Exact Quantum Records: Spectral Capacity, Local Readout, and Flag Obstructions

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Abstract

We correct and extend a previous note on finite information and spatial atoms. We define localized records using local measurements and joint preparation, clarify which unitaries preserve restricted admissible preparations, and give complete proofs of the resulting packing and bundle statements. For a semibounded Hamiltonian with a discrete spectrum tending to infinity, we prove an exact one-shot zero-error capacity formula under a per-codeword mean-energy bound and unrestricted measurements. For an oscillator it is $\log_2\lfloor 1 + 2E/(\hbar\omega)\rfloor$. For unrestricted rank-$d$ families, maximal codes form an ordered flag space; global codes correspond to reductions from $U(d)$ to $U(1)^d$. A finite-volume atomicity conclusion still requires a uniform information-density bound and record richness. No result here establishes those physical premises or a fundamental unit of space.

Keywords: quantum state discrimination; energy constraint; local observables; vector bundles; homotopy quotient; measure atomicity.

1. Introduction and corrections

The preceding article [1] distinguishes finite information from spatial atomicity and proves several conditional results. Two qualifications sharpen its statements. First, its Definition 5.1 specifies global joint preparation and decoding but does not explicitly require each label to be recoverable in its named subregion. Its packing inequality is valid for the global code it defines; spatial localization requires additional local effects. Section 4 supplies such a definition and proves the corresponding theorem.

Second, Section 8 of [1] invokes code data from a definition that permits restricted preparations and measurements, but its full $U(d)$ action and dimension-only existence criterion require unrestricted data, or suitable invariant constraints. Section 6 gives the unrestricted theorem and a resource-preserving formulation. These are corrections of the interpretation and hypotheses, not a claim that the earlier article derived a physical lattice.

We also prove an energy-constrained spectral formula in Section 3. Its proof uses elementary spectral ordering and a Fourier basis. We do not assert historical priority or an exhaustive novelty search. The main capacity results concern one-shot exact classical messages; measurement apparatus, reference systems, preshared entanglement and repeated uses are excluded unless included explicitly in the model. Standard quantum information and vector-bundle background may be found in [2,3].

2. Exact codes and entropy

A **density operator** is a positive trace-class operator of trace one. An m-message **exact code** is a family $\rho_1, \dots, \rho_m$ of admissible density operators together with an allowed POVM $(M_1, \dots, M_m)$ such that

$$M_j \geq 0, \qquad \sum_{j=1}^{m} M_j = I, \qquad \operatorname{tr}(M_j \rho_i) = \delta_{ij}.$$

Assume at least one code exists. Write N_0 for the supremum of finite attainable message counts and $C_0 = \log_2 N_0$, allowing $+\infty$.

For a unital C*-algebra A, the same definition uses normalized positive states ω_i and effects $M_j \in A$, with $\omega_i(M_j) = \delta_{ij}$. The notation N_0, C_0 applies to the explicitly declared admissible states and POVMs. The Hilbert-space support and spectral proofs below use density operators; the local packing theorem applies to these algebraic states.

Lemma 2.1 (orthogonal supports). The supports of distinct states in an exact code are orthogonal. Conversely, finitely many density operators with pairwise orthogonal supports can be exactly distinguished by some POVM; its admissibility is an additional condition.

Proof. The exact decoding equations give $\operatorname{tr}((I - M_i)\rho_i) = 0$ and $\operatorname{tr}(M_i\rho_j) = 0$ for $j \neq i$. If $A, \rho \geq 0$ and $\operatorname{tr}(A\rho) = 0$, then $A^{1/2}\rho^{1/2} = 0$. Thus M_i is the identity on the support of ρ_i and zero on the support of ρ_j. These eigenspaces are orthogonal. Conversely, take the support projections and add the complementary projection to one outcome. □

Proposition 2.2 (dimension and error). On $\mathbb{C}^d$, any exact code has $m \leq d$. If every state and POVM is allowed, $N_0 = d$. If m states have uniform prior and average error at most $\epsilon < 1$, then $m \leq d/(1 - \epsilon)$.

Proof. The supports in Lemma 2.1 are nonzero and orthogonal; a basis attains d messages. For the error bound, $0 \leq \rho_i \leq I$ and $\sum_i M_i = I$ imply

$$1 - \epsilon \leq \frac{1}{m} \sum_i \operatorname{tr}(M_i\rho_i) \leq \frac{1}{m} \sum_i \operatorname{tr} M_i = \frac{d}{m}.$$

Rearrange. □

Let $s_2(\rho) = -\operatorname{tr}(\rho \log_2 \rho)$, with $0 \log 0 = 0$.

Proposition 2.3 (entropy and dimension). For nonzero H,

$$\sup_{\rho \in \mathcal{D}(H)} s_2(\rho) < \infty \quad \Longleftrightarrow \quad \dim H < \infty.$$

In dimension d the supremum is $\log_2 d$.

Proof. For eigenvalues p_i with $r \leq d$ nonzero terms, Jensen's inequality gives $\sum p_i \log_2(1/p_i) \leq \log_2 r \leq \log_2 d$. The state I/d attains equality. If H is infinite-dimensional, the uniform mixture of any n orthonormal vectors has entropy $\log_2 n$, which is unbounded. □

Proposition 2.4 (entropy bounds codes under mixing). Suppose the admissible set contains the uniform mixture of every admissible exact code and each admissible state's entropy is at most B. Then $C_0 \leq B$.

Proof. Orthogonal supports give

$$s_2\left(\frac{1}{m} \sum_i \rho_i\right) = \log_2 m + \frac{1}{m} \sum_i s_2(\rho_i) \geq \log_2 m.$$

The mixture is admissible, so $\log_2 m \leq B$ for every code. □

The mixing hypothesis cannot be dropped: all pure states on an infinite-dimensional space have entropy zero, yet include orthogonal codes of arbitrary finite size.

3. Exact capacity under an energy constraint

Let H be a nonnegative self-adjoint Hamiltonian on a nonzero separable Hilbert space, with a complete eigenbasis and eigenvalues $0 \leq e_0 \leq e_1 \leq \cdots$, repeated according to multiplicity. Either the list has finite length d, or it is infinite with $e_k \to \infty$. The energy of a state is $\operatorname{tr}(\rho H)$ in the nonnegative quadratic-form sense. We allow every state of energy at most E and every POVM on this system, with no ancillary resource. Let $N_0(E)$ be the one-shot exact message count.

Theorem 3.1 (spectral capacity). For $E \geq e_0$,

$$N_0(E) = \max\left\{ m \geq 1 :\ m \leq d \text{ if } d < \infty,\ \frac{1}{m}\sum_{k=0}^{m-1} e_k \leq E \right\}.$$

In the infinite case the maximum is finite because $e_k \to \infty$.

Proof. Let $\rho_1, \ldots, \rho_m$ be an exact code. Orthogonal supports and $0 \leq \rho_i \leq I$ give $Q = \sum_i \rho_i \leq I$ and $\operatorname{tr} Q = m$. In an eigenbasis of H put $q_k = \langle k|Q|k\rangle$. Then $0 \leq q_k \leq 1$, $\sum_k q_k = m$, and

$$\sum_k e_k q_k = \sum_i \operatorname{tr}(\rho_i H) \leq mE.$$

If $d < \infty$, orthogonal supports also give $m \leq d$. Set $a = e_{m-1}$. Since $\sum_{k\geq m} q_k = \sum_{k<m}(1 - q_k)$, and all terms below are nonnegative,

$$\sum_k e_k q_k - \sum_{k<m} e_k = \sum_{k\geq m}(e_k - a)q_k + \sum_{k<m}(a - e_k)(1 - q_k) \geq 0.$$

The required average-energy inequality follows.

Conversely, for every m meeting that inequality define

$$|\psi_j\rangle = \frac{1}{\sqrt{m}}\sum_{k=0}^{m-1} e^{2\pi i jk/m}|k\rangle, \qquad 0 \leq j < m.$$

The Fourier vectors are orthonormal, and each has mean energy $(\sum_{k<m} e_k)/m \leq E$. Projections onto these vectors, with the orthogonal complement assigned to one outcome, form an allowed exact POVM. Thus the upper bound is attained. □

Corollary 3.2 (oscillator). For $H = \hbar\omega N$, $\omega > 0$, with ground energy subtracted,

$$N_0(E) = \left\lfloor 1 + \frac{2E}{\hbar\omega} \right\rfloor, \qquad C_0(E) = \log_2 \left\lfloor 1 + \frac{2E}{\hbar\omega} \right\rfloor.$$

Proof. Here $\frac{1}{m}\sum_{k<m} e_k = \hbar\omega(m-1)/2$. Apply Theorem 3.1. □

This expectation constraint is not a hard spectral cutoff: a code state may have support on levels above E. The result also does not constrain the energy or physical realizability of the measurement device.

For comparison, the maximum von Neumann entropy at mean occupation $\bar{n} = E/(\hbar\omega)$ is

$$g(\bar{n}) = (\bar{n} + 1)\log_2(\bar{n} + 1) - \bar{n}\log_2 \bar{n}.$$

Indeed, dephasing in the energy basis does not decrease entropy, and nonnegativity of classical relative entropy against the geometric distribution $(1 - q)q^k$, $q = \bar{n}/(1 + \bar{n})$, yields this upper bound; the diagonal Gibbs state attains it. At $\bar{n} = 0$ the ground state is the only admissible state. At $\bar{n} = 1$, for example, $g(1) = 2$ but $C_0 = \log_2 3$. Thus an entropy upper bound need not be tight for one-shot exact messages.

4. Local records and packing

Let $R_1, \ldots, R_n \subseteq R$. Let each local unital C-*algebra embed into a global unital* C-algebra $\mathcal{A}(R)$, and let Ω_R and the allowed global POVMs be explicitly specified. For $m_i \geq 1$, a **locally readable joint record** consists of local POVMs $\{M_a^{(i)}\}_{a=1}^{m_i}$ in $\mathcal{A}(R_i)$ and, for every tuple $\mathbf{a} = (a_1, \ldots, a_n)$, a global admissible state $\omega_{\mathbf{a}} \in \Omega_R$ such that

$$\omega_{\mathbf{a}}(M_{a_i}^{(i)}) = 1 \quad \text{for all } i.$$

Assume the effects belonging to different local POVMs commute and the product POVM is allowed globally. This includes both joint preparation of all label tuples and a local readout requirement.

Theorem 4.1 (local packing). If the global exact capacity satisfies $C_0(R) \le B(R) < \infty$, then every locally readable joint record satisfies

$$\sum_{i=1}^{n} \log_2 m_i \le B(R).$$

If each record has at least $b_0 > 0$ bits, then $n \le \lfloor B(R)/b_0 \rfloor$.

Proof. For each outcome tuple $\mathbf{b}$, set $M_{\mathbf{b}} = \prod_i M_{b_i}^{(i)}$. These operators are positive by commutativity and sum to I. For commuting positive contractions, $I - \prod_i E_i \le \sum_i (I - E_i)$. Taking $E_i = M_{a_i}^{(i)}$ and applying $\omega_{\mathbf{a}}$ shows $\omega_{\mathbf{a}}(M_{\mathbf{a}}) = 1$. Normalization makes all other joint-outcome probabilities zero. Thus the allowed states and product POVM form an exact code of size $\prod_i m_i$, proving the inequality; divide by b_0 for the bound on n. □

Pairwise commutativity alone does not imply the existence of the joint preparations or the admissibility of their product measurement. In [1], the packing argument remains correct as global counting; the explicit local requirements above are needed to infer localized records.

5. Geometry requires additional premises

On $X = \mathbb{R}^3$ with ordinary volume, assign $\mathcal{A}(R) = \mathbb{C} \oplus \mathbb{C}$ to each nonempty bounded region, with identity inclusions, and scalars to the empty region. With all states and POVMs allowed, each nonempty region has exactly one bit of classical capacity, all observables commute, and balls of arbitrarily small positive volume exist. The shared bit is not independently preparable in disjoint regions. This proves finite positive regional capacity alone does not imply a positive volume gap.

Theorem 5.1 (conditional volume bound). If $b_0, \kappa > 0$ and every record-bearing region in a given class satisfies

$$b_0 \le C_0(R) \le B(R) \le \kappa \mu(R),$$

then $\mu(R) \ge v_0 = b_0/\kappa$ for every region in that class.

Proof. Combine the inequalities and divide by κ. □

Theorem 5.2 (conditional finite measure algebra). Suppose $0 < \mu(X) < \infty$ and every positive-measure measurable $A \subseteq X$ obeys the hypotheses of Theorem 5.1 with the same constants. Then the measure algebra is generated by at most $\lfloor \mu(X)/v_0 \rfloor$ positive atoms.

Proof. Every positive-measure set has measure at least v_0. Start with the partition $\{X\}$ and split any positive part that is not an atom into two positive measurable parts. A partition with n positive parts has $nv_0 \le \mu(X)$, so only finitely many splits are possible and the final parts are atoms. Every measurable subset meets each atom in a null or full-measure subset, and therefore is, modulo null sets, a union of those atoms. □

These premises fail for ordinary atomless Euclidean volume if every positive-measure set is required to carry a nonzero record under the uniform density bound. Measure atomicity does not imply equal atoms or a minimum distance between underlying points. Atoms of measures 1 and $\sqrt{2}$ also show that a positive gap need not imply a common volume unit whose integer multiples contain both measures.

6. Bundle codes and the homotopy quotient

Let Y be a nonempty paracompact Hausdorff space, and $V \to Y$ a rank-d Hermitian bundle. In this section all continuous density endomorphisms and POVM endomorphisms satisfying the code equations are allowed. A **global code** consists of such endomorphisms defined over all of Y; a local code is defined on an open neighborhood.

Theorem 6.1 (maximal codes and line decompositions). Local m-message codes exist everywhere exactly when $m \le d$. A global d-message code exists exactly when V splits orthogonally into d complex line subbundles.

Proof. Local orthonormal frames provide codes for $m \le d$, while Lemma 2.1 excludes $m > d$ in any fiber. For a global d-code, the d nonzero pairwise orthogonal supports in each d-dimensional fiber are lines. Each density operator is the corresponding rank-one projection. Continuity makes its image a line subbundle: locally, applying the projection to a fixed nonzero vector in its image gives a nonvanishing section. These lines are mutually orthogonal and span V. Conversely, their orthogonal projections are continuous and form both the code states and decoding POVM. □

Proposition 6.2 (flag moduli). With ordered labels and unrestricted data,

$$\mathrm{Code}_d(\mathbb{C}^d) \cong U(d)/U(1)^d, \qquad \mathrm{Code}_d(\mathbb{C}^d)//U(d) \simeq BU(1)^d.$$

The map to $BU(d)$ is induced by the diagonal inclusion $U(1)^d \hookrightarrow U(d)$.

Proof. In a maximal exact code, each state is the rank-one projection onto its support. Every POVM effect acts as the identity on its own line and as zero on all other lines; since those lines span the fiber, the effect equals the same projection. Hence a code is exactly an ordered orthogonal decomposition into lines. $U(d)$ acts transitively and the stabilizer is $U(1)^d$, giving the first homeomorphism. For the second, the Borel construction $EU(d) \times_{U(d)} U(d)/U(1)^d$ is $EU(d)/U(1)^d$, a model for $BU(1)^d$. □

By the classification of numerable bundles [3], Theorem 6.1 is equivalently the existence, up to homotopy, of a lift of the classifying map $Y \to BU(d)$ through $BU(1)^d \to BU(d)$. The associated flag bundle has a section precisely when such a reduction exists. Compatible code endomorphisms on an open cover glue because endomorphisms and their pointwise equations glue. The groupoid stack of local Hermitian bundles, code data and unitary isomorphisms embeds as a 1-truncated object in $\mathrm{Sh}_\infty(Y)$ [8]; the topological classifying space $BU(d)$ above is not being identified with a constant sheaf. This is descent over the parameter space; partial trace over a subsystem is a different operation and can destroy exact distinguishability.

For example, the complex rank-two Hopf bundle over S^4 has nonzero c_2 [4]. As $H^2(S^4;\mathbb{Z}) = 0$, a sum of two line bundles would have $c_2 = 0$. It therefore has local two-message codes but no global continuous two-message code.

For restricted systems, retain a triple $(H, \Omega, \mathcal{M})$ of a Hilbert space, an admissible state set and an admissible POVM collection. A unitary equivalence must transport both Ω and $\mathcal{M}$ by conjugation. The automorphism group of a fixed resource structure consists only of unitaries preserving those sets. For instance, an admissible set containing only two computational-basis states is not preserved by a Hadamard rotation. Consequently, the full $U(d)$ quotient and dimension-only existence criterion above apply to unrestricted data, not to arbitrary resource restrictions.

7. Conditional gravitational illustration and scope

Assume, for a spherical system, both $C_0(R) \le 2\pi E r/(\hbar c \ln 2)$ and $E \le c^4 r/(2G)$. If $C_0(R) \ge b_0 > 0$, direct substitution gives

$$r \ge \ell_P \sqrt{\frac{b_0 \ln 2}{\pi}}, \qquad \ell_P^2 = \frac{\hbar G}{c^3}.$$

The first inequality is an assumed operational capacity interpretation of an entropy bound, and the second is a Schwarzschild comparison; their simultaneous applicability at Planck scale is not established. Bekenstein's bound and Casini's relative-entropy formulation have their own hypotheses and interpretations [5,6]. Continuum quantum field theories commonly use type III local algebras under suitable conditions [7]; the finite-dimensional bundle model is not a derivation of such algebras.

The spectral formula concerns exact one-shot classical discrimination under a per-state expectation bound. The local packing theorem assumes jointly admissible preparations and measurement. The measure theorem assumes a uniform density bound and richness on every positive measurable set. The bundle theorem assumes unrestricted continuous code data. None proves those assumptions physically, establishes durable storage, or derives spatial discreteness from quantum mechanics. The attached finite-dimensional computations test examples and implementation arithmetic, not the general proofs or experimental claims.

Authorship and automation. The responsible submitting researcher is identified publicly as Anonymous author. The research question and publication request were supplied by that researcher; OpenAI ChatGPT/Codex assisted in drafting the manuscript and numerical checks. The submitting researcher retains responsibility for reviewing the claims.

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