# Induction and Completion in Group-Algebra Semantics

Finite Quotients, Higher Coherence, and Partial Evaluation

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November 2026

## Abstract

A finite group law has a finite verification procedure, but the resulting classification problem changes with the object, its presentation, and the data retained by a functor. We distinguish effective bounded enumeration, classification of finite simple factors, reconstruction from compatible finite quotients, and finite-stage descent in higher categories. Completed group algebras equipped with their continuous Hopf structure retain the profinite group. Their continuous duals are filtered unions of finite function Hopf algebras. Even computable towers of finite cyclic groups have an undecidable isomorphism problem for a fixed infinite limit. Finite coskeletal Kan tables give an effective classification of $\pi$-finite spaces. Finite coherent tests on pro-objects descend individually, while unbounded families can require arbitrarily fine stages. For partial evaluation, we prove that compatibility of the action and effect observations with quotient maps makes finite residual programs commute with refinement. A reference implementation checks this compatibility on a nonabelian tower. The results give a criterion for when finite symmetry data suffice for a given computation. **Keywords.** Profinite groups, group algebras, Lawvere theories, $\infty$-categories, partial evaluation, interpreter composition.

## 1 Classification depends on the data

Klein's Erlangen program specifies a space and a group of transformations, then studies the properties invariant under that action [1]. This suggests a practical question. If a theory or a computation uses finite symmetries, how much does an inventory of those symmetries determine? The answer changes as we pass from a group to an action, from the action to a linear representation, or from finite quotients to a compatible higher system. We use *classification* with the equivalence relation and the input model made explicit. A finite table, a finite presentation, a profinite tower, and a homotopy-coherent diagram provide different input models.

There are two distinct finiteness claims in the motivating question. One concerns the specification of the operations. The other concerns the objects and morphisms that the operations generate. The Lawvere theory $\mathbb{T}_{\mathrm{Grp}}$ is the opposite of the category of finitely generated free groups, with objects the finite powers of its distinguished generator. A map $n \to m$ is an $m$-tuple of words on $n$ letters, and composition is simultaneous substitution. A model in a category $\mathcal{E}$ with finite products is a product-preserving functor $\mathbb{T}_{\mathrm{Grp}} \to \mathcal{E}$. It sends the operations $2 \to 1$, $1 \to 1$, and $0 \to 1$ to multiplication, inverse, and unit. The equations are equalities of composites of arrows in $\mathbb{T}_{\mathrm{Grp}}$. This describes the syntax independently of the target category [10, 11]. On $\mathsf{Set}$ the same data give a specific finitary monad $X \mapsto UF(X)$, whose unit inserts variables and whose multiplication substitutes words. Its algebras are groups. The Lawvere theory is the reusable specification and the monad is formed on its chosen base; forming an analogous free-model monad on another base requires the corresponding free models.

In an ordinary group presentation $\langle S \mid R \rangle$, a relation $u = v$ is a pair in $F(S) \times F(S)$ and imposes $uv^{-1} = 1$. The presented group is the quotient by the normal closure of these differences. Elements of that closure have finite expressions as products of conjugates of relators and their inverses. This yields finite certificates for true equations, provided relators can be enumerated. A corresponding terminating procedure for every false equation need not exist. The distinction persists for finite syntax. In the internal language of a topos, the same algebraic presentation can express group objects. Decidability or external finiteness of every model does not follow from that internal formulation.

**Results.** Finite order bounds yield terminating enumeration; simple composition factors describe a different, structural classification. The completed group algebra and its continuous dual recover every profinite group when equipped with the appropriate Hopf structure. The dual gives an exact sense in which finite stages still generate an algebra by a filtered union. At the higher level, descent controls individual finite observations, while one stage for an entire compatible family requires more. Finally, a residual evaluator at one quotient can agree with projection of the evaluator at every finer quotient. The paper proves this for actions, observations and control decisions that respect refinement, and tests it on a concrete tower

with deliberately incompatible observations as negative controls.

**Relation to prior work.** The journal treatments of higher representation theory and higher Galois theory develop continuous actions and pro-space tests [31, 32]. Its treatment of higher e-graphs and Futamura projections considers coherence in specialization [33]. Here the comparison is between quotient stages for an explicitly given group algebra and an effectful interpreter. The undecidability result concerns executable presentations of those stages, and the specialization result gives an exact residual comparison under compatible observations.

**Conventions.** Products of groups and all tensor products are taken over the specified base. Profinite groups carry their usual compact Hausdorff topology. We explicitly indicate when a coefficient field is discrete and when a tensor product is completed. In higher statements limits, colimits, and mapping spaces are homotopy-coherent. We use one universe for small shapes and a larger one for categories of models as needed.

## 2 Finite tables, finite presentations, and symmetries

The Lawvere theory in Section 1 can have nonconstant internal models. In the topos of $C_2$-sets, give $C_3$ either the trivial action or the action by inversion. Both carry internal group structures with the same underlying ordinary group. Their global elements number three and one, respectively, so they are distinct as internal groups. For an explicit finite carrier in $\mathsf{Set}$, the first comparison is exhaustive.

**Proposition 2.1** (Enumeration on a bounded carrier)**.** *For every $n \geq 1$, there is a terminating algorithm returning one canonical table for every group of order $n$. Given two explicit group tables, isomorphism is decidable.*

*Proof.* List the $n^{n^2}$ binary operations on $\{1, \dots, n\}$ in lexicographic order. Test associativity on all triples, check for a two-sided identity, and check for two-sided inverses. Act on each passing table by the finite permutation group $S_n$ and retain the lexicographic minimum of its orbit. A permutation carries one table to another precisely when it is a group isomorphism. Tables of differing size can be rejected immediately. □

Running Proposition 2.1 at each $n$ enumerates finite groups without duplicates. The classification of finite simple groups provides a structural theorem about the composition factors that can occur [2]. The Jordan–Hölder theorem fixes their multiset for a given finite group. The structures of the extensions joining the factors still matter. The sequence with kernel $C_2$ and quotient $C_2$ has both the split form $C_2 \times C_2$ and the nonsplit form $C_4$. A split extension with identified kernel and quotient and a chosen homomorphic section is determined by the resulting conjugation action of the quotient on the kernel. For an abelian kernel $A$ and prescribed quotient action of $Q$, arbitrary extensions with fixed identifications of $A, Q$ are classified by $H^2(Q; A)$. Indeed a normalized section $s$ gives $f(q, r) = s(q)s(r)s(qr)^{-1}$ in $A$, and associativity gives

$$f(q, r) + f(qr, t) = q \cdot f(r, t) + f(q, rt).$$

Replacing the section adds a coboundary. Conversely, this identity gives associativity for $(a, q)(b, r) = (a + q \cdot b + f(q, r), qr)$ and normalization provides the identity. The resulting formula also realizes the two extensions above. The October 2026 Axiom Math release formalizes the exhaustiveness direction of the classification in Lean. Its main theorem states that every finite simple group is isomorphic to a group in the specified family [8, 9]. The machine-checked Feit–Thompson theorem gives a separate result, solvability of finite groups of odd order [3]. We cite these developments as scope checks for the formalization claim.

Element orders constrain the order of the ambient group by Lagrange's theorem. If $a$ and $b$ commute and their orders $r, s$ are coprime, then $ab$ has order $rs$. Indeed $(ab)^k = 1$ implies $a^k = b^{-k}$ lies in $\langle a \rangle \cap \langle b \rangle$, a subgroup whose order divides both $r$ and $s$. For $S_3$, a transposition and a 3-cycle give element orders 2 and 3. Every element has one of the cycle types $1^3$, $2\,1$, and 3, so the order-six group has no order-six element.

**Finite syntax and proof search.** An arbitrary finite group admits a finite presentation with symbols $x_g$ for its elements and relations $x_e = 1$ and $x_g x_h = x_{gh}$. Evaluation of the symbols in $G$ and multiplication of the corresponding symbols construct inverse homomorphisms. The free one-generator group $\mathbb{Z}$ gives an infinite group with a finite presentation. Every group is a filtered union of its finitely generated subgroups. Every group is also a filtered colimit of finitely presented groups obtained by taking finite subsets of the generators and relators in a presentation, including every generator mentioned in the chosen relators. The second colimit follows by the universal property of presentations and its transition maps may identify elements.

**Proposition 2.2** (Finite-promise conversion)**.** *There is an algorithm on finite presentations that terminates with a canonical multiplication table when the presented group is finite.*

*Proof.* For a presentation $\langle S \mid R \rangle$ enumerate finite group tables $Q$ and tuples assigning each $s \in S$ an element $\varphi(s)$ of $Q$. Test the relators in $Q$. For each passing tuple, search over choices of words $w_q$ with $\varphi(w_q) = q$ and over

finite certificates of the equations

$$w_1 = 1, \qquad w_q w_r = w_{qr}, \qquad w_{\varphi(s)} = s$$

in the presented group. Such certificates can be enumerated as products of conjugates of relators and their inverses in $F(S)$. Dovetail all searches. A successful tuple yields inverse homomorphisms by the displayed equations. If the presented group is finite, its table and representatives occur in the search and all required equations have finite certificates. Canonicalize the output by Proposition 2.1. □

Thus positive equality of words is semidecidable from a finite presentation. For a residually finite group one can also search for a finite quotient separating unequal words by the same assignments to finite tables. The two searches give a word-problem algorithm under this promise. Without a promise, the Adian–Rabin theorem implies that no algorithm decides triviality of a finitely presented group [4, Theorem 8.4]. Even existence of a nontrivial finite quotient is undecidable by Bridson–Wilton [5]. Finite terms and finite witnesses of provable relations coexist with unbounded search for a witness and undecidable questions about the presented object.

# 3 What group algebras and their modules classify

Fix a field $k$. For a discrete group $G$, the group algebra $k[G]$ is the vector space of finite linear combinations of basis vectors $[g]$, with $[g][h] = [gh]$. Every representation $G \to \mathrm{GL}_k(V)$ extends uniquely by linearity to a unital module structure on $V$, and restriction along $g \mapsto [g]$ is the inverse construction. For any unital $k$-algebra $B$, this linear extension gives a natural bijection

$$\mathrm{Hom}_{k\text{-alg}}(k[G], B) \cong \mathrm{Hom}_{\mathrm{Grp}}(G, B^\times).$$

Consequently $k[-]$ preserves colimits, and for every $\langle S \mid R\rangle$ it yields a presentation by generators $s, s^{-1}$ with relations for the inverse pairs and $r = 1$ for each $r \in R$. For finite $G$ it has dimension $|G|$. Conversely, the basis for an infinite $G$ is infinite even when its presentation is finite.

**Proposition 3.1** (Dependence on input and base)**.** *Over $\mathbb{F}_q$, finite group algebras of dimension $n$ can be enumerated canonically as associative unital algebras, and their isomorphism problem on explicit multiplication tables is decidable. For any fixed field $k$, isomorphism to $k$ among group algebras given by finite group presentations is undecidable.*

*Proof.* In a chosen vector-space basis, multiplication is a tensor of $n^3$ structure constants and the unit is a vector. Enumerate the finite group tables of order $n$, linearize them, and canonicalize the tensor and unit under the finite group $\mathrm{GL}_n(\mathbb{F}_q)$. A linear isomorphism preserving the recorded data is precisely an algebra isomorphism. For the undecidability result, construct the finite presentation of $k[G]$ described above. The group basis gives $k[G] \cong k$ exactly when $G = 1$, so a decision algorithm would decide the Adian–Rabin problem. □

**Semisimple coefficients.** If $|G|$ is invertible in $k$, an ordinary projection onto an invariant subspace can be averaged over $G$ to obtain an equivariant projection. Maschke's theorem and the splitting-field form of Artin–Wedderburn give, for a splitting field $k$,

$$k[G] \cong \prod_{i=1}^{r} M_{d_i}(k), \qquad d_i = \dim_k(V_i), \qquad \sum_i d_i^2 = |G|,$$

where $V_i$ run over the irreducibles [6]. The multiset $(d_i)$ classifies the split semisimple algebra. Every finite-dimensional representation is classified within that algebra by the multiplicity vector of its simple summands. For example, $C_4$ and $C_2 \times C_2$ have four complex one-dimensional characters each. Their complex group algebras both have underlying algebra $\mathbb{C}^4$, and the representation categories are equivalent as $\mathbb{C}$-linear categories. Their groups differ in exponent.

**Modular coefficients.** In characteristic $p$, $k[C_{p^a}] \cong k[t]/(t^{p^a})$ by setting $t = g - 1$. Finite-dimensional modules correspond to nilpotent operators $T$ with $T^{p^a} = 0$, and the indecomposables are the individual nilpotent Jordan blocks of sizes $1, \dots, p^a$. For $C_p \times C_p$, the algebra is $k[x, y]/(x^p, y^p)$, where the variables are the two group generators minus 1. Here representation dimension is unbounded even among indecomposables, as the following modules show.

**Example 3.2** (Unbounded modular indecomposables)**.** For $n \geq 1$, let $J$ be the size-$n$ nilpotent Jordan block, and make $k^n \oplus k^n$ a $k[C_p \times C_p]$-module by $x(a, b) = (0, a)$ and $y(a, b) = (0, Ja)$. These operators have all products of length two zero and satisfy the defining algebra relations. Commutation with $x$ forces an endomorphism to have block form $\left(\begin{smallmatrix} A & 0 \\ C & A \end{smallmatrix}\right)$, and commutation with $y$ requires $AJ = JA$. The cyclicity of $J$ implies that its commutant is $k[J] \cong k[t]/(t^n)$, a local algebra. An idempotent endomorphism therefore has $A = 0$ or $A = I$. Its lower-left idempotency equation $CA + AC = C$ then forces $C = 0$. There are only the two trivial idempotents, so the module is indecomposable for every $n$.

Characteristic two also gives finite, nonisomorphic 2-groups whose associative group algebras over every characteristic-two field are isomorphic [7]. In particular, the algebra may distinguish the groups in one pair

and identify the groups in another. For the earlier pair $C_4, C_2 \times C_2$, the radical modulo its square in characteristic two has dimension one for $k[t]/(t^4)$ and dimension two for $k[x,y]/(x^2,y^2)$, so those particular algebras are distinct.

**Proposition 3.3** (Reconstruction from the Hopf structure)**.** *Equip $k[G]$ with $\Delta[g] = [g] \otimes [g]$, $\epsilon[g] = 1$, and $S[g] = [g^{-1}]$. Then the group-like elements recover $G$, and every Hopf $k$-algebra map $k[G] \to k[H]$ arises uniquely from a group homomorphism $G \to H$. For finite $G$, the tensor category of finite-dimensional representations equipped with its forgetful tensor functor to vector spaces also recovers $G$.*

*Proof.* A group-like element $a = \sum a_g[g]$ satisfies $\Delta a = a \otimes a$ and $\epsilon a = 1$. The off-diagonal coefficients require $a_g a_h = 0$ for $g \neq h$. Since $k$ is a field, exactly one coefficient is nonzero, and the counit sets it to 1. This proves the first two assertions. For the last, put $A = k[G]$ and let $F$ be the forgetful functor. If $\eta$ is a natural endomorphism of $F$, put $a = \eta_A(1)$ using the finite regular module. Naturality on the $A$-linear maps $A \to M$ given by $b \mapsto bm$ yields $\eta_M(m) = am$. The tensor and unit conditions force $\Delta a = a \otimes a$ and $\epsilon a = 1$ by testing the regular module on both factors and evaluating at $1 \otimes 1$, and by testing the unit representation. Conversely, such an $a$ acts as a tensor automorphism. Thus $\mathrm{Aut}^{\otimes}(F) \cong G$. □

The data retained by the category of representations matter. A linear module-category equivalence can lose information even about the algebra, as $k$ and $M_d(k)$ have equivalent linear module categories. The group reconstruction above uses the monoidal structure and the specified fiber functor.

# 4 Profinite groups and continuous linearization

## 4.1 Finite quotients and their compatibility

Write $U \trianglelefteq_o G$ when $U$ is an open normal subgroup of a profinite group $G$. The quotients $G/U$ are finite, and their transition maps send $gV$ to $gU$ when $V \subseteq U$. The limit is taken with the initial topology of the quotient projections. Throughout this section, homomorphisms between profinite groups are continuous. For background on these constructions see [12].

**Proposition 4.1.** *The canonical map $G \to \varprojlim_U G/U$ is a topological isomorphism. For profinite groups $G, H$,*

$$\mathrm{Hom}_{\mathrm{cont}}(G,H) \cong \varprojlim_{V \trianglelefteq_o H} \varinjlim_{U \trianglelefteq_o G} \mathrm{Hom}(G/U, H/V). \quad (1)$$

*In particular the category of profinite groups is equivalent to* Pro(FinGrp).

*Proof.* The open normal subgroups form a neighborhood basis of the identity, so they separate points. A compatible tuple of cosets has the finite intersection property and therefore has a representative in $G$ by compactness. Compactness and Hausdorffness give the topological isomorphism. A continuous map to $H/V$ has open kernel and hence factors through $G/U$ for some $U$. Two finite-stage maps agreeing on $G$ agree after precomposition with the projection from a common refinement $G/(U \cap U')$. Taking the limit over $V$ proves (1). To use this description for arbitrary cofiltered diagrams of finite groups, replace each stage by the eventual image of the transition maps into it. These images stabilize by finiteness and cofilteredness, the replacement is isomorphic as a pro-object, and the resulting projections from the compact limit are surjective by the finite intersection property. Thus every such pro-object is represented by its compact limit and finite quotient system. □

Every finite target observes some single finite source quotient. A map with an infinite target instead supplies coherent maps for all finite target quotients, and the necessary source quotient can grow with the target. For instance the identity of an infinite profinite group cannot factor through a finite group. This is a direct consequence of the order of the limit and colimit in (1).

**Example 4.2.** For any set $S$ of primes, set $G_S = \prod_{p \in S} C_p$. The element whose coordinates are generators topologically generates $G_S$, because its powers project surjectively to every finite subproduct by the Chinese remainder theorem. The group admits a continuous quotient $C_p$ exactly when $p \in S$. The forward implication follows because a continuous map to $C_p$ factors through a finite subproduct. This gives continuum many topologically singly generated isomorphism types. Only countably many finite descriptions over a countable alphabet are available, even if the descriptions are allowed to specify algorithms for constructing quotient towers. The cardinality argument concerns all profinite groups, including those with no effectiveness condition.

**Example 4.3.** For every infinite set $I$, the finite continuous quotient types of $C_2^I$ are precisely the finite elementary abelian 2-groups. Nevertheless, the vector space $\mathrm{Hom}_{\mathrm{cont}}(C_2^I, C_2)$ over $\mathbb{F}_2$ is $\bigoplus_I \mathbb{F}_2$, since each continuous character depends on finitely many coordinates. Its dimension distinguishes groups for different cardinalities of $I$. Thus the transition system contains information absent from the set of isomorphism types of its stages.

**Proposition 4.4.** *Two topologically finitely generated profinite groups with the same finite continuous quotient types are isomorphic as profinite groups.*

*Proof.* For a topologically finitely generated profinite group $H$, there are finitely many continuous maps into any finite group, since their values on a finite dense

generating set determine them. For each positive integer $n$, intersect the kernels of all maps to groups of order at most $n$, taking a representative per isomorphism type. The result is an open characteristic subgroup $K_n$, and these subgroups are cofinal among open normal subgroups. If $G$ has the same finite quotient types, each finite set of continuous surjections $G \to H/K_n$ is nonempty. Postcomposition gives an inverse system of these sets. Any finite list of compatibility conditions is realized by starting with a surjection at a sufficiently high stage and projecting it down. Compactness gives compatible surjections at every stage, yielding $G \to H$. This map is surjective because its image is compact and projects onto all $H/K_n$. Reversing the argument gives $H \to G$.

For a surjective continuous endomorphism of a topologically finitely generated profinite group, inverse image permutes the finite set of open normal subgroups of any fixed index. Its kernel is therefore contained in every open normal subgroup. Such an endomorphism is injective. Applying this to the composite of the two surjections shows that $G \to H$ is an isomorphism. □

This invariant uses a complete set of quotient types. The proposition does not give a criterion for stopping an isomorphism test after examining finitely many quotients. The examples $G_S$ already permit agreement on all quotient types of any specified bounded order, with disagreement at a larger prime. A different issue is whether a discrete group can be recovered from its profinite completion. Bridson and Grunewald construct nonisomorphic finitely presented residually finite groups with isomorphic completions [14]. For topologically finitely generated profinite groups themselves, the strong completeness theorem of Nikolov and Segal identifies the open subgroups with the subgroups of finite index, so the abstract group structure determines the topology in that setting [13].

## 4.2 Completed group algebras

Fix a finite field $k$. A profinite $k$-algebra is a Hausdorff topological $k$-algebra presented as an inverse limit of finite discrete $k$-algebras. Define

$$k[[G]] = \varprojlim_{U \trianglelefteq_o G} k[G/U], \qquad I_U = \ker\big(k[[G]] \to k[G/U]\big). \tag{2}$$

The $I_U$ form a basis of open ideals at zero. Write $[g]$ for the compatible tuple of the basis elements $[gU]$. This is the finite-field case of pseudocompact linearization [15, 16]. Its topology is the one specified by the group quotient system in (2). For a commutative profinite coefficient ring $R$, one can also retain coefficient precision explicitly, by taking the inverse limit of $(R/J)[G/U]$ over both open ideals $J$ of $R$ and open normal subgroups $U$ of $G$.

**Proposition 4.5.** *For any profinite $k$-algebra $A$ there is a natural bijection*

$$\mathrm{Hom}_{k\text{-}alg,cont}(k[[G]], A) \cong \mathrm{Hom}_{\mathrm{cont}}(G, A^\times),$$

*where the unit group $A^\times$ has its induced topology. The ordinary group algebra of the underlying abstract group maps injectively and densely into $k[[G]]$.*

*Proof.* For a continuous map into the units of a finite algebra, the kernel is open and the usual group-algebra adjunction gives a map from some $k[G/U]$. For general $A$, use this extension in every finite algebra quotient. The units of $A$ are the compatible tuples of units since their inverses are compatible, and the extensions are compatible by uniqueness. Explicitly, the finite sums in the symbols $[g]$ surject onto each $k[G/U]$, hence are dense. The same fact gives uniqueness of the extension into $A$. To prove injectivity of the ordinary group algebra, separate any finitely many distinct support elements into distinct cosets in a common finite quotient. A nonzero linear combination supported on these elements survives in its group algebra. □

With its topological Hopf structure, $k[[G]]$ recovers $G$. Use the completed tensor product given by the inverse limit of the finite tensor products, and extend $\Delta([g]) = [g] \otimes [g]$, $\varepsilon([g]) = 1$, and the antipode $[g] \mapsto [g^{-1}]$ continuously. If $a$ satisfies $\Delta(a) = a \otimes a$ and $\varepsilon(a) = 1$, its image in every $k[G/U]$ is group-like. Write an image as $\sum_x a_x[x]$. The off-diagonal coefficients give $a_x a_y = 0$ for $x \neq y$, so the field property and the counit force exactly one coefficient, equal to 1. The resulting elements of the groups $G/U$ are compatible. Consequently the group of group-like elements of $k[[G]]$, with its subspace topology, is canonically isomorphic to $G$. In particular continuous Hopf algebra maps correspond to continuous group homomorphisms. Indeed a Hopf map must send each $[g]$ to a group-like element, and conversely a group homomorphism supplies a unique continuous algebra map by Proposition 4.5. That map preserves the Hopf structure by density of the group basis's linear span.

## 4.3 Two continuity regimes for modules

Let $\mathrm{Rep}_k^{\mathrm{disc}}(G)$ be the category of $k$-vector spaces with discrete topology and continuous linear $G$-action. Continuity is equivalent to each vector having an open stabilizer. Let $\mathrm{Rep}_k^{\mathrm{fd,cont}}(G)$ be its full subcategory on finite-dimensional representations. In the category of discrete $k[[G]]$-modules, continuity means that each vector is annihilated by some $I_U$.

**Proposition 4.6.** *There are natural equivalences of ordinary categories*

$$\mathrm{Rep}_k^{\mathrm{disc}}(G) \simeq \mathrm{Mod}_{k[[G]]}^{\mathrm{disc,cont}},$$
$$\mathrm{Rep}_k^{\mathrm{disc}}(G) \simeq \mathrm{Ind}\Big(\mathrm{Rep}_k^{\mathrm{fd,cont}}(G)\Big).$$

*The finite-dimensional category is the directed union of the categories of finite-dimensional $k[G/U]$-modules along inflation (restriction of scalars through the quotient algebra maps).*

*Proof.* An open stabilizer has finite index, so the orbit of each vector is finite and spans a finite-dimensional invariant subspace. On any such subspace, intersect the open stabilizers of a basis. The result is the kernel of the restricted action, an open normal subgroup. Extend the action on the subspace through the corresponding finite group algebra. These extensions agree on intersections by passing to a common group quotient. Conversely the map $g \mapsto [g]$ gives a continuous action on any discrete-continuous module, and equivariant linear maps commute with the extension on these finite spans. This also proves the claim about inflation.

Every discrete-continuous representation is the filtered union of its finite-dimensional invariant subspaces. To see that the latter are finitely presentable in the whole category, lift images of a finite basis along a filtered colimit. At the chosen stage the lifted vectors span, together with their orbits, a finite-dimensional subrepresentation. Choose a common finite quotient through which both it and the domain act. Equivariance of the lifted linear map is expressed by finitely many linear equations, one per basis vector and element of the finite quotient. Equations holding in the colimit hold together at a later stage. Equality of two maps is likewise tested on a finite basis. Filtered colimits are computed on vector spaces with the induced continuous action. This establishes the claimed Ind description. □

There is also a compact regime. A profinite continuous module here has a neighborhood basis of open invariant subspaces of finite codimension and is complete and Hausdorff. It is the inverse limit of its finite quotient modules, each of which has action factoring through a finite continuous quotient of $G$. Conversely such inverse limits, with their inverse-limit topology, give continuous modules. A continuous module map to a finite discrete quotient factors through a finite quotient of its source, since its kernel is open. This gives the analogue of (1) and identifies the compact regime with $\mathrm{Pro}(\mathrm{Rep}_k^{\mathrm{fd,cont}}(G))$. One way to see the distinction concretely is to put the topology of (2) on the regular module. It is a profinite continuous module. For infinite $G$, the regular module with discrete topology is excluded from Proposition 4.6 because the annihilator of 1 is zero and each $I_U$ contains $[u] - [1]$ for some $u \neq 1$. For the pseudocompact module formulation see [15].

## 4.4 A computation for $\mathbb{Z}_p$

Take $k = \mathbb{F}_p$. The quotient $\mathbb{Z}_p/p^n\mathbb{Z}_p$ is cyclic of order $p^n$ with generator $g_n$ the class of 1. Characteristic $p$ gives

$$\begin{aligned}\mathbb{F}_p[C_{p^n}] &\cong \mathbb{F}_p[X]/(X^{p^n} - 1)\\ &\cong \mathbb{F}_p[T]/(T^{p^n}), \qquad T = g_n - 1.\end{aligned}$$

The transition maps reduce modulo the smaller powers of $T$. Consequently $\mathbb{F}_p[[\mathbb{Z}_p]] \cong \mathbb{F}_p[[T]]$ as topological algebras. If $M$ is finite-dimensional, an action through the $n$th stage is equivalent to a linear endomorphism $N$ satisfying $N^{p^n} = 0$, with $g_n$ acting by $1 + N$. The structure theorem for finitely generated modules over the PID $k[T]$ gives its unique direct sum decomposition into blocks $k[T]/(T^r)$ with $1 \leq r \leq p^n$. Over the whole continuous finite-dimensional category the only bound on $r$ is finiteness. This is a case where the isomorphism types can be described via finite combinatorial data while the completed algebra is an inverse limit. Polynomial truncation gives every finite observation of a power series, with compatibility supplied by reduction. The series can have infinitely many nonzero coefficients.

The dense inclusion of the ordinary group algebra in this completion is proper. Write $[a]$ for an additive group element $a \in \mathbb{Z}_p$ in the ordinary group basis. The element $[0] + [1] - [2]$ maps to $1 - T - T^2$, which has constant term 1 and hence a power-series inverse. It has no inverse in the ordinary group algebra. If it had one, the combined finite supports of the element and its inverse would lie in a finitely generated subgroup of $\mathbb{Z}_p$. Such a subgroup is torsion-free abelian and hence free abelian of finite rank. Its group algebra is Laurent polynomial, whose units are scalar monomials. To see the latter fact, order the exponents lexicographically. In a product the least and greatest exponents are the sums of the respective extreme exponents. If their difference is zero, each factor has support a singleton. The three distinct exponents of $[0] + [1] - [2]$ rule out this possibility. Thus completion changes even elementary invertibility questions.

The bare completed algebra can also lose group information even for infinite groups. Take $k = \mathbb{F}_5$ and compare $\mathbb{Z}_5 \times C_4$ with $\mathbb{Z}_5 \times C_2 \times C_2$. For each finite factor $Q$, completion gives $k[[T]] \otimes_k k[Q]$ because $k[Q]$ is finite-dimensional and the quotients $C_{5^n} \times Q$ are cofinal. Each of the two $k[Q]$ is isomorphic as an algebra to $k^4$, since the linear factors of $X^2 - 1$ and $X^4 - 1$ in $k$ are distinct and exhaust the polynomials. The completed algebras are therefore isomorphic as topological algebras to $k[[T]]^4$. The groups are distinguished by the presence of an element of order four. The Hopf structures recover the distinction by group-like elements.

**Theorem 4.7** (Effective towers and isomorphism)**.** *Consider programs promised to present an inverse sequence of finite cyclic groups with explicit surjective transition maps. Even with the promise that the limit is infinite, there is no algorithm deciding whether the limit is isomorphic to a fixed $\mathbb{Z}_p$. For the same input class and $k = \mathbb{F}_p$, there is no algorithm deciding whether its completed group algebra is isomorphic as a topological $k$-algebra to $k[[\mathbb{Z}_p]]$.*

*Proof.* Fix a prime $q \neq p$. Given a machine $E$, define $\epsilon_n = 1$ if $E$ halts within the first $n$ steps and $\epsilon_n = 0$ otherwise. Output the cyclic group of order $p^n q^{\epsilon_n}$ at stage $n$, using residue maps. These tables and transition maps can be computed by bounded simulation, and the indicated orders divide one another. The limit is $\mathbb{Z}_p$ if $E$ never halts

and $\mathbb{Z}_p \times C_q$ if it does, by the Chinese remainder theorem and eventual constancy of the $q$-component. The two limits are distinguished by the existence of the quotient $C_q$.

In the nonhalting case the completed algebra is $k[[T]]$, which is local. In the halting case the quotient system is eventually of the form $C_{p^n} \times C_q$, so by finite-dimensionality of $k[C_q]$ the completed algebra is $k[[T]] \otimes_k k[C_q]$. The element $q^{-1} \sum_{g \in C_q} [g]$ gives a nonzero idempotent distinct from the identity in that algebra. A local ring has no such idempotent. Each proposed isomorphism algorithm would therefore decide halting. □

The obstruction in Theorem 4.7 concerns programs specifying entire quotient towers. Each stage is fully inspectable. Restrictions on the description language or additional certificates for the shape of the limit can yield decidable subclasses, as the explicit module blocks above illustrate for the fixed additive group $\mathbb{Z}_p$.

### 4.5 Tensor reconstruction

Fix a finite field $k$ and put $A_U = k[G/U]$ for each open normal subgroup $U$ of $G$. The representation-theoretic invariant that retains this diagonal can be stated without explicitly mentioning a Hopf algebra. Let $\mathcal{R}_G$ be the tensor category of finite-dimensional continuous $k$-linear representations on discrete spaces and let $\omega_G : \mathcal{R}_G \to \mathrm{Vect}_k^{\mathrm{fd}}$ be the forgetful functor. Give $\mathrm{Aut}^{\otimes}(\omega_G)$ the topology of pointwise convergence on the finite sets $\mathrm{GL}_k(\omega_G(V))$. One may choose a small skeleton for this product.

**Proposition 4.8** (Tensor fiber functor)**.** *The action maps induce an isomorphism of topological groups*

$$G \simeq \mathrm{Aut}^{\otimes}(\omega_G).$$

*An equivalence of these tensor categories compatible with the fiber functors therefore induces an isomorphism of the profinite groups.*

*Proof.* For a natural endomorphism $T$ of the underlying linear functor, put $a_U = T_{A_U}(1)$ using the regular $A_U$-module viewed as a $G$-representation. Naturality with respect to stage quotient maps gives a compatible family $(a_U)_U$. A representation $V$ factors through some $G/U$ because its action has finite discrete image and the kernel is open. For each $v \in V$, the $A_U$-linear map $A_U \to V$ sending 1 to $v$ forces $T_V(v) = a_U v$. Thus the endomorphism is determined by its family. Conversely every such family acts naturally on all objects, since any two finite quotient stages have a common refinement.

Tensor compatibility on $A_U \otimes A_U$, evaluated at $1 \otimes 1$, gives $\Delta(a_U) = a_U \otimes a_U$. Unit compatibility gives $\varepsilon(a_U) = 1$. By the reconstruction from group-like elements, the family is $([gU])_U$ for a unique $g \in G$, which acts by a tensor natural automorphism. Each finite representation evaluation is continuous on $G$. The reverse identification is continuous because $T \mapsto T_{A_U}(1)$ is a finite evaluation for every $U$. □

## 5 The inductive dual and enriched reconstruction

Fix a finite field $k$ with the discrete topology and a profinite group $G$. Write $A_G = k[[G]]$ for the completed algebra and $A_G^\vee$ for its continuous linear dual into $k$. Let $C_{\mathrm{lc}}(G, k)$ be the algebra of locally constant functions with pointwise operations. The standing coefficient hypothesis makes $A_G$ profinite as a topological ring. The same dual description of functions works with a general discrete field under the inverse-limit linear topology [17, §7.2].

**Theorem 5.1** (Inductive recovery on the dual side)**.** *There are natural identifications of discrete commutative Hopf algebras*

$$A_G^\vee \cong C_{\mathrm{lc}}(G, k) \cong \varinjlim_{N \trianglelefteq_o G} k^{G/N}, \tag{3}$$

*where the transition maps in the colimit are pullbacks along quotient maps. The contravariant assignment $G \mapsto C_{\mathrm{lc}}(G, k)$ is fully faithful on continuous group homomorphisms. Its essential image consists of filtered colimits of finite Hopf algebras $k^Q$ for finite groups $Q$, with injective Hopf transition maps, up to isomorphism.*

*Proof.* A continuous linear functional on $A_G$ has a kernel containing the kernel of some finite quotient projection. This is because the latter kernels form a neighborhood basis at zero. It therefore factors through the finite vector space $k[G/N]$, whose dual is $k^{G/N}$. The resulting transitions are pullbacks. Conversely, if $f : G \to k$ is locally constant, each $g \in G$ has a right coset neighborhood $gN_g$ on which $f$ is constant, with $N_g$ open normal. Compactness gives a finite subcover. The intersection of the finitely many corresponding $N_g$ works uniformly, and $f$ factors through that quotient. The same argument for $G \times G$ gives

$$C_{\mathrm{lc}}(G, k)^{\otimes 2} \cong C_{\mathrm{lc}}(G \times G, k),$$

by finite-dimensional function algebra tensor products and the cofinal use of the same $N$ on both factors. The coproduct is $f \mapsto [(g, h) \mapsto f(gh)]$, the counit evaluates at 1, and the antipode precomposes with inversion. They are dual to the corresponding maps on finite group algebras.

For a finite set $X$, each unital $k$-algebra map $k^X \to k$ is evaluation at a unique point. Orthogonality allows at most one coordinate idempotent to map to 1. Their images sum to 1, so one must do so, since the only idempotents of a field are 0 and 1. Thus a character on $C_{\mathrm{lc}}(G, k)$ corresponds to a compatible system of points in $G/N$, or equivalently to an element of $G$. Give the set of characters the topology of pointwise evaluation into the discrete field.

The pullback of evaluation is continuous, and indicators of cosets recover the inverse-limit topology. The product of characters is induced by the coproduct and becomes group multiplication. A Hopf map $C_{\mathrm{lc}}(H,k) \to C_{\mathrm{lc}}(G,k)$ induces a continuous group map $G \to H$ by precomposition of characters. Evaluation at all $g$ detects the functions, so the Hopf map equals pullback along that group map.

Finally, the finite version of this argument identifies injective Hopf maps $k^Q \to k^{Q'}$ with surjective group homomorphisms $Q' \to Q$. A filtered diagram of the claimed form is therefore the pullback system of a cofiltered diagram of finite groups with surjective maps. The inverse limit has surjective projections by compactness, and their kernels are cofinal among open normal subgroups. The first part of the proof identifies the colimit with the algebra of functions on that limit. □

The theorem describes the inductive structure of the functions, not of the points. In the case $G = \mathbb{Z}_p$, an indicator for a residue class modulo $p^n$ appears at stage $n$. Its pullback to stage $n+1$ is a sum of the $p$ finer indicators. An arbitrary compatible $p$-adic element requires the full tower. This is an explicit duality between the construction of finite observations and their consistent assignments. The coproduct matters. For example, as bare commutative $k$-algebras $k^{C_4}$ and $k^{C_2 \times C_2}$ are both $k^4$. The coproduct on characters distinguishes the two group laws even in characteristic two.

# 6 Higher groups and coherent algebraic structure

Use the Lawvere theory from Section 1. A model of a finite-product theory $T$ in an $\infty$-category $\mathcal{C}$ with finite products is a product-preserving functor $T \to \mathcal{C}$. A relation between composites in $T$ constrains the corresponding images, with the coherence supplied by the functor. Badzioch's rigidification theorem gives a model in simplicial groups for the space-valued homotopy theory of the theory of groups [18]. The simplicial direction can carry substantial information even when the original group operations are given by a finite signature.

A complementary description uses group objects in an $\infty$-topos $\mathcal{X}$. A group object $G_\bullet$ is a reduced groupoid object. Its degree zero is terminal, the Segal maps give $G_n \simeq G_1^n$, and the shear map $(d_2, d_1)\colon G_2 \to G_1 \times G_1$ is an equivalence. The latter expresses division, with $(x,y) \mapsto (x, xy)$ in an ordinary group. Write $G$ for $G_1$ when no confusion results. Every groupoid object of an $\infty$-topos is effective, so the geometric realization $BG = |G_\bullet|$ fits into

$$G \simeq * \times_{BG} *, \qquad \mathrm{Grp}(\mathcal{X}) \simeq \mathcal{X}_{*,\mathrm{conn}}.$$

Here $\mathcal{X}_{*,\mathrm{conn}}$ consists of pointed objects whose basepoint is an effective epimorphism. Its inverse equivalence sends the pointed object to the Čech nerve of its point. In spaces this is the equivalence between grouplike $E_1$-spaces and pointed connected spaces. For a discrete group its delooping is a connected 1-type, and for a group object with higher homotopy its delooping can have arbitrarily high homotopy groups [19, 20].

**Proposition 6.1** (Filtered assembly of models)**.** *Let $\mathcal{C}$ admit filtered colimits that commute with finite products. For a small finite-product theory $T$, filtered colimits in $\mathrm{Fun}^{\times}(T,\mathcal{C})$ are computed pointwise. In an $\infty$-topos, the corresponding assertion for reduced groupoid objects holds in the simplicial functor category.*

*Proof.* Colimits in the ambient functor category are pointwise. Each product comparison for the pointwise colimit is an equivalence because filtered colimits commute with finite products and all the stagewise comparisons are equivalences. The reduced, Segal, and shear comparisons are preserved for the same reason, using exactness of filtered colimits in an $\infty$-topos. □

This proposition establishes an inductive construction once a filtered diagram of models has been supplied. A different question asks whether a model has a presentation by stages belonging to a given class. For example, an ordinary group expressed as a filtered colimit of finite groups is locally finite. A finite tuple of elements lifts to one stage and generates a subgroup contained in its finite image. Conversely, the finite subgroups of a locally finite group form a filtered poset with union the whole group. Thus even for 0-truncated group objects, the category of finite groups generates only a restricted class under filtered colimits. The finitely presented group $\mathbb{Z}$ lies outside that class.

## 6.1 Postnikov data and higher isomorphism types

A space is $\pi$-finite if it has finitely many components, its homotopy groups are finite, and there is a finite truncation bound. The homotopy groups and their actions describe successive Postnikov layers. The layers are attached by $k$-invariants, with local coefficients when the fundamental group acts nontrivially. For a pointed connected 2-type the data are a group $Q = \pi_1$, a $Q$-module $A = \pi_2$, and a class in $H^3(Q;A)$, considered under isomorphisms of the group and its module. Equivalently these classify the corresponding loop 2-groups under equivalence [22]. Even finite layers require the attachment data.

**Example 6.2** (Two inequivalent higher groups with the same homotopy groups)**.** Let $Q = C_2$, let $A = \mathbb{F}_2$ with trivial action, and take

$$X_\epsilon = \mathrm{hofib}\big(\kappa_\epsilon : BC_2 \longrightarrow K(A,3)\big), \qquad \epsilon \in \{0,1\},$$

where $\kappa_0$ is null and $\kappa_1$ represents the nonzero class in $H^3(C_2;\mathbb{F}_2)$. The familiar periodic free resolution gives

one copy of $\mathbb{F}_2$ in every cohomological degree. Indeed applying $\mathrm{Hom}_{\mathbb{F}_2[C_2]}(-,\mathbb{F}_2)$ to the periodic resolution with differential multiplication by $g-1$ gives zero differentials. The fiber sequences give $\pi_1(X_\epsilon)=C_2$, $\pi_2(X_\epsilon)=A$, and no higher homotopy groups. The action is trivial because the extension is a pullback of the path fibration of the simply connected $K(A,3)$. The projection $X_\epsilon \to BC_2$ is its first Postnikov projection. For $\epsilon=0$ it has a section. A homotopy section for $\epsilon=1$ would supply a nullhomotopy of $\kappa_1$, so it has none. Truncation is natural in equivalences, hence $X_0 \not\simeq X_1$. Delooping gives $\Omega X_0 \not\simeq \Omega X_1$ as group objects even though their homotopy groups and actions agree.

**Proposition 6.3** (Effective enumeration of finite homotopy types)**.** *The equivalence classes of $\pi$-finite spaces admit an effective enumeration by finite simplicial presentations. Equivalence of two given presentations in this enumeration is decidable.*

*Proof.* A finite presentation here means a finite table of simplices and structure maps in degrees at most $r$, extended by the $r$-coskeleton. The simplicial identities in the finite table are decidable. The coskeleton in each fixed degree is a finite limit of finite sets, so its simplices and faces remain computable. A horn of dimension at least $r+2$ has the same $r$-skeleton as the simplex, so it has a unique filler in an $r$-coskeletal simplicial set. Consequently the Kan condition requires a finite search in dimensions at most $r+1$. A levelwise finite Kan complex has finitely many components and finite homotopy groups, since classes in degree $m$ have representatives among its $m$-simplices with constant boundary. Coskeletality forces the homotopy groups above degree $r$ to vanish by uniqueness of a simplex with prescribed boundary. Conversely, a minimal Kan model of each $\pi$-finite space is levelwise finite and coskeletal [21, Lemma 7.2.4]. Thus the search exhausts the desired types.

If $Y$ is $s$-coskeletal, a map $X \to Y$ is exactly a compatible map of truncated tables in degrees at most $s$. The same test determines maps $X \times \Delta^1 \to Y$ with prescribed endpoints. For Kan complexes $X, Y$, an equivalence is witnessed by simplicial maps in both directions and simplicial homotopies of the two composites to the identities. A mapping complex with Kan target is Kan, so any two maps in the same component are joined by a simplicial edge. These witnesses can all be found or ruled out by finite search, which also allows the enumeration to discard duplicate equivalence classes. □

The finite input here describes all simplicial degrees through the coskeleton. The cardinality of the resulting complex, its compactness as an object of spaces, and the size of an inverse system of such types are separate questions.

# 7 Finite tests of inverse systems

Write $\mathcal{S}$ for spaces and $\mathcal{S}^\pi$ for its full subcategory of $\pi$-finite spaces. A profinite space is an object of $\mathrm{Pro}(\mathcal{S}^\pi)$ [21]. The direction of its indexing matters. If $P=\{X_i\}_{i\in I}$ and $Q=\{Y_j\}_{j\in J}$ are represented by cofiltered diagrams in an essentially small $\infty$-category $\mathcal{C}$, then

$$\mathrm{Map}_{\mathrm{Pro}(\mathcal{C})}(P,Q) \simeq \lim_{j\in J} \operatorname*{colim}_{i\in I^{\mathrm{op}}} \mathrm{Map}_{\mathcal{C}}(X_i,Y_j). \tag{4}$$

This formula follows from the dual presheaf model for $\mathrm{Ind}(\mathcal{C}^{\mathrm{op}})$. Filtered colimits of the representables are computed pointwise, and mapping out of a colimit yields a limit [19, 21]. In $\mathrm{Ind}(\mathcal{C})$ the dual variance makes a constant source compact with respect to filtered colimits. In a pro-category the constant object is instead used as the target of a map out of a cofiltered limit.

**Theorem 7.1** (Finite coherent tests of a pro-object)**.** *Let $P=\{X_i\}$ as in (4), and let $p$ be a finite diagram of targets in $\mathcal{C}$ indexed by a finite simplicial set $K$. The natural comparison is an equivalence of spaces*

$$\begin{aligned}\operatorname*{colim}_{i\in I^{\mathrm{op}}} \lim_{k\in K} \mathrm{Map}_{\mathcal{C}}(X_i,p(k)) \\ \simeq \lim_{k\in K} \mathrm{Map}_{\mathrm{Pro}(\mathcal{C})}(P, c\,p(k)).\end{aligned} \tag{5}$$

*Here $c$ denotes the constant pro-object functor. In particular, for $Y\in\mathcal{C}$ and any finite simplicial set $L$,*

$$\begin{aligned}\operatorname*{colim}_{i\in I^{\mathrm{op}}} \mathrm{Map}_{\mathcal{S}}(|L|, \mathrm{Map}_{\mathcal{C}}(X_i,Y)) \\ \simeq \mathrm{Map}_{\mathcal{S}}(|L|, \mathrm{Map}_{\mathrm{Pro}(\mathcal{C})}(P, cY)).\end{aligned} \tag{6}$$

*Thus a finite family of maps together with specified homotopies of finite shape can be represented at a common stage, up to equivalence. Equalities of the resulting components become equalities after transition to a later stage.*

*Proof.* Insert (4) on the right of (5). Filtered colimits of spaces commute with finite limits [24]. For (6), take a constant target diagram at $Y$ of shape $L$. The limit of the constant space $V$ indexed by $L$ is $\mathrm{Map}_{\mathcal{S}}(|L|,V)$. Finally, $\pi_0$ preserves colimits of spaces, and an equality in a filtered colimit of sets is witnessed at a common stage. A relative version follows by taking the fiber of the restriction map along a map of finite shapes $L_0 \to L$, with the boundary condition chosen at a stage. Such a fiber is another finite limit, so the relative witness descends after refinement. □

The same finite bound has a direct diagrammatic form for filtered colimits of $\infty$-categories. If $L$ is finite and $\mathcal{D}_i$ is a filtered diagram of $\infty$-categories, the canonical map

$$\operatorname*{colim}_i \mathrm{Fun}(L,\mathcal{D}_i) \longrightarrow \mathrm{Fun}\Big(L, \operatorname*{colim}_i \mathcal{D}_i\Big) \tag{7}$$

is an equivalence. One reduces to a strict diagram over a directed poset in the quasicategory model. The underlying filtered colimit of simplicial sets computes the

$\infty$-categorical colimit [25]. An $n$-simplex of the functor quasicategory has shape $L \times \Delta^n$, which is finite. It therefore factors through a stage, and equality of two such representatives is witnessed at a common stage [23]. This form applies when an interpreter comparison is a diagram of functors and transformations. The shape must include whichever commutativities and higher cells the comparison uses.

### 7.1 Two failures of uniform finite-stage descent

**Example 7.2** (Truncated with finite homotopy, yet noncompact)**.** The space $BC_2 \simeq \mathbb{RP}^\infty$ is $\pi$-finite. As a space, $BC_2$ is the filtered colimit of its finite CW skeleta $\mathbb{RP}^n$. If its identity factored up to homotopy through $\mathbb{RP}^n$, the induced endomorphism of $H^m(BC_2; \mathbb{F}_2)$ would factor through zero for $m > n$. This contradicts the nonvanishing just computed. In particular the map

$$\operatorname*{colim}_n \operatorname{Map}(BC_2, \mathbb{RP}^n) \longrightarrow \operatorname{Map}(BC_2, BC_2)$$

misses the identity component. A bound on homotopy groups of the test object is distinct from compactness of the test object in spaces.

**Example 7.3** (Finite observation of a higher tower)**.** Let $P = \{K(V_n, 2)\}_{n \geq 1}$, where $V_n = \mathbb{F}_2^n$ and the bonding maps are projections. The tower is an object of $\operatorname{Pro}(\mathcal{S}^\pi)$. By (4), the components of its maps to the constant $K(\mathbb{F}_2, 2)$ form

$$\operatorname*{colim}_n H^2(K(V_n, 2); \mathbb{F}_2) \cong \bigoplus_{r \geq 1} \mathbb{F}_2.$$

Here the Hurewicz and universal coefficient theorems give $H^2(K(V_n, 2); \mathbb{F}_2) \cong \operatorname{Hom}(V_n, \mathbb{F}_2)$, and transition maps append zeros. Every coordinate observation therefore factors through a finite stage. If the whole identity of $P$ factored through $cK(V_n, 2)$ via the stage projection for some $n$, coordinate $n+1$ would then factor through $V_n$, contrary to its nonzero value on its own basis vector and its vanishing on $V_n$. More generally any factorization of the identity through this constant object has its first factor represented at some stage $m$ by (4) and would force every coordinate to be supported at most at $m$. The compatibility in the limit over target stages is essential.

## 8 Group algebras and higher local systems

Let $R$ be a commutative ring spectrum, so $\operatorname{Mod}_R$ is symmetric monoidal with tensor product $\otimes_R$. For a grouplike $E_1$-space $G$, define the $E_1$ group algebra over $R$ by

$$R[G] = R \wedge \Sigma^\infty_+ G.$$

The suspension spectrum functor with disjoint basepoint is strong symmetric monoidal from cartesian spaces to spectra. It sends the multiplication on $G$ to the relevant $E_1$ product, and the map $G \to *$ to an augmentation. This construction refines the ordinary group algebra. For an ordinary commutative ring $k$ and a discrete group $G$, one has $(Hk)[G] \simeq H(k[G])$ since the underlying spectrum is the coproduct of copies of $Hk$ indexed by $G$, with the prescribed basis multiplication.

**Theorem 8.1** (Spectral group algebras describe actions)**.** *There is an equivalence of $\infty$-categories over* $\operatorname{Mod}_R$

$$\operatorname{Mod}_{R[G]} \simeq \operatorname{Fun}(BG, \operatorname{Mod}_R),$$

*where the forgetful functor corresponds to evaluation at the chosen point of $BG$.*

*Proof.* Evaluation admits a left adjoint by left Kan extension from the point. Its composite with evaluation sends $M$ to the copower of $M$ by $\operatorname{Map}_{BG}(*, *) \simeq G$. This is $R[G] \otimes_R M$, with monad multiplication induced by composition of loops. Evaluation preserves colimits, which are pointwise, and detects equivalences because every point in the connected groupoid $BG$ is equivalent to the chosen point. The monadicity theorem identifies these functors with algebras for $R[G] \otimes_R (-)$, namely left modules over $R[G]$ [20]. □

**Example 8.2** (A finite higher homotopy group gives infinite homological amplitude)**.** Let $G = K(C_p, 1)$, with its group structure from looping $K(C_p, 2)$, and let $R = H\mathbb{F}_p$. Then

$$\pi_q(R[G]) \cong H_q(BC_p; \mathbb{F}_p) \cong \mathbb{F}_p \qquad (q \geq 0).$$

Indeed, write $\mathbb{F}_p[C_p] \cong \mathbb{F}_p[u]/(u^p)$ with $u = g - 1$. The periodic resolution of the trivial module has differentials alternately multiplication by $u$ and $u^{p-1}$, exact by their annihilator ideals. Tensoring over the group algebra with the trivial module makes both differentials zero and computes the displayed group homology. Thus a $\pi$-finite higher group can yield a group algebra with a nonvanishing homotopy group in every nonnegative degree.

For a cofiltered diagram $\{G_i\}$ of higher groups, applying $R[-]$ gives a pro-object $\{R[G_i]\}$ of $E_1$ algebras. One can also form the inverse limit algebra inside $\operatorname{Alg}_{E_1}(\operatorname{Mod}_R)$. Its underlying module is the homotopy inverse limit of the underlying modules. For a tower, the fiber formula for the inverse limit of spectra, using $1 -$ shift on their product, gives the exact sequence

$$0 \longrightarrow \varprojlim_i{}^1 \pi_{q+1} R[G_i] \longrightarrow \pi_q\Big(\varprojlim_i R[G_i]\Big) \longrightarrow \varprojlim_i \pi_q R[G_i] \longrightarrow 0.$$

Consequently the pro-object, the algebra obtained by taking the limit, and the separate inverse limits of homotopy groups give different specifications. At finite stages Theorem 8.1 identifies the module categories with categories

of local systems. A pullback of such a local system along a stage map corresponds to restriction of scalars, by naturality of action on loops. To specify continuous higher actions of a tower one must also choose a continuity condition, such as finite-stage factorization for finite tests. The comparison of this continuous module theory with modules over the inverse limit algebra requires a separate argument.

# 9 Specialization across finite quotients

Partial evaluation specializes a program to its static inputs and leaves a residual program for the dynamic inputs [26]. An interpreter can itself be the program being specialized, with guest code treated as static. Truffle is one example of this architecture [28]. The question here is whether two transformations commute: specializing an interpreter and changing the finite quotient in which it runs. A sound statement has to track the effects the interpreter exposes, in addition to its returned value. It also has to distinguish a natural family of denotations from a particular representation of residual code.

## 9.1 A criterion that includes state and observations

Let $I$ be a directed poset of precision levels and write $p_{ji} : V_j \to V_i$ for the transition on an indexed family $V$ when $j \geq i$. Let $A_i, X_i, Y_i, S_i$ be static inputs, dynamic inputs, outputs, and states. Let $L_i$ be a monoid of observations and assume that its transitions are monoid maps. Put $U_i = X_i \times S_i$ and $V_i = Y_i \times S_i \times L_i$, with transitions on products taken componentwise. A deterministic interpreter with state and ordered observations is a family of functions

$$F_i : A_i \times U_i \longrightarrow V_i.$$

In particular, the observation component can be a word in an alphabet of events, so that multiplication is concatenation. This is state-and-writer semantics in the sense of monadic accounts of effects [27]. We say that $F$ respects the quotients when

$$p^V_{ji} F_j(a, u) = F_i\big(p^A_{ji} a, p^U_{ji} u\big) \quad (j \geq i). \tag{8}$$

For surjective input transitions, the candidate factor at level $i$ is forced by the left side. It exists exactly when the left side is constant on the fibers of $p^A_{ji} \times p^U_{ji}$. This is a test on the state and trace outputs as well as the returned value.

At any one level, effectful arrows from $X$ to $Y$ have type $X \times S \to Y \times S \times L$. To compose $f$ and $g$, apply $g$ to the value and state returned by $f$, and multiply observations in execution order. The identity returns the input and state and emits the monoid unit. Static parameters can be bundled into the input or treated as fixed external parameters; an interpreter that passes a static parameter to a downstream interpreter passes its projected value at each lower level.

**Theorem 9.1** (Specialization and composition)**.** *A family of interpreters satisfying (8) is closed under effectful composition with any such family on matching types. If $a = (a_i) \in \varprojlim_i A_i$ is coherent, then the residual functions*

$$\mathrm{res}_i(u) = F_i(a_i, u)$$

*satisfy $p^V_{ji} \mathrm{res}_j = \mathrm{res}_i p^U_{ji}$. They induce the specialization at $a$ of $\varprojlim_i F_i$. For continuous stage functions and transitions this induced function is continuous for the inverse-limit topologies.*

*Proof.* For composition, write $f_j(x, s) = (y, t, \ell)$ and $g_j(y, t) = (z, v, m)$. Compatibility sends the two evaluations to the corresponding evaluations on projected inputs. The resulting observations at level $i$ are $p^L_{ji}(\ell) p^L_{ji}(m) = p^L_{ji}(\ell m)$. The final projected value and state agree for the same reason. For specialization, substitute $a_j$ in (8) and use $p^A_{ji}(a_j) = a_i$. The limit map is then coordinatewise specialization. Its continuity follows from continuity of its composites with the stage projections. □

The condition is stable under interpreter pipelines. For example, an instruction decoder can feed a group-algebra evaluator whose resulting state is read by a trace interpreter. Each component must satisfy (8) for the data it receives. Combining two stateful components on a shared state uses an execution order; exchanging the order would need an additional commutation identity. In particular, on a noncommutative group algebra, successive updates $v \mapsto av$ and $v \mapsto bv$ yield $bav$. Reversing them yields $abv$, which can differ already on $v = 1$.

Continuous finite observation has a weaker form of locality that is useful when designing precision levels.

**Lemma 9.2** (Finite observation)**.** *Let $U = \varprojlim_i U_i$ be an inverse limit of nonempty finite discrete sets with surjective transitions. Every continuous $h : U \to D$ into a finite discrete set factors through $U_j$ for some $j$.*

*Proof.* Each point of $U$ lies in a cylinder on which $h$ is constant. A cylinder specified using finitely many stages can be expressed using one stage by directedness. A finite subcover exists by compactness. Choose a common stage $j$ above the finitely many stages in that cover. Points with the same $j$-coordinate lie in the same cylinders from the subcover, so have the same image. The projection to $U_j$ is surjective, hence defines the desired factor uniquely. □

For a continuous function between profinite limits, the lemma can be applied after projecting to any given finite output stage $i$. The input stage $j$ may need higher

precision than $i$. For instance the function

$$\mathbb{Z}_2 \longrightarrow \{0,1\}, \qquad x \longmapsto \left\lfloor \frac{x \bmod 4}{2} \right\rfloor$$

is continuous and separates 0 from 2, which have the same class modulo 2. If this value is emitted in the trace, replacing a modulo-4 interpreter with a modulo-2 interpreter loses the observation even when the returned value is constant. Testing exact equality with 0 on $\mathbb{Z}_2$ is a stronger failure. Every neighborhood of 0 contains a nonzero $2^j$, so the test is discontinuous when its result is given the discrete topology.

There is a homotopy-coherent form of the specialization identity. In $\mathrm{Fun}(I^{\mathrm{op}}, \mathcal{S})$, take diagrams $A, U, V$, an arrow $F : A \times U \to V$, and a section $a : 1 \to A$. Here $\mathcal{S}$ denotes spaces, and the space of such sections is $\mathrm{Map}(1, A) \simeq \varprojlim_i A_i$. The residual arrow is composition of $F$ with $a \times \mathrm{id}_U$, using $1 \times U \simeq U$. Restriction along any indexing functor preserves the terminal object and products, since these are computed pointwise. It therefore preserves the residual arrow up to the canonical coherent equivalence. An $E_1$-monoid of observations supplies associative concatenation with all its coherences when the effectful composition above is interpreted in spaces. Coherence of $F$ and $a$ is part of the input. Independent choices of stagewise denotations do not supply it.

For an actual compiler, equality of interpreted outputs is the general conclusion. Equality of residual syntax under reduction requires a choice of representation and specialization rules that themselves respect reduction. A compiler can, for instance, optimize a branch at one quotient using a relation that does not hold at higher precision. The following compiler uses a canonical affine block representation whose coefficients reduce exactly.

## 9.2 A dihedral convolution interpreter

For $n \geq 1$, take

$$D_n = (\mathbb{Z}/2^n) \rtimes C_2, \qquad R_n = \mathbb{F}_3[D_n],$$

where the nontrivial element of $C_2$ acts by inversion. Write $r = (1, 0)$ and $s = (0, 1)$ at each stage, so $srs = r^{-1}$. The underlying group multiplication is

$$(u, e)(v, f) = \big(u + (-1)^e v,\ e + f\big).$$

Reduction acts on the rotation coordinate and is the identity on $C_2$. On $R_n$ it sends each basis element to the corresponding lower-stage basis element and sums coefficients when they collide. This gives a unital ring map $q_{ji} : R_j \to R_i$. Indeed, it preserves products of basis elements because the underlying group map preserves products, and bilinearity extends the identity to all pairs. The augmentation $\epsilon_n : R_n \to \mathbb{F}_3$ is the sum of coefficients and satisfies $\epsilon_i q_{ji} = \epsilon_j$. The group data form the tower for $\mathbb{Z}_2 \rtimes C_2$. At every stage the left regular module $R_n$ is the state space.

A program is a finite list of instructions of four types. $\mathsf{Mix}(a)$ replaces $v$ by $av$, and $\mathsf{Deposit}(b)$ replaces $v$ by $v + b$, for $a, b \in R_n$. $\mathsf{Tag}(g)$ appends the event $\mathrm{tag}(g)$, for $g \in D_n$. $\mathsf{Signal}$ appends the event $\mathrm{signal}(\epsilon_n(v))$. The last two instructions preserve state. The observation monoid is the free monoid on the disjoint union $D_n \amalg \mathbb{F}_3$, with the two kinds of event distinguished. It reduces by lowering group tags and preserving scalar events. All four instruction semantics respect the quotients by the ring and augmentation identities. The compiler takes the instruction list and its operands as static data, and takes the initial module vector as dynamic data.

Between signals, the compiler summarizes a block as $(A, B, T)$, meaning

$$v \longmapsto Av + B, \qquad \text{with observations } T.$$

It starts with $(1, 0, ())$, then applies

$$\begin{aligned} \mathsf{Mix}(a)\colon (A, B, T) &\longmapsto (aA, aB, T), \\ \mathsf{Deposit}(b)\colon (A, B, T) &\longmapsto (A, B + b, T), \\ \mathsf{Tag}(g)\colon (A, B, T) &\longmapsto (A, B, T\ \mathrm{tag}(g)). \end{aligned}$$

At a signal it emits the current block followed by the signal, and starts a fresh block. It emits the remaining block at the end. Each rule preserves the denotation of the prefix accumulated so far by associativity and distributivity. Tags can be grouped because they depend only on the static operands and all state-changing instructions in the block are silent. A signal reads the current state, so maintaining its position between residual blocks preserves its dependence on both the static prefix and the dynamic vector. Induction on instruction lists proves that executing the residual program returns the same state and trace as interpreting the source program.

If a block is lowered by mapping its coefficients with $q_{ji}$ and lowering every tag, the lowering commutes with each compiler rule. Lowering preserves signals, identity blocks and block boundaries. Thus, in this fixed residual representation, lowering the specialized program gives exactly the specialization of the lowered program. An extra simplifier that compiles zero-weight affine updates as constant assignments, preserving their tags, would need a separate comparison at the level of denotations. A nonzero higher-stage weight can reduce to zero.

Consider this program, reading the left expression before the right on each row:

| | |
|---|---|
| $\mathsf{Mix}(1 + r)$ | $\mathsf{Deposit}(s)$ |
| $\mathsf{Tag}(r^5)$ | $\mathsf{Mix}(1 + s)$ |
| $\mathsf{Signal}$ | $\mathsf{Mix}(1 - r^2)$ |
| $\mathsf{Tag}(rs)$ | $\mathsf{Deposit}(2r^{-3}s)$ |
| $\mathsf{Mix}(s)$ | $\mathsf{Signal}$. |

The two nonidentity residual blocks have affine coefficients

$$A_1 = (1+s)(1+r) = 1 + r + s + r^{-1}s,$$
$$B_1 = (1+s)s = 1 + s,$$
$$A_2 = s(1-r^2) = s - r^{-2}s,$$
$$B_2 = 2sr^{-3}s = 2r^3.$$

The output on $v$ is $A_2(A_1v + B_1) + B_2$, with observations in the order

$$\mathrm{tag}(r^5), \quad \mathrm{signal}(\epsilon_n(v) + 2), \quad \mathrm{tag}(rs), \quad \mathrm{signal}(2).$$

The first signal evaluates in $\mathbb{F}_3$ using $\epsilon_n(A_1) = 4 = 1$ and $\epsilon_n(B_1) = 2$. The second uses $\epsilon_n(A_2) = 0$ and $\epsilon_n(B_2) = 2$. At level 1 the relation $r^2 = 1$ annihilates $A_2$ and makes the final state $2r$ for every input. The first signal still depends on the input, hence a specialization that keeps just the final state loses an observable. At level 2, $s$ and $r^{-2}s$ are distinct basis elements, so the higher-stage coefficient survives. Indeed, $A_2A_1 = A_2(1 + r + s + r^{-1}s)$ has a nonzero coefficient of $s$ at level 2. The final state at that stage depends on the dynamic vector.

The reference implementation `demo.py` runs with the Python standard library. It checks the displayed blocks and compares source and residual execution at levels 1 through 5 on zero and every basis input (129 checks). It compares the residual syntax and projected traces and states across every strict transition among those levels (402 input checks). It also checks reduction of basis products (20,032 checks). For the displayed program the final state and dynamic signal values are affine functions of the input and all event positions and group tags are fixed, so zero and the basis determine the tested functions at those levels. Additional controls check a pair of quotient-equal inputs with distinct second-bit traces, a pair of level-1 inputs with the same final state and distinct signals, and distinct level-2 final states for the latter pair. The algebraic proof covers all levels. The runtime cost depends on the size of the specialized convolution kernel at the chosen level.

# 10 What a symmetry classification determines in physics

Fix a finite group $G$ and a unitary action on a finite-dimensional complex Hilbert space $V$. Complete reducibility gives

$$V \cong \bigoplus_\lambda W_\lambda \otimes \mathbb{C}^{m_\lambda},$$

with $W_\lambda$ distinct irreducibles. If a Hamiltonian $H$ commutes with the action, Schur's lemma gives

$$H = \bigoplus_\lambda I_{W_\lambda} \otimes H_\lambda, \qquad H_\lambda = H_\lambda^*.$$

The group, action, and specified invariance constrain the blocks, while entries of the Hermitian matrices remain dynamical parameters. For a concrete example take $G = C_2$ and $V = \mathbb{C}^2$ with generator acting by $\left(\begin{smallmatrix} 0 & 1 \\ 1 & 0 \end{smallmatrix}\right)$. All matrices $aI + b\left(\begin{smallmatrix} 0 & 1 \\ 1 & 0 \end{smallmatrix}\right)$ with $a, b \in \mathbb{R}$ are invariant. The splitting of the energy levels is $2|b|$, which can vary even with the representation fixed. One still needs dynamics, observables, and a state or boundary condition for predictions. This argument applies to finite-dimensional representations. Symmetries by continuous Lie groups, higher gauge fields, or spacetime groupoids require their own kinds of actions and dynamical data.

In quantum kinematics, even a fixed finite group can act projectively. For $C_2 \times C_2$, the classes of the two matrices

$$X = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \qquad Z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

in the projective unitary group commute and have order two. Every choice of unitary lifts is obtained by multiplying the two matrices by phases and still anticommutes. The projective action admits no ordinary unitary lift, illustrating the role of a central extension or a twisted group algebra.

A finite gauge group also leaves discrete choices. In the oriented $2+1$-dimensional Dijkgraaf–Witten setting, a group cocycle in $H^3(BG; U(1))$ supplies the topological action [29, 30]. For $G = C_2$ written additively, the normalized function

$$\omega(a, b, c) = (-1)^{abc}, \qquad a, b, c \in \mathbb{F}_2,$$

is a cocycle. The four-input exponent of its coboundary is

$$bcd + (a+b)cd + a(b+c)d + ab(c+d) + abc = 0 \pmod 2.$$

For any normalized $U(1)$-valued 2-cochain $\nu$, its coboundary at $(1,1,1)$ is

$$\frac{\nu(1,1)\nu(1,0)}{\nu(0,1)\nu(1,1)} = 1,$$

whereas $\omega(1,1,1) = -1$. Normalized cochains compute ordinary group cohomology, so $[\omega] \neq [1]$. The same $G$ and the same ordinary group algebra over a fixed $k$ admit distinct choices of action class. This conclusion concerns classification of the gauge data. It does not assert that cohomology classes classify systems with arbitrary dynamics and observables.

# 11 Conclusions

With an order bound and explicit finite tables, classification is effective. Presentations demand separate algorithmic hypotheses, and a factor inventory must be

supplemented by extension data. For profinite groups the inverse system supplies compatible finite quotients. Completed Hopf structure retains this system's group, and dualizing gives a filtered union of finite function algebras. Higher coherent semantics supports finite-stage reasoning for compact tests. It also allows systems whose unrestricted families of tests require arbitrarily fine stages.

For a compiler, the relevant claim is operational. A residual layer produced at a chosen quotient may be reused at that quotient when refinement preserves every observation used by the specialization. A composition of layers requires that property at their interfaces, including states, effects, and control decisions. This provides explicit hypotheses for correctness. The supplied executable is a finite reference model for those hypotheses. It establishes neither performance results for an industrial runtime nor total decidability of program equivalence.

**Reproducibility and authorship.** The accompanying source archive contains the manuscript and reference implementation. The executable checks concrete instances of the specialization diagrams. The proofs specify the general hypotheses independently of those checks. The submitting author uses an anonymous byline. AI assistance was used in literature retrieval, drafting, and building the reference implementation.

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