 Framed Manifolds and the Main Method **34**
8.1 The construction . . . 34
8.2 Suspension and stability . . . 36
8.3 What the theorem says about spaces . . . 36

9 Degree, Linking and the Hopf Invariant **38**
9.1 Framed points: Hopf's theorem . . . 38
9.2 Linking . . . 38
9.3 The Hopf invariant . . . 39
9.4 Framed circles and $\pi_3(S^2)$ . . . 40

IV The First Stems **42**

10 Rotations and the First Stem **43**
10.1 Quaternions and rotations . . . 43
10.2 Framed circles . . . 44
10.3 The J-homomorphism . . . 45

11 The Arf Invariant and the Second Stem **47**
11.1 A quadratic form on a framed surface . . . 47
11.2 Surgery . . . 48
11.3 History, and what the form means . . . 49

12 Stable Stems as the Homotopy of the Unit **51**
12.1 Two rings . . . 51
12.2 The table . . . 53
12.3 What became of the method . . . 53

13 Assessment **55**
13.1 What the language did 55
13.2 What it did not do 55
13.3 How much is higher 56
13.4 Programme 56

A Toolkit **58**

B Concordance with the Source **60**

C Computations **62**

D Solutions to the Exercises **64**

References **70**

Part I

Complexes

Chapter 1

Complexes as Presheaves

The first monograph of the source builds everything from one kind of object, the finite simplicial complex, and one kind of map, the simplicial map. This chapter says what category that is, what is wrong with it, and what the standard repair is. The polyhedron of a complex turns out to be a colimit, the nerve of a cover a limit of sorts, and the awkwardness of products the reason simplicial sets were invented.

1.1 Simplices, complexes, polyhedra

Definition 1.1. An ***abstract simplicial complex*** K is a finite set V of *vertices* with a family of non-empty subsets, the *simplices*, closed under passing to non-empty subsets and containing every singleton. A simplex with $r+1$ vertices has dimension r. A ***simplicial map*** $K \to L$ is a map of vertex sets taking simplices to simplices; it need not be injective on a simplex. This is the category SCx.

A simplex σ has a geometric realization $|\sigma|$, the convex hull of its vertices placed in general position, and a face $\tau \subseteq \sigma$ gives an inclusion $|\tau| \subseteq |\sigma|$. The source defines a complex as a set of simplices in a Euclidean space that meet properly, and its polyhedron as their union [source, I, §2]. Categorically:

Proposition 1.2 (The polyhedron is a colimit)**.** [reformulation; cf. [Spa66]] *Regard K as a poset under inclusion. Then*

$$|K| \;=\; \operatorname*{colim}_{\sigma \in K} |\sigma|$$

in topological spaces, and $K \mapsto |K|$ is a functor $\mathsf{SCx} \to \mathsf{Top}$: a simplicial map is realized by the map that is affine on each simplex.

The colimit says exactly what "meeting properly" says: two simplices are glued along their common face and nowhere else. That the abstract colimit can be drawn in a Euclidean space of small dimension is the first theorem of the source.

Theorem 1.3 (Realization)**.** [classical; [source, I, §1–2, Thms. 1–3]] *A complex of dimension r has a realization by straight simplices in $\mathbb{R}^{2r+1}$: place the vertices in general position, so that every $2r+2$ of them are affinely independent.*

Proof. Two simplices have at most $2r+2$ vertices between them. If these are affinely independent, the convex hulls of two subsets of them meet in the convex hull of the common part. □

Points on the moment curve $t \mapsto (t, t^2, \dots, t^n)$ are in general position in $\mathbb{R}^n$, by the Vandermonde determinant; for seven points and $n = 5$ all seven six-point subsets are affinely independent, so every 2-complex on seven vertices, the torus of Figure 1.3 included, sits in $\mathbb{R}^5$. [C2] The number $2r+1$ returns in Chapter 7 as Whitney's embedding dimension, for the same reason: two r-dimensional things in a space of dimension $2r+1$ do not meet in general.

1.2 What is wrong with complexes: products

The category SCx has products, but they are the wrong ones. A vertex of $K \times L$ is a pair of vertices, and a set of pairs is a simplex when both projections are. For two edges this gives four vertices of which *every* subset is a simplex: a tetrahedron, where a square was wanted.

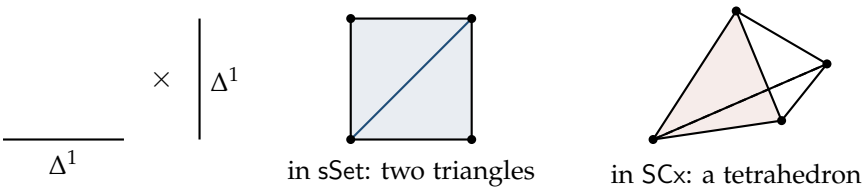


Figure 1.1. *The product of two edges. The categorical product of simplicial complexes fills in everything; the product of simplicial sets is the square, cut along the diagonal that the ordering of the vertices selects.*

The source meets the same difficulty when it needs $K \times [0,1]$ and resolves it by hand, with an explicit subdivision of each prism [source, I, §14]; see Chapter 5. The systematic repair is to remember an order on the vertices and to allow degenerate simplices.

Definition 1.4. Let Δ be the category of the finite ordinals $[n] = \{0 < \cdots < n\}$ and monotone maps. A ***simplicial set*** is a functor $\Delta^{\mathrm{op}} \to \mathsf{Set}$; these form the category sSet. An ordered complex K gives the simplicial set whose n-simplices are the monotone maps $[n] \to V$ with image a simplex.

Theorem 1.5. [classical; [Mil57, GoJa99]] (a) *Geometric realization* $|\cdot| : \mathsf{sSet} \to \mathsf{Top}$ *is the left Kan extension of* $[n] \mapsto |\Delta^n|$ *along the Yoneda embedding; on ordered complexes it agrees with Proposition 1.2.* (b) *It preserves finite products, in compactly generated spaces.* (c) *The non-degenerate* $(p+q)$*-simplices of* $\Delta^p \times \Delta^q$ *are the* $\binom{p+q}{p}$ *shuffles.*

The counts are 2, 3, 6, 10 for $(p,q) = (1,1),(2,1),(2,2),(3,2)$. [C4] The middle one is the prism over a triangle cut into three tetrahedra, which is the cylindric construction of the source. A simplicial set is a presheaf, so sSet has all limits and colimits, computed pointwise; a complex is a presheaf on its own poset of simplices. That is the sense of the chapter's title.

1.3 Nerves

Definition 1.6. The ***nerve*** of a finite family of sets $\{U_i\}_{i\in I}$ is the complex on I whose simplices are the subfamilies with non-empty intersection.

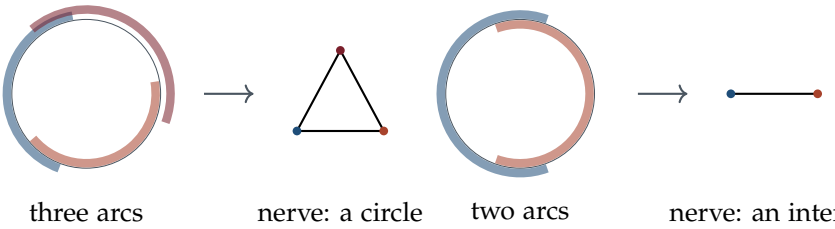


Figure 1.2. *Two covers of a circle. In the first every non-empty intersection is an arc and the nerve is a circle. In the second the two arcs meet in two pieces, the nerve does not see this, and it is contractible.*

Theorem 1.7 (Nerve theorem)**.** [classical; [Bor48, Ler45]] *If every non-empty intersection of members of a finite open cover of a paracompact space is contractible, the space is homotopy equivalent to the nerve.*

In the language of Theorem 1.5: a space is the homotopy colimit of the diagram of intersections of a cover, and when the intersections are contractible the diagram is the constant diagram on the nerve. For the cover of $|K|$ by the open stars of its vertices the nerve is K itself, since stars meet exactly when their vertices span a simplex; this holds for the six complexes below. [C3] Nerves are how the source passes from a compact metric space to complexes in its dimension theory [source, I, §3], and how data analysis passes from a point cloud to a complex [ADTML]. We return to the order of a cover in Chapter 4.

1.4 Six complexes

Computations in Part One are made on six surfaces. Each is a ***pseudo-manifold***: every edge lies in exactly two triangles, or at most two for the band. [C1]

complex	vertices, edges, triangles	χ	closed
boundary of a tetrahedron	4, 6, 4	2	yes
icosahedron	12, 30, 20	2	yes
torus	7, 21, 14	0	yes
projective plane	6, 15, 10	1	yes
Klein bottle	16, 48, 32	0	yes
Möbius band	5, 10, 5	0	no

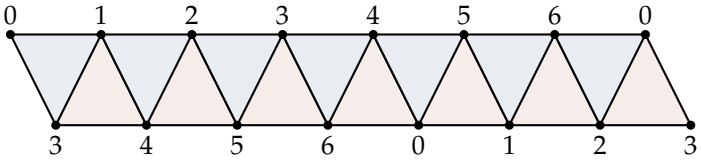


Figure 1.3. *The torus on seven vertices. The strip shows all fourteen triangles* $\{i, i+1, i+3\}$ *and* $\{i, i+2, i+3\}$*, indices modulo 7. The two ends are glued, and the top edge is glued to the bottom edge with a shift by three. Every pair of vertices is joined by an edge.*

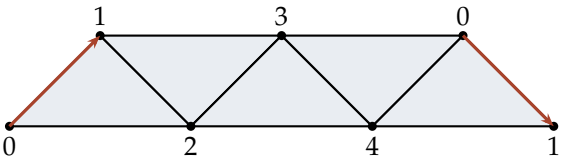


Figure 1.4. *The Möbius band on five vertices: the triangles* $\{i, i+1, i+2\}$*, indices modulo* 5*. The two marked edges are the same edge* $\{0, 1\}$*, and gluing them requires a half twist.*

The torus is the smallest possible: a closed surface with V vertices has $E = 3(V - \chi)$ edges, at most $\binom{V}{2}$, and for $\chi = 0$ this forces $V \geq 7$. Equality means that the complete graph K_7 is drawn on the torus, which is why maps on a torus can need seven colours.

Notes

The source's first chapter also contains the linear algebra of general position, which we have compressed into one proof. For simplicial sets see [GoJa99]; for the comparison of the three kinds of product, [Mil57]. The nerve theorem has many forms; the one stated is Borsuk's. The seven-vertex torus is due to Möbius and was realized with flat faces in $\mathbb{R}^3$ by Császár.

Exercises

Exercise 1.1. Show that a closed triangulated surface with Euler characteristic χ has at least $\frac{1}{2}\left(7 + \sqrt{49 - 24\chi}\right)$ vertices, and evaluate the bound for the sphere, the projective plane and the torus.

Exercise 1.2. List the three tetrahedra of $\Delta^2 \times \Delta^1$ as chains of pairs, and check that each triangle in the interior of the prism lies in exactly two of them.

Exercise 1.3. Cover the boundary of a tetrahedron by its four closed faces. Find the nerve and the order of the cover, and say whether the nerve theorem applies.

Exercise 1.4. Why can six points on the moment curve in $\mathbb{R}^3$ not be used to realize the complete 2-complex on six vertices? Which pairs of simplices are the obstruction?

Chapter 2

Homology as a Functor

The source calls them Betti groups. To a complex it attaches chains, to chains a boundary, and to the boundary the quotient of cycles by boundaries. This chapter records the functors involved and the one statement that explains why the construction is forced: chains are the free abelian group on a space.

2.1 Chains

Fix an order on the vertices of K. Let $C_r(K)$ be the free abelian group on the r-simplices, and

$$\partial[v_0,\dots,v_r]=\sum_{i=0}^{r}(-1)^i[v_0,\dots,\widehat{v_i},\dots,v_r],\qquad v_0<\cdots<v_r.$$

Proposition 2.1. [classical; [source, I, §4]] *$\partial\partial=0$, and $K\mapsto C_*(K)$ is a functor from* SCx *to chain complexes of abelian groups: a simplicial map sends a simplex to its image with the sign of the permutation that sorts it, or to 0 if two vertices collide.*

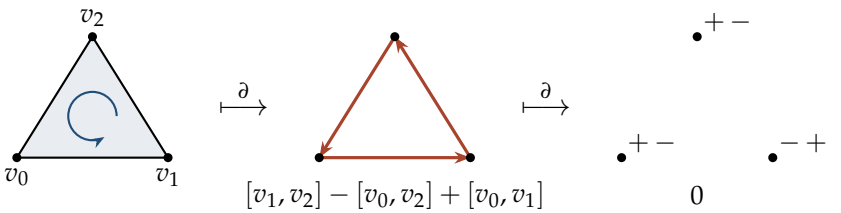


Figure 2.1. *The boundary of an oriented triangle is the cycle of its three edges, each vertex of which is the end of one edge and the beginning of the next: the boundary of a boundary cancels in pairs.*

[H1] The ***homology groups*** are $H_r(K)=Z_r/B_r$ with $Z_r=\ker\partial$ and $B_r=\operatorname{im}\partial$. They are finitely generated, so $H_r\cong\mathbb{Z}^{\beta_r}\oplus T_r$ with T_r finite; β_r is the ***Betti number*** and the orders of the cyclic summands of T_r the ***torsion coefficients***. Both are read off the Smith normal form of the boundary matrices: $\beta_r=\operatorname{rank}C_r-\operatorname{rank}\partial_r-\operatorname{rank}\partial_{r+1}$, and the torsion of H_r is given by the invariant factors of ∂_{r+1} that exceed one.

complex	H_0, H_1, H_2 over $\mathbb{Z}$	β over $\mathbb{F}_2$	β over $\mathbb{F}_3$	χ
sphere	$\mathbb{Z},\ 0,\ \mathbb{Z}$	$1,0,1$	$1,0,1$	2
torus	$\mathbb{Z},\ \mathbb{Z}^2,\ \mathbb{Z}$	$1,2,1$	$1,2,1$	0
projective plane	$\mathbb{Z},\ \mathbb{Z}/2,\ 0$	$1,1,1$	$1,0,0$	1
Klein bottle	$\mathbb{Z},\ \mathbb{Z}\oplus\mathbb{Z}/2,\ 0$	$1,2,1$	$1,1,0$	0
Möbius band	$\mathbb{Z},\ \mathbb{Z},\ 0$	$1,1,0$	$1,1,0$	0

[H2, H4] The last column does not depend on the field, and the others do.

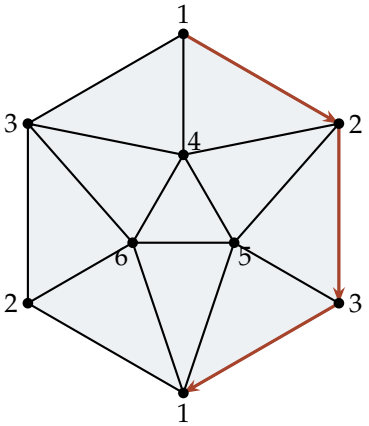


Figure 2.2. *The projective plane on six vertices: a hexagon with opposite boundary points identified. The path $1 \to 2 \to 3 \to 1$ is a cycle z. The boundary of the sum of the ten triangles, all oriented alike, runs round the hexagon and is $2z$. So z is not a boundary but $2z$ is: a torsion class, visible over $\mathbb{F}_2$ and invisible over $\mathbb{F}_3$ and over the rationals.*

2.2 Why chains: the free abelian group on a space

The zero-dimensional case is elementary and shows the pattern.

Proposition 2.2. [classical; [source, I, §5, Thms. 6–8]] *$H_0(K)$ is the free abelian group on the set $\pi_0(K)$ of components.*

[H5] The functor π_0 is left adjoint to the inclusion of sets into complexes, and the free abelian group is left adjoint to the forgetful functor, so H_0 is a composite of left adjoints: it is the universal way of turning a complex into an abelian group. The higher groups are the same statement with sets replaced by homotopy types.

Theorem 2.3 (Dold–Kan and Dold–Thom). [classical; [Dol58, GoJa99, DoTh58]] (a) *Normalized chains give an equivalence between simplicial abelian groups and non-negatively graded chain complexes.* (b) *For a simplicial set X let $\mathbb{Z}[X]$ be the free simplicial abelian group on it. Then $H_n(X) = \pi_n \mathbb{Z}[X]$.*

Theorem 2.4 (Chains as a left adjoint). [reformulation; cf. [Lur17]] *Let $\mathcal{S}$ be the ∞-category of homotopy types and $\mathcal{D}(\mathbb{Z})$ the derived ∞-category of abelian groups. The functor $C_*\colon \mathcal{S} \to \mathcal{D}(\mathbb{Z})$ is the unique colimit-preserving functor sending a point to $\mathbb{Z}$. It is left adjoint to the functor that sends a chain complex to its underlying space.*

So the chain complex is not a device that happens to produce invariants. It is the free module over $\mathbb{Z}$ on a homotopy type, as $\mathbb{Z}[S]$ is the free module on a set S, and the boundary operator is what freeness looks like after Theorem 2.3. Part Two will make the same construction with $\mathbb{Z}$ replaced by another ring, and Chapter 12 compares the two.

2.3 Euler and Poincaré

Theorem 2.5 (Euler–Poincaré). [classical; [source, I, §6, Thm. 9]] *If α_r is the number of r-simplices of K, then $\sum_r (-1)^r \alpha_r = \sum_r (-1)^r \beta_r$, and the same holds with Betti numbers over any field.*

Proof. Rank is additive on the short exact sequences $0 \to Z_r \to C_r \to B_{r-1} \to 0$ and $0 \to B_r \to Z_r \to H_r \to 0$, and the ranks of the B_r cancel in the alternating sum. □

[H3] The proof uses one property of rank, additivity, and the Euler characteristic is the universal invariant with that property: it is the class of $C_*(K)$ in the Grothendieck group of perfect complexes, which is $\mathbb{Z}$. Additivity for unions is the same fact one level down: $\chi(A \cup B) = \chi(A) + \chi(B) - \chi(A \cap B)$. [H7] In Chapter 6 the Euler characteristic becomes the trace of the identity, and Theorem 2.5 a special case of a formula for traces.

2.4 Cones, suspensions, products

Proposition 2.6. [classical; [source, I, §8]] (a) *A cone has the homology of a point.* (b) *The suspension ΣK, two cones glued along K, has $\tilde{H}_{r+1}(\Sigma K) \cong \tilde{H}_r(K)$.*

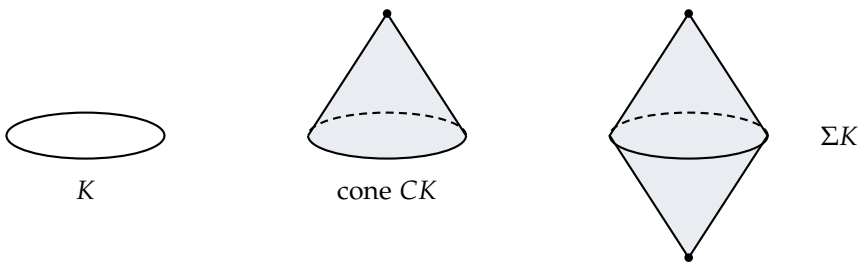


Figure 2.3. *A complex, its cone and its suspension. The cone kills all homology; the suspension moves it up by one dimension.*

The cones on the six complexes are acyclic; the suspension of the torus has Betti numbers $(1,0,2,1)$, and the suspension of the projective plane has $H_2 = \mathbb{Z}/2$ and nothing else in positive degrees. [H6] Statement (b) says that homology cannot tell a complex from its suspension except by a shift. Invariants with this property are called *stable*; homotopy groups do not have it, and the second monograph of the source is about the range of dimensions in which they acquire it (Chapter 8).

Theorem 2.7 (Künneth). [classical; [Hat02]] *Over a field,* $H_*(K \times L) \cong H_*(K) \otimes H_*(L)$.

With the product triangulation of Theorem 1.5, the torus $(\mathbb{Z}/3)^2$ has Betti numbers $(1,2,1)$, and the 3-torus $(\mathbb{Z}/4)^3$, with 64 vertices and 384 tetrahedra, has $(1,3,3,1)$ over $\mathbb{F}_2$, $\mathbb{F}_3$ and a large prime field. [H8] In the language of Theorem 2.4, C_* is symmetric monoidal: the free module on a product is the tensor product of the free modules.

Notes

The source works throughout with coefficients in the integers and in $\mathbb{Z}/m$ and proves the structure theorem for finitely generated abelian groups along the way. The universal coefficient theorem, which explains the middle columns of the table, is in [Hat02]. Theorem 2.4 is the form in which homology enters higher algebra [Lur17]; a concrete version is that C_* is characterized by excision and its value on a point.

Exercises

Exercise 2.1. The cellular chain complex of the Klein bottle with one vertex, two edges a, b and one face has $\partial_2 = (2,0)^{\mathsf{T}}$ and $\partial_1 = 0$. Compute the homology and compare with the table.

Exercise 2.2. Using the table, verify for the projective plane and the Klein bottle that $\beta_r(\mathbb{F}_p) = \beta_r + t_r(p) + t_{r-1}(p)$, where $t_r(p)$ is the number of torsion coefficients of H_r divisible by p.

Exercise 2.3. Find the homology of the suspension of the Klein bottle, and its Betti numbers over $\mathbb{F}_2$.

Exercise 2.4. Prove that $\chi(K \times L) = \chi(K)\chi(L)$ from Theorem 2.7, and check it for the 3-torus above from the numbers of its simplices.

Chapter 3

Invariance

The groups of Chapter 2 were defined from a triangulation. The central theorem of the first monograph is that they depend only on the polyhedron, and in fact only on its homotopy type. The source proves this without singular chains, by three constructions: simplicial approximation, barycentric subdivision and the cone. Each is a functor or a natural transformation, and the proof is the statement that they fit together.

3.1 Simplicial approximation

For a vertex v of K the ***open star*** $\operatorname{St} v$ is the union of the interiors of the simplices containing v. Stars are an open cover of $|K|$ whose nerve is K (Theorem 1.7).

Definition 3.1. A simplicial map $\varphi\colon K \to L$ is a ***simplicial approximation*** of a continuous map $f\colon |K| \to |L|$ if $f(\operatorname{St} v) \subseteq \operatorname{St} \varphi(v)$ for every vertex v of K.

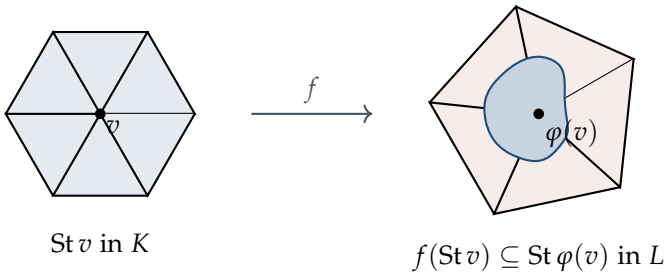


Figure 3.1. *The star condition. If the image of every star of K lies in a star of L, choosing such a star for each vertex defines a simplicial map, and $f(x)$ and $\varphi(x)$ always lie in a common simplex.*

Theorem 3.2 (Simplicial approximation). [classical; [source, I, §7, Thm. 10]] (a) *If the star condition holds, φ is simplicial and homotopic to f by the straight-line homotopy.* (b) *For every continuous $f\colon |K| \to |L|$ there is an m such that f has a simplicial approximation $K^{(m)} \to L$ on the m-th barycentric subdivision of K.*

Part (b) is quantitative. Let λ be a Lebesgue number of the cover $\{f^{-1}\operatorname{St} w\}$ of $|K|$. The star condition holds as soon as stars of $K^{(m)}$ have diameter less than λ, and each subdivision of an r-dimensional complex multiplies the mesh by

at most $r/(r+1)$. So

$$m \;\geq\; \frac{\log(2\,\mathrm{mesh}\,K/\lambda)}{\log((r+1)/r)}$$

subdivisions suffice. For a triangle the mesh after $0,\dots,4$ subdivisions is 1.414, 0.745, 0.478, 0.283, 0.174, with ratios 0.527, 0.641, 0.592, 0.615, all below $\frac{2}{3}$, and the number of triangles is 6^m. [I4]

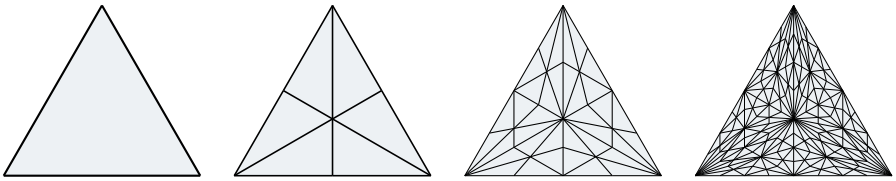


Figure 3.2. *A triangle and its first three barycentric subdivisions. The pieces become thin as well as small, which is why the guaranteed factor $\frac{2}{3}$ per step is not attained; what the proof needs is only that the mesh tends to zero.*

The approximation carries the homotopy class, and too coarse a complex carries the wrong one. Map a polygon with N sides to a triangle by sending each vertex to the vertex of the triangle whose third of the circle contains the image under $z \mapsto z^d$. For $N = 60$ the simplicial map has degree d for $d = -2,\dots,3$. For $N = 8$ and $d = 3$ the rule still defines a simplicial map, since on a triangle every vertex map is simplicial, but its degree is 2: the stars of the octagon are mapped onto arcs of 270°, larger than the stars of the triangle, and the map constructed is not an approximation of anything in particular. [I5]

3.2 Subdivision is a functor

Definition 3.3. The ***barycentric subdivision*** $\mathrm{sd}\,K$ is the complex whose vertices are the simplices of K and whose simplices are the chains $\sigma_0 \subset \sigma_1 \subset \cdots \subset \sigma_k$: the nerve of the poset of simplices.

Proposition 3.4. [reformulation; cf. [GoJa99]] (a) sd *is a functor* $\mathsf{SCx} \to \mathsf{SCx}$, *and* $|\mathrm{sd}\,K| \cong |K|$ *naturally, the vertex σ going to the barycentre of $|\sigma|$.* (b) *An n-simplex is divided into $(n+1)!$ simplices, one for each maximal flag of faces.* (c) $\mathrm{sd}\,K$ *is canonically ordered, by dimension, so* sd *factors through simplicial sets.* (d) *Sending the vertex σ of* $\mathrm{sd}\,K$ *to any vertex of σ is a simplicial approximation of the identity* $|\mathrm{sd}\,K| \to |K|$.

The counts in (b) are $2, 6, 24, 120$. [C5] Statement (c) is how Chapter 1 embedded complexes in simplicial sets: an unordered complex becomes ordered after

one subdivision. For (d), the map sending σ to its last vertex is a chain map $C_*(\operatorname{sd} K) \to C_*(K)$ and is onto in top homology for the torus, the projective plane and the Klein bottle. [I2] On simplicial sets subdivision has a right adjoint Ex, and iterating it gives Kan's functorial replacement of a simplicial set by one in which every horn has a filler [Kan57]: the passage from complexes to homotopy types.

3.3 The cone is a contraction

Proposition 3.5 (Conic construction). [classical; [source, I, §8]] *On the cone CK with apex a define $D[v_0,\dots,v_r] = (-1)^{r+1}[v_0,\dots,v_r,a]$ for simplices not containing a, and $D = 0$ on the others. Then*

$$\partial D + D\partial = \mathrm{id} - \varepsilon,$$

where ε is zero in positive degrees and sends a 0-chain c to (sum of coefficients of c) times a.

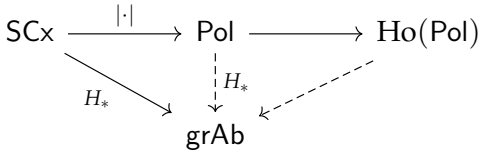
[I3] This is Proposition 2.6(a) with the reason attached: a cone is acyclic because the identity is chain homotopic to the projection on a point. Simplices are cones. So any two natural chain maps defined on simplices that agree in degree zero are chain homotopic, by building the homotopy one dimension at a time and filling each step with D. This argument, the method of acyclic models [EiMa53], is what the source carries out by hand for the subdivision operator.

Theorem 3.6 (Invariance under subdivision). [classical; [source, I, §11, Thm. 15]] *There is a natural chain map $C_*(K) \to C_*(\operatorname{sd} K)$, sending a simplex to the sum of its pieces with their induced orientations. It is a chain homotopy inverse of the map of Proposition 3.4(d). Hence $H_*(\operatorname{sd} K) \cong H_*(K)$.*

The subdivided projective plane has 31 vertices, 90 edges and 60 triangles, and homology $\mathbb{Z}, \mathbb{Z}/2, 0$; likewise for the torus and the Klein bottle. [I1]

3.4 The theorem

Theorem 3.7 (Invariance). [classical; [source, I, §12, Thms. 16–17]] *A continuous map $f\colon |K| \to |L|$ induces homomorphisms $f_*\colon H_r(K) \to H_r(L)$, defined by any simplicial approximation on any sufficiently fine subdivision, functorially. Homeomorphic polyhedra have isomorphic homology groups.*



The theorem is the left dashed arrow: homology of complexes factors through the category Pol of polyhedra and continuous maps. The right dashed arrow, through the homotopy category, is Chapter 5. The modern route to both introduces the singular complex Sing X, whose n-simplices are all continuous maps $|\Delta^n| \to X$; Sing is right adjoint to realization, $H_*(\operatorname{Sing} X)$ is defined for every space with no choices, and the content of Theorem 3.7 becomes the statement that the unit $K \to \operatorname{Sing}|K|$ induces an isomorphism on homology. The three constructions of this chapter are exactly what that proof uses: approximation to make singular simplices simplicial, subdivision to make them small, and the cone to compare.

Notes

The source's order is: approximation, cone, subdivision, Sperner's lemma, invariance; we have moved Sperner's lemma to the next chapter. For acyclic models see [EiMa53]; for Ex^∞, [Kan57, GoJa99]. The estimate of the number of subdivisions is implicit in every proof of Theorem 3.2 and is used in [ADTML] to bound the size of a piecewise linear network realizing a homotopy class.

Exercises

Exercise 3.1. Show that the diameter of a simplex of the barycentric subdivision of an r-simplex of diameter d is at most $\frac{r}{r+1}d$. How many subdivisions of the triangle of [I4] guarantee mesh below 0.01 by this bound, and how many triangles is that?

Exercise 3.2. Explain the degree 2 found for the octagon: list the images of the eight vertices under the rule for $d = 3$ and count the signed number of times the edge from the last third to the first is covered.

Exercise 3.3. Verify $\partial D + D\partial = \mathrm{id}$ on a 1-simplex $[v_0, v_1]$ of the base of a cone.

Exercise 3.4. Let K have f-vector $(\alpha_0, \alpha_1, \alpha_2)$. Find the f-vector of $\operatorname{sd} K$ and check that the Euler characteristic is unchanged. Apply it to the projective plane.

Part II

Maps

Chapter 4

Sperner, Brouwer, Dimension

In the middle of its invariance proof the source stops for a digression: a combinatorial lemma about labellings of a subdivided simplex, and two consequences, the fixed point theorem and the invariance of dimension [source, I, §3, §10]. The lemma is a degree computation, and the consequences are what one gets from knowing that a functor does not vanish.

4.1 The lemma

Let K be a subdivision of an n-simplex Δ with vertices $e_0, \dots, e_n$. A ***Sperner labelling*** assigns to each vertex of K one of the labels $0, \dots, n$ so that a vertex lying in the face spanned by $e_{i_0}, \dots, e_{i_k}$ gets one of the labels $i_0, \dots, i_k$. A cell of K is ***complete*** if its vertices carry all $n+1$ labels.

Theorem 4.1 (Sperner). [classical; [source, I, §10]] *The number of complete cells is odd. Counted with the sign $+1$ if the labelling preserves the orientation of the cell and -1 if it reverses it, the number of complete cells is exactly* 1.

Proof. A labelling is a simplicial map $\lambda\colon K \to \Delta$, and the condition says that it maps each face of Δ to itself. On chains, λ sends the sum $[K]$ of the oriented cells to $c\,[\Delta]$, where c is the signed count. Since λ commutes with ∂, the restriction of λ to the boundary sends $\partial[K]$ to $c\,\partial[\Delta]$, so c is also the signed count on each facet, which is a Sperner labelling of dimension $n-1$. In dimension 0 the count is 1. □

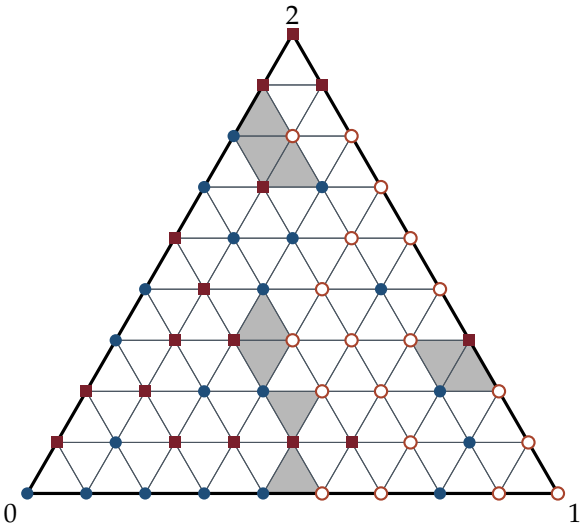


Figure 4.1. *A Sperner labelling of a triangle divided into* 81 *cells: discs, rings and squares for the labels* 0, 1, 2. *A vertex on a side carries one of the two labels of the ends of that side. The* 9 *complete cells are shaded; their number is odd.*

For 200 random labellings of the triangular grid of side 12 the number of complete cells ranged from 17 to 47, always odd, and the signed count was always 1; on the side of the triangle carrying the labels 0 and 1 the signed number of edges labelled 01 was always 1. [S1, S2] For twenty labellings of the second barycentric subdivision of a tetrahedron, with 576 cells, there were between 29 and 65 complete cells and the signed count had absolute value 1. [S3] The proof is Stokes' theorem for a simplicial map: the degree of a map of a ball that preserves the boundary sphere is the degree on the boundary. The induction on dimension in the proof is the induction that computes H_{n-1} of a sphere by suspension (Proposition 2.6).

The parity half of the proof is also an algorithm. Enter the triangle through an edge of the boundary labelled 01; each cell entered through such an edge is either complete or has exactly one other such edge to leave by. The walk cannot cycle and can leave the triangle only through another boundary 01-edge, of which there is an odd number. Finding the end of a path of this kind, given its beginning, is the defining problem of the complexity class PPAD [Pap94].

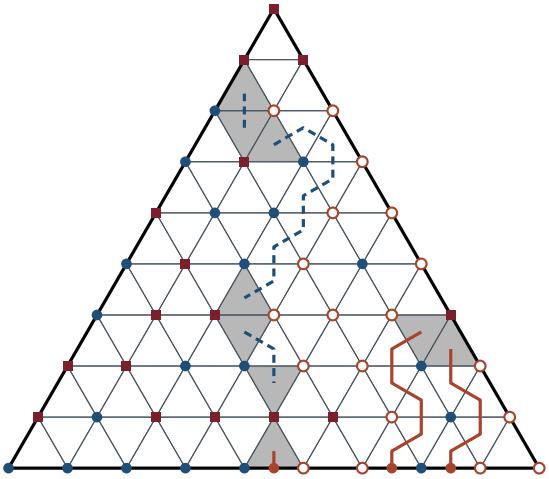


Figure 4.2. *The labelling of Figure 4.1 with its walks. A walk passes from cell to cell through edges labelled* 01*. The 3 walks that enter through the bottom side (solid) end in 3 complete cells; the other 6 complete cells are joined in pairs by walks that never reach the boundary (dashed). Hence the parity.*

4.2 No retraction, and fixed points

Theorem 4.2. [classical; [source, I, §10]] (a) *There is no continuous retraction of a simplex onto its boundary.* (b) *(Brouwer) Every continuous map of a simplex to itself has a fixed point.*

$$\partial\Delta \hookrightarrow \Delta \xrightarrow{r} \partial\Delta \qquad \mathbb{Z} \longrightarrow 0 \longrightarrow \mathbb{Z}$$

Apply $\tilde{H}_{n-1}$ to the left diagram to get the right one, which cannot commute. This is the whole use of a functor: a retraction would be a factorization of an identity through zero. Sperner's lemma gives the same conclusion without homology, and effectively: label a vertex x of a fine subdivision by an index i with $x_i > 0$ and $f(x)_i \le x_i$ in barycentric coordinates; a complete cell is one near which no coordinate can increase, and so contains an approximately fixed point. For one map of the triangle, the point at the centre of a complete cell of the grid of side n is moved by 0.059, 0.014 and 0.0026 for $n = 8, 32, 128$. [S4]

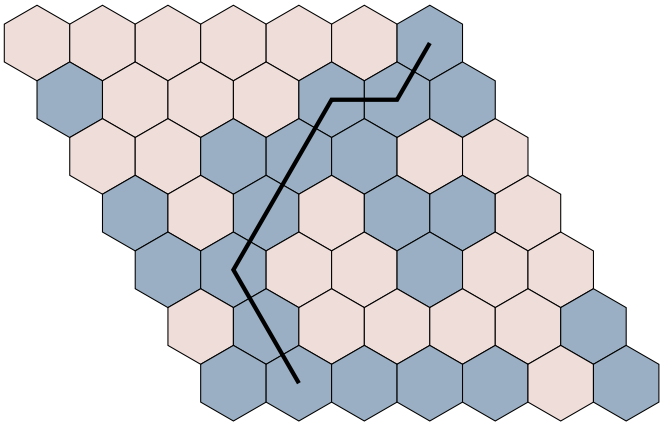


Figure 4.3. *A filled Hex board. The dark player joins the top side to the bottom, the light player the left to the right. Here the dark player has a chain, drawn; the other cannot then have one, and one of the two always does.*

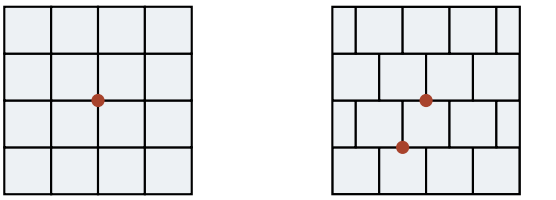
The game of Hex gives a third form of the same statement. On a rhombic board of hexagons filled with two colours, exactly one of the two players has a chain joining their pair of opposite sides; that it cannot be both is the Jordan curve theorem in a simple case, and that it must be one is equivalent to Brouwer's theorem in the plane [Gal79]. In 300 random fillings of the 9×9 board there was always exactly one winner, the first player in 156 of them. [S5]

4.3 Dimension

The ***order*** of a finite cover is the largest number of its members with a common point, so the nerve of a cover of order k has dimension $k-1$. A compact metric space has ***dimension at most*** n if it has covers by arbitrarily small closed sets of order at most $n+1$ [source, I, §3]; equivalently, if it can be mapped to n-dimensional nerves with arbitrarily small fibres.

Theorem 4.3 (Lebesgue; Brouwer)**.** [classical; [source, I, §10, Thms. 13–14]] (a) *A cover of an n-simplex by closed sets, none of which meets all the facets, has order at least $n+1$.* (b) *A polyhedron $|K|$ has dimension* $\dim K$*. In particular $\mathbb{R}^m$ and $\mathbb{R}^n$ are not homeomorphic for $m \neq n$.*

Idea. (a) Assign to each set of the cover a vertex of the simplex opposite a facet it misses; a fine Sperner labelling compatible with this assignment has a complete cell, whose vertices lie in $n+1$ different sets, and a limit point lies in all of them. (b) The stars of the vertices of a fine subdivision, slightly shrunk, give small covers of order $n+1$; by (a) there are none of smaller order. □



square grid: order 4 brick pattern: order 3

Figure 4.4. *Two covers of the plane by small closed squares. Four squares meet at each corner of the grid; shifting alternate rows leaves only triple points. By Theorem 4.3 no cover of the plane by small sets does better: the plane has dimension two.*

[S6] The theorem says that dimension is a property of the nerves that a space admits, and so a property of the space as a limit of polyhedra. The same idea gives the source's embedding theorem, that a compact metric space of dimension r embeds in $\mathbb{R}^{2r+1}$ [source, I, §3, Thm. 5]: approximate by an r-dimensional nerve, realize the nerve in general position by Theorem 1.3, and pass to the limit.

Notes

Sperner's lemma dates from 1928; the signed form is in [Bro90]. The source derives Brouwer's theorem and the invariance of dimension from it and returns to fixed points with homological tools in its last section; see Chapter 6. The equivalence with Hex is Gale's [Gal79]. For the complexity of finding a complete cell, and of computing equilibria of games by the same method, see [Pap94] and [GIHC]; consequences for learning systems are drawn in [ADTML].

Exercises

Exercise 4.1. Prove Sperner's lemma in dimension one directly, in its signed form.

Exercise 4.2. In dimension two, show that the number of complete cells has the parity of the number of boundary edges labelled 01, by counting pairs (cell, edge of the cell labelled 01) in two ways.

Exercise 4.3. Deduce Theorem 4.2*(b)* from *(a)*.

Exercise 4.4. A cover of the interval $[0,1]$ by closed intervals of length less than 1 has order at least two. Give a cover of order exactly two, and a cover of the square $[0,1]^2$ by sets of diameter less than $\frac{1}{2}$ of order three.

Chapter 5

Homotopy and the Cylinder

Two maps are homotopic if one can be deformed into the other. The third chapter of the source shows that homology cannot tell homotopic maps apart, by one construction: a triangulation of the cylinder $K \times [0,1]$ and a chain homotopy read off from it. This chapter records that construction, says what kind of structure a chain homotopy is, and ends with the map that homology cannot see at all.

5.1 The cylindric construction

Order the vertices of K and write v and v' for the bottom and top copies of a vertex in $K \times [0,1]$. By Theorem 1.5*(c)* the prism over a simplex $[v_0, \dots, v_r]$ is the union of the $r+1$ simplices $[v_0, \dots, v_i, v'_i, \dots, v'_r]$.

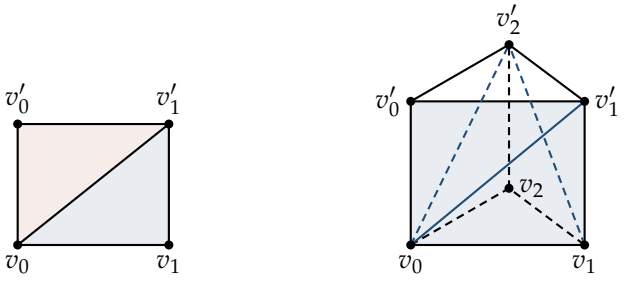


Figure 5.1. *The prisms over an edge and over a triangle. The three tetrahedra are* $[v_0, v'_0, v'_1, v'_2]$, $[v_0, v_1, v'_1, v'_2]$ *and* $[v_0, v_1, v_2, v'_2]$*; the added diagonals are in colour.*

Theorem 5.1 (Prism operator). [classical; [source, I, §14]] *Define* $P\colon C_r(K) \to C_{r+1}(K \times [0,1])$ *by* $P[v_0, \dots, v_r] = \sum_{i=0}^{r} (-1)^i [v_0, \dots, v_i, v'_i, \dots, v'_r]$. *Then*

$$\partial P + P\partial = i_1 - i_0,$$

where i_0, i_1 *are the inclusions of the bottom and the top.*

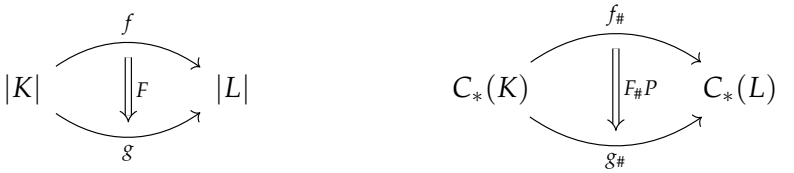
[P1] The cylinder on the projective plane has homology $\mathbb{Z}, \mathbb{Z}/2, 0, 0$, that of the projective plane, and the cylinder on the Möbius band has Betti numbers $(1,1,0,0)$. [P2]

Theorem 5.2 (Homotopy invariance). [classical; [source, I, §13–15, Thms. 20–24]] *Homotopic maps of polyhedra induce the same homomorphisms of homology groups. Homotopy equivalent polyhedra have isomorphic homology groups.*

Proof. A homotopy is a map $F\colon |K| \times [0,1] \to |L|$ with $Fi_0 = f$ and $Fi_1 = g$. Approximate it simplicially; then $F_\# P$ is a chain homotopy from $f_\#$ to $g_\#$, and on a cycle z the difference $g_\# z - f_\# z = \partial F_\# P z$ is a boundary. □

The Möbius band and its core circle both have Betti numbers $(1,1)$; a disc has those of a point. [P4] A combinatorial form: two simplicial maps are ***contiguous*** if the images of each simplex under the two together span a simplex. Any two maps $\operatorname{sd} K \to K$ that send each vertex σ to a vertex of σ are contiguous, and they induce the same map on homology; this was checked for twelve random pairs on four complexes. [P3]

5.2 What a chain homotopy is



A homotopy is a 2-cell between maps, and the theorem sends it to a 2-cell between chain maps. The construction continues: a homotopy between homotopies gives a map $C_r(K) \to C_{r+2}(L)$ with the analogous identity, and so on. Spaces with maps, homotopies and higher homotopies form an ∞-category, as do chain complexes with chain maps, chain homotopies and higher ones, and the assertion that all of these constructions are compatible is:

Theorem 5.3. [reformulation; cf. [Lur17]; Theorem 2.4] *Chains extend to a functor of ∞-categories $C_*\colon \mathcal{S} \to \mathcal{D}(\mathbb{Z})$, and homology is its composite with $H_*\colon \mathcal{D}(\mathbb{Z}) \to \mathsf{grAb}$.*

Theorem 5.2 is the shadow of this on homotopy categories. The prism operator itself is a product: $P\sigma$ is the product of σ with the fundamental chain of the interval, under the shuffle map $C_*(K) \otimes C_*(L) \to C_*(K \times L)$ of Eilenberg and Zilber [EiZi53]. The identity of Theorem 5.1 is the Leibniz rule for that product, since the boundary of the interval is top minus bottom.

5.3 Degree

For a map $f\colon S^n \to S^n$ the homomorphism f_* on $H_n(S^n) = \mathbb{Z}$ is multiplication by an integer, the ***degree*** [source, I, §15]. By Theorem 5.2 it is a homotopy invariant.

Simplicially it is the signed number of cells mapped onto a given cell, which is how the degree of the labelling was counted in Theorem 4.1.

Proposition 5.4. [classical] (a) $\deg(g \circ f) = \deg g \cdot \deg f$. (b) *The antipodal map of S^n has degree $(-1)^{n+1}$.* (c) *The suspension of a map has the same degree.*

For (b), on the boundaries of the cross-polytopes of dimensions $2, 3, 4$ the antipodal map acts on top homology by $1, -1, 1$. [L6] Statement (c) is Proposition 2.6*(b)* applied to maps. The converse of homotopy invariance for spheres, that maps of the same degree are homotopic, is Hopf's theorem; it is not accessible to the methods of Part One and is proved in Chapter 9 with framed points.

5.4 What homology cannot see

Homology is a functor on the homotopy category, but it is far from faithful. The simplest witness is the ***Hopf map*** $h\colon S^3 \to S^2$, $(z_1, z_2) \mapsto z_1/z_2$. Since S^3 and S^2 have no homology in a common positive degree, h_* is the same as the map induced by a constant, in every degree and with every coefficient group. Yet h is not homotopic to a constant (Theorem 9.6). The invariant that detects it is not a property of the homomorphisms h_* but of how the chains fit together: the preimages of two points are two circles in S^3, and they are linked.

In the language of Theorem 5.3: the functor C_* is faithful on neither objects nor morphisms, and what it forgets is measured by replacing the coefficient ring $\mathbb{Z}$ with a larger one through which every such functor factors. That ring is the subject of Part Two.

Notes

The source's cylindric construction is the classical prism operator; the Eilenberg–Zilber theorem [EiZi53] is its general form. That chain complexes form an ∞-category whose 2-cells are chain homotopies is standard; [Lur17] is the reference for the statement that C_* is a functor of ∞-categories and for the universal property of Theorem 2.4.

Exercises

Exercise 5.1. Verify $\partial P + P\partial = i_1 - i_0$ on an edge $[v_0, v_1]$.

Exercise 5.2. Show that contiguous simplicial maps have homotopic realizations.

Exercise 5.3. Show that a map $S^n \to S^n$ without fixed points has degree $(-1)^{n+1}$, by exhibiting a homotopy to the antipodal map.

Exercise 5.4. Compute the map induced by the Hopf map on $H_r(\,\cdot\,; A)$ for every r and every abelian group A.

Chapter 6

Traces and Fixed Points

The first monograph ends with a theorem that turns the algebra of the preceding chapters into a statement about points: a map of a polyhedron whose induced homomorphisms have non-zero alternating trace has a fixed point [source, I, §16]. The alternating trace is a trace in the categorical sense, and its properties are those of any trace.

6.1 The Hopf trace formula

Theorem 6.1 (Hopf). [classical; [source, I, §16]] *Let φ be a chain map of a finite complex of finitely generated free abelian groups to itself. Then*

$$\sum_{r} (-1)^r \operatorname{tr}(\varphi \mid C_r) \;=\; \sum_{r} (-1)^r \operatorname{tr}(\varphi_* \mid H_r \otimes \mathbb{Q}).$$

Proof. The trace is additive on short exact sequences of modules with compatible endomorphisms. Apply this to $0 \to Z_r \to C_r \to B_{r-1} \to 0$ and $0 \to B_r \to Z_r \to H_r \to 0$ as in the proof of Theorem 2.5. $\square$

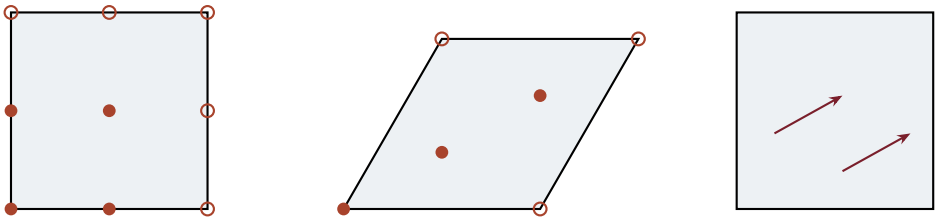
The common value for $\varphi = f_\#$ is the ***Lefschetz number*** $L(f)$. With φ the identity the formula is the Euler–Poincaré formula, so $L(\mathrm{id}) = \chi$. [L3] For the 42 automorphisms $x \mapsto ax + b$ of the seven-vertex torus of Figure 1.3 the two sides were computed independently and agree. [L1]

automorphism of the torus	order	fixed points	L
$x \mapsto x + b$, $b \neq 0$	7	none	0
$x \mapsto -x + b$	2	4	4
$x \mapsto 2x + b$, $4x + b$	3	3	3
$x \mapsto 3x + b$, $5x + b$	6	1	1

Theorem 6.2 (Lefschetz). [classical; [source, I, §16, Thm. 27]] *If $f\colon |K| \to |K|$ is continuous and $L(f) \neq 0$, then f has a fixed point.*

Proof. If f has no fixed point it moves every point by at least some $\epsilon > 0$. Subdivide until the mesh is small compared with ϵ and approximate f simplicially; then no simplex is mapped to itself, every diagonal entry of every matrix of $f_\#$ is zero, and $L(f) = 0$ by the left-hand side of Theorem 6.1. $\square$

The proof uses the chain side of the formula to see that the number vanishes and the homology side to see that it is an invariant. For a simplicial map the chain side counts the simplices mapped onto themselves, with signs: each of the automorphisms above with $L \neq 0$ fixes a simplex, and the six translations fix none. [L2]



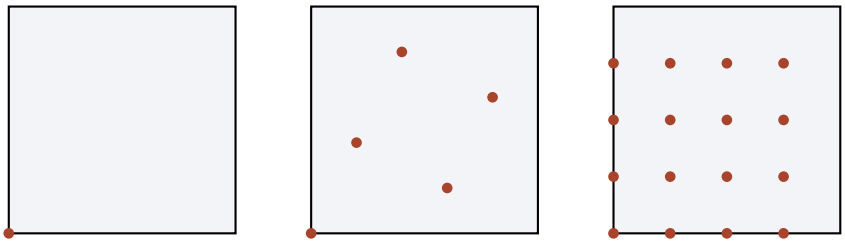
$x \mapsto -x$: $L = 4$ | a third of a turn: $L = 3$ | translation: $L = 0$

Figure 6.1. *Fixed points on a torus, drawn as a parallelogram with opposite sides identified; hollow marks are repetitions of a point already shown. A half turn fixes four points, a third of a turn on the hexagonal torus fixes three, a translation none.*

6.2 Examples

Proposition 6.3. [classical] (a) *For $f\colon S^n \to S^n$, $L(f) = 1 + (-1)^n \deg f$.* (b) *For the map of the torus $\mathbb{R}^2/\mathbb{Z}^2$ induced by an integer matrix A, $L = \det(I - A)$, and if this is not zero the number of fixed points is its absolute value.* (c) *Every self-map of a polyhedron with the rational homology of a point has $L = 1$.* (d) *$L(f \times g) = L(f)L(g)$.*

(a) The antipodal map has $L = 0$ and no fixed point, consistently with Proposition 5.4. [L6] (b) was checked on forty random matrices by counting the solutions of $Ax = x$ on the torus. [L4] (c) covers the simplex, which gives Brouwer's theorem again, and also the projective plane, whose rational homology vanishes in positive degrees although its integral homology does not: all sixty automorphisms of the six-vertex projective plane of Figure 2.2 have $L = 1$ and fix a point. [L8] (d) [L9]



$n = 1$: 1 point | $n = 2$: 5 points | $n = 3$: 16 points

Figure 6.2. *Points of the torus, drawn as a square with opposite sides identified, fixed by the n-th iterate of the cat map. The corner is one point. Their number is the absolute value of a Lefschetz number.*

Applied to the iterates of a map, (b) counts periodic points. For the map of the torus given by $A = \left(\begin{smallmatrix} 2 & 1 \\ 1 & 1 \end{smallmatrix}\right)$ the number of points with $A^n x = x$ is $|\det(I - A^n)| = \operatorname{tr} A^n - 2$, that is $1, 5, 16, 45, 121, 320, 841, 2205$ for $n = 1, \dots, 8$. [L5] The generating function is rational,

$$\exp \sum_{n \geq 1} \frac{N_n}{n} t^n = \frac{(1-t)^2}{1 - 3t + t^2},$$

because each N_n is an alternating sum of traces of powers: the homology of the torus, with the action of A, is a complete invariant of the counting problem.

6.3 The trace in a symmetric monoidal category

Definition 6.4. An object X of a symmetric monoidal category with unit $\mathbf{1}$ is ***dualizable*** if there are an object $X^\vee$ and maps $\eta \colon \mathbf{1} \to X \otimes X^\vee$, $\epsilon \colon X^\vee \otimes X \to \mathbf{1}$ satisfying the triangle identities. The ***trace*** of $f \colon X \to X$ is the composite

$$\mathbf{1} \xrightarrow{\eta} X \otimes X^\vee \xrightarrow{f \otimes \mathrm{id}} X \otimes X^\vee \xrightarrow{\simeq} X^\vee \otimes X \xrightarrow{\epsilon} \mathbf{1},$$

an element of $\operatorname{End}(\mathbf{1})$.

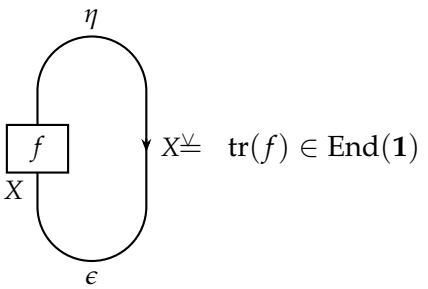


Figure 6.3. *The trace as a closed loop: create a pair, apply f to one strand, annihilate. For a matrix this is the sum of the diagonal entries; for a chain complex it is the alternating sum, the signs coming from the symmetry.*

Theorem 6.5 (Dold–Puppe). [literature; [DoPu80, PoSh14]] (a) *In $\mathcal{D}(\mathbb{Z})$ the dualizable objects are the perfect complexes, and the trace of an endomorphism of one is the alternating sum of Theorem 6.1.* (b) *A symmetric monoidal functor preserves dualizable objects and traces.* (c) *For a finite complex K the suspension spectrum $\Sigma^\infty_+ K$ is dualizable in spectra, and the trace of $\Sigma^\infty_+ f$ in $\pi_0 \mathbb{S} = \mathbb{Z}$ is the fixed point index of f.* (d) *Traces are additive on cofibre sequences [Mayo1].*

Read together: the fixed point index is a trace in spectra by (c); chains are a symmetric monoidal functor from spectra to $\mathcal{D}(\mathbb{Z})$ by Theorem 2.7; so by (b) the index equals the trace of $f_\#$, which by (a) is the Lefschetz number. That

is the Lefschetz–Hopf theorem with the proof sorted into its functorial parts. The step from (a) to the fixed points, which in the source is the argument with small simplices, is here the computation of a trace in a category where the unit has endomorphisms $\pi_0 S$. Part Two is about the rest of the homotopy of that unit.

Additivity (d) is the trace form of $\chi(A \cup B) = \chi(A) + \chi(B) - \chi(A \cap B)$. For a simplicial map preserving a subcomplex A, $L(f) = L(f|_A) + L(f \text{ on } K/A)$. The rotation of the icosahedron by a fifth of a turn about an axis through two vertices has $L = 2$: the two poles contribute 1 each and the rest contributes 0, as no other simplex is preserved. [L7]

Notes

The source proves Theorem 6.2 by the argument given and derives the theorems of Brouwer and of Euler–Poincaré–Hopf from it. The categorical trace is due to Dold and Puppe [DoPu80]; for its generalizations and for the role of additivity see [May01, PoSh14]. The zeta function of the cat map is the simplest case of a general rationality statement for maps whose periodic points are counted by Lefschetz numbers.

Exercises

Exercise 6.1. Prove that the trace is additive: if φ preserves a submodule $M' \subseteq M$ of a free module with free quotient, then $\operatorname{tr} \varphi = \operatorname{tr}(\varphi|_{M'}) + \operatorname{tr}(\varphi \text{ on } M/M')$.

Exercise 6.2. Find the four fixed points of $x \mapsto -x$ on the torus $\mathbb{R}^2/\mathbb{Z}^2$, and compute L from the action on homology.

Exercise 6.3. Show that $N_n = \operatorname{tr} A^n - 2$ for the cat map, and derive the generating function.

Exercise 6.4. Show that every continuous self-map of the real projective space of even dimension has a fixed point, and give a fixed-point-free map in every odd dimension.

Part III

Framed Manifolds

Chapter 7

General Position

The second monograph opens with a chapter on smooth manifolds that is, by the author's account, more detailed than its applications require [source, II, Introduction]. Its three results are an embedding theorem, a theorem on critical values, and a description of the singularities that a smooth map cannot avoid. All three are instances of one count of dimensions, and that count is the engine of everything that follows. The categorical setting for smooth manifolds was laid out in an earlier volume of this series [DTHC]; here we recall only what the main method needs.

7.1 The count

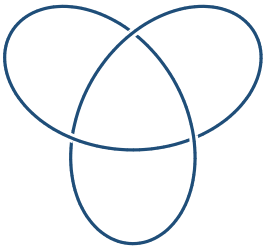
Two submanifolds of dimensions a and b of an N-dimensional manifold are ***in general position*** if at each common point their tangent spaces span the whole tangent space; they then meet in a manifold of dimension $a + b - N$, which is empty if this number is negative. A family of conditions of codimension c imposed on a k-parameter family is, generically, satisfied on a set of dimension $k - c$.

target of M^k	condition	codimension	generic picture
$\mathbb{R}^N$, $N \geq 2k+1$	two points with the same image	N, in dimension $2k$	embedding
$\mathbb{R}^{2k}$	the same	$2k$	isolated double points
$\mathbb{R}^{2k-1}$	rank less than k	k	isolated cross-caps
$\mathbb{R}$	derivative zero	k	isolated critical points

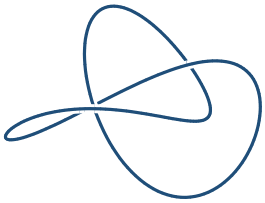
Theorem 7.1 (Whitney; Pontryagin). [classical; [source, II, §2, Thm. 2; §4]] *A compact smooth manifold of dimension k embeds in $\mathbb{R}^{2k+1}$ and immerses in $\mathbb{R}^{2k}$ with transverse double points. A generic map to $\mathbb{R}^{2k-1}$ is an immersion except at isolated points, near which it has the form of the* Whitney umbrella*; a generic function has only non-degenerate critical points.*

The source proves the first statement by starting from an embedding in a large Euclidean space and projecting along a direction chosen outside a set of

measure zero, one dimension at a time; the directions to avoid are the chords and tangents of the manifold, which form a set of dimension at most $2k+1$. The number $2k+1$ is that of Theorem 1.3, for the same reason. A curve in space shows the next two rows of the table.



from above: 3 double points



another direction

Figure 7.1. *Two projections of a trefoil knot to the plane. A generic projection of a curve has finitely many transverse double points, drawn with the nearer strand unbroken; a curve in the plane, unlike a curve in space, cannot be freed of them by a small perturbation.*

The trefoil has three double points when projected along its axis and between three and six along twelve random directions; the number is finite for almost every direction and cannot be made zero. [M1] For surfaces in $\mathbb{R}^3$ the unavoidable singularity is the map $(x, y) \mapsto (x, xy, y^2)$: its differential has rank two except at the origin, and it identifies $(0, y)$ with $(0, -y)$, so that a line of double points ends at the singular point. [M3]

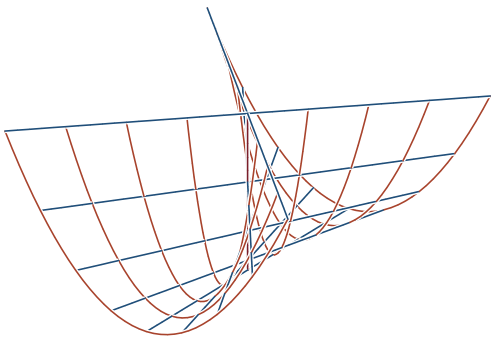


Figure 7.2. *The Whitney umbrella, the image of the square $|x|, |y| \le 1$ under $(x, y) \mapsto (x, xy, y^2)$, drawn by the images of the coordinate lines. The two sheets cross along a segment, in the third colour, which ends at the one point where the map is not an immersion.*

7.2 Sard's theorem

Theorem 7.2 (Sard). [classical; [source, II, §3]] *The set of critical values of a smooth map has measure zero. In particular, for a smooth map $f\colon M \to N$ almost every point p of N is a* regular value, *and then $f^{-1}(p)$ is a submanifold of dimension* $\dim M - \dim N$.

The mechanism is that near a critical point a map compresses more than linearly. For $f(x) = x^3 - 3x$ the set where $|f'| < \delta$ has length of order δ and its image has length $\delta^2/6 + O(\delta^3)$: the ratio to δ^2 is 0.16855, 0.16685, 0.16669 for $\delta = 10^{-1}, 10^{-2}, 10^{-3}$. [M2] Sard's theorem is what makes the next chapter possible: it is the supply of regular values.

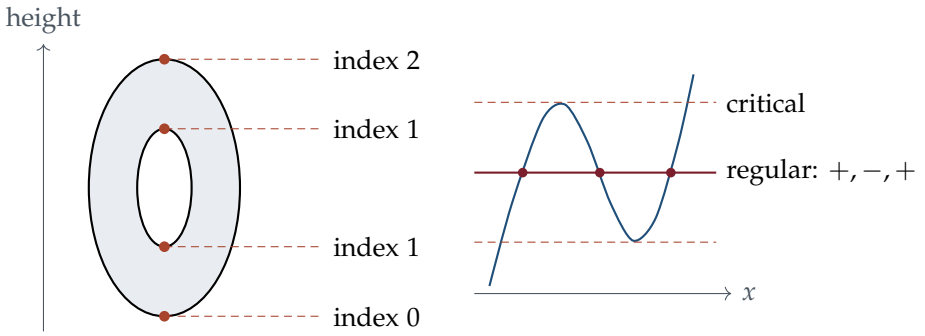


Figure 7.3. *Left: the height function on an upright torus has four non-degenerate critical points, of index* $0, 1, 1, 2$. *Right: over a regular value the preimages are isolated and carry signs; their signed number, here* 1, *does not change until a critical value is crossed, and then it changes by a cancelling pair.*

The alternating sum $1 - 2 + 1$ of the numbers of critical points is the Euler characteristic of the torus. [M4] For a rational map of degree three of the Riemann sphere the number of preimages was 3 over each of three values. [M5]

7.3 Why general position is categorical

Transversality is the condition under which a pullback of manifolds exists and is a manifold of the expected dimension [DTHC, Ch. 4]. The preimage of a regular value is the pullback of f along the inclusion of a point; the intersection of two submanifolds is the pullback of their inclusions. General position theorems say that the maps for which these pullbacks exist are dense, and, in their relative form, that a path between two such maps can itself be put in general position. The main method needs three such statements:

1. every map of spheres is homotopic to a smooth map with a given point as regular value;
2. every homotopy between two such maps can be made smooth with the same point as regular value;
3. a k-dimensional submanifold of $\mathbb{R}^N$ with $N \geq 2k + 2$ is unknotted: any two embeddings are isotopic.

The third is the first row of the table applied to an isotopy, which has one more dimension. It is the reason that homotopy groups of spheres stabilize: once the codimension exceeds the dimension by two, there is no room for the ambient space to matter.

Notes

For the theory behind the table, jets and the Thom transversality theorem, and for the derived point of view in which non-transverse pullbacks also exist, see [DTHC] and the references there. The source's proof of the embedding theorem by projection is close to Whitney's original one; its section on typical singularities contains the normal forms for double points, critical points and cross-caps.

Exercises

Exercise 7.1. Show that the set of directions of chords of a compact k-manifold in $\mathbb{R}^N$ is the image of a manifold of dimension $2k$, and deduce the dimension in which a generic projection is injective.

Exercise 7.2. For the Whitney umbrella $F(x, y) = (x, xy, y^2)$, find all pairs of distinct points with the same image and the points where $\mathrm{d}F$ has rank less than two.

Exercise 7.3. Show that for $f(x) = x^3 - 3x$ the image of $\{|f'| < \delta\}$ has length $\delta^2/6$ to leading order.

Exercise 7.4. Show that the matrices of rank r form a submanifold of codimension $(k - r)(N - r)$ in the $k \times N$ matrices, and recover the third row of the table.

Chapter 8

Framed Manifolds and the Main Method

The problem of the second monograph is the classification of maps $S^{n+k} \to S^n$ up to homotopy. Its method, which the source calls simply the main method [source, II, §6], replaces a map by the preimage of a point, a k-dimensional manifold, together with one extra datum. The classification of maps becomes a classification of manifolds, and the homotopy groups of spheres become the bordism groups of framed manifolds.

8.1 The construction

Definition 8.1. A ***framed submanifold*** of $\mathbb{R}^{n+k}$ is a closed smooth submanifold M^k with a *frame* $U = (u_1, \dots, u_n)$: at each point n linearly independent vectors normal to M, depending smoothly on the point. Two framed submanifolds are ***framed cobordant*** if there is a compact framed submanifold W^{k+1} of $\mathbb{R}^{n+k} \times [0,1]$, meeting the two ends orthogonally, in the given framed manifolds.

The source says "homologous" for cobordant and writes Π_n^k for the set of classes [source, II, §7]; we write $\Omega_k^{\mathrm{fr}}(\mathbb{R}^{n+k})$.

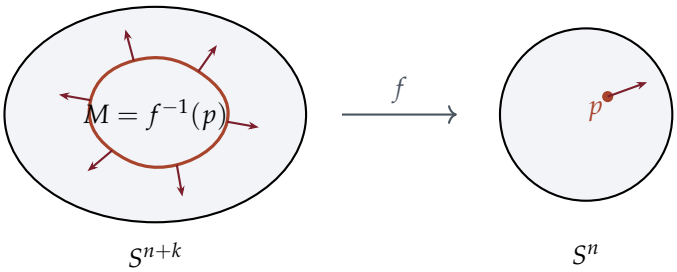


Figure 8.1. *From a map to a framed manifold. The preimage of a regular value p is a submanifold, and the differential of f identifies each normal space of M with the tangent space at p; a basis of that tangent space pulls back to a frame.*

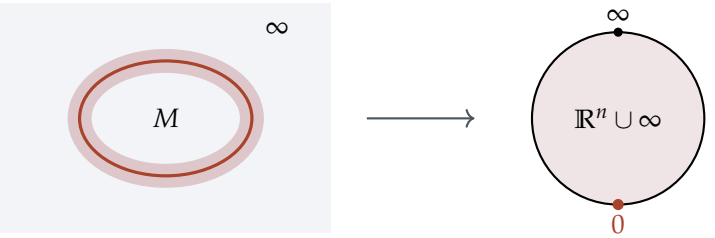


Figure 8.2. *From a framed manifold to a map. The frame identifies a tubular neighbourhood of M with $M \times \mathbb{R}^n$; project it to $\mathbb{R}^n$, and send everything outside the tube to the point at infinity of $S^n = \mathbb{R}^n \cup \infty$. The core M goes to the origin.*

Theorem 8.2 (Pontryagin). [classical; [source, II, §6, Thm. 10]] *The two constructions are inverse to each other up to homotopy and framed cobordism:*

$$\pi_{n+k}(S^n) \;\cong\; \Omega_k^{\mathrm{fr}}(\mathbb{R}^{n+k}).$$

The sum of homotopy classes corresponds to the disjoint union of framed manifolds placed in two half-spaces.

Structure of the proof. By the three statements of Chapter 7: a map can be made smooth with p regular, which gives the framed manifold; a homotopy can be made smooth with p regular, and its preimage of p is a framed cobordism; and a map is determined up to homotopy by its behaviour near the preimage of a point, since the complement of a point of S^n is contractible. □

For $k = 0$ a framed manifold is a finite set of points with a frame at each, and up to deformation a frame is a sign. The signed number of preimages of a regular value of a circle map with a folded graph was a for maps of degree $a = -2, \dots, 3$. [F1] In the other direction, the map built from framed points in the plane, with frames of signs $(+)$, $(-)$ and $(+,+,-)$, has degree 1, -1 and 1, computed by integrating the pulled-back area form. [F3]

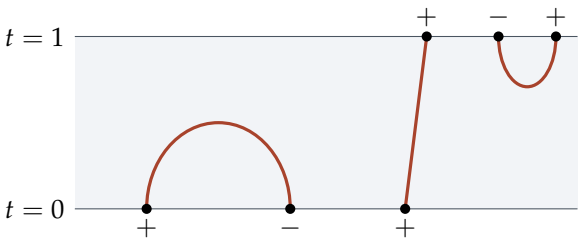


Figure 8.3. *A framed cobordism between framed 0-manifolds: arcs join points of opposite sign at the same end or points of the same sign at opposite ends. The signed count, 1 at both ends, is the only invariant: this is $\pi_n(S^n) = \mathbb{Z}$.*

8.2 Suspension and stability

A framed submanifold of $\mathbb{R}^{n+k}$ is a framed submanifold of $\mathbb{R}^{n+k+1}$ with one more frame vector, the new coordinate direction. On maps this is the suspension.

Theorem 8.3 (Freudenthal, in Pontryagin's form)**.** [classical; [source, II, §8, Thm. 11]] *The suspension $E\colon \Omega^{\mathrm{fr}}_k(\mathbb{R}^{n+k}) \to \Omega^{\mathrm{fr}}_k(\mathbb{R}^{n+k+1})$ is onto for $n \geq k+1$ and an isomorphism for $n \geq k+2$.*

The suspension of $z \mapsto z^d$ has degree d as a map of the 2-sphere, for $d = 1, 2, 3, -2$. [F2] The proof is general position: in the stated ranges a framed manifold, respectively a framed cobordism, can be pushed into a hyperplane and its frame straightened so that the last vector is the new coordinate direction. The common value for large n is the ***stable stem*** π^{s}_k, the group Ω^{fr}_k of framed k-manifolds with no reference to an ambient space: by the third statement of Chapter 7 the embedding is unique, and a frame becomes a trivialization of the stable normal bundle.

8.3 What the theorem says about spaces

Theorem 8.2 is a statement about sets of components. The spaces behind it are the mapping spaces $\Omega^{n+k}S^n$, and the construction applies to families.

Theorem 8.4. [literature; [Seg73, McD75, BaPr72]] (a) *$\Omega^n S^n$ is the group completion of the space of finite subsets of $\mathbb{R}^n$: a point of it is a configuration of points with signs, in which a positive and a negative point may collide and annihilate.* (b) *As $n \to \infty$ this becomes the group completion of $\coprod_m B\Sigma_m$, the classifying space of finite sets and bijections.*

A finite set is what one adds to form the natural numbers, and the natural numbers complete to the integers. Theorem 8.4*(b)* says that if finite sets are kept together with their bijections and the completion is done up to coherent homotopy, the result is not $\mathbb{Z}$ but a space $\Omega^\infty\mathbb{S}$ whose components are $\mathbb{Z}$ and whose higher homotopy groups are the stable stems:

Theorem 8.5 (Framed bordism is the homology of the sphere)**.** [reformulation; cf. [BaPr72, Lur17]] *The infinite loop space $\Omega^\infty\mathbb{S}$ of the **sphere spectrum** is the free grouplike commutative monoid in spaces on one generator. The homotopy groups of $\mathbb{S}$ are the stable stems, and for a space X the groups $\pi_k(\Sigma^\infty_+ X)$ are the bordism groups of framed k-manifolds with a map to X.*

Compare Theorem 2.4. The integers are the free abelian group on a point and chains are the free $\mathbb{Z}$-module on a space. The sphere spectrum is the

free "abelian group up to coherent homotopy" on a point, and $\Sigma^\infty_+ X$ is the free $\mathbb{S}$-module on a space. Its cycles are framed manifolds, as the cycles of Chapter 2 were simplicial chains. Pontryagin's theorem is the case $X = \mathrm{pt}$: the coefficient groups of this homology theory, unlike those of ordinary homology, are not concentrated in degree zero.

The first homotopy group can be read off Theorem 8.4*(b)* directly: the fundamental group of the group completion of $\coprod B\Sigma_m$ is the abelianization of the symmetric groups, which has order two, the sign. [T1] Geometrically a loop of configurations is a braid of world-lines, and a transposition is a full twist of a frame. This is $\pi_1^{\mathrm{s}} = \mathbb{Z}/2$, proved by the source in another way (Chapter 10).

Remark 8.6 (The cobordism hypothesis). [literature; [BaDo95, Lur09, GrPa21]] Framed manifolds with boundaries and corners, in dimensions up to n, form a symmetric monoidal (∞, n)-category, and the cobordism hypothesis asserts that it is the free one with duals on a single object. In dimension one the object is a positively framed point, its dual the negative one, the arcs of Figure 8.3 are the unit and counit of the duality, and the closed loop of Figure 6.3 is a framed circle: the trace of the identity is the value of the theory on a circle. Passing to invertible theories recovers Theorem 8.5. A complete proof, in a more general geometric form, is due to Grady and Pavlov.

Notes

The method was published in 1938 and 1950 [source, II, refs. 1, 2] and extended by Thom to unframed manifolds, with the sphere replaced by the Thom space of a universal bundle; that general form, and its place among the functors of differential topology, is treated in [DTHC]. Theorem 8.4 is due to Segal and McDuff in finite dimensions and to Barratt, Priddy and Quillen in the limit.

Exercises

Exercise 8.1. Show that the space of frames at a point of $\mathbb{R}^n$ has two components, and that the Pontryagin–Thom map of a single framed point has degree ± 1 accordingly.

Exercise 8.2. Show geometrically that the sum on $\Omega^{\mathrm{fr}}_k(\mathbb{R}^{n+k})$ is commutative for $n \geq 2$.

Exercise 8.3. Describe the inverse of a framed manifold (M, U) in $\Omega^{\mathrm{fr}}_k(\mathbb{R}^{n+k})$ and a framed cobordism from the sum to the empty manifold.

Exercise 8.4. For which n does Theorem 8.3 give the stable value of $\pi_{n+1}(S^n)$ and of $\pi_{n+2}(S^n)$?

Chapter 9

Degree, Linking and the Hopf Invariant

With the main method in hand the source classifies maps in the two lowest cases where a complete answer is an integer: maps to a sphere of the same dimension, by the degree, and maps $S^3 \to S^2$, by an invariant of Hopf [source, II, §9–10, §13]. Both are numbers attached to framed manifolds of dimension zero and one.

9.1 Framed points: Hopf's theorem

Theorem 9.1 (Hopf). [classical; [source, II, §9, Thms. 12–13]] *Two maps of a connected closed oriented n-manifold to S^n are homotopic if and only if they have the same degree, and every integer occurs. In particular $\pi_n(S^n) \cong \mathbb{Z}$.*

Proof. By Theorem 8.2 a map is a finite set of signed points, and two points of opposite sign can be joined by an arc in the manifold and cancelled, as in Figure 8.3; what remains is $|d|$ points of one sign. Two such configurations with the same d are related by moving points, which is a cobordism. □

This is the converse to the homotopy invariance of the degree that Chapter 5 could not prove. The degree of Part One, defined through H_n, and the degree of Part Two, a signed count of framed points, are the same number: the comparison map from framed bordism to homology (Chapter 12) is an isomorphism in dimension zero.

9.2 Linking

Two disjoint closed oriented curves A, B in $\mathbb{R}^3$ have a ***linking number***, which the source calls the looping coefficient [source, II, §10]: the signed number of intersections of A with a surface bounded by B.



Figure 9.1. *The linking number counts, with signs, the points where one curve pierces a surface spanning the other.*

Proposition 9.2 (Gauss). [classical] *The linking number is the degree of the map $A \times B \to S^2$, $(a,b) \mapsto (a-b)/|a-b|$, and so equals*

$$\operatorname{lk}(A,B) = \frac{1}{4\pi} \oint_A \oint_B \frac{(a-b)\cdot(\mathrm{d}a \times \mathrm{d}b)}{|a-b|^3}.$$

It is symmetric, and invariant under deformations that keep the curves disjoint.

The integral gives 1 for the Hopf link, 0 for two distant circles and 2 for the two components of the $(2,4)$ torus link, to four or more digits. [K1] For curves of dimension k in $\mathbb{R}^{2k+1}$ the same definitions apply, with $\operatorname{lk}(A,B) = (-1)^{k+1}\operatorname{lk}(B,A)$.

9.3 The Hopf invariant

Definition 9.3. Let $f\colon S^{2k+1} \to S^{k+1}$ be smooth and p, q two regular values. The ***Hopf invariant*** is $\gamma(f) = \operatorname{lk}(f^{-1}(p), f^{-1}(q))$. For a framed submanifold (M^k, U) of $\mathbb{R}^{2k+1}$ it is the linking number of M with a copy of itself pushed off along the frame.

Theorem 9.4. [classical; [source, II, §10]] (a) *γ is well defined and is a homomorphism $\pi_{2k+1}(S^{k+1}) \to \mathbb{Z}$.* (b) *$\gamma = 0$ if k is even.* (c) *For $g\colon S^{k+1} \to S^{k+1}$ of degree d, $\gamma(g \circ f) = d^2\gamma(f)$; for $e\colon S^{2k+1} \to S^{2k+1}$ of degree d, $\gamma(f \circ e) = d\,\gamma(f)$.*

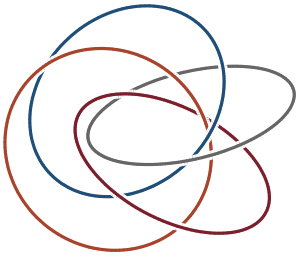


Figure 9.2. *Four fibres of the Hopf map, the preimages of four points on a circle of latitude of S^2, shown in $\mathbb{R}^3$ after stereographic projection. Each is a round circle, they lie on a torus, and each pair is linked exactly once.*

For the Hopf map $h(z_1, z_2) = z_1/z_2$ the preimage of a point is a great circle of S^3, and any two of them link once: $\gamma(h) = 1$. [K2, K5] For $z^2 \circ h$ and $z^3 \circ h$ the preimages of two points are two and three circles each, and the total linking is 4 and 9; for $h \circ e$ with $e(z_1, z_2) = (z_1^d, z_2)$ normalized, the preimages are torus knots linking d times, for $d = 1, 2, 3$. [K2, K3] The asymmetry in (c), quadratic on one side and linear on the other, is the sign that γ is not an invariant of the induced maps on homology. It is the simplest invariant that is quadratic in the chains: if the cofibre of f has cells in dimensions $k+1$ and $2k+2$ with dual classes a and b, then $a^2 = \gamma(f)\, b$.

Remark 9.5 (What sees it). [literature; [Man06]] The cup product is part of the structure of cochains as a commutative algebra up to coherent homotopy. The chain complex of Chapter 2 forgets that structure, and with it the Hopf invariant; the cochain algebra with it remembers, for simply connected spaces of finite type, the whole homotopy type.

9.4 Framed circles and $\pi_3(S^2)$

For $k = 1$ the framed manifolds are framed links in $\mathbb{R}^3$. A frame on a single unknotted circle is determined up to deformation by the number t of turns it makes about the circle, and $\gamma = t$: the linking number of the circle with its push-off is $0, 1, 2, -3$ for $t = 0, 1, 2, -3$. [K4]

Theorem 9.6. [classical; [source, II, §13, Thm. 19]] *The Hopf invariant is an isomorphism $\pi_3(S^2) \cong \mathbb{Z}$, and the Hopf map is a generator.*

Structure of the proof. The Hopf map is a fibre bundle with fibre a circle, so homotopies can be lifted [source, II, §12]: every map $S^3 \to S^2$ is homotopic to $h \circ e$ for some $e\colon S^3 \to S^3$, and e is determined by its degree (Theorem 9.1). By Theorem 9.4*(c)*, $\gamma(h \circ e) = \deg e$. □

In terms of framed links: every framed link in $\mathbb{R}^3$ is framed cobordant to a single unknotted circle, and the number of turns of its frame is the sum of all self-linkings and all pairwise linkings, each pair counted in both orders.

Remark 9.7 (Hopf invariant one). [classical; [Ada60]] The complex numbers give a map $S^3 \to S^2$ of Hopf invariant one; the quaternions and octonions give maps $S^7 \to S^4$ and $S^{15} \to S^8$. By a theorem of Adams there are no others: γ takes the value 1 on $\pi_{2k+1}(S^{k+1})$ only for $k = 0, 1, 3, 7$. For other odd k its image is the even integers.

Notes

The source defines the invariant for framed manifolds and proves its properties geometrically, including the vanishing in the range where the suspension theorem applies [source, II, §11]. The integral formula for the Hopf invariant, $\int \alpha \wedge \mathrm{d}\alpha$ with $\mathrm{d}\alpha$ the pullback of the area form, is Whitehead's; it is the continuous form of the count in Definition 9.3.

Exercises

Exercise 9.1. Compute the linking number of the Hopf link from Figure 9.1 by counting the crossings of one curve over the other in the diagram, and state the rule.

Exercise 9.2. Show that the preimages under h of the points 0 and ∞ are the circles $z_1 = 0$ and $z_2 = 0$ of S^3, and that under stereographic projection they become a circle and a line through its centre.

Exercise 9.3. Explain the factor d^2 in Theorem 9.4*(c)* for $g(z) = z^d$ by counting pairs of fibres.

Exercise 9.4. Show that $\gamma = 0$ when k is even, using the symmetry of the linking number.

Part IV

The First Stems

Chapter 10

Rotations and the First Stem

For $k = 1$ and $n \geq 3$ a framed manifold is a collection of circles in a space of dimension at least four, where nothing is knotted or linked. All that is left is the frame, and a frame along a circle is a loop of rotations. The source therefore pauses to compute the fundamental group of the rotation group, with quaternions [source, II, §12], and then reads off $\pi_{n+1}(S^n)$ [source, II, §14].

10.1 Quaternions and rotations

Theorem 10.1. [classical; [source, II, §12]] *For a unit quaternion q the map $v \mapsto qvq^{-1}$ is a rotation of the space of imaginary quaternions. This is a homomorphism $S^3 \to \mathrm{SO}(3)$, onto, with kernel ± 1. Hence $\mathrm{SO}(3)$ is the real projective space $\mathbb{RP}^3$, and $\pi_1(\mathrm{SO}(3)) \cong \mathbb{Z}/2$.*

Associativity, multiplicativity of the norm, and the properties of the homomorphism were checked on random rational quaternions, exactly. [R1, R2] The cellular chain complex of $\mathbb{RP}^3$ is $\mathbb{Z} \xleftarrow{0} \mathbb{Z} \xleftarrow{2} \mathbb{Z} \xleftarrow{0} \mathbb{Z}$, with homology $\mathbb{Z}, \mathbb{Z}/2, 0, \mathbb{Z}$. [R4]

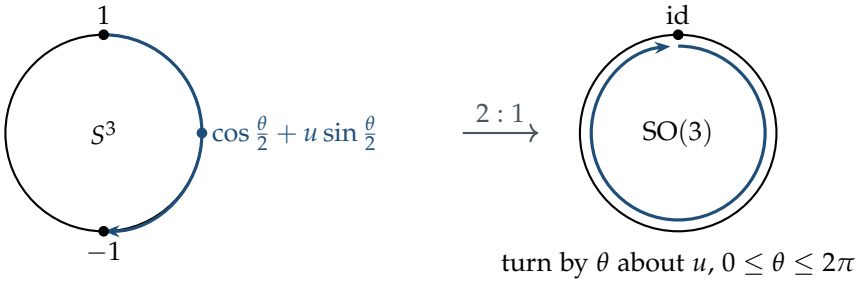
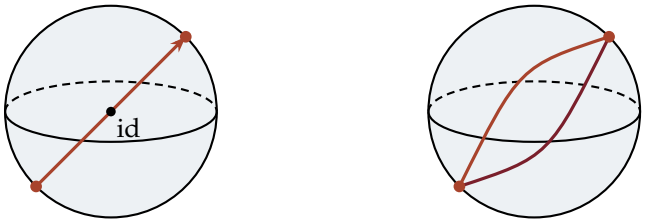


Figure 10.1. *A full turn about an axis is a closed loop of rotations. Its lift to the unit quaternions runs half way round a great circle, from 1 to −1, and is not closed; the lift of two full turns is.*

Numerically, following the quaternion continuously along a loop of rotation matrices, the product of a turn of $a \cdot 2\pi$ about one axis and $b \cdot 2\pi$ about another returns to its start when $a + b$ is even and to the opposite quaternion when it is odd. [R3]



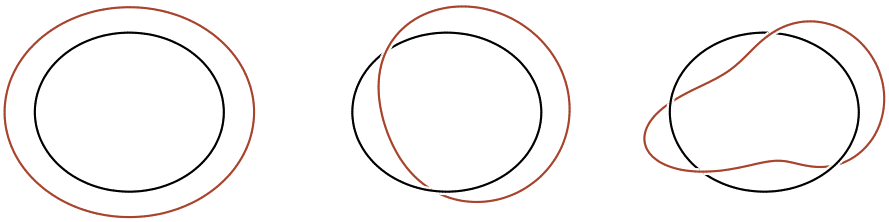
a diameter is a closed loop twice the loop bounds a disc

Figure 10.2. $\mathrm{SO}(3)$ *as a ball of radius* π*: the point at distance* θ *from the centre in the direction* u *is the turn by* θ *about* u*, and antipodal points of the boundary are the same rotation. A diameter is a loop that cannot be contracted; the loop traversed twice can be moved off the boundary and contracted.*

Proposition 10.2. [classical; [source, II, §12, Thm. 17]] *The map* $\mathrm{SO}(n+1) \to S^n$ *sending a rotation to the image of a fixed vector is a fibre bundle with fibre* $\mathrm{SO}(n)$*, and has the covering homotopy property. Consequently* $\pi_1(\mathrm{SO}(n)) \cong \mathbb{Z}/2$ *for all* $n \geq 3$*, while* $\pi_1(\mathrm{SO}(2)) \cong \mathbb{Z}$*.*

10.2 Framed circles

A frame of the normal bundle of a circle in $\mathbb{R}^{n+1}$, compared with a reference frame, is a loop in $\mathrm{SO}(n)$. For $n = 2$ its class is the integer of Theorem 9.6. For $n \geq 3$ only its parity survives: the frame that turns t times in one normal plane has class t mod 2. [R6]



$t = 0$: class 0 $t = 1$: class 1 $t = 2$: class 0

Figure 10.3. *A circle (black) and its push-off along the first vector of a frame that turns* t *times (colour). In* $\mathbb{R}^3$ *the two curves link* t *times and* t *is an invariant. In* $\mathbb{R}^4$ *and beyond the curves can be pulled apart, and only the parity of* t *remains, as the class of the frame in* $\pi_1(\mathrm{SO}(n))$*.*

Theorem 10.3. [classical; [source, II, §14, Thm. 21]] *For* $n \geq 3$*,* $\pi_{n+1}(S^n) \cong \mathbb{Z}/2$*. The class of a framed* 1*-manifold is the number of its components whose frame is not the one induced from a spanning disc, modulo two.*

Structure of the proof. A circle in $\mathbb{R}^{n+1}$ bounds an embedded disc, unique up to isotopy. The disc, pushed into $\mathbb{R}^{n+1} \times [0,1]$, is a null-cobordism of the circle with one particular frame; so a circle with that frame represents zero. There are two frames up to deformation by Proposition 10.2, the sum of two circles

with the other frame is a single circle with the first, by a saddle cobordism, and it remains to see that the other frame does not bound. That is detected by the invariant δ of the source [source, II, §14], which we describe next. □

There is a subtlety of bookkeeping that the source treats with care. If the frame is completed by the unit tangent vector to a frame of $\mathbb{R}^{n+1}$ along the circle, it becomes a loop in $\mathrm{SO}(n+1)$, and for the circle that bounds a disc this loop is the *non-trivial* one: the tangent vector turns once. The invariant is therefore the class of the completed loop plus one for each component, and with this normalization it is additive and vanishes on boundaries.

Remark 10.4 (The first stem is the Hopf invariant modulo two). [classical] The suspension $\pi_3(S^2) \to \pi_4(S^3)$ is onto by Theorem 8.3, and on framed circles it is the passage from t to t mod 2. So $\pi_4(S^3)$ is generated by the suspension η of the Hopf map, and $2\eta = 0$. The kernel is generated by the class of Hopf invariant two, the Whitehead square of the identity.

10.3 The J-homomorphism

The construction that turned a loop of rotations into a framed circle is general.

Definition 10.5. For a map $\alpha\colon S^k \to \mathrm{SO}(n)$ let $J(\alpha)$ be the class of the standard sphere $S^k \subset \mathbb{R}^{n+k}$ with its standard frame twisted by α. This is a homomorphism $J\colon \pi_k(\mathrm{SO}(n)) \to \pi_{n+k}(S^n)$, and stably $J\colon \pi_k(\mathrm{SO}) \to \pi_k^{\mathrm{s}}$.

Theorem 10.6. [literature; [Ada66, Qui71]] *The stable homotopy groups $\pi_k(\mathrm{SO})$ are $\mathbb{Z}/2, 0, \mathbb{Z}, 0, 0, 0, \mathbb{Z}$ for $k = 1, \dots, 7$ and repeat with period 8 thereafter (Bott). J is injective on the groups $\mathbb{Z}/2$, and its image in π^{s}_{4m-1} is cyclic of order the denominator of $B_{2m}/4m$, where B_{2m} is a Bernoulli number.*

Theorem 10.3 says that J is an isomorphism for $k = 1$. The orders in the second statement are $24, 240, 504, 480, 264, 65520, 24, 16320$ for $m = 1, \dots, 8$. [R5] The first two are the orders of the whole groups π_3^{s} and π_7^{s}. The image of J is the part of the stable stems that comes from framing a sphere in a non-standard way; the rest is represented by framed manifolds that are not spheres, of which the next chapter has the first example.

Notes

The source computes π_1 of the rotation groups through quaternions and the covering homotopy theorem for the bundles $\mathrm{SO}(n+1) \to S^n$. The identification of the image of J is due to Adams, completed by Quillen's proof of the Adams conjecture. That π_1^{s} is the sign of a permutation was noted after

Theorem 8.5; the two descriptions agree because a transposition of two points in a high-dimensional space is realized by a rotation through a half turn, and the frame it leaves behind has made one full turn.

Exercises

Exercise 10.1. Show that $v \mapsto qvq^{-1}$ for $q = \cos\frac{\theta}{2} + u\sin\frac{\theta}{2}$ is the rotation by θ about the unit imaginary quaternion u.

Exercise 10.2. Deduce $\pi_1(\mathrm{SO}(n)) \cong \pi_1(\mathrm{SO}(3))$ for $n \geq 3$ from the bundles of Proposition 10.2.

Exercise 10.3. Describe a framed cobordism from two circles with the non-bounding frame to one circle with the bounding frame.

Exercise 10.4. Compute the denominators of $B_{2m}/4m$ for $m = 1, 2, 3$ from $B_2 = \frac{1}{6}$, $B_4 = -\frac{1}{30}$, $B_6 = \frac{1}{42}$.

Chapter 11

The Arf Invariant and the Second Stem

The last section of the source computes $\pi_{n+2}(S^n)$ [source, II, §15]. A framed manifold is now a surface, and the method is to simplify the surface by cutting handles until a sphere is left. Whether a handle can be cut is decided by the invariant of the preceding chapter, applied to the circles on the surface; the obstruction to cutting all of them is an invariant of quadratic forms over the field of two elements.

11.1 A quadratic form on a framed surface

Let (M^2, U) be a framed surface in $\mathbb{R}^{n+2}$ and C a simple closed curve on M. The frame U restricted to C, together with the normal to C in M, is a frame of C in $\mathbb{R}^{n+2}$. Let $q(C) \in \mathbb{Z}/2$ be the class of this framed circle in $\pi_{n+2}(S^{n+1})$ under Theorem 10.3: it is 0 if the frame is the one that extends over a disc.

Theorem 11.1. [classical; [source, II, §15]] *$q(C)$ depends only on the class of C in $H_1(M;\mathbb{Z}/2)$, and*

$$q(x+y) = q(x) + q(y) + x \cdot y,$$

where $x \cdot y$ is the intersection number modulo two.

So q is not linear: it is a ***quadratic refinement*** of the intersection form. On a surface of genus g the intersection form on $H_1(M;\mathbb{Z}/2) = \mathbb{F}_2^{2g}$ is the standard symplectic form, and its quadratic refinements are the 2^{2g} functions determined by arbitrary values on a symplectic basis $a_1, b_1, \ldots, a_g, b_g$.

Definition 11.2. The ***Arf invariant*** of q is $\operatorname{Arf}(q) = \sum_{i=1}^{g} q(a_i)q(b_i) \in \mathbb{Z}/2$.

Theorem 11.3 (Arf). [classical; [Arf41]] (a) $\operatorname{Arf}(q)$ *does not depend on the symplectic basis, and two quadratic refinements are equivalent under the symplectic group if and only if they have the same Arf invariant.* (b) *There are $2^{g-1}(2^g+1)$ refinements with $\operatorname{Arf} = 0$ and $2^{g-1}(2^g-1)$ with $\operatorname{Arf} = 1$.* (c) $\operatorname{Arf}(q)$ *is the value that q takes more often: q equals 1 on $2^{2g-1} - 2^{g-1}$ classes if $\operatorname{Arf} = 0$ and on $2^{2g-1} + 2^{g-1}$ if $\operatorname{Arf} = 1$.* (d) Arf *is additive for orthogonal sums.*

genus	refinements	Arf $= 0$	Arf $= 1$
1	4	3	1
2	16	10	6
3	64	36	28

[A1, A2, A3] For genus two the symplectic group has order 720, and its orbits on the sixteen refinements have sizes 10 and 6. [A4] On a torus the single odd form is the one with $q(a) = q(b) = q(a+b) = 1$.

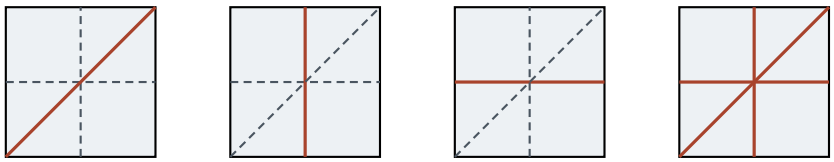


Figure 11.1. *The four quadratic refinements on a torus, drawn as a square with opposite sides identified. The curves a (horizontal), b (vertical) and $a+b$ (diagonal) are solid where $q = 1$ and dashed where $q = 0$. Three forms have one solid curve; they are permuted by the symmetries of the torus. The fourth has all three, and nothing can be cut.*

11.2 Surgery

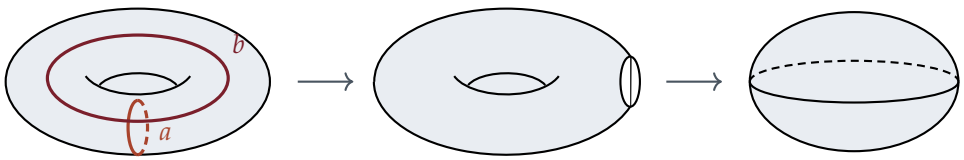


Figure 11.2. *Surgery, which the source calls reconstruction. A curve a on the surface with $q(a) = 0$ carries the frame that extends over a disc. Cut along a and glue in two parallel copies of that disc, with the extended frame: the result is a framed surface of genus one less, framed cobordant to the original. If $q(a) = 1$ the frame does not extend and the handle cannot be cut.*

Proposition 11.4. [classical; [source, II, §11, §15]] *Surgery on a class a with $q(a) = 0$ replaces H_1 by $a^{\perp}/\langle a \rangle$ with the induced form, and does not change the Arf invariant. A form with* Arf $= 0$ *can be reduced to genus zero. A form with* Arf $= 1$ *can be reduced to a torus on which q is 1 on all three non-zero classes, and no further.*

This was verified for all 84 forms of genus at most three, by carrying out the reduction. [A5]

Theorem 11.5. [classical; [source, II, §15, Thm. 24]] *For $n \geq 2$, $\pi_{n+2}(S^n) \cong \mathbb{Z}/2$. The class of a framed surface is the Arf invariant of its quadratic form, and the non-zero class is represented by the torus $S^1 \times S^1$ with the product of two copies of the non-bounding frame of the circle: $\eta^2 \neq 0$.*

Structure of the proof. By Proposition 11.4 a framed surface is framed cobordant to a sphere or to the odd torus. A framed sphere is framed cobordant to zero, since $\pi_2(\mathrm{SO}(n)) = 0$ and so all frames on it are equivalent. The Arf invariant is a cobordism invariant, by the analysis of the critical levels of a framed 3-manifold [source, II, §15, Thms. 22–24], so the odd torus is not zero. □

11.3 History, and what the form means

The first announcement of this result, in 1938, gave the answer zero; the correct value was published in 1950 [source, II, refs. 1, 2]. The slip was to treat q as if it were linear. If it were, it would vanish on a half-dimensional subspace, every handle could be cut, and the group would be trivial. The cross term $x \cdot y$ in Theorem 11.1 is what prevents this, and it has a structural meaning.

Remark 11.6 (Why quadratic). [classical] A class $x \in H_1(M; \mathbb{Z}/2)$ on a framed surface determines a framed circle, hence an element $q(x)$ of π_1^{s}. So q is the map on first framed bordism induced by collapsing M to a point, restricted to classes coming from homology, and it would be linear if the framed bordism of M split as the homology of M with coefficients in π_0^{s} and π_1^{s} separately. It does not split: the sphere spectrum is not a product of Eilenberg–MacLane spectra, and the first obstruction, its first k-invariant, is the Steenrod square Sq^2. The Arf invariant of a framed surface is that k-invariant, seen through a manifold.

Remark 11.7 (Spin structures). [literature; [Joh80, Ati71]] A stable framing of a surface induces a spin structure, and q depends only on it: the quadratic refinements of the intersection form correspond exactly to the 2^{2g} spin structures on the surface. The Arf invariant is the parity of the dimension of the space of holomorphic sections of the corresponding square root of the canonical bundle, and is the isomorphism from the spin bordism group of surfaces to $\mathbb{Z}/2$. The counts of Theorem 11.3*(b)* are the numbers of even and odd theta characteristics.

Remark 11.8 (The Kervaire invariant). [literature; [Ker60, Bro69, HHR16, LWX24]] The same construction on framed manifolds of dimension $4m + 2$, with the middle homology in place of H_1, gives the Kervaire invariant. Browder showed that it can be non-zero only in dimensions $2^{j+1} - 2$. Framed manifolds with invariant one exist in dimensions $2, 6, 14, 30, 62$; Hill, Hopkins and Ravenel showed there are none in dimensions 254 and above. The preprint of Lin, Wang and Xu establishes existence in dimension 126, the final case left by those results. The dimension-two case is Theorem 11.5.

Notes

The source obtains the quadratic property of q and its invariance by direct geometric arguments on surfaces in general position and on the level sets of a three-dimensional cobordism. Arf's classification dates from 1941. For the relation with spin structures see Johnson [Joh80]. The history of the correction is as usually told; the two notes are the first two references of the source's second monograph.

Exercises

Exercise 11.1. List the four quadratic refinements in genus one with their values on a, b, $a+b$, and verify Theorem 11.3*(b)* and *(c)*.

Exercise 11.2. Prove the counts in Theorem 11.3*(b)* by induction on the genus from *(d)*.

Exercise 11.3. Show that if q were linear and the intersection form non-degenerate, there would be a class $a \neq 0$ with $q(a) = 0$ whenever $g \geq 1$, and that surgery could be continued to genus zero. Where does the argument fail for the odd torus?

Exercise 11.4. Let q_1, q_2 be the odd forms on two tori. Compute the Arf invariant of the form on the genus-two surface that is their orthogonal sum, and interpret the result in π_2^{s}.

Chapter 12

Stable Stems as the Homotopy of the Unit

The two monographs of the source were written decades apart and share no notation. This chapter states the sense in which they are one theory with two coefficient rings, and places Pontryagin's results in the table of stable stems that later methods filled in.

12.1 Two rings

A ***spectrum*** is the stable analogue of a space, an object on which suspension has been made invertible; spectra form a symmetric monoidal ∞-category Sp whose unit is the sphere spectrum $\mathbb{S}$ of Theorem 8.5. Every abelian group A gives an Eilenberg–MacLane spectrum HA, and $H\mathbb{Z}$ is a commutative ring spectrum whose modules are the objects of $\mathcal{D}(\mathbb{Z})$.

$$\mathcal{S} \xrightarrow{\Sigma^\infty_+} \mathrm{Sp} \xrightarrow{H\mathbb{Z}\otimes_{\mathbb{S}} -} \mathcal{D}(\mathbb{Z}), \qquad \mathcal{S} \xrightarrow{C_*} \mathcal{D}(\mathbb{Z})$$

Theorem 12.1. [reformulation; cf. [Lur17, EKMM97]] (a) $\mathbb{S}$ *is the initial commutative ring spectrum: every ring spectrum receives a unique map from it, as every ring receives one from* $\mathbb{Z}$. (b) *The* ***Hurewicz map*** $\mathbb{S} \to H\mathbb{Z}$ *is an isomorphism on* π_0 *and is zero on* π_k *for* $k > 0$. (c) *The triangle above commutes: chains are obtained from the suspension spectrum by extension of scalars along* $\mathbb{S} \to H\mathbb{Z}$.

	over $\mathbb{Z}$ (Part One)	over $\mathbb{S}$ (Part Two)
free module on a space X	chains $C_*(X)$	suspension spectrum $\Sigma^\infty_+ X$
its homotopy groups	homology $H_*(X)$	stable homotopy $\pi^{\mathrm{s}}_*(X_+)$
a cycle	a simplicial chain with zero boundary	a framed manifold with a map to X
a boundary	the boundary of a chain	the boundary of a framed manifold
the case $X = \mathrm{pt}$	$\mathbb{Z}$ in degree 0	the stable stems
suspension	shifts the degree	shifts the degree
trace of a self-map	Lefschetz number	fixed point index
degree 0	signed count of points	signed count of points
degree 1	nothing	$\mathbb{Z}/2$: one full turn of a frame
degree 2	nothing	$\mathbb{Z}/2$: the Arf invariant

The first seven rows are the same construction carried out twice; the last three are where the two rings differ. Statement (c) says that Part One can be recovered from Part Two by forgetting, and (b) says what is forgotten: everything in positive degrees. The Hopf map, which Chapter 5 found invisible to homology, is visible to $\mathbb{S}$ after one suspension, as the generator of π^{s}_1.

The traces of Chapter 6 sit on the row where the two agree. The Euler characteristic of a compact manifold M is the trace of the identity of $\Sigma^\infty_+ M$, an element of $\pi^{\mathrm{s}}_0 = \mathbb{Z}$; it is represented by the composite $\mathbb{S} \to \Sigma^\infty_+ M \to \mathbb{S}$ of the Pontryagin–Thom collapse of an embedding with the projection to a point [BeGo75]. For a polyhedral surface this is the classical statement that the angle defects add up to $2\pi\chi$: 4π for a random tetrahedron and 0 for a polyhedral torus. [T2]

12.2 The table

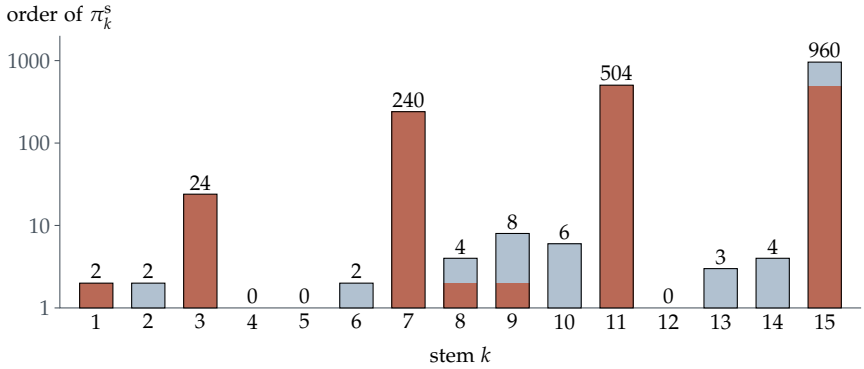


Figure 12.1. *Orders of the stable stems π_k^s for $1 \le k \le 15$, on a logarithmic scale; a group of order one is marked 0. The darker part of each bar is the image of the J-homomorphism. The source reaches $k = 2$; the large groups in degrees $3, 7, 11, 15$ are almost entirely framed spheres.*

k	0	1	2	3	4	5	6	7	8
π_k^s	$\mathbb{Z}$	$\mathbb{Z}/2$	$\mathbb{Z}/2$	$\mathbb{Z}/24$	0	0	$\mathbb{Z}/2$	$\mathbb{Z}/240$	$(\mathbb{Z}/2)^2$
generator	ι	η	η^2	ν			ν^2	σ	$\eta\sigma, \varepsilon$

Here η, ν, σ are the stable classes of the three Hopf maps of Remark 9.7. The source proves the columns $k = 0, 1, 2$ (Theorems 9.1, 10.3 and 11.5) and reports that Rokhlin obtained $k = 3$ by the same method, while "for $k > 3$ the problem so far defies solution by our method" [source, II, Introduction]. The remaining columns, and several dozen more, were found by the algebraic methods the source attributes to the French school [Ser51, Tod62, Rav86]: Serre proved that all the groups with $k > 0$ are finite, and the Adams spectral sequence computes them from the cohomology of the Steenrod algebra.

Two features of the table were already visible to the geometric method. The groups in degrees 3 and 7 are the images of J (Theorem 10.6), of orders 24 and 240: framed spheres. The classes η^2 and ν^2 in degrees 2 and 6 are not in the image of J; they are products of spheres with product frames, and they are detected by the Arf and Kervaire invariants (Remark 11.8).

12.3 What became of the method

The main method computes homotopy groups from manifolds. It stalled at $k = 3$ because the classification of framed manifolds up to cobordism requires, in each dimension, a way to simplify an arbitrary manifold, and beyond surfaces the obstructions to simplification are as hard as the groups being sought. Fifteen years later the direction of the argument was reversed.

Kervaire and Milnor [KeMi63] applied surgery to framed manifolds whose boundary is a homotopy sphere, took the stable stems as known from algebra, and deduced the classification of smooth structures on spheres: the group of exotic spheres in dimension k is, up to the image of J and the Kervaire invariant, the stable stem π_k^s. The dictionary of Theorem 8.2 has been read from right to left ever since.

In the other direction, the statement that framed manifolds are the homology theory of the initial ring has been extended from groups to categories: the cobordism hypothesis (Remark 8.6) is Theorem 8.5 with manifolds with corners in place of closed manifolds and duality data in place of inverses.

Notes

For spectra and the smash product see [EKMM97, Lur17]; for the stable stems, [Rav86]. The values in the table and in Figure 12.1 are quoted from the literature and not computed here; the orders of the image of J are computed from Bernoulli numbers. [R5] Rokhlin's computation of π_3^s is the origin of his theorem on the signature of four-manifolds.

Exercises

Exercise 12.1. Show that the map $\pi_k^s(X_+) \to H_k(X)$ induced by $\mathbb{S} \to H\mathbb{Z}$ sends a framed manifold $g\colon M \to X$ to $g_*[M]$, and that it is an isomorphism for $k = 0$.

Exercise 12.2. It is known that $\eta^3 \neq 0$. Using the table, show that $\eta^3 = 12\nu$.

Exercise 12.3. Prove that the angle defects of a closed polyhedral surface with triangular faces add up to $2\pi\chi$.

Exercise 12.4. Which entries of the table for $k \leq 8$ are entirely in the image of J, according to Theorem 10.6?

Chapter 13

Assessment

13.1 What the language did

It identified the two monographs as one construction. Chains are the free $\mathbb{Z}$-module on a space (Theorem 2.4); the suspension spectrum is the free $\mathbb{S}$-module (Theorem 8.5); the first is obtained from the second by extension of scalars (Theorem 12.1). Simplicial cycles and framed manifolds are the cycles of the two theories, and the table in Chapter 12 lists the parallel, row by row.

It said why the first monograph is easy and the second is hard. The coefficients of the first theory, the homotopy groups of $H\mathbb{Z}$, are $\mathbb{Z}$ in degree zero. Those of the second are the stable stems. Everything the source proves in Part One is linear algebra over a ring whose homotopy is trivial; everything in Part Two is the first three homotopy groups of a ring whose homotopy is not.

It sorted the proof of the fixed point theorem into functors. The fixed point index is a trace in spectra, the Lefschetz number a trace in $\mathcal{D}(\mathbb{Z})$, and they agree because a symmetric monoidal functor preserves traces and the two units have the same π_0 (Theorem 6.5).

It explained an error. The quadratic form on a framed surface fails to be linear because the sphere spectrum is not a product of Eilenberg–MacLane spectra (Remark 11.6). The announcement of 1938 amounts to the assumption that it is.

It placed the constructions of Part One. The polyhedron is a colimit, subdivision a functor with a right adjoint, the cone a contraction, the prism a product with an interval, and invariance the statement that chains extend to a functor on homotopy types (Propositions 1.2, 3.4 and 3.5 and Theorems 5.1 and 5.3).

13.2 What it did not do

It proved nothing. Every theorem here is in the source or in the literature. The proofs given are those that are short; the others are indicated by their structure. In particular the general position arguments of Chapter 7, on which the whole of Part Two rests, and the analysis of cobordisms of surfaces in Chapter 11, are the source's and are not reproduced.

It did not compute a stable stem. The higher-categorical description of $\mathbb{S}$ as a free object is not a method of calculation. The values in Figure 12.1 come from

the Adams spectral sequence and its successors. The script accompanying this volume checks the invariants that detect the first three stems, on examples; it does not and could not verify that they are complete.

It did not replace the geometry. That a framed circle in $\mathbb{R}^4$ has an invariant in $\mathbb{Z}/2$ is, in the language of this volume, the statement $\pi_1^{\mathrm{s}} = \mathbb{Z}/2$; one still has to see why, and the source's reason, a loop of rotations that lifts to an open path of quaternions, is the proof.

13.3 How much is higher

chapters	categorical content	higher?
1	presheaves, colimits, Kan extension	no
2, 3	functors to chain complexes; adjunctions	implicitly: chains as a left adjoint of ∞-categories
4	a functor that does not vanish	no
5	homotopies as 2-cells	yes, and this is where it starts
6	traces in symmetric monoidal categories	only through the comparison with spectra
7	pullbacks	no (see [DTHC] for the derived version)
8–11	the first three homotopy groups of the sphere spectrum	yes: the object exists only as an ∞-categorical one
12	modules over two ring spectra	yes

Compared with the earlier volumes of the series [GIHC, ESCHC, DTHC, EGHC, PDEHC], this is the one in which the higher structure is least optional. The source develops homology using chain complexes. The sphere spectrum describes the framed bordism theory of the second monograph. Its infinite loop space is the free grouplike commutative monoid in spaces on a point. Pontryagin's method represents its homotopy classes by framed manifolds.

13.4 Programme

[programme] *(a)* Carry the dictionary of Chapter 12 one step further and treat Rokhlin's computation of π_3^{s} in the same style, with the e-invariant and the signature in place of the Arf invariant. *(b)* Write Part One over $\mathbb{S}$ directly: a simplicial complex with a map to a space gives a class in stable homotopy, and the Hurewicz map is then visibly the passage from Part Two to Part One. *(c)* Make the script compute framed bordism invariants of triangulated surfaces with combinatorial framings, so that Theorem 11.1 is checked on complexes and not only on the algebra of forms.

Notes

The companion paper [ADTML] applies Part One, and the rotation groups of Chapter 10, to machine learning: obstructions to continuous representations, invariants computed by networks on complexes, and counts of fixed points and of class boundaries.

Appendix A

Toolkit

Categories. SCx: simplicial complexes and simplicial maps. sSet: simplicial sets, presheaves on the category Δ of finite non-empty ordinals. Top: spaces. $\mathcal{S}$: the ∞-category of homotopy types, obtained from either sSet or Top by inverting weak equivalences. $\mathcal{D}(\mathbb{Z})$: the derived ∞-category of abelian groups, chain complexes with quasi-isomorphisms inverted. Sp: spectra.

Kan extension. Given $F\colon \mathsf{C} \to \mathsf{E}$ with E cocomplete, the left Kan extension along the Yoneda embedding is the unique colimit-preserving functor from presheaves on C to E that restricts to F. Geometric realization is the case $\mathsf{C} = \Delta$, $F[n] = |\Delta^n|$.

Smith normal form. Every integer matrix is equivalent, by invertible integer row and column operations, to a diagonal matrix with entries $d_1 \mid d_2 \mid \dots$. The number of non-zero d_i is the rank, and those greater than one are the torsion coefficients of the cokernel.

Chain homotopy. Chain maps φ, ψ are chain homotopic if $\psi - \varphi = \partial D + D\partial$ for some D raising degree by one. Chain homotopic maps induce the same map on homology.

Dualizable objects. In a symmetric monoidal category, X is dualizable when $- \otimes X$ has a right adjoint of the form $- \otimes X^\vee$. Finite-dimensional vector spaces, perfect complexes and the suspension spectra of finite complexes are the dualizable objects of their categories.

Framings. A framing of a submanifold is a trivialization of its normal bundle; a stable framing is a trivialization of the normal bundle of an embedding in a Euclidean space of large dimension, and is independent of the embedding. A framing of the normal bundle determines one of the stable tangent bundle and conversely.

Spectra. A spectrum is a sequence of pointed spaces E_n with equivalences $E_n \simeq \Omega E_{n+1}$. Its homotopy groups are $\pi_k(E) = \pi_{k+n}(E_n)$, for any $n \geq -k$.

The sphere spectrum has $E_n = \operatorname{colim}_m \Omega^m S^{n+m}$, and the Eilenberg–MacLane spectrum $H\mathbb{Z}$ has $E_n = K(\mathbb{Z}, n)$.

Quadratic forms over $\mathbb{F}_2$. A function q on an $\mathbb{F}_2$-vector space with $q(x+y) - q(x) - q(y)$ bilinear. When the bilinear form is non-degenerate the space has even dimension and q is classified by its Arf invariant.

Appendix B

Concordance with the Source

The source is [Pon86]. Roman numerals denote its two monographs.

source	here	form
I §1 Euclidean space; general position	Theorem 1.3	one proof
I §2 simplex, complex, polyhedron	Proposition 1.2 and Theorem 1.5	the polyhedron as a colimit; simplicial sets
I §3 dimension theory	Theorems 1.7 and 4.3	nerves; order of a cover
I §4 homology groups	Proposition 2.1	chains as a functor
I §5 components; H_0	Proposition 2.2	a composite of left adjoints
I §6 Betti numbers; Euler–Poincaré	Theorem 2.5	additivity of rank
I §7 simplicial maps and approximations	Theorem 3.2	with the number of subdivisions
I §8 conic construction	Propositions 2.6 and 3.5	a chain contraction
I §9 barycentric subdivision	Proposition 3.4	a functor; the nerve of a poset
I §10 covering of a simplex; Sperner; Brouwer	Theorems 4.1 and 4.2	a degree computation
I §11 invariance under subdivision	Theorem 3.6	acyclic models
I §12 invariance	Theorem 3.7	factorization through polyhedra
I §13–14 homotopy; cylindric construction	Theorems 5.1 and 5.2	the prism as a product
I §15 invariants of maps	Proposition 5.4 and Theorem 5.3	degree; chains on homotopy types
I §16 fixed points	Theorems 6.1, 6.2 and 6.5	categorical trace
II §1–2 manifolds; embedding	Theorem 7.1	the dimension count
II §3 Sard; general position	Theorem 7.2	with a quantitative example
II §4 typical singularities	Theorem 7.1	table of codimensions
II §5 smooth approximation	Chapter 7, end	three statements used later

source	here	form
II §6 the main method	Theorem 8.2	the two constructions
II §7 the group Π_n^k	Definition 8.1 and Theorem 8.5	framed bordism; the sphere spectrum
II §8 suspension	Theorem 8.3	stability as general position
II §9 degree; Hopf's theorem	Theorem 9.1	framed points
II §10 looping coefficient; Hopf invariant	Proposition 9.2 and Theorem 9.4	linking of preimages
II §11 reconstruction	Proposition 11.4	surgery
II §12 quaternions; covering homotopy; rotations	Theorem 10.1 and Proposition 10.2	π_1 of the rotation groups
II §13 $\pi_3(S^2)$	Theorem 9.6	through the Hopf fibration
II §14 $\pi_{n+1}(S^n)$	Theorem 10.3	framed circles; the invariant δ
II §15 $\pi_{n+2}(S^n)$	Theorems 11.1 and 11.5	the Arf invariant

Not in the source and added here: simplicial sets and nerves beyond dimension theory; the universal property of chains; the categorical trace; the configuration-space and free-object descriptions of the sphere spectrum; the J-homomorphism; spin structures; the Kervaire invariant; the comparison of the two coefficient rings. In the source and treated only by citation: the linear algebra of general position, the structure theorem for finitely generated abelian groups, the proofs of the embedding theorem and of Sard's theorem, the normal forms of singularities, and the geometric proofs of Theorems 11.1 and 11.5.

Appendix C

Computations

The companion script `verify-topology.py` uses only the Python standard library, a fixed seed, and exact integer or rational arithmetic wherever a statement is combinatorial. It performs 63 checks and runs in about twenty seconds. Homology is computed by Smith normal form over the integers and by Gaussian elimination over prime fields; traces on homology by explicit bases of cycles modulo boundaries over the rationals; linking numbers by the Gauss integral with the midpoint rule; degrees by integrating pulled-back area forms. A check that passes on an instance supports a statement and does not prove it.

checks	content
C1–C5	the six model complexes; general position on the moment curve; nerve of the star cover; shuffles; flags
H1–H8	$\partial\partial = 0$; homology with torsion; Euler–Poincaré; mod-p Betti numbers and universal coefficients; H_0; cones and suspensions; additivity of χ; product triangulations of tori
I1–I5	invariance under subdivision; the last-vertex map; the cone contraction; mesh of subdivisions; degrees of simplicial approximations
S1–S6	Sperner in dimensions two and three, signed; boundary degree; approximate fixed points; Hex; order of covers
P1–P4	the prism identity; cylinders; contiguous maps; homotopy equivalent complexes
L1–L9	Hopf trace formula; fixed simplices; $L(\mathrm{id}) = \chi$; torus maps; periodic points of the cat map; antipodal maps; additivity; the projective plane; products
M1–M5	double points of projections; quantitative Sard; the umbrella; Morse points on a torus; degree over different regular values
F1–F3	signed preimages; suspension; the collapse map of framed points
K1–K5	Gauss linking integrals; Hopf invariants $1, 4, 9$ and $1, 2, 3$; framings of a circle in space; properties of the Hopf map
R1–R6	quaternions; the double cover; lifting loops of rotations; $\mathbb{RP}^3$; Bernoulli denominators; framings modulo two
A1–A5	counts of quadratic refinements; majority vote; additivity; orbits of $Sp(4, \mathbb{F}_2)$; surgery reduction

checks	content
T1–T2	abelianization of symmetric groups; angle defects

Not computed. The values of the stable stems beyond those detected by the invariants above; the orders in Figure 12.1 are quoted. The statement that the invariants are complete, that is, that framed manifolds with the same invariant are cobordant, is the content of the source's theorems and is not checked.

Figures. The figures were drawn by a separate plotting script, not part of the bundle, and are embedded in the source as coordinates. The Sperner labelling, the Hex board, the points of the cat map, the knot projections, the Hopf fibres and the framed circles are computed from the same formulas as the checks; the remaining figures are schematic.

Points of contact with the source. No numerical example of the source was found to disagree with a computation. Two matters of convention are worth recording. The source's invariant δ of a framed circle includes one unit per component, for the reason explained after Theorem 10.3; a reader comparing with other accounts, where the bounding frame is the reference, should expect that shift. And the source's "homology of framed manifolds" is cobordism, not the homology of Part One.

Appendix D

Solutions to the Exercises

Chapter 1

1.1. Each triangle has three edges and each edge lies in two triangles, so $3F = 2E$, and $V - E + F = \chi$ gives $E = 3(V - \chi)$. From $E \le \binom{V}{2}$ we get $V^2 - 7V + 6\chi \ge 0$, hence the bound. It is 4, 6, 7 for $\chi = 2, 1, 0$, attained by the tetrahedron and by the two complexes of the chapter.

1.2. Writing a vertex as (vertex of the triangle, level): $[00, 01, 11, 21]$, $[00, 10, 11, 21]$, $[00, 10, 20, 21]$. The triangle $\{00, 11, 21\}$ is in the first two, $\{00, 10, 21\}$ in the last two; every other triangle of the three lies on the boundary of the prism.

1.3. Any two faces share an edge, any three a vertex, and the four have empty intersection. The nerve is the boundary of a tetrahedron on the four faces, again a sphere. The order is 3, the least possible for a surface by Theorem 4.3. The intersections are contractible, and the conclusion of the nerve theorem holds (for a closed cover of a polyhedron by subcomplexes it applies as stated).

1.4. Five points in $\mathbb{R}^3$ are never affinely independent, and by Radon's theorem they split into two sets whose convex hulls meet. For points $t_1 < \cdots < t_5$ on the moment curve the split is $\{t_1, t_3, t_5\}$ and $\{t_2, t_4\}$: the triangle 135 meets the edge 24. These are disjoint simplices of the complete 2-complex, which is therefore not realized. No other placement helps: in any embedding of the complete graph on six vertices in $\mathbb{R}^3$ two disjoint triangles are linked.

Chapter 2

2.1. ∂_2 is injective, so $H_2 = 0$; $H_1 = \mathbb{Z}^2/\langle(2, 0)\rangle = \mathbb{Z}/2 \oplus \mathbb{Z}$; $H_0 = \mathbb{Z}$. This is the table's row.

2.2. Projective plane: $\beta = (1, 0, 0)$ and $t_1(2) = 1$, so over $\mathbb{F}_2$ we get $(1, 0 + 1, 0 + 1) = (1, 1, 1)$ and over $\mathbb{F}_3$, $(1, 0, 0)$. Klein bottle: $\beta = (1, 1, 0)$, $t_1(2) = 1$, giving $(1, 2, 1)$ and $(1, 1, 0)$.

2.3. $H_0 = \mathbb{Z}$, $H_1 = 0$, $H_2 = \mathbb{Z} \oplus \mathbb{Z}/2$, $H_3 = 0$; over $\mathbb{F}_2$ the Betti numbers are $(1, 0, 2, 1)$.

2.4. $\sum_n (-1)^n \sum_{i+j=n} \beta_i(K)\beta_j(L) = \chi(K)\chi(L)$. For the 3-torus, $64 - 448 + 768 - 384 = 0 = 0^3$.

Chapter 3

3.1. A simplex of the subdivision has vertices at barycentres of nested faces, and the distance between the barycentres of $\sigma \subset \tau$ is at most $\frac{r}{r+1}$ times the diameter of τ: the barycentre of τ is a weighted mean of that of σ and that of the complementary face, with weight at most $\frac{r}{r+1}$ on the latter. From $1.414\,(2/3)^m < 0.01$ we need $m \geq 13$, which is $6^{13} \approx 1.3 \times 10^{10}$ triangles; by the observed ratios, about ten subdivisions would do.

3.2. The images of the vertices are in the thirds $0, 1, 2, 0, 1, 2, 0, 1$. Going round, the edge from 2 to 0 is covered at steps 3 and 6, and the closing step runs from 1 back to 0, against the orientation. So the edge $[2, 0]$ is covered twice and the degree is 2.

3.3. $D[v_0, v_1] = [v_0, v_1, a]$, with boundary $[v_1, a] - [v_0, a] + [v_0, v_1]$; and $D\partial[v_0, v_1] = D[v_1] - D[v_0] = -[v_1, a] + [v_0, a]$. The sum is $[v_0, v_1]$.

3.4. Vertices $\alpha_0 + \alpha_1 + \alpha_2$; edges $2\alpha_1 + 6\alpha_2$ (pairs of nested faces); triangles $6\alpha_2$. The alternating sum is $\alpha_0 - \alpha_1 + \alpha_2$. For $(6, 15, 10)$ this gives $(31, 90, 60)$.

Chapter 4

4.1. Along the interval the labels $\lambda(v_i) \in \{0, 1\}$ start at 0 and end at 1, and $\sum_i (\lambda(v_{i+1}) - \lambda(v_i)) = 1$; each term is the sign of an edge labelled 01 or 10, or zero.

4.2. A complete cell has one edge labelled 01, a cell with labels $0, 0, 1$ or $0, 1, 1$ has two, any other has none; so the number of pairs has the parity of the number of complete cells. An interior edge lies in two cells and a boundary edge in one; so it also has the parity of the number of boundary 01-edges, which all lie on one side and are odd in number by the previous exercise.

4.3. If f had no fixed point, the map sending x to the point where the ray from $f(x)$ through x leaves the simplex would be a continuous retraction onto the boundary.

4.4. If the order were one, the interval would be a disjoint union of finitely many closed sets, hence equal to one of them, of length less than one. The intervals $[0, \frac{1}{2}]$ and $[\frac{1}{2}, 1]$ have order two. For the square use bricks of side 0.3 in the pattern of Figure 4.4; their diameter is 0.43.

Chapter 5

5.1. $P[v_0, v_1] = [v_0, v_0', v_1'] - [v_0, v_1, v_1']$, with boundary $[v_0', v_1'] + [v_0, v_0'] - [v_1, v_1'] - [v_0, v_1]$, and $P\partial[v_0, v_1] = [v_1, v_1'] - [v_0, v_0']$. The sum is $[v_0', v_1'] - [v_0, v_1]$.

5.2. For x in a simplex σ, both $f(x)$ and $g(x)$ lie in the simplex spanned by $f(\sigma) \cup g(\sigma)$, and so does the segment between them: $(1-t)f + tg$ is a homotopy.

5.3. If $f(x) \neq x$ for all x, the segment from $f(x)$ to $-x$ misses the origin, and normalizing $(1-t)f(x) - tx$ gives a homotopy from f to the antipodal map, of degree $(-1)^{n+1}$.

5.4. S^3 has homology only in degrees 0 and 3, S^2 only in 0 and 2. So the induced map is the identity of A in degree 0 and zero otherwise, exactly as for a constant map.

Chapter 6

6.1. In a basis of M' extended to one of M the matrix is block triangular, and the trace is the sum of the traces of the diagonal blocks.

6.2. $-x \equiv x$ means $2x \in \mathbb{Z}^2$: the points $(0,0)$, $(\frac{1}{2},0)$, $(0,\frac{1}{2})$, $(\frac{1}{2},\frac{1}{2})$. On homology the map is $1, -I, 1$ in degrees $0, 1, 2$, so $L = 1 + 2 + 1 = 4$.

6.3. $\det(I - A^n) = 1 - \operatorname{tr} A^n + \det A^n = 2 - \operatorname{tr} A^n < 0$. With $\lambda = \frac{3+\sqrt{5}}{2}$, $\sum N_n t^n/n = -\log(1-\lambda t) - \log(1 - t/\lambda) + 2\log(1-t)$, and $(1-\lambda t)(1 - t/\lambda) = 1 - 3t + t^2$.

6.4. In even dimension the rational homology is that of a point, so $L = 1$ for every map. In dimension $2m-1$ multiplication by i on $\mathbb{C}^m = \mathbb{R}^{2m}$ induces a map of projective space with no fixed point, since a fixed point would be a real eigenvector of a quarter turn.

Chapter 7

7.1. The chords are the image of $(M \times M) \setminus$ diagonal, of dimension $2k$, under $(x, y) \mapsto (x - y)/|x - y|$, and the tangent directions the image of the unit tangent bundle, of dimension $2k - 1$. In S^{N-1} these have measure zero when $2k < N - 1$; so one can project injectively from $\mathbb{R}^N$ to $\mathbb{R}^{N-1}$ as long as $N - 1 \geq 2k + 1$.

7.2. $F(x,y) = F(x',y')$ forces $x = x'$, $xy = xy'$, $y^2 = y'^2$: either $y = y'$ or $x = 0$, $y' = -y$. The 2×2 minors of $\mathrm{d}F$ are x, $2y$, $2y^2$, which vanish together only at the origin.

7.3. Near $x = 1$, $|3x^2 - 3| < \delta$ is $|x - 1| < \delta/6$ to first order and $f(x) - f(1) \approx 3(x-1)^2 \leq \delta^2/12$; the same at $x = -1$. The image is two intervals of that length.

7.4. Near a matrix with an invertible $r \times r$ block A, written $\left(\begin{smallmatrix} A & B \\ C & D \end{smallmatrix}\right)$, the rank is r exactly when $D - CA^{-1}B = 0$: $(k-r)(N-r)$ equations. For $N = 2k-1$ and $r = k-1$ this is k, the dimension of the manifold.

Chapter 8

8.1. Frames at a point are the invertible matrices, with two components by the sign of the determinant. Near the point the collapse map is the linear map of the frame followed by a map of degree one, so its degree is that sign.

8.2. There is an isotopy of the ambient space, which needs only two dimensions of room, exchanging two disjoint balls by translations; its trace is a framed cobordism between the two orders of summation.

8.3. Reflect M and its frame in the hyperplane separating the half-spaces. Rotating M about that hyperplane through the extra coordinate of $\mathbb{R}^{n+k} \times [0,1]$, by a half turn, sweeps out a framed manifold whose boundary is M together with its reflection.

8.4. For $k = 1$ the suspension is an isomorphism for $n \geq 3$, so $\pi_4(S^3)$ is already stable, and $\pi_3(S^2) \to \pi_4(S^3)$ is onto. For $k = 2$ it is an isomorphism for $n \geq 4$ and onto from $n = 3$; the source obtains the value for $n \geq 2$ directly.

Chapter 9

9.1. Orient both curves and give each crossing the sign of the rotation from the upper strand to the lower. The linking number is the sum of the signs of the crossings where A passes over B; in Figure 9.1 there is one such crossing.

9.2. $h = 0$ means $z_1 = 0$ and $h = \infty$ means $z_2 = 0$. Projecting from $(0,0,0,1)$, the circle $z_2 = 0$ becomes the unit circle of the plane $x_3 = 0$, and the circle $z_1 = 0$, which passes through the centre of projection, becomes the x_3-axis.

9.3. The preimage of a value under $z^d \circ h$ is the union of the fibres over its d roots. Two values give $d \times d$ pairs of fibres, each linking once with the same sign.

9.4. For k even $\operatorname{lk}(A,B) = -\operatorname{lk}(B,A)$. Since γ does not depend on the pair of regular values, $\gamma = \operatorname{lk}(f^{-1}p, f^{-1}q) = \operatorname{lk}(f^{-1}q, f^{-1}p) = -\gamma$.

Chapter 10

10.1. q commutes with u, so u is fixed. For $v \perp u$ we have $uv = -vu = u \times v$ and $uvu = v$, so with $c = \cos\frac{\theta}{2}$, $s = \sin\frac{\theta}{2}$: $qvq^{-1} = (c+us)v(c-us) = (c^2 - s^2)v + 2cs\, u \times v$.

10.2. The exact sequence $\pi_2(S^n) \to \pi_1(\mathrm{SO}(n)) \to \pi_1(\mathrm{SO}(n+1)) \to \pi_1(S^n)$ has vanishing ends for $n \geq 3$.

10.3. Take a pair of pants in $\mathbb{R}^{n+1} \times [0,1]$ with two circles at the bottom and one at the top. A frame on it restricts to frames on the three circles whose classes, in the source's normalization, add up; two non-bounding circles give $1 + 1 = 0$, the bounding frame on the third.

10.4. $\frac{1}{6} \cdot \frac{1}{4} = \frac{1}{24}$, $\frac{1}{30} \cdot \frac{1}{8} = \frac{1}{240}$, $\frac{1}{42} \cdot \frac{1}{12} = \frac{1}{504}$.

Chapter 11

11.1. The values $(q(a), q(b), q(a+b))$ are $(0,0,1)$, $(0,1,0)$, $(1,0,0)$ with $\operatorname{Arf} = 0$ and $(1,1,1)$ with $\operatorname{Arf} = 1$. So three and one, as $2^0(2 \pm 1)$; and $q = 1$ on one class in the first three and on three in the last.

11.2. Adding a torus, $N_0(g+1) = 3N_0(g) + N_1(g)$. With the stated values this is $2^{g-1}\big(3(2^g+1) + (2^g-1)\big) = 2^g(2^{g+1}+1)$, and similarly for N_1.

11.3. A linear form on a space of dimension $2g \geq 2$ has a kernel of dimension at least $2g-1$, so some $a \neq 0$ has $q(a) = 0$; the induced form on $a^\perp / \langle a \rangle$ is again linear and the process continues. On the odd torus $q(a+b) = 1 \neq q(a) + q(b)$: the form is not linear, and it has no non-zero class with $q = 0$.

11.4. $\operatorname{Arf} = 1 + 1 = 0$. The connected sum of two odd tori is framed cobordant to zero: $2\eta^2 = 0$.

Chapter 12

12.1. A framing orients M, and the composite $\Sigma^\infty_+ M \to \Sigma^\infty_+ X \to H\mathbb{Z} \otimes \Sigma^\infty_+ X$ carries the class of M to $g_*[M]$. For $k = 0$ both groups are free on the components of X, with the signed count of points in each.

12.2. $2\eta = 0$, so η^3 has order two in $\pi^{\mathrm{s}}_3 = \mathbb{Z}/24$, whose only element of order two is 12ν.

12.3. The angles of each triangle add up to π, so the sum of the defects is $2\pi V - \pi F$; and $3F = 2E$ gives $V - E + F = V - \frac{F}{2}$.

12.4. $k = 1, 3, 7$ entirely; $k = 8$ in part, a subgroup of order two; $k = 2$ and 6 not at all.

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