Actions of Toposes on Toposes

Geometric actions, internal symmetry, and quotient constructions

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Abstract. A monoidal Grothendieck topos can act on a topos by a geometric morphism out of their bicategorical product. We compare this notion with discrete groups acting by geometric autoequivalences, internal groups acting on objects, and groupoid actions in an ∞-topos. We prove how a discrete parameter topos recovers a weak group action and explain why the equivariant topos is a quotient in the direction of geometric morphisms. Classifying topoi, localic actions, and a point-free example show what is retained by generalized points, torsors, and descent. We review the canonical isotropy action on topoi relative to a base. We also distinguish the action of a discrete group from the multiplication of its classifying topos and give the higher bar construction with its slice universal property.

Keywords. Grothendieck topoi, geometric morphisms, actions, classifying topoi, torsors, groupoid objects, descent.

1 The geometric meaning of an action

An action of a group on a space consists of a parameter object, an evaluation, and composition laws. The same description makes sense in the bicategory of Grothendieck topoi and geometric morphisms. There is also a distinct internal description in which the group and the acted-on space are objects of a fixed topos. The two descriptions can interact through classifying topoi and descent. The parameter object, the direction of inverse image, and the kind of quotient determine the relationship in a given example.

Use **Top** for the bicategory of Grothendieck topoi, with arrows directed as geometric morphisms $f : \mathcal{X} \to \mathcal{Y}$. Thus $f^* : \mathcal{Y} \to \mathcal{X}$ is left exact and left adjoint to f_*. By a geometric transformation we mean a natural transformation between inverse images in the displayed order. Invertible geometric transformations are the coherence cells in our actions. We fix universes large enough for the sites occurring below. Products mean bicategorical products in this geometric direction. The terminal topos is Set. For the product-of-sites description see [8, 10].

Definition 1.1 (A monoidal topos and its action)**.** A monoidal topos in this sense is a pseudomonoid $\mathcal{T}$ for the product of **Top**. It has geometric morphisms

$$m : \mathcal{T} \times \mathcal{T} \to \mathcal{T}, \qquad e : \mathrm{Set} \to \mathcal{T}$$

with invertible associativity and unit cells satisfying the pentagon and triangle. A left action on $\mathcal{X}$ is a geometric morphism $a : \mathcal{T} \times \mathcal{X} \to \mathcal{X}$ with invertible cells $a(m \times 1) \simeq a(1 \times a)$ and $a(e \times 1) \simeq 1_{\mathcal{X}}$ satisfying the action coherences. Product comparisons are understood.

This is the usual action definition in a cartesian bicategory. In inverse-image notation the evaluation is $a^* : \mathcal{X} \to \mathcal{T} \times \mathcal{X}$, with the associativity comparison between the two composites out of $\mathcal{X}$. It is important here to use the geometric product before taking inverse image. The cartesian product of the underlying categories of sheaves typically gives a different topos in **Top**.

For an exponentiable $\mathcal{X}$ in the bicategorical sense, an action can be curried to a morphism $\mathcal{T} \to [\mathcal{X}, \mathcal{X}]$. The endomorphism topos then has a multiplication induced by composition, and the action cells say that this map preserves it coherently. The evaluation definition works without this exponential hypothesis. Exponentiability of Grothendieck topoi is a separate property studied by Johnstone and Joyal [5].

1.1 Generalized points determine the action

Write $\mathrm{Geom}(\mathcal{B}, \mathcal{X})$ for the category of geometric morphisms and transformations. An action evaluates on each test topos $\mathcal{B}$ by the functor

$$\begin{aligned} \mathrm{Geom}(\mathcal{B}, \mathcal{T}) \times \mathrm{Geom}(\mathcal{B}, \mathcal{X}) &\longrightarrow \mathrm{Geom}(\mathcal{B}, \mathcal{X}), \\ (t, x) &\longmapsto a\langle t, x\rangle. \end{aligned}$$

The action cells induce the corresponding cells on all these functors. In particular, $t : \mathcal{B} \to \mathcal{T}$ acts on $\mathcal{B} \times \mathcal{X}$ over $\mathcal{B}$ by the pair $\langle \pi_{\mathcal{B}}, a\langle t\pi_{\mathcal{B}}, \pi_{\mathcal{X}}\rangle\rangle$. If $\mathcal{T}$ has an inverse morphism with the usual group identities up to coherent invertible cells, this endomorphism over $\mathcal{B}$ is an equivalence. The inverse is induced by the inverse generalized point, using the unit and multiplication cells.

Proposition 1.2 (Test-topos description)**.** *The evaluation a and its coherence cells are equivalent to the operation on geometric morphisms from every test topos, with its coherence cells and pseudonaturality under precomposition.*

Proof. A product gives a pseudonatural equivalence between the pair of input hom-categories and $\mathrm{Geom}(\mathcal{B}, \mathcal{T} \times \mathcal{X})$. At $\mathcal{B} = \mathcal{T} \times \mathcal{X}$, apply the operation to its two projections. The resulting geometric morphism is a. For a pair (t, x) with test topos $\mathcal{B}$, pseudonaturality under $\langle t, x\rangle$

identifies its evaluation with $a\langle t, x\rangle$. Evaluating transformations at the same projections recovers the transformation between action morphisms. Applying this argument with the products of two and three parameter factors recovers the coherence cells and their equations. This is the bicategorical Yoneda argument. □

Testing only at Set can fail entirely. Let L be the locale of the complete Boolean algebra of regular open subsets of $\mathbb{R}$. Its Boolean algebra is atomless since any nonempty regular open contains a nonempty smaller interval with closure inside it. A locale point $h : \mathcal{O}(L) \to 2$ preserves arbitrary joins and finite meets. Set $u = \bigvee\{v : h(v) = 0\}$, so $h(u) = 0$. Every nonzero element below $\neg u$ has image 1. Two disjoint nonzero elements below $\neg u$ would violate preservation of meets. Thus $\neg u$ would be an atom. There is no such point.

The topos $\mathcal{X} = \mathrm{Sh}(L \amalg L)$ has no ordinary points. The identity and the swap of its two summands give two different actions of the two-element discrete group (realized geometrically as in the next subsection). On ordinary points the actions return the same empty description. With test topos $\mathrm{Sh}(L)$, the inclusion of the first summand is sent to itself or to the second summand. Pullback of the subterminal selecting the first summand distinguishes those two morphisms. For the locale and sheaf correspondence see [4, 8].

1.2 Discrete parameters and composition

For a set I write $D_I = \mathrm{Set}^I$, the sheaf topos of the discrete locale on I. Its underlying category is a product of copies of Set. In **Top**, however, it is their coproduct. More generally a product of categories $\prod_i \mathcal{X}_i$ is a coproduct of topoi in the geometric direction: a left exact left adjoint into it is a family of left exact left adjoints into the individual factors. Given their right adjoints, the right adjoint of the family is the product of their values in the common source category.

Proposition 1.3 (Discrete parameter topos)**.** *There is an equivalence in* **Top**

$$D_I \times \mathcal{X} \simeq \coprod_{i \in I} \mathcal{X}.$$

For a discrete monoid M, the multiplication on M induces one on D_M. Actions of D_M on $\mathcal{X}$ are weak monoid actions by geometric endomorphisms. For a group G these are weak group actions by geometric autoequivalences.

Proof. A morphism $\mathcal{B} \to D_I$ is classified by a partition of its terminal object, $1_{\mathcal{B}} = \coprod_{i \in I} U_i$, into disjoint subterminal objects. To see this, apply inverse image to the subterminal supported at each coordinate. In the other direction the inverse image sends (S_i) to $\coprod_i S_i \cdot U_i$, where $S_i \cdot U_i$ denotes the coproduct of S_i copies of U_i. It has right adjoint $A \mapsto (\mathrm{Hom}_{\mathcal{B}}(U_i, A))_i$. The disjointness and pullback stability of coproducts give left exactness. A natural transformation gives inequalities $U_i \le V_i$ between the two partitions. Each V_i is the coproduct of its intersections with all the U_j, of which only U_i remains. Hence the partitions coincide, and on copowers the transformation is fixed by its restriction to the U_i.

Disjoint stable coproducts give an equivalence $\mathcal{B} \simeq \prod_i \mathcal{B}/U_i$. A morphism $\mathcal{B} \to \mathcal{X}$ therefore restricts to a family of morphisms $\mathcal{B}/U_i \to \mathcal{X}$, and the restrictions determine it. A map $\mathcal{B} \to \coprod_i \mathcal{X}$ has exactly the same data: pull back the subterminals selecting its summands and take the induced maps on the pieces. These descriptions are natural in $\mathcal{B}$ and also identify transformations. This proves the product formula.

In particular $D_{I \times J} \simeq D_I \times D_J$. The action evaluation for D_M is therefore specified by its restrictions a_m to the summands, and the cells restrict to $a_{mn} \simeq a_m a_n$ and the identity at the unit. Conversely these data assemble by the coproduct property. For a group the component at g^{-1} supplies the inverse to the component at g. □

In particular D_G and the classifying topos of G-sets serve different purposes. The former parametrizes group elements. The latter classifies torsors and retains their automorphisms. For trivial action on Set, a geometric quotient is the classifying topos, as Section 2 explains.

1.3 The quotient convention

For a geometric action, the action nerve has terms $\mathcal{T}^n \times \mathcal{X}$. The first face at degree one is the action, and the other is projection. The higher faces compose parameters or act with one of them. The simplicial identities for a weak action hold up to coherent isomorphism. One can specify its geometric quotient by a bicategorical codescent property. A geometric morphism from the quotient to $\mathcal{Y}$ amounts to $f : \mathcal{X} \to \mathcal{Y}$ equipped with an *invertible* comparison $fa \simeq f\,\mathrm{pr}_{\mathcal{X}}$, subject to unit and composition. This specifies the quotient when the indicated universal object exists. The invertibility convention matters for monoids. For the trivial action of the additive monoid $\mathbb{N}$ on Set, invertible comparisons are one automorphism of the selected point of $\mathcal{Y}$, with its powers. They are equivalently $\mathbb{Z}$-actions. The quotient for this convention is the classifying topos of $\mathbb{Z}$-sets by the trivial-action case of the next section. Allowing noninvertible comparisons instead gives actions of a single endomorphism. The classifying topos for these data is the category of sets with an endomorphism: an inverse image into it consists of a left exact cocontinuous functor to sets and a natural endomorphism, since limits and colimits of such diagrams are computed pointwise. The forgetful point of this topos has the natural endomorphism supplied by the structure map of each set. Its value on the set of natural numbers with successor is not invertible, so this point with its comparison gives only a lax cocone.

2 External actions by geometric automorphisms

Let G be a small discrete group and $\mathcal{E}$ a Grothendieck 1-topos. An external action of G is a pseudofunctor from the one-object groupoid BG to geometric morphisms, with value $\mathcal{E}$ at the object. Write ρ_g for the resulting geometric autoequivalence. Composition satisfies $\rho_g\rho_h \simeq \rho_{gh}$, with the usual associativity and unit coherences. In the direction of inverse images it is convenient to index by $T_g = \rho^*_{g^{-1}}$. Then there are coherent isomorphisms $\mu_{g,h} : T_gT_h \to T_{gh}$ and $\epsilon : \mathrm{id} \to T_1$.

The equivariant category $\mathcal{E}^G$ consists of pairs (X, θ) where $\theta_g : T_gX \to X$ are isomorphisms and

$$\theta_{gh}\mu_{g,h,X} = \theta_g T_g(\theta_h), \qquad \theta_1\epsilon_X = \mathrm{id}_X. \tag{1}$$

Morphisms commute with all θ_g. Thus the underlying object need only be identified with each translate, and the identifications themselves are part of the object. A strict action is the case in which the composition and unit comparisons are identities.

Proposition 2.1 (Quotient and fixed-point variance)**.** *The category $\mathcal{E}^G$ is a Grothendieck topos. Its forgetful functor $U : \mathcal{E}^G \to \mathcal{E}$ is the inverse image of a geometric morphism $q : \mathcal{E} \to \mathcal{E}^G$. With the canonical isomorphisms $q\rho_g \simeq q$, this is a pseudocolimit of the action in geometric direction. In inverse-image direction it is the pseudolimit of the T_g in the 2-category of topoi and left exact left adjoints.*

Proof. Each T_g is an equivalence and so preserves all limits and colimits. The underlying limit or colimit of an equivariant diagram inherits its equivariance by applying T_g to the defining cones. Coherence follows by the universal property, and U creates these constructions and reflects isomorphisms. Disjointness of coproducts, pullback stability of colimits, and effectiveness of equivalence relations can therefore be checked in $\mathcal{E}$. For a separating set of objects A of $\mathcal{E}$, use the equivariant objects

$$L(A) = \coprod_{g\in G} T_gA.$$

The structure maps move summands by the $\mu_{h,g}$. A map $L(A) \to X$ is equivariant precisely when it is determined by its component $A \to UX$ at the identity summand, using ϵ_A. Its other components are $T_gA \xrightarrow{T_g(f)} T_gUX \xrightarrow{\theta_g} UX$. Since ordinary maps out of the chosen A's separate maps in $\mathcal{E}$, the $L(A)$'s separate equivariant maps. Giraud's characterization gives the topos claim [8, Ch. VII]. The object $\prod_g T_gA$, with factors permuted using the same coherences, is right adjoint to U. Indeed an equivariant map to that product is likewise determined by its identity component $f : UX \to A$, with g-component $T_g(f)\theta_g^{-1}$. Thus U is a left exact left adjoint.

For a test topos $\mathcal{F}$, an inverse image into $\mathcal{E}^G$ is a functor $S : \mathcal{F} \to \mathcal{E}^G$ preserving finite limits and small colimits. Composing with U gives $F : \mathcal{F} \to \mathcal{E}$, together with natural isomorphisms $T_gF \simeq F$ obeying (1). Conversely these data lift F to $\mathcal{E}^G$, and preservation is checked by the created limits and colimits. A colimit-preserving functor between Grothendieck topoi has a right adjoint by the adjoint functor theorem for locally presentable categories. Compatible natural transformations are exactly transformations of the lifts. This gives the pseudolimit property. Reversing geometric arrows exchanges the cone with a cocone and gives the claimed pseudocolimit. □

The adjunction $L \dashv U$ also gives an algebraic description. The monad on $\mathcal{E}$ sends A to $\coprod_g T_gA$, with unit at the identity summand and multiplication carrying (g, h) to gh using $\mu_{g,h}$. An algebra is a family of maps $\beta_g : T_gX \to X$ satisfying (1). Even if invertibility is omitted from this definition, the equation for $h = g^{-1}$ gives a right inverse to β_g after identifying $T_gT_{g^{-1}}X$ with X. The equation with g and g^{-1} reversed gives a left inverse by whiskering with T_g and using the coherences. Thus the monad's algebras are exactly equivariant objects. For the trivial action the monad is the copower by the set G, and its algebras are actions of the constant internal group. For a nontrivial action, its summands transport the object before mapping it back.

Similarly an equivariant geometric morphism $f : \mathcal{A} \to \mathcal{E}$ has an inverse-image functor with coherent identifications $S_gf^* \simeq f^*T_g$, where S_g acts on $\mathcal{A}$. Applying f^* to the underlying object and using these identifications defines $\mathcal{E}^G \to \mathcal{A}^G$. It preserves finite limits and colimits under the forgetful functors, hence induces a geometric morphism $\mathcal{A}^G \to \mathcal{E}^G$.

2.1 A site calculation and its examples

Suppose the action is realized on a small site (C, J) by strict functors $a_g : C \to C$ preserving the topology. Then T_g acts by precomposition with $a^{\mathrm{op}}_{g^{-1}}$. Form the Grothendieck construction $C \rtimes G$, with objects those of C and arrows $c \to d$ pairs $(g, u : a_gc \to d)$. Their composite with $(h, v) : d \to e$ is $(hg, v \circ a_h(u))$. Give this category the topology generated by J-covers in the identity copy of C.

A presheaf on $C \rtimes G$ is a presheaf on C with the equivariance in (1). To see this, factor (g, u) through the canonical isomorphism $c \to a_gc$ followed by the identity-indexed arrow u. Restriction along these canonical isomorphisms encodes the θ_g's, and factorization and the cocycle law recover all restrictions. A covering sieve S on d in C extends to the sieve of all (g, u) with $u \in S$ in the larger category. The extended covering sieves form a topology since pullback along each canonical isomorphism is transported by a site automorphism, and pullback and transitivity for the identity-indexed arrows use J. In

particular, to test transitivity for all arrows of an extended sieve it suffices to test the identity-indexed arrows, since each other arrow factors through one of those by an isomorphism. A matching family for such a sieve is determined by its entries along the identity-indexed arrows. The entry along (g, u) must be obtained from the entry along $(1, u)$ by restriction along $c \to a_g c$, and this assignment satisfies the remaining compatibilities by the same factorization. Therefore the sheaf condition is exactly the J-sheaf condition on C, and this site presents $\mathcal{E}^G$.

For the trivial action this gives $\mathcal{E}^G \simeq \mathrm{Fun}(BG, \mathcal{E})$. It is the geometric product $\mathcal{E} \times \mathbf{B}G$, where $\mathbf{B}G$ denotes the topos of G-sets. Indeed in this case $C \rtimes G = C \times BG$ and the topology is the product-site topology with the trivial topology on BG [10, Ch. II]. In particular the trivial action on **Set** has quotient $\mathbf{B}G$. A limit in geometric direction of the constant action on the terminal topos **Set** would instead be **Set**. The quotient includes a G-action on every object.

For a discrete group acting on a space X by homeomorphisms, apply the site construction to opens with their usual covering families. Equivariant objects are sheaves on X with lifts of the action, or equivalently equivariant local homeomorphisms to X. This is the classifying topos of the action groupoid $G \times X \rightrightarrows X$ [9]. For the transitive discrete space $X = G/H$, taking the fiber over the identity coset recovers an H-set A. Its inverse construction has total space $G \times_H A$ over G/H. Hence the quotient topos is $\mathbf{B}H$ although the orbit set has one point. In this example the remaining isotropy determines the entire quotient.

3 Internal actions and relative classifying topoi

Let $\mathcal{E}$ be a Grothendieck topos. An internal group G is a group object in $\mathcal{E}$. Write $B_{\mathcal{E}}G$ for the topos of left G-objects in $\mathcal{E}$. Its projection $\pi : B_{\mathcal{E}}G \to \mathcal{E}$ has inverse image the trivial-action functor and direct image the invariants functor. The latter assigns to a G-object the subobject on which its action agrees with the projection. In this setting a right G-torsor is a right G-object T such that $T \to 1$ is an epimorphism and

$$T \times G \longrightarrow T \times T, \qquad (t, g) \longmapsto (t, tg)$$

is an isomorphism. The epimorphism requires local inhabitation. Together the two conditions give a local identification with the regular action.

3.1 The universal torsor and change of base

The regular left G-object G has a commuting right action of the trivial G-object π^*G. With this right action it is a π^*G-torsor, denoted U_G. For a geometric morphism $f : \mathcal{F} \to \mathcal{E}$, pullback of U_G gives an equivalence of groupoids

$$\mathrm{Geom}^{\simeq}_{/\mathcal{E}}(\mathcal{F}, B_{\mathcal{E}}G) \simeq \mathrm{Tors}_{\mathcal{F}}(f^*G). \tag{2}$$

The domain takes the isomorphism groupoid of geometric morphisms over $\mathcal{E}$, with transformations written in the direction of inverse images. The comparison over $\mathcal{E}$ is a specified isomorphism. This is the relative classifying property [1, Thm. 2.3].

To see its construction, let T be a right f^*G-torsor. On a left G-object A, the associated inverse image is $T \times^{f^*G} f^*A$, the quotient identifying (tg, a) with (t, ga). After pullback along $T \twoheadrightarrow 1$ the torsor has a section, so the contracted product becomes the underlying f^*A in that slice. Finite limits are therefore preserved, since this pullback is conservative. Colimit preservation follows from the contracted-product coequalizer and the preservation of colimits by products in a topos. Thus the functor has a right adjoint by the adjoint functor theorem for presentable categories. On trivial actions it agrees with f^*, and on the regular left object its value is T. For the other composite, apply an inverse image over $\mathcal{E}$ to the natural presentation of A as $U_G \times^{\pi^*G} \pi^*A_0$, where π^*A_0 bears the pullback of the original action as an internal action of π^*G. It gives the contracted product with the pulled-back torsor. This presentation also identifies natural isomorphisms over the base with torsor isomorphisms.

For any f as above there is a pullback of topoi in the bicategorical sense:

$$\mathcal{F} \times_{\mathcal{E}} B_{\mathcal{E}}G \simeq B_{\mathcal{F}}(f^*G).$$

Indeed, a test topos over $\mathcal{F}$ sees the same pulled-back group and the same torsor groupoid on either side. Relative transformations between the corresponding inverse images reduce to equivariant torsor maps, which are isomorphisms because they are locally maps of regular torsors. See [1, Cor. 2.4].

3.2 Groupoid actions and descent

For an internal groupoid $R \rightrightarrows U$ with source s and target t, an action is an object $A \to U$ together with transport along arrows. Equivalently it is an isomorphism $s^*A \simeq t^*A$ over R that is the identity on units and satisfies the composition law over composable pairs. Write $B_{\mathcal{E}}R$ for the category of these objects. Limits and colimits are computed on the anchored objects in $\mathcal{E}/U$, because s^*, t^* preserve them. Transport then inherits the laws from the diagram. The slice's universal colimits and effective equivalence relations give the same properties for action objects. To see the generator condition, use the free action on $A \to U$, $R \times_{s,U} A \to U$, with anchor given by t. Free actions on slice generators detect maps by this adjunction and the faithful forgetful functor. Giraud's criterion makes $B_{\mathcal{E}}R$ a Grothendieck topos. Its projection

to $\mathcal{E}$ has inverse image sending C to $U \times C$ with transport trivial in the second factor.

One can use the same contracted-product argument to classify right f^*R-torsors. Here T has an anchor to f^*U, is locally inhabited, and the action induces an isomorphism $T \times_{f^*U} f^*R \simeq T \times T$, using the target of the arrow in the fiber product. The universal torsor is the arrow object with left transport by composition and right multiplication along its source anchor. After locally choosing a section of T with anchor u, its contracted product with a left action is the pullback of that action's underlying object along u. This proves the finite-limit condition in the classification as before. For the colimit condition, first form the fiber product over f^*U in the slice and then take the quotient in $\mathcal{F}$. Pullbacks and the functor composing structure maps preserve colimits. Reconstruction uses the natural isomorphism $R \times^R A \simeq A$ by evaluation. The identity at the anchor of a represents its inverse on the class of a.

For example, for the action groupoid $G \ltimes X$ of a left G-object X, an anchored object transports exactly when it is an equivariant map to X. Hence

$$B_{\mathcal{E}}(G \ltimes X) \simeq (B_{\mathcal{E}}G)/X.$$

A map from $\mathcal{F}$ over $\mathcal{E}$ into this quotient topos is specified by a right f^*G-torsor T and an equivariant map $T \to f^*X$, with $a(tg) = g^{-1}a(t)$. This follows because a map to the slice also supplies a section of the associated object $T \times^{f^*G} f^*X$. When $\mathcal{E} = \mathrm{Set}$ and $X = G/H$ for a subgroup H, taking the fiber above the identity coset yields an equivalence of this slice with the topos of H-sets. Induction $G \times^H (-)$ gives an inverse, since its fibers are transported by left translations and the ambiguities in the transport are exactly the H-action.

There is also an explicit atlas for $B_{\mathcal{E}}R$. Regard R as a left action via its target map. The slice $(B_{\mathcal{E}}R)/R$ is equivalent to $\mathcal{E}/U$: take the fiber of an equivariant map over the identity arrows, and reconstruct its fiber over r by transport along r from the identity of $s(r)$. The projection of this slice is an étale surjection since $R \to 1$ is epic in the action topos. Its double overlap is the slice over $R \times R$ in $B_{\mathcal{E}}R$. Under the first projection and the identity-fiber equivalence, it becomes $\mathcal{E}/R$: the pair $(1_u, r)$ is specified by an arrow with target u. The overlap maps are given by the two anchors of r, and the cocycle on triple overlaps composes arrows. Effective descent along this atlas recovers the transport description of $B_{\mathcal{E}}R$.

For an epimorphism $U \to V$ the Čech groupoid $U \times_V U \rightrightarrows U$ gives $B_{\mathcal{E}}(U \times_V U \rightrightarrows U) \simeq \mathcal{E}/V$. Indeed, its arrows identify the objects over U by exactly the descent cocycle. Quotient the resulting equivalence relation on an action object to descend it to V. Any local representative of a quotient element can be transported to a specified U-coordinate, and that representative is then unique. Pullback to U therefore recovers the original object's action. In particular refining the object space of an internal groupoid along an epimorphism preserves its classifying topos by this descent argument. This is a basic Morita invariance mechanism. For continuous and localic groupoids, object and arrow spaces have topology and the actions used in the classifying topos are on equivariant sheaves. The geometric hypotheses and the descent results for those settings are developed in [9, 10, 6].

Example 3.1 (A sheaf of groups with trivial global group)**.** Over $S^1 = \mathbb{R}/\mathbb{Z}$, let G be the locally constant sheaf with fiber $\mathbb{Z}/3$ and monodromy $a \mapsto -a$. It is represented as an étale group bundle by identifying (t, a) with $(t+1, -a)$ in $\mathbb{R} \times \mathbb{Z}/3$. A global section is fixed by inversion, so the group of global sections is trivial. Pullback of $B_{\mathrm{Sh}(S^1)}G$ over any point gives the topos of $\mathbb{Z}/3$-sets by base change. This topos has a nontrivial regular action that is noninitial in the action topos and has no equivariant global element. In Set every noninitial object has a global element. The global section group has classifying topos Set. The relative classifying topos thus retains the local group and its transport.

4 Concrete monoidal topoi and actions

An ordinary group G produces two different topoi. The topos $\mathrm{Sh}(G_{\mathrm{disc}})$ is the sheaf topos of the discrete locale with set of points G. Multiplication of the discrete locale gives a monoidal multiplication of this topos, with inverse induced by $g \mapsto g^{-1}$. By contrast, write $\mathbf{B}G$ for the topos of left G-sets. It classifies right G-torsors, and the ordinary group law on G does not in general induce a multiplication of $\mathbf{B}G$. The distinction already appears over **Set**: the discrete locale has one geometric point for every element of G, whereas $\mathbf{B}G$ has one point up to isomorphism, with automorphism group G.

4.1 From locales to monoidal topoi

The construction Sh preserves products of locales as products of topoi. One explanation uses the localic reflection: for a Grothendieck topos $\mathcal{U}$, maps $\mathcal{U} \to \mathrm{Sh}(L)$ classify frame maps

$$\mathcal{O}(L) \longrightarrow \mathrm{Sub}_{\mathcal{U}}(1).$$

Here a frame map preserves arbitrary joins and finite meets. The frame of opens of a locale product is the coproduct of the two frames. Hence giving the classifying data for a map into the sheaves on a product is equivalent to giving them for each factor. The same statement holds for the comparisons of maps: they are determined by the order on the images of opens, which can be checked on the two factor frames [8, Ch. IX, §5, Prop. 3]. Thus if a group locale K acts on a locale L, the multiplication and action induce

$$\mathrm{Sh}(K) \times \mathrm{Sh}(L) \longrightarrow \mathrm{Sh}(L)$$

with the required associativity and unit. The construction is pointfree and applies to localic group actions used in topos descent [6, Chs. V–VIII].

For example, ordinary real addition and real addition with the discrete topology give different group topoi. Both have $(\mathbb{R}, +)$ as their group of geometric points. Their subterminal frames distinguish them. For the discrete topology the frame is a Boolean power set. For the usual topology, a proper nonempty open interval has no open complement in the connected real line. The two topoi therefore cannot be equivalent. Both act on the sheaf topos of the usual real line by translations. The two actions agree on individual ordinary points, while their parameter topoi encode different families of translations.

A simple monoid that is not a group is the Sierpiński locale Σ, whose frame is free on one generator. A geometric map $\mathcal{U} \to \mathrm{Sh}(\Sigma)$ classifies a subobject of 1 in $\mathcal{U}$ [12]. The product $\mathrm{Sh}(\Sigma)^2$ classifies two such subobjects. The universal operation $(U, V) \mapsto U \cap V$, together with the unit $1 \hookrightarrow 1$, defines a monoidal structure on $\mathrm{Sh}(\Sigma)$. The operation commutes with inverse image functors because they preserve finite limits. On the product topos $\mathrm{Sh}(\Sigma)^r$ it defines an action by

$$(U, V_1, \ldots, V_r) \longmapsto (U \cap V_1, \ldots, U \cap V_r).$$

Applying first U and then W gives the same family as applying $U \cap W$. The point classifying $0 \hookrightarrow 1$ sends every tuple to the one consisting of zero subobjects. For $r \geq 1$ it cannot act by an equivalence since this map identifies distinct tuples already over **Set**. This example has a noninvertible action in geometric morphisms, expressed entirely by geometric logical operations.

4.2 When a torsor classifier is monoidal

The classifying property of $\mathbf{B}G$ identifies geometric maps $\mathcal{U} \to \mathbf{B}G$ with right G-torsors in $\mathcal{U}$; G is interpreted as the constant group [8, Ch. VIII, Thm. 2.7]. For an ordinary group H, a right G-torsor in $\mathbf{B}H$ is a right G-torsor of sets equipped with a commuting left H-action. Choose an element p of the torsor and write $h \cdot p = p\phi(h)$. Commutation of the actions gives $\phi(hh') = \phi(h)\phi(h')$. Replacing p by pg conjugates ϕ by g^{-1}. Conversely, any homomorphism $\phi : H \to G$ gives a torsor by taking the regular right G-set with left H-action by ϕ. Thus maps $\mathbf{B}H \to \mathbf{B}G$ are classified up to isomorphism by homomorphisms $H \to G$ up to conjugation. A map induced by ϕ has inverse image given by restriction of action along ϕ. Composition of the geometric maps therefore agrees with composition of the inducing homomorphisms.

There is a product equivalence $\mathbf{B}G \times \mathbf{B}H \simeq \mathbf{B}(G \times H)$. Indeed, the product of a right G-torsor and a right H-torsor is a right $(G \times H)$-torsor. Conversely, a $(G \times H)$-torsor P recovers the two torsors by quotienting by the H- and G-actions respectively. The reconstructed product identifies with P after a covering over which the torsor is trivial, hence globally by descent. The two constructions are natural in the test topos, so represent the same product.

Proposition 4.1. *The topos $\mathbf{B}G$ admits a monoidal structure for the Cartesian product of Grothendieck topoi if and only if G is abelian. For abelian G the standard multiplication makes $\mathbf{B}G$ a group object, with all laws interpreted up to the canonical equivalences of products.*

Proof. A multiplication is classified by a homomorphism $\mu : G \times G \to G$. The unit map $\mathbf{Set} \to \mathbf{B}G$ is unique up to isomorphism, since right G-torsors in sets are isomorphic to G. The two unit comparisons imply that each restriction of μ to a factor is conjugate to the identity. Both restrictions are therefore surjective. Their images commute elementwise, since the factors of $G \times G$ do. Thus any two elements of G commute.

When G is abelian, $(g, h) \mapsto gh$ is a homomorphism. Its associativity and unit equalities induce the required coherences by restriction of actions. The inverse homomorphism $g \mapsto g^{-1}$ similarly supplies inversion. □

There are actions of such group topoi that are visible on automorphisms of points. Let A be abelian and let $\lambda : A \to Z(H)$ be a homomorphism into the center of an arbitrary group H. The map

$$A \times H \longrightarrow H, \qquad (a, h) \longmapsto \lambda(a)h$$

is a homomorphism. It gives a left action $\mathbf{B}A \times \mathbf{B}H \to \mathbf{B}H$ because the unit and associativity equalities hold on these homomorphisms. Conversely, an action morphism with invertible unit comparison is represented by a homomorphism $\phi : A \times H \to H$ whose restriction to H is conjugate to the identity. After conjugating the whole homomorphism this restriction is the identity, so $\phi(a, 1)$ must lie in $Z(H)$. This recovers λ at the level of action morphisms. Equivalences that also keep track of all chosen coherence data require their own comparison.

Taking geometric points of this action, each of the three classifying topoi involved has just one isomorphism class of point. But after choosing the regular torsors as representatives, the action on their automorphisms is the displayed map $(a, h) \mapsto \lambda(a)h$. In particular, λ can be read off by setting $h = 1$. The groupoid of points thus retains information lost on passing to mere isomorphism classes, even for these atomic topoi.

5 Internal actions in an ∞-topos

An internal group can act on an object of an ∞-topos. In this setting the quotient is again an object, and the category of equivariant objects is itself a slice ∞-topos. The construction uses effective groupoids and descent for slices [7, §6.1.3]. The resulting interpretation of actions and torsors is developed in [11, §§3–4.1].

5.1 Action groupoids and effective descent

Let $\mathcal{X}$ be an ∞-topos. Present a group object G by a reduced simplicial groupoid $G_\bullet$ in $\mathcal{X}$. Thus $G_0 = 1$ and the Segal maps identify G_n with G^n, where $G = G_1$. Write $BG = |G_\bullet|$ for the geometric realization. Groupoid effectivity identifies $G_\bullet$ with the Čech nerve of the effective epimorphism $1 \to BG$. In particular, $G \simeq 1 \times_{BG} 1$ with its loop group structure. The equivalence between group objects and pointed connected objects is given by this delooping and loop construction [7, Thm. 6.1.0.6 and Lemma 7.2.2.11].

One definition of an action on an object X is a cartesian transformation $U_\bullet \to G_\bullet$ with $U_0 = X$. Here cartesian means that the naturality square for each simplicial operator is a homotopy pullback. In particular $U_n \simeq X \times G^n$ by using the initial vertex in $[n]$. The two vertex maps $U_1 \rightrightarrows X$ are then the projection and the action. For an ordinary right action, for example, these take (x, g) to x and xg. Composition and the higher compatibility of the action are encoded by the simplicial maps. In this description the Segal and invertibility conditions for $U_\bullet$ are obtained by pulling back those for $G_\bullet$: a string of arrows over $(g_1, \dots, g_n)$ is determined by a starting object. Thus $U_\bullet$ is an action groupoid. The category $\mathrm{Act}_G(\mathcal{X})$ uses all natural transformations over $G_\bullet$ as its equivariant maps.

Proposition 5.1 (The slice of actions)**.** *The functor that sends an action groupoid to its realized map over BG gives an equivalence*

$$\mathrm{Act}_G(\mathcal{X}) \simeq \mathcal{X}_{/BG}.$$

The image of an action on X has source the homotopy quotient $X//G = |[n] \mapsto X \times G^n|$ with the action bar structure. Its fiber over $1 \to BG$ is X.

Proof. For each morphism $Y \to BG$, pull back $G_\bullet \to BG$ along it. The result is a cartesian transformation to $G_\bullet$ with degree-zero object $Y \times_{BG} 1$. Its realization is Y: colimits in an ∞-topos are stable under pullback. Conversely let $U_\bullet \to G_\bullet$ be a cartesian transformation and realize both diagrams. The cartesian descent criterion for colimit diagrams makes the augmented transformation $U_\bullet \to G_\bullet$ cartesian as well [7, Thm. 6.1.3.9(4) and Prop. 6.1.3.10]. In particular

$$\begin{array}{ccc} U_0 & \longrightarrow & |U_\bullet| \\ \downarrow & & \downarrow \\ 1 & \longrightarrow & BG \end{array}$$

is a homotopy pullback, and the same holds with U_n and G_n in place of U_0 and 1. These comparisons identify the original diagram with the one pulled back from its realization. They are natural in transformations over $G_\bullet$, so the equivalence covers maps and their higher coherences. □

By taking the slice version of descent one can also express the result as

$$\mathcal{X}_{/BG} \simeq \lim_{[n]\in\Delta} \mathcal{X}_{/G_n}.$$

An object of this limit is an object over each stage together with coherent identifications along the pullback functors. This describes the action without requiring a chosen model of the quotient. A geometric inverse image commutes with these particular action constructions: it preserves finite limits and geometric realizations, so it carries the action groupoid, the homotopy quotient, and the fiber square to their counterparts for the pulled-back group and object.

5.2 What a family of quotient points specifies

For a parameter object T, a map $T \to BG$ pulls back the universal effective epimorphism to a G-torsor $P \to T$. A torsor here is an effective epimorphism equipped with a right action such that $(p, g) \mapsto (p, pg)$ identifies $P \times G$ with $P \times_T P$. Its action groupoid is the Čech nerve of $P \to T$, so realization recovers T and supplies a map to BG. Proposition 5.1 makes these constructions inverse. More generally, a map $T \to X//G$ specifies the torsor induced by its composite with BG together with an equivariant map $P \to X$. Indeed a lift of $T \to BG$ to $X//G$ is a map in the slice, which under the proposition is the asserted equivariant map. Thus the fibers of the mapping-space map to $\mathrm{Map}_{\mathcal{X}}(T, BG)$ record equivariant maps for each torsor. This is the family-level refinement of an orbit space.

There is also an object of homotopy fixed points. Given $Y = X//G \to BG$, the pullback along $q : BG \to 1$ has a right adjoint Π_q since $\mathcal{X}$ is locally cartesian closed. Its value $\Pi_q(Y)$ represents sections of Y parametrized by an arbitrary object:

$$\mathrm{Map}_{\mathcal{X}}(T, \Pi_q Y) \simeq \mathrm{Map}_{\mathcal{X}_{/BG}}(T \times BG, Y).$$

Under Proposition 5.1, $T \times BG$ corresponds to the trivial action on T, because the quotient of the trivial bar construction is its product with BG. Consequently $\Pi_q(Y)$ represents equivariant maps out of objects carrying the trivial action. It is denoted X^{hG}. The homotopy orbit object $X//G$ and the homotopy fixed-point object X^{hG} therefore come from the left and right adjoints to the trivial-action functor. Indeed, the left adjoint to q^* takes a map over BG to its domain, forgetting the structure map. In spaces these two objects specialize to the colimit and limit of the coherent G-diagram, respectively.

Example 5.2 (Reflection of a circle)**.** In spaces let C_2 act on S^1 by complex conjugation. The coarse orbit space is an interval. The homotopy quotient has a fibration sequence $S^1 \to S^1//C_2 \to BC_2$. It admits a section because $1 \in S^1$ is a fixed point. The fundamental group

sequence is a split extension of C_2 by $\mathbb{Z}$, using $\pi_2(BC_2) = 0$. The nonidentity element acts on $\mathbb{Z}$ by inversion since conjugation reverses the orientation of S^1. It follows that

$$S^1//C_2 \simeq B(\mathbb{Z} \rtimes C_2).$$

Here both the base and the fiber are aspherical, so the long exact homotopy sequence makes the total space aspherical as well. The quotient has infinite dihedral fundamental group, a distinction invisible to the interval of orbits.

5.3 Comparison with external symmetry

An external action on an ∞-topos assigns geometric autoequivalences to elements of a group and coherence to their composition. The associated category of equivariant objects is the limit of the resulting diagram of categories of inverse images. The action groupoid above uses objects and arrows internal to a single $\mathcal{X}$. The two constructions meet for a trivial external action of a discrete group H. Use the constant internal group $\underline{H} = \coprod_{h \in H} 1$. By disjoint universal coproducts, $\mathcal{X}_{/\underline{H}^n} \simeq \prod_{H^n} \mathcal{X}$. Slice descent then gives $\mathcal{X}_{/B\underline{H}} \simeq \mathrm{Fun}(B\underline{H}, \mathcal{X})$: the cosimplicial limit of these products imposes the coherent action along the nerve of H. This identification supplies an underlying internal object group for that particular category of equivariant objects.

6 Isotropy actions and crossed topoi

A topos carries an intrinsic source of symmetry. For a Grothendieck topos $\mathcal{E}$ and an object U, write $p_U : \mathcal{E}/U \to \mathcal{E}$ for the slice projection, with inverse image $A \mapsto A \times U \to U$. Funk, Hofstra, and Steinberg prove that the presheaf of groups

$$U \longmapsto \mathrm{Aut}(p_U^*)$$

is represented by a group object $Z_{\mathcal{E}}$ [2, §4.1]. The restrictions of this presheaf pull natural automorphisms back along maps of slices. The identity of $Z_{\mathcal{E}}$ therefore gives a natural automorphism of $p_{Z_{\mathcal{E}}}^*$. Reading its component at A over $Z_{\mathcal{E}}$ gives the canonical action $Z_{\mathcal{E}} \times A \to A$, with the order of factors chosen for left actions and multiplication given by composition. Every morphism of $\mathcal{E}$ intertwines these actions by naturality. The action on the group object itself is conjugation.

On a fixed base $\mathcal{E}$, a *crossed topos* of Funk, Hofstra, and Steinberg is described by an internal crossed module $\partial : K \to Z_{\mathcal{E}}$ using the canonical $Z_{\mathcal{E}}$-action on K. Equivariance of ∂ follows from naturality, and the Peiffer identity says that $\partial(k)$ acts on K by conjugation by k. These are the internal conditions for algebras of their isotropy monad on grouped topoi [2, §2 and Cor. 5.10]. The action induced on each A restricts the canonical one along ∂. Objects on which this induced action is trivial form a topos whose inclusion in $\mathcal{E}$ is the inverse image of a connected atomic isotropy quotient [2, Prop. 4.17].

Example 6.1 (Isotropy of the topos of G-sets)**.** For a discrete group G and $\mathcal{E} = \mathbf{B}G$, the group $Z_{\mathcal{E}}$ is G with its conjugation action. Indeed, writing a component of an automorphism of p_U^* as $(x, u) \mapsto (\alpha_X(x, u), u)$, put $z_u = \alpha_G(1, u)$ on the left regular G-set. Naturality along $G \to X,\ g \mapsto gx$ forces $\alpha_X(x, u) = z_u x$. Equivariance of the components gives $z_{hu} = h z_u h^{-1}$. Conversely this condition makes the displayed formula an equivariant natural automorphism. Global sections of the internal isotropy group form the centre of G, which can be trivial even for a nontrivial G. The quotient by the full isotropy action is **Set**, since its objects are the sets with trivial G-action.

Henry's localic isotropy $I_{\mathcal{E}} \to \mathcal{E}$ extends the construction to all geometric probes $f : \mathcal{F} \to \mathcal{E}$. Its group of sections after base change along f is $\mathrm{Aut}(f^*)$. As a topos over $\mathcal{E}$, $I_{\mathcal{E}}$ is the pseudopullback of the diagonal of $\mathcal{E}$ along itself. A test morphism into this pullback gives two morphisms to $\mathcal{E}$ and two invertible comparisons between them. Composing one comparison with the inverse of the other gives a morphism and its automorphism. Composition gives the group law over $\mathcal{E}$, and $Z_{\mathcal{E}}$ is its internal group of points [3, Props. 2.2–2.3 and 6.1]. This group acts on every topos $Y \to \mathcal{E}$ in the pseudo-slice. On test morphisms it changes the specified comparison between a map through Y and the map to $\mathcal{E}$ by composing with the isotropy automorphism [3, Prop. 2.5].

For a locally positive (overt) internal localic group K and an isotropy action $K \to I_{\mathcal{E}}$, the quotient $\mathcal{E} \to \mathcal{E}_K$ has inverse image the inclusion of objects with trivial induced action and is connected atomic. Conversely every connected atomic morphism $f : \mathcal{E} \to \mathcal{F}$ is such a quotient by the relative isotropy $I_{\mathcal{E}/\mathcal{F}}$, which consists of automorphisms becoming trivial over $\mathcal{F}$ [3, Props. 5.4 and 5.9]. Local positivity is a hypothesis of this atomicity assertion. Arbitrary localic isotropy quotients give hyperconnected morphisms [3, §4].

7 Comparing ordinary and higher quotients

The central action of Section 4 admits a direct quotient calculation at both levels. Let A be abelian, H a discrete group, and $\chi : A \to Z(H)$ a homomorphism into its center. Write $\mathbf{B}H$ for the topos of left H-sets, and BH for the pointed connected classifying space. At the topos level the action of $\mathbf{B}A$ on $\mathbf{B}H$ is induced by the group homomorphism $(a, h) \mapsto \chi(a)h$. At the space level the same homomorphism gives an action of the group object BA on BH. One can take the usual nerves of the one-object groupoids, in which products and the action are given levelwise by the group laws.

Proposition 7.1 (A central-action quotient)**.** *With invertible comparison cells, the geometric quotient of* $\mathbf{B}H$ *by* $\mathbf{B}A$ *is* $\mathbf{B}(H/\chi(A))$*. The quotient space* $BH//BA$ *is connected and has homotopy groups*

$$\pi_1(BH//BA) \simeq H/\chi(A), \qquad \pi_2(BH//BA) \simeq \ker \chi.$$

Its homotopy groups in degrees greater than two vanish.

Proof. Put $K = H/\chi(A)$. A geometric morphism $f : \mathbf{B}H \to \mathcal{Y}$ has inverse image $F : \mathcal{Y} \to H$-Set. A comparison between the action and projection on this functor is an isomorphism between two functors into $(A \times H)$-Set, given by restricting the H-actions along $(a, h) \mapsto \chi(a)h$ and $(a, h) \mapsto h$ respectively. On the second functor A acts trivially. An equivariant isomorphism can exist only if $\chi(A)$ acts trivially on each $F(Y)$. The unit condition forces the comparison restricted to H to be the identity. That restriction is faithful on morphisms, so the only possible comparison is the identity. It satisfies the composition condition whenever $\chi(A)$ acts trivially.

Thus a cocone is precisely a left exact cocontinuous F taking values in the full subcategory of H-sets on which $\chi(A)$ acts trivially. This subcategory is K-Set, and its inclusion creates finite limits and small colimits. A transformation of cocones is exactly a transformation of the resulting functors into it. This gives the geometric universal property of $\mathbf{B}K$.

For the higher calculation, the bar description gives a fiber sequence $BH \to BH//BA \to B^2A$. The boundary $A = \pi_2(B^2A) \to \pi_1(BH) = H$ in its long exact sequence is χ up to the choice of boundary sign. Indeed, the orbit map $BA \to BH$ of the base point is $B\chi$ and is equivariant for translation acting on BA. It induces a map over B^2A from $BA//BA \simeq *$ to $BH//BA$. The connecting homomorphism for the first fiber sequence identifies $\pi_2(B^2A)$ with $\pi_1(BA)$. Naturality gives the stated boundary in the second. Exactness yields the displayed groups, since BH is a 1-type and B^2A is a 2-type. □

If χ is trivial, the geometric quotient is $\mathbf{B}H$. The space construction gives $BH \times B^2A$. In particular, for H trivial it still retains A in degree two. This example compares a specific codescent construction in Grothendieck topoi with a homotopy quotient in spaces. Each universal property uses its own category of test objects.

8 Which description to use

A monoidal topos acting by geometric morphisms can be evaluated on all test topoi. Discrete parameter topoi specialize to families of geometric automorphisms, and the topos of equivariant objects realizes their geometric quotient. Internal actions have a fixed base topos and are represented by relative classifying topoi or their slices. They retain torsors and local transport. Isotropy supplies a canonical symmetry attached to a topos itself. Passing to higher topoi replaces ordinary descent isomorphisms by coherent groupoid objects and permits higher homotopy in the quotient. The descriptions agree where their hypotheses identify the same descent data, as with ordinary discrete groups, and the central-action calculation specifies one place where their invariants differ.

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