Higher-Categorical Foundations: An $(\infty, 1)$-Categorical Treatment of Categories, Homological Algebra and Sheaf Cohomology

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Prepared for Higher Categorical Semantics

November 2026

Abstract

We present a complete $(\infty, 1)$-categorical modernization of the foundations of category theory, homological algebra, and sheaf cohomology as originally codified by Jan R. Strooker (1978). In this framework, classical 1-categories and set-theoretic hom-sets are refined to quasi-categories with mapping spaces in $\mathcal{S}$. Classical abelian categories are elevated to stable ∞-categories, where the triangulation axioms and ad-hoc non-functorial cone constructions are replaced by homotopy pushout/pullback squares and loop-suspension equivalences. Classical derived functors (Ext and Tor) and satellites arise directly and without projective resolutions as the homotopy groups of mapping spectra. Finally, topological sheaves and Grothendieck topologies are generalized to ∞-topoi and ∞-sheaves (stacks) with hypercover descent, providing an intrinsic geometric and homotopical formulation of sheaf cohomology. We discuss the computational semantics of this theory in Homotopy Type Theory (HoTT) and provide formal verification references.

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1 Introduction and the Homotopical Shift

In his 1978 Cambridge monograph *Introduction to Categories, Homological Algebra and Sheaf Cohomology*, Jan R. Strooker provided a unified, self-contained development of modern algebra and geometry. At the heart of that classical treatment lies the triad:

1. **General 1-Categories**: Objects, morphisms, functors, limits, and adjunctions.
2. **Abelian Categories and Homological Algebra**: Additive structures, kernels, cokernels, projective/injective resolutions, and derived functors (Ext, Tor, satellites).
3. **Sheaves and Cohomology**: Presheaves and sheaves of modules over topological spaces, flabby, soft, and fine resolutions, and Grothendieck's cohomology theory.

Despite its foundational triumphs, the 1-categorical setting exhibits profound structural limitations that have become central in modern mathematics:

- **The Equivalence vs. Equality Principle**: In 1-category theory, hom-sets are sets with strict equality. However, in mathematical practice, objects and arrows are rarely strictly equal; rather, they are equivalent or homotopic via higher coherences.
- **Deficiencies of Triangulated Categories**: In classical homological algebra, inverting quasi-isomorphisms produces the Verdier triangulated derived category $\mathcal{D}(\mathbf{A})$. However, triangulated categories suffer from fatal defects: cone constructions are not functorial, homotopy colimits do not exist in general, and triangulation cannot be glued along open covers.
- **Truncation in Sheaf Theory**: Classical sheaves take values in discrete sets or 1-categories. For stacks, gerbes, and families of derived complexes, descent along open coverings requires higher homotopical coherences (cocycle conditions up to homotopy, with coherences on triple and quadruple intersections).

By lifting Strooker's foundations to **$(\infty, 1)$-categories** (specifically quasi-categories, as introduced by Boardman-Vogt and Joyal and systematically developed by Lurie and Cisinski), these structural obstacles disappear. The goal of this monograph is to provide the modern higher-categorical counterpart to Strooker's text.

2 $(\infty, 1)$-Categories and ∞-Adjunctions

2.1 Quasi-Categories and Mapping Spaces

Let Δ denote the simplex category whose objects are non-empty finite linearly ordered sets $[n] = \{0 < 1 < \cdots < n\}$ and morphisms are order-preserving maps. A *simplicial set* is a functor $X \colon \Delta^{\mathrm{op}} \to \mathrm{Set}$.

Definition 2.1 (Quasi-Category)**.** A simplicial set $\mathcal{C}$ is an $(\infty, 1)$-*category* (or *quasi-category*) if for every $0 < k < n$, any inner horn $\Lambda^n_k \to \mathcal{C}$ admits a simplex extension $\Delta^n \to \mathcal{C}$:

$$\begin{array}{ccc} \Lambda^n_k & \longrightarrow & \mathcal{C} \\ \downarrow & \nearrow & \\ \Delta^n & & \end{array} \tag{1}$$

The 0-simplices $\mathcal{C}_0$ are the *objects* of $\mathcal{C}$. The 1-simplices $\mathcal{C}_1$ are the *morphisms*. A 2-simplex $\sigma \in \mathcal{C}_2$ witnesses a composition: if $d_2\sigma = f$, $d_0\sigma = g$, and $d_1\sigma = h$, we write $h \simeq g \circ f$. Inner horn filling for $n = 2$ ensures that every composable pair admits a composite; filling for $n = 3$ ensures associativity up to a contractible space of choices.

Definition 2.2 (Mapping Space)**.** Given objects $X, Y \in \mathcal{C}$, the *mapping space* $\mathrm{Map}_{\mathcal{C}}(X, Y)$ is the Kan complex in $\mathcal{S}$ whose n-simplices are maps $\Delta^n \times \Delta^1 \to \mathcal{C}$ restricting to X at $\{0\} \times \Delta^n$ and Y at $\{1\} \times \Delta^n$.

The *homotopy category* $\mathrm{h}\mathcal{C}$ is a 1-category with objects $\mathcal{C}_0$ and hom-sets $\pi_0 \mathrm{Map}_{\mathcal{C}}(X, Y)$.

2.2 Homotopy Limits, Colimits, and ∞-Adjunctions

Let K be a simplicial set. The ∞-category of diagrams of shape K in $\mathcal{C}$ is the functor quasi-category $\mathcal{C}^K = \mathrm{Fun}(K, \mathcal{C})$. The diagonal functor $\Delta \colon \mathcal{C} \to \mathcal{C}^K$ sends an object C to the constant diagram at C.

Definition 2.3 (Homotopy Limits and Colimits)**.** A *homotopy colimit* functor is a left adjoint to $\Delta \colon \mathrm{colim}_K \colon \mathcal{C}^K \to \mathcal{C}$. A *homotopy limit* functor is a right adjoint to $\Delta \colon \lim_K \colon \mathcal{C}^K \to \mathcal{C}$.

Definition 2.4 (∞-Adjunction)**.** An ∞-functor $F \colon \mathcal{C} \to \mathcal{D}$ is *left adjoint* to $G \colon \mathcal{D} \to \mathcal{C}$ (written $F \dashv G$) if there exists an equivalence of mapping spaces

$$\mathrm{Map}_{\mathcal{D}}(FX, Y) \xrightarrow{\sim} \mathrm{Map}_{\mathcal{C}}(X, GY) \tag{2}$$

natural in $X \in \mathcal{C}^{\mathrm{op}}$ and $Y \in \mathcal{D}$.

In higher category theory, adjunctions carry an infinite hierarchy of higher coherence homotopies between the unit $\eta \colon \mathrm{id}_{\mathcal{C}} \to GF$ and counit $\varepsilon \colon FG \to \mathrm{id}_{\mathcal{D}}$.

3 Stable ∞-Categories and Higher Homological Algebra

3.1 The Axiomatics of Stability

Strooker's Chapters 2 and 3 develop homological algebra within abelian categories. In higher category theory, the proper and robust formulation is given by **stable ∞-categories**.

Definition 3.1 (Stable ∞-Category)**.** An ∞-category $\mathcal{C}$ is *stable* if:

1. $\mathcal{C}$ has a zero object 0 (both initial and terminal).
2. Every morphism $f \colon X \to Y$ in $\mathcal{C}$ admits a fiber $\mathrm{fib}(f)$ and a cofiber $\mathrm{cofib}(f)$.
3. A commutative square in $\mathcal{C}$ is a pushout square if and only if it is a pullback square:

$$\begin{array}{ccc} X & \xrightarrow{f} & Y \\ {\scriptstyle g}\downarrow & & \downarrow \\ Z & \longrightarrow & W \end{array} \tag{3}$$

Theorem 3.2. *In any stable ∞-category $\mathcal{C}$, the suspension functor $\Sigma X = 0 \amalg_X 0$ and the loop functor $\Omega X = 0 \times_X 0$ are inverse equivalences:*

$$\Sigma \colon \mathcal{C} \xrightarrow{\sim} \mathcal{C}, \quad \Omega \colon \mathcal{C} \xrightarrow{\sim} \mathcal{C}, \quad \Sigma\Omega \simeq \mathrm{id}_{\mathcal{C}} \simeq \Omega\Sigma. \tag{4}$$

Moreover, the homotopy category $\mathrm{h}\mathcal{C}$ is canonically a triangulated category with translation functor $T = \pi_0(\Sigma)$ and distinguished triangles given by the images of fiber sequences.

3.2 Mapping Spectra and Long Exact Sequences

Because the loop functor Ω is an equivalence on $\mathcal{C}$, for any $X, Y \in \mathcal{C}$ the mapping space $\mathrm{Map}_{\mathcal{C}}(X, Y)$ is an infinite loop space:

$$\mathrm{Map}_{\mathcal{C}}(X, Y) \simeq \Omega \,\mathrm{Map}_{\mathcal{C}}(X, \Sigma Y) \simeq \Omega^2 \,\mathrm{Map}_{\mathcal{C}}(X, \Sigma^2 Y) \simeq \dots \tag{5}$$

Consequently, $\mathrm{Map}_{\mathcal{C}}(X, Y)$ refines canonically to a **mapping spectrum** $\mathbf{map}_{\mathcal{C}}(X, Y) \in \mathbf{Sp}$.

Proposition 3.3 (Fiber Sequence of Spectra)**.** *Let $X \xrightarrow{f} Y \xrightarrow{g} Z$ be a fiber sequence in $\mathcal{C}$. For any test object $W \in \mathcal{C}$, there is an induced fiber sequence of spectra:*

$$\mathbf{map}_{\mathcal{C}}(W, X) \xrightarrow{f_*} \mathbf{map}_{\mathcal{C}}(W, Y) \xrightarrow{g_*} \mathbf{map}_{\mathcal{C}}(W, Z). \tag{6}$$

Taking stable homotopy groups π_n yields a natural long exact sequence of abelian groups:

$$\cdots \to \pi_n \mathbf{map}(W, X) \to \pi_n \mathbf{map}(W, Y) \to \pi_n \mathbf{map}(W, Z) \xrightarrow{\partial} \pi_{n-1} \mathbf{map}(W, X) \to \dots \tag{7}$$

3.3 Derived Functors Without Injective Resolutions

In classical homological algebra (Strooker, Chapter 3), computing $\mathrm{Ext}^n_R(M, N)$ requires:

1. Finding an injective resolution $0 \to N \to I^0 \to I^1 \to \dots$
2. Applying $\mathrm{Hom}_R(M, -)$ to the complex.
3. Computing the n-th cohomology group of the resulting cochain complex.

In the stable ∞-categorical setting, let R be a ring. The derived ∞-category $\mathcal{D}(R)$ is defined as the stabilization of the ∞-category of simplicial R-modules (or the dg-nerve of the category of chain complexes):

$$\mathcal{D}(R) \simeq \mathrm{Mod}_R(\mathbf{Sp}). \tag{8}$$

Theorem 3.4. *For any R-modules M, N:*

$$\mathrm{Ext}^n_R(M, N) \cong \pi_{-n} \mathbf{map}_{\mathcal{D}(R)}(M, N) \cong [M, \Sigma^n N]_{\mathrm{h}\mathcal{D}(R)}. \tag{9}$$

Furthermore, the derived tensor product is the relative smash product in $\mathrm{Mod}_R(\mathbf{Sp})$*:*

$$\mathrm{Tor}^R_n(M, N) \cong \pi_n(M \otimes^{\mathbb{L}}_R N). \tag{10}$$

Theorem 3.5 (Canonical Snake Lemma)**.** *Let $0 \to A \to B \to C \to 0$ be a short exact sequence of R-modules. In $\mathcal{D}(R)$, this induces a fiber sequence*

$$A \to B \to C \xrightarrow{\partial} \Sigma A. \tag{11}$$

The connecting homomorphism $\delta \colon \mathrm{Hom}_R(W, C) \to \mathrm{Ext}^1_R(W, A)$ is the canonical boundary map of the fiber sequence. No diagram chasing or choices of injective resolutions are required.

4 ∞-Topoi and Higher Sheaf Cohomology

4.1 ∞-Sites and Hypercover Descent

Let X be a topological space, and let $\mathbf{O}(X)$ be its poset of open subsets.

Definition 4.1 (∞-Sheaf)**.** A functor $\mathscr{F} \colon \mathbf{O}(X)^{\mathrm{op}} \to \mathcal{S}$ is an *∞-sheaf* (or *stack*) if for every open $U \subseteq X$ and every hypercovering $U_\bullet \to U$, the canonical restriction map is an equivalence in $\mathcal{S}$:

$$\mathscr{F}(U) \xrightarrow{\sim} \lim_{[n] \in \Delta^{\mathrm{op}}} \mathscr{F}(U_n). \tag{12}$$

The ∞-category of ∞-sheaves $\mathrm{Sh}_\infty(X)$ is an **∞-topos** satisfying the Giraud-Lurie axioms: presentability, universal colimits, disjoint coproducts, and effective groupoid objects.

4.2 Cohomology as Homotopy Groups of Global Sections

Let $\mathscr{A}$ be an abelian sheaf on X. We associate to $\mathscr{A}$ the Eilenberg-MacLane spectrum sheaf $\mathbf{H}\mathscr{A} \in \mathrm{Sh}_\infty(X; \mathbf{Sp})$, characterized by $\pi_0(\mathbf{H}\mathscr{A}) \cong \mathscr{A}$ and $\pi_k(\mathbf{H}\mathscr{A}) = 0$ for $k \neq 0$.

Definition 4.2 (Higher Sheaf Cohomology)**.** The n-th sheaf cohomology group of X with coefficients in $\mathscr{A}$ is given by:

$$H^n(X; \mathscr{A}) \cong \pi_{-n}\Gamma(X; \mathbf{H}\mathscr{A}) \cong \pi_0 \operatorname{Map}_{\mathrm{Sh}_\infty(X)}(*, \mathbf{B}^n\mathscr{A}), \tag{13}$$

where $\Gamma(X; -) = \operatorname{Map}_{\mathrm{Sh}_\infty(X)}(*, -)$ is the global sections functor.

4.3 Model-Theoretic Interpretation of Classical Sheaf Classes

Strooker (Chapter 4) identifies three key classes of acyclic sheaves:

1. **Flabby Sheaves (Faisceaux Flasques)**: Restriction maps $\mathscr{F}(X) \to \mathscr{F}(U)$ are surjective.
2. **Soft Sheaves (Faisceaux Mous)**: Sections over closed subsets extend globally.
3. **Fine Sheaves (Faisceaux Fins)**: The sheaf of endomorphisms admits partitions of unity subordinate to any locally finite cover.

Theorem 4.3. *In the projective model category of simplicial presheaves* $\mathbf{sPSh}(X)$ *equipped with the local Jardine-Bousfield model structure:*

1. *A sheaf $\mathscr{F}$ is flabby if and only if it is fibrant in the injective model structure. Consequently, the global section functor preserves fiber sequences of flabby sheaves directly without derived replacement.*
2. *Soft and fine sheaves on paracompact Hausdorff spaces are acyclic because their higher cohomology groups vanish, meaning their mapping spectra are* 0*-truncated.*
3. *The Godement canonical resolution is the explicit construction of a fibrant replacement via the adjunction between sheaves on X and sheaves on the discrete underlying set of points $|X|$.*

5 Higher Categorical Semantics and Type Theory

The conceptual culmination of higher category theory is its internal synthesis with **Homotopy Type Theory (HoTT)** and Martin-Löf dependent type theory:

Strooker (1978)	$(\infty,1)$**-Category Theory**	**Homotopy Type Theory**
Set / Discrete Object	∞-Groupoid / Space $X \in \mathcal{S}$	Type $A\colon \mathcal{U}$
Element $x \in A$	Point $* \xrightarrow{x} X$	Term $a\colon A$
Equality $x = y$	Path Space $\operatorname{Map}_A(x, y)$	Identity Type $\mathrm{Id}_A(x, y)$
Isomorphism $A \cong B$	Equivalence $A \simeq B$	Type Equivalence $A \simeq B$
Category $\mathbf{C}$	Quasi-Category $\mathcal{C}$	Complete Segal Type
Functor $F\colon \mathbf{C} \to \mathbf{D}$	∞-Functor $F \in \mathrm{Fun}(\mathcal{C}, \mathcal{D})$	Type Function $F\colon C \to D$
Abelian Category $\mathbf{A}$	Stable ∞-Category $\mathcal{C}$	Spectrum Type
Sheaf Cohomology $H^n(X; \mathscr{F})$	Homotopy Group $\pi_{-n}\Gamma(X; \mathscr{F})$	Truncated Dependent Sum

Under Voevodsky's **Univalence Axiom**, $(A \simeq B) \simeq (A =_{\mathcal{U}} B)$, equivalent mathematical objects are identically equal inside the type-theoretic universe. Strooker's categorical principles are thereby realized not as external metalogical conventions, but as theorems of an internal computational logic.

6 Conclusion

The modernization of Strooker's *Introduction to Categories, Homological Algebra and Sheaf Cohomology* into higher category theory demonstrates that modern $(\infty, 1)$-categories do not replace classical foundations, but rather fulfill their structural promises. By replacing discrete sets with homotopy types, abelian categories with stable ∞-categories, and 1-sheaves with ∞-topoi, the entire edifice of homological algebra and sheaf cohomology attains intrinsic coherence and computational realization.

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