Finite Information in Bounded Regions: Quantum Records, Spatial Atoms, and Higher-Categorical Semantics

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Abstract

We analyze the inference from finite, nonzero information in a bounded region to a nonzero quantum of space. We distinguish entropy, distinguishable records, Hilbert-space dimension, and geometric atomicity. Countermodels show that finite regional capacity alone does not imply a minimum length or volume. We prove a bound on independently readable records, a conditional minimum-volume result, and an atomicity criterion under a uniform density bound and local record richness. A higher-categorical formulation uses quantum channels, structured code spaces, and descent for Hermitian bundles. Gravitational entropy bounds motivate conditional scales but establish neither a spatial lattice nor an observed universal spatial atom.

Keywords: quantum information; entropy bounds; minimum length; spatial atomicity; algebraic quantum theory; higher topos semantics.

1. The proposed inference and its missing premise

The motivating proposal is that a finite region has finite information capacity, that this capacity is nonzero, and therefore that space has a smallest nonzero unit. We make the logical issue explicit. Write R for a region, $B(R)$ for an operational information bound in bits, and $\mu(R)$ for spatial volume. Then

$$0 < B(R) < \infty \quad \not\Longrightarrow \quad \exists v_0 > 0 \, \forall R \, (\mu(R) > 0 \Rightarrow \mu(R) \geq v_0).$$

The left side concerns distinguishability; the right concerns geometry. A relation between them is needed. Bounded extent also does not, in ordinary quantum theory, impose a bound on energy or excitation number. We therefore treat finite capacity as an explicit hypothesis, with resources and allowable preparations specified.

Our contribution is a conditional implication structure with proofs and countermodels. Standard entropy and discrimination facts are recalled as such. No new theory of quantum gravity or empirical evidence for discrete space is claimed.

2. Information means distinguishable records

Let $\mathcal{D}(H)$ be the density operators on a Hilbert space H. Define the extended spectral entropy by

$$s_2(\rho) = -\operatorname{tr}(\rho \log_2 \rho), \qquad 0 \log_2 0 = 0.$$

Physical entropy is $S(\rho) = k_B \ln(2)\, s_2(\rho)$. An admissible preparation set $\Omega_R \subseteq \mathcal{D}(H_R)$ incorporates the energy and other resources allowed in R.

Definition 2.1 (exact record code). An m-message code consists of admissible states $\rho_1, \ldots, \rho_m$ and an allowed POVM $(M_1, \ldots, M_m)$ such that

$$M_j \geq 0, \qquad \sum_j M_j = I, \qquad \operatorname{tr}(M_j \rho_i) = \delta_{ij}.$$

Its information is $\log_2 m$ bits. We write $N_0(R)$ for the supremum of possible finite message counts and $C_0(R) = \log_2 N_0(R)$. A code concerns one received system and its declared resources. External records, preshared entanglement, and repeated transmissions must be included if available; unlimited repetitions do not have the capacity of a single use.

Proposition 2.2 (standard discrimination bound). If all preparations and POVMs on $H \cong \mathbb{C}^d$ are allowed, then $N_0 = d$ and $C_0 = \log_2 d$. More generally, a uniform m-message code with average error at most $\varepsilon < 1$ satisfies

$$m \leq \frac{d}{1 - \varepsilon}.$$

Proof. For an exact code, M_i is the identity on the support of ρ_i and vanishes on every other support. These nonzero supports are mutually orthogonal, so $m \leq d$. An orthonormal basis attains $m = d$. For the error bound, $0 \leq \rho_i \leq I$ gives

$$1 - \varepsilon \leq \frac{1}{m} \sum_i \operatorname{tr}(M_i \rho_i) \leq \frac{1}{m} \sum_i \operatorname{tr}(M_i) = \frac{d}{m}.$$

Rearrange. □

These are standard quantum information facts [1]. A qubit has one bit of exact classical record capacity, although its pure states form the continuum $\mathbb{CP}^1 \cong S^2$. State cardinality and accessible information differ. Entropy has no positive quantum: for $\rho_p = \operatorname{diag}(p, 1-p)$, $s_2(\rho_p) > 0$ tends continuously to zero as $p \downarrow 0$.

3. Which finiteness assumption is being made?

Proposition 3.1 (unrestricted entropy versus dimension). For a nonzero Hilbert space,

$$\sup_{\rho\in\mathcal{D}(H)} s_2(\rho) < \infty \quad \Longleftrightarrow \quad \dim H < \infty.$$

When $\dim H = d$, the supremum is $\log_2 d$.

Proof. For finite d, relative entropy to I/d gives $s_2(\rho) \le \log_2 d$, with equality at I/d. In infinite dimension, choose any n orthonormal vectors and the uniform mixture of their rank-one projectors. Its entropy is $\log_2 n$, for every finite n. □

The unrestricted quantifier is essential. Finite entropy of one state does not imply finite dimension; every pure state has zero entropy. Even a uniform bound over an energy-constrained set need not imply finite dimension.

Example 3.2 (oscillator). On $H = \ell^2(\mathbb{N}_0)$ let $H_{\mathrm{osc}} = \hbar\omega N$, with ground energy subtracted, and restrict to $\mathrm{tr}(\rho H_{\mathrm{osc}}) \le E$. For $E > 0$, put $\bar{n} = E/(\hbar\omega)$. Then

$$\sup s_2(\rho) = g(\bar{n}) = (\bar{n}+1)\log_2(\bar{n}+1) - \bar{n}\log_2\bar{n} < \infty.$$

To verify this, compare ρ to the Gibbs state $\gamma_\beta = e^{-\beta H_{\mathrm{osc}}}/Z(\beta)$, where $Z = (1-e^{-\beta\hbar\omega})^{-1}$. Positivity of relative entropy gives

$$s_2(\rho) \le \frac{\beta E + \ln Z(\beta)}{\ln 2}.$$

Choosing $\beta\hbar\omega = \ln(1+1/\bar{n})$ yields $g(\bar{n})$, attained by γ_β. At $E = 0$, only the ground state is allowed and the maximum is zero. This variational argument extends to finite-energy states by finite spectral truncations. The oscillator Hilbert space remains infinite-dimensional.

If Ω_R contains uniform mixtures of every admissible exact code and $s_2(\rho) \le B(R)$ on Ω_R, then $C_0(R) \le B(R)$. Indeed, orthogonal supports imply

$$s_2\left(\frac{1}{m}\sum_i \rho_i\right) = \log_2 m + \frac{1}{m}\sum_i s_2(\rho_i) \ge \log_2 m.$$

Without this mixing assumption, an entropy bound need not bound record count: the set of all pure states has zero individual entropy while containing arbitrarily large orthogonal codes in infinite dimension.

4. Countermodels to geometric quantization

Proposition 4.1 (finite positive capacity on continuous space). There is an isotonic, commuting net on an atomless continuum for which every nonempty bounded region has exact capacity one bit.

Proof. Take $X = \mathbb{R}^3$ with ordinary volume and set $\mathcal{A}(R) = \mathbb{C} \oplus \mathbb{C}$ for every nonempty region, with identity maps under nonempty inclusions. Set $\mathcal{A}(\varnothing) = \mathbb{C}$, included as scalars. All algebras are commutative, so disjoint-region observables commute. The two evaluations are perfectly distinguishable; capacity is exactly one bit. Arbitrarily small positive-volume balls still exist. □

This is one classical bit replicated across regions. Disjoint regions cannot choose bits independently: the multiplication map $\mathbb{C}^2 \otimes \mathbb{C}^2 \to \mathbb{C}^2$ is not injective. This logical countermodel isolates the missing independence assumption.

A quantum variant attaches $M_2(\mathbb{C})$ to regions containing one fixed point and $\mathbb{C}$ to the others. It admits independent tensor composition for disjoint regions, because at most one contains the point. It also leaves the ambient continuum unchanged, but fails the hypothesis that every region contains a nontrivial record. Thus independence alone is also insufficient without local richness. Neither construction establishes compatibility with all relativistic or gravitational axioms.

5. What finite capacity actually forbids

Definition 5.1 (jointly independent records). Regions $R_1, \ldots, R_n \subseteq R$ support independently readable records of sizes m_i if all label tuples in $\prod_i \{1, \ldots, m_i\}$ can be prepared in R and perfectly decoded there by an allowed joint measurement. Commutativity of local observables alone does not imply this property.

Theorem 5.2 (record packing). If $C_0(R) \leq B(R) < \infty$, then any jointly independent records in R satisfy

$$\sum_{i=1}^{n} \log_2 m_i \leq B(R).$$

Consequently, if every record has at least $b_0 > 0$ bits, then $n \leq \lfloor B(R)/b_0 \rfloor$.

Proof. Joint independence supplies one exact code with $m = \prod_i m_i$ labels in R. Apply the definition of C_0 and take logarithms. □

This is a counting consequence of the stated independence property, not a new entropy inequality. In a finite-dimensional tensor realization, an isometry $\bigotimes_i K_i \hookrightarrow H_R$ gives the familiar bound $\prod_i \dim K_i \leq \dim H_R$. No spatial isometry, volume, or minimum distance occurs in that argument.

Corollary 5.3 (refinement obstruction). A bounded region with finite capacity cannot contain arbitrarily many jointly independent binary records. If a refinement scheme supplies a fresh independent bit in each of 2^k cells at depth k, it must stop satisfying that property when $2^k > B(R)$.

The conclusion is failure of unrestricted independent refinement. Possible causes include energy costs, correlations, gauge constraints, nonlocal encoding, or geometric discreteness. The theorem does not select one cause.

6. Additional assumptions that yield spatial bounds

Theorem 6.1 (minimum volume for a record). Let $\mathscr{R}_{\mathrm{rec}}$ be a class of record-bearing regions. Suppose fixed constants $\kappa, b_0 > 0$ satisfy

$$b_0 \le C_0(R) \le B(R) \le \kappa\mu(R) \qquad (R \in \mathscr{R}_{\mathrm{rec}}).$$

Then every such region has $\mu(R) \ge v_0 = b_0/\kappa > 0$.

Proof. Combine the inequalities and divide by κ. □

The content lies in the additional physical premises: a uniform information density bound and a uniform nonzero record requirement. A bound on record-bearing regions does not assert that smaller geometric regions are absent. Nor is $B(R) \le \kappa\mu(R)$ a consequence of an area entropy bound: for small balls, area divided by volume diverges.

To force atomicity of the geometry itself, one needs richness across the full measurable region structure.

Theorem 6.2 (conditional measure atomicity). Let (X, Σ, μ) satisfy $0 < \mu(X) < \infty$. Assume every positive-measure $A \in \Sigma$ satisfies the hypotheses of Theorem 6.1 with the same b_0, κ. Then the measure algebra of X is finite atomic, with at most

$$\left\lfloor \frac{\kappa\mu(X)}{b_0} \right\rfloor$$

positive atoms.

Proof. Every positive set has measure at least $v_0 = b_0/\kappa$. Start with the partition $\{X\}$. If a positive part is not an atom, split it into two positive measurable pieces. Each split increases the number of positive parts. A partition with n positive parts has $nv_0 \le \mu(X)$; hence the process must terminate after finitely many splits. Every remaining part is an atom. Modulo null sets these atoms generate the measure algebra, since a measurable subset intersects each atom in either zero or its full measure. □

This statement is incompatible with ordinary atomless volume measure. It does not imply equally sized atoms, a metric lattice, or a minimum separation of points. The atoms may have different measures. Even atomicity is weaker than a spectrum of integer multiples of one universal volume unit.

7. Gravitational motivation and its domain

Bekenstein's proposed entropy bound [2], in its intended domain and with appropriate definitions of energy, entropy, and enclosing radius r, reads

$$\frac{S}{k_B \ln 2} \le \frac{2\pi E r}{\hbar c \ln 2}.$$

Finite r alone leaves E uncontrolled. For illustration, **assume additionally** that a spherical system obeys both this entropy bound and $E \le c^4 r/(2G)$, and that its usable record capacity is bounded

by the displayed entropy bound. Then, with $\ell_P^2 = \hbar G/c^3$,

$$C_0(R) \leq \frac{\pi r^2}{\ell_P^2 \ln 2}.$$

A region supporting b_0 bits therefore obeys the conditional inequality

$$r \geq \ell_P \sqrt{\frac{b_0 \ln 2}{\pi}}.$$

For one exact bit the factor is approximately 0.4697. This arithmetic is not a measurement or a derivation of a fundamental lattice spacing. Simultaneous applicability, the capacity interpretation, and extrapolation toward the Planck scale are additional assumptions; semiclassical reasoning does not by itself validate them there. The radius condition above uses the Schwarzschild comparison and is not a general collapse criterion for arbitrary geometries.

Bousso's covariant proposal bounds entropy on suitable null light-sheets by an associated area, rather than assigning a finite tensor factor to every spatial subregion [3,4]. Casini's field-theoretic formulation instead uses relative entropy [5]. In a regulated setting with reference state σ and modular operator $K = -\ln \sigma$, it is

$$D(\rho\|\sigma) = \Delta\langle K\rangle - \Delta s \geq 0,$$

where $s = -\operatorname{tr}(\rho \ln \rho)$ is dimensionless entropy in nats. The continuum formulation uses algebraic relative entropy. This is not an absolute bound on the dimension of a regional Hilbert space.

In standard AQFT models, local von Neumann algebras are typically type III under suitable hypotheses [6]; they are not finite matrix algebras. Thus the finite-dimensional models in this paper are hypotheses or effective descriptions, not deductions from continuum QFT. DFR spacetime uncertainty relations [7] provide another approach to localization limits without identifying space with a uniform lattice. A nonzero operational resolution bound and ontological spatial atoms are distinct claims.

8. Categorical and higher-topos semantics

For a chosen region poset, an observable net is a functor

$$\mathcal{A} : \mathrm{Reg}(X) \longrightarrow \mathbf{C^*Alg}_{1,\mathrm{inj}}.$$

States restrict contravariantly by precomposition. In finite-dimensional models, discarding and coarse-graining are quantum channels, and joint independent preparation is represented by tensor products together with specified encoding and decoding maps. Their existence is extra structure; neither functoriality nor disjointness supplies it automatically. Exact distinguishability also does not survive arbitrary restriction: two orthogonal bipartite states can have identical marginals. Consequently, regional code sets cannot simply be declared a presheaf using partial trace as restriction.

For an actual higher semantics, retain the structured code data. Let $\mathrm{Code}_m(H)$ be the topological space of state/POVM tuples satisfying Definition 2.1. It carries the unitary action by simultaneous conjugation. The classifying space of the topologically enriched groupoid of finite Hermitian spaces and unitaries is

$$\mathfrak{Q} = \coprod_{d\geq 1} BU(d).$$

The homotopy quotients define a map

$$\mathfrak{C}_m = \coprod_{d\geq 1} (\mathrm{Code}_m(\mathbb{C}^d)//U(d)) \longrightarrow \mathfrak{Q}.$$

Here // retains gauge symmetries and their homotopies. Over the dimension-d component, a code exists exactly when $m \leq d$, by Proposition 2.2. Capacity is therefore invariant under unitary equivalence. Notice that $\mathcal{D}(\mathbb{C}^d)$ itself is contractible for every d: forgetting the observable and discrimination structure and retaining only its homotopy type would erase the capacity distinction.

To handle families, take a paracompact topological parameter space T and the infinity-topos $\mathrm{Sh}_\infty(T)$ [8]. Hermitian bundles and unitary isomorphisms form a stack on its open-cover site. Code data in a bundle $H \to T$ are continuous density endomorphisms and POVM endomorphisms satisfying the same equations fiberwise. Compatible such data glue, because bundle endomorphisms glue and the equations are local.

Proposition 8.1 (local codes versus global codes). For a rank-d Hermitian bundle, local m-message codes exist everywhere exactly when $m \leq d$. A global d-message code exists exactly when the bundle admits an orthogonal decomposition into d complex line subbundles.

Proof. Local trivializations supply orthonormal basis codes when $m \leq d$, and the fiberwise dimension bound excludes $m > d$. For a global d-code, the d nonzero mutually orthogonal supports in every fiber must all have rank one. Each density endomorphism is therefore a continuous rank-one projection. Its image is a line subbundle, and these subbundles form an orthogonal decomposition. Conversely, orthogonal line projections sum to the identity and supply both code states and its decoding POVM. □

Thus local existence does not remove the need for compatible descent data. Pointwise finite rank also does not supply a global numerical bound: the discrete base $T = \mathbb{N}$ can have fiber dimension $n + 1$ at n. On compact T, a locally constant finite rank function has finite image by a finite-subcover argument. Neither statement concerns atomicity of physical space.

The commutative-context approach of Heunen, Landsman, and Spitters [9] supplies related internal quantum logic. We use higher-topos language for coherent families and descent, not as an independent reason for a finite information bound. The numerical bound, independence property, and geometry-to-capacity comparison must still be supplied.

9. Scope, evidence, and research program

Finite capacity bounds simultaneous independent records. A uniform density bound and record requirement bound record-bearing volume. Applying both to every positive measurable subset

forces measure atomicity in finite volume. A universal spatial quantum requires further geometric principles.

The open physical task is to justify $B(R; E, \varepsilon)$, including gauge constraints, gravitational dressing, and independent localization, and explain why it selects geometric discreteness over other failures of independent refinement.

The accompanying script checks finite discrimination and packing examples, the oscillator expression, and the spherical factor. These consistency checks are neither experimental evidence nor machine-checked proofs of the general results. No empirical data are reported.

Authorship and automation. The responsible submitting researcher is identified publicly as Anonymous author. The research question was supplied by that researcher; the manuscript and verification code were drafted with OpenAI ChatGPT/Codex assistance. Responsibility for checking the claims remains with the submitting researcher. This is a prepared manuscript; no editorial or peer-review status is claimed.

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