Counterexample-Guided Abstraction Refinement in Higher Toposes

Homotopy fibers, effective finite algorithms, and certified semantics

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Abstract. A may abstraction can assemble a bad trace from separately realizable steps even when their witnesses cannot compose. We model the residual obligation as the homotopy fiber of a map of bad-path objects. Its propositional support is stable under reindexing, while the fiber records transport, compatibility, and higher identifications. We prove a composition law for successive observations, a path-blocking theorem for exactification of visited cells, and a layered predicate-refinement theorem using certified forward and backward predicates. The specialization to sets recovers the existential quotient and path-feasibility test of classical safety CEGAR. An executable Lean 4 implementation returns either a safety proof or a concrete finite bad walk and decides the lift of a given abstract walk. A separate native HoTT Lean development checks fiber, transport, and logical reflection constructions with proof-relevant identity types. Computed case studies exhibit the scope of each construction.

Keywords. CEGAR, homotopy type theory, higher topos, model checking, propositional truncation, abstract interpretation, formal verification.

1 Introduction

A verifier often has access to a finite observational model of a much larger transition system. A safety property can be checked on that model by looking for a path from an initial state to a bad state. When the observational model contains the image of every concrete transition, the absence of a bad abstract path proves concrete safety. Presence leads to another question. Does the chosen abstract path come from one concrete execution, or does it concatenate edges witnessed by incompatible concrete states?

Counterexample-guided abstraction refinement (CEGAR) makes that question an algorithmic step [6, 7]. The answer matters even for a three-node abstraction. Suppose s enters one representative of a middle cell and an edge to t leaves another. Each abstract edge has a concrete witness. The two edge witnesses compose exactly when the concrete states match or a suitable compatibility path connects them. This paper uses that compatibility problem as the organizing principle.

An abstraction with higher state and observation types raises a further distinction. A candidate may have a type of realizations with multiple connected components, nontrivial automorphisms, or obstructions to selecting compatible realizations across a cover. The existence of a realization, expressed by propositional truncation, forgets this information. It still gives the proper existential proposition for a may semantics. We retain the type before truncating it and use its pullback and composition laws to organize refinements.

1.1 Principal results

For a finite word of labels, a graph with state, edge, initial, and bad-state objects has a bad-path object obtained by a finite homotopy limit. An observation induces a map of bad-path objects. If b is an abstract bad path in a context U, the concrete lifts form its homotopy fiber over U. The first result proves that both the fiber and its support commute with substitution in U. The second identifies the fiber of a composite observation with the dependent sum of the finer fibers over the intermediate one. This explains the difference between refuting a coarse trace and refuting one of its refined descendants.

The finite specialization establishes two effective refinements. One isolates every state in every visited abstract cell. It removes every descendant of the particular spurious path and strictly increases the number of nonempty cells. The other attaches a certified reachable predicate to each layer of a fixed labelled path. Its endpoint condition reduces to a separation between exact prefix and suffix sets. This second construction explains the role of interpolants and exposes the extra assumptions needed to reuse a separator at a single location.

The Lean 4 project implements a proof-carrying saturation algorithm, the exact may quotient, a tabulating lift checker, and the first refinement loop. The same project implements a finite first-order language with state variables, equality, transitions, and quantifiers. Its evaluation is proved equivalent to its relational semantics and respects renaming through binders. The native HoTT project checks homotopy fibers, reindexing, transport, the composition equivalence, and the distinction between two lifts into the circle. It also implements a Boolean evaluator with a Type-valued interpretation and reflection in the HoTT kernel. These are two formalizations with different kernels and theorem statements.

1.2 Relation to prior work

Clarke, Grumberg, Jha, Lu, and Veith develop the original refinement architecture for symbolic model checking [6, 7]. The overview [36] gives the same broad sequence

of abstraction, counterexample analysis, and refinement. The existential image of a transition relation also belongs to the abstract-interpretation account of sound approximations to least fixed points [8, 11]. Predicate abstraction [12], lazy abstraction [14], and interpolation-based refinement [28, 29] provide three relevant ways to organize precision. The safety loop studied here is the finite-path fragment of that literature. It permits early proof of safety at a coarse abstraction and uses the exact concrete model as a termination bound. Other properties can need a different abstraction contract.

For the higher interpretation, the relevant standard machinery is the internal language of an ∞-topos, stable image factorization, and the path calculus of homotopy type theory [26, 16, 33, 34]. Complete Segal spaces give a separate account of composition for directed systems [31]. The groupoid and simplicial interpretations explain how identity types can carry structure [15, 19, 4]. This paper applies those tools to a specifically verification-theoretic obligation. Its higher constructions use existing limits, images, and identity types. Its finite theorems identify which aspects of a refinement policy those constructions justify.

1.3 Reading guide and evidence

Sections 2–7 define the semantic objects and compare their specializations. Sections 8–15 develop algorithms, proofs, and the study protocol. The three subsequent parts contain concrete finite runs, layered predicate certificates, and higher examples. The companion archive includes the two formalizations, the data and generators for the tables, and the manuscript source.

The manuscript and its companion were developed with AI assistance. Kernel-checked claims are identified by declaration and toolchain. Categorical arguments elsewhere are mathematical proofs in the text. The computed families are finite constructions illustrating those proofs. Their counts and timings do not estimate performance on industry verification workloads.

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II. Semantics

The residual obligation in sets, in an ∞-topos, and in the internal language of dependent type theory.

2 Classical CEGAR as a safety procedure

Fix a set X, a set of labels L, initial states $I \subseteq X$, bad states $B \subseteq X$, and relations $T_\ell \subseteq X \times X$ for $\ell \in L$. Dead ends are allowed. An execution is a finite list of states with a labelled edge between successive states. The empty word has one state and can already witness a violation when $I \cap B$ is inhabited.

For $w = \ell_1 \cdots \ell_m$, write

$$P_w(G) = \big\{(x_0, \dots, x_m) \mid x_0 \in I, \quad x_m \in B, \\ (x_{i-1}, x_i) \in T_{\ell_i} \ (1 \le i \le m)\big\}.$$

Define $P(G) = \coprod_{w \in L^*} P_w(G)$. Safety is $P(G) = \varnothing$. This is a semantic property of finite bad prefixes [1, 2]. When labels are only execution annotations and every word is allowed, replace all T_ℓ by their union. A concrete counterexample in the union graph can be labelled by choosing a label for each edge from its finite witnesses. A label-sensitive specification instead requires a word constraint or a monitor.

2.1 May abstraction

Let $q : X \to A$ be an observation. It can be a quotient by a partition, a vector of predicates, or a map into a carrier that includes unrealized keys. Define $I_q = q(I)$, $B_q = q(B)$, and $T_\ell^q = (q \times q)(T_\ell)$ as direct images. The abstract graph $q_!G$ has state set A and these predicates. Its initial, bad, and edge facts are all existential.

Proposition 2.1 (Simulation and minimality)**.** *Projection of node lists gives $P_w(G) \to P_w(q_!G)$. Among graphs on A whose initial, bad, and edge relations contain the images of G, the may abstraction is least under inclusion.*

Proof. Apply q to the endpoints and each edge of a concrete path. The images satisfy the required relations by their definition. Every graph with the indicated preservation property contains each point of the three direct images. This proves leastness componentwise. □

A concrete bad path therefore implies an abstract one. The reverse implication would say that any collection of existential edge witnesses with matching abstract endpoints can be chosen with matching concrete endpoints. That assertion fails for elementary reasons.

Example 2.2 (Independent witnesses)**.** Take states s, u, v, t, initial s, bad t, and edges $s \to u$ and $v \to t$. Put u and v into one cell M and leave s, t in singleton cells S, T. The abstract path S, M, T has two different middle representatives. Its concrete prefix set at M is $\{u\}$; its concrete suffix set is $\{v\}$. Their intersection is empty. Splitting M removes the abstract path. If the edge $u \to t$ is added, the same abstract path has a concrete lift, and the algorithm returns s, u, t. The abstract edge set in this comparison is identical.

2.2 Adjunction and failure of compositional exactness

In the powerset order, direct image and inverse image satisfy

$$q_!(U) \subseteq V \quad \Longleftrightarrow \quad U \subseteq q^*(V).$$

The induced closure $U \mapsto q^* q_!(U)$ replaces a subset by the union of all its intersecting cells. This equality of the adjunction is independent of surjectivity. If q is surjective, $q_! q^*$ is the identity. If not, it intersects V with the image of q. The statement applies to edges after replacing q with $q \times q$. Thus the finite semantics uses the best one-step abstraction in the given domain of subsets of A [8, 11, 30].

Let $R, T \subseteq X \times X$ and take $T \circ R$ in execution order. A direct check gives

$$(q \times q)_!(T \circ R) \ \subseteq \ (q \times q)_!(T) \circ (q \times q)_!(R).$$

To prove it, project a common middle witness y to $q(y)$. The larger relation allows two different middle witnesses in the same cell. Example 2.2 gives strict inclusion. Its concrete relations $R = \{(s, u)\}$ and $T = \{(v, t)\}$ compose to the empty relation. Their images compose to $\{(S, T)\}$. A sufficient condition for equality is that, for each concrete x, z, any common observed midpoint of an R-successor of x and a T-predecessor of z has a concrete common midpoint for x, z. A stronger, easy-to-check condition is that R be saturated in its second coordinate: whenever xRy and $q(y) = q(y')$ one has xRy'. The dual condition on the first coordinate of T works by the same reasoning.

2.3 Feasibility as a join

Write the candidate $b = (a_0, \dots, a_m)$. Its feasible prefix sets are

$$F_0 = I \cap q^{-1}(a_0), \\ F_i = q^{-1}(a_i) \cap \mathrm{post}_{T_{\ell_i}}(F_{i-1}). \tag{1}$$

Its feasible suffix sets are

$$D_m = B \cap q^{-1}(a_m), \\ D_i = q^{-1}(a_i) \cap \mathrm{pre}_{T_{\ell_{i+1}}}(D_{i+1}). \tag{2}$$

The notation $\mathrm{post}_R(S)$ means $\{y \mid \exists x \in S,\ xRy\}$ and $\mathrm{pre}_R(S)$ means $\{x \mid \exists y \in S,\ xRy\}$. The latter is an existential predecessor, distinct from a universal weakest precondition.

Lemma 2.3 (Cut criterion)**.** *For every $0 \le i \le m$, $F_i \cap D_i$ is the set of possible states at index i of a lift of b. In particular b lifts exactly when $F_m \cap B$ is inhabited.*

Proof. Induction on i in (1) gives a prefix witness for each member of F_i and shows that every concrete prefix ends there. The reverse induction on (2) gives a suffix witness for each member of D_i. A prefix and suffix ending and starting at the same x_i concatenate, with x_i counted once. A concrete lift projects to both witnesses. At m, the condition $x_m \in q^{-1}(a_m)$ already holds for $x_m \in F_m$. □

For a fixed word, this is the relational join of the rows for all edges, constrained by initial and bad predicates and by observations. Each two-column edge projection can be nonempty while the join is empty. Keeping a predecessor for each reached endpoint constructs an actual witness rather than just counting feasible endpoints.

2.4 Finite reachability and sound termination

On a finite graph define $U_0 = \varnothing$ and $U_{j+1} = U_j \cup I \cup \mathrm{post}_T(U_j)$, with T the union of labels. The iteration increases until it stabilizes. Every insertion has a finite path proof, and induction on a path shows that its last state belongs to any inductive invariant containing I. Consequently a stable set is the exact reachable set. Disjointness from B is a safety certificate. The finite kernel implementation uses a one-at-a-time worklist with the same invariant.

The standard CEGAR loop first checks abstract reachability. An abstract safety proof transfers by Proposition 2.1. If an abstract counterexample is found, its witness set is queried. A nonempty witness set gives a concrete counterexample. An empty one justifies a refinement. Termination requires a separately specified progress measure. Even finite systems can be fed an ineffective policy that returns the same abstraction. The bulk exactification policy in Section 8 supplies a strict bound on the number of refinement steps.

3 Bad-path objects in an ∞-topos

Let $\mathcal{E}$ be a Grothendieck ∞-topos. All pullbacks in this section are homotopy pullbacks, and equivalences refer to equivalences in $\mathcal{E}$. The constructions use finite limits, relative (-1)-truncation, and, for the union of all words, small coproducts. They therefore apply in other settings that supply these operations with the stated stability. Size conditions for universes concern the choice of internal language and do not enter the proofs of finite-limit identities.

Definition 3.1 (Boundary graph)**.** A graph with boundary in $\mathcal{E}$ consists of an object X, objects I, B equipped with maps $I \to X$, $B \to X$, and an object $T_\ell \to X \times X$ for each $\ell \in L$. The two projections of the last map are s_ℓ, t_ℓ. A morphism of graphs consists of a map on states and maps on the edge and boundary objects with coherent commuting squares.

Boundary maps can carry witness data. For example, two initial witnesses with the same state can encode distinct initialization histories. The ordinary relational interpretation takes all three types of structure maps to be monomorphisms.

For $w = \ell_1 \cdots \ell_m$, take the limit of the zigzag

$$I \longrightarrow X \longleftarrow T_{\ell_1} \longrightarrow X \longleftarrow \cdots \longleftarrow T_{\ell_m} \longrightarrow X \longleftarrow B.$$

For $m = 0$, set $P_\epsilon = I \times_X B$. All stated operations are on the indexed family P_w; a countable coproduct over w is optional for fixed-word semantics. When working with all words we use an external small label alphabet, with the empty word included.

3.1 The image construction

An arrow $f : Y \to Z$ factors as an effective epimorphism followed by a monomorphism. We write $\mathrm{im}(f) \hookrightarrow Z$ for the monomorphism. As a family over Z it is the relative propositional truncation of f. For a state observation $q : X \to A$ put

$$I_q = \mathrm{im}(I \to X \to A), \qquad B_q = \mathrm{im}(B \to X \to A),$$
$$T_\ell^q = \mathrm{im}(T_\ell \to X^2 \to A^2).$$

There is a graph morphism $G \to q_!G$ whose components are the image factor maps. Up to equivalence in each slice, $q_!G$ is the least relational graph on A through which all its state, edge, and boundary maps factor. Indeed, factoring through a monomorphism is the universal property of the image [26, 33].

This produces for each word a comparison

$$p_w : P_w(G) \longrightarrow P_w(q_!G).$$

It has two different meanings of image. The image of each edge or boundary map is already used in $q_!G$. The image of p_w is the subobject of coherent bad paths within the object of abstract bad paths. The latter can be strictly smaller. This is the categorical failure of commuting edgewise image with the path limit. The four-state example gives a strict inequality in a discrete specialization.

Definition 3.2 (Residual and support)**.** For $b : U \to P_w(q_!G)$, the residual $F_b \to U$ is the projection from

$$F_b = U \times_{P_w(q_!G)} P_w(G).$$

Its support $S_b \hookrightarrow U$ is the relative image of this projection. A chosen realization is a section of F_b over U. A refutation in U is a map $F_b \to 0_U$ in the slice, where 0_U is its initial object.

At a point in spaces, the fiber contains a concrete bad path together with an identification between its observation and the candidate. A refutation can be phrased equivalently as a map $S_b \to 0_U$. A proof that the support is the whole context means $F_b \to U$ is an effective epimorphism. Such a proof makes no choice of section.

Theorem 3.3 (Substitution)**.** *For any $v : V \to U$ there are canonical equivalences over V*

$$v^* F_b \simeq F_{bv}, \qquad v^* S_b \simeq S_{bv}.$$

They are compatible with identities and composition in the base. In addition, a refutation restricts along every v.

Proof. The first equivalence is the pullback pasting law, applied to $V \to U \to P_w(q_! G)$ and p_w. The factorization $F_b \to S_b \hookrightarrow U$ pulls back to an effective epimorphism followed by a monomorphism. In an ∞-topos both classes are pullback-stable. Uniqueness of the factorization identifies the middle object with S_{bv}. The equivalences are induced by universal properties and carry the corresponding coherence for multiple pullbacks. Since initial objects in the slices are stable under pullback, a map to 0_U gives one to 0_V. □

The distinction between relative image stability and absolute truncation is material. For a nontrivial discrete group H, in spaces the pullback $* \times_{BH} *$ is equivalent to the discrete space H. Taking connected components gives H. Taking connected components first turns the two maps into maps $* \to *$ and gives $*$. For (-1)-truncation a simpler counterexample is the pullback of the two distinct inclusions $* \to \{0, 1\} \leftarrow *$. Its truncation is empty, while the pullback of the truncated diagram is terminal. Thus neither example threatens the slice-wise statement of Theorem 3.3.

3.2 Universal safe transfer

A convenient internal safety type for a fixed word is $\mathsf{Safe}_w(G) = (P_w(G) \to 0)$. If all words are in scope, the corresponding global type is $P(G) \to 0$. Equivalently it is a dependent product of the individual refutations. The choice of the empty type has the same effect as first truncating $P(G)$, by elimination into propositions.

Proposition 3.4 (Safety under forward comparison)**.** *Every map $p : P \to Q$ gives a map $(Q \to 0) \to (P \to 0)$. If p is an effective epimorphism, this map is an equivalence of propositions.*

Proof. Composition gives the first map. If p is an effective epimorphism, Q is the support of p relative to Q. A map $P \to 0$ can be viewed as a map from $P \to Q$ to 0_Q. The latter is a proposition in the slice. The image adjunction extends that map across the support, giving $Q \to 0$. Both types are propositions, so the two implications exhibit equivalence. □

3.3 Ordinary sets as a specialization

Embed sets as 0-truncated spaces. Their finite limits in spaces are set-theoretic limits. A homotopy pullback of maps between discrete spaces consists of strict matching pairs with unique identifications. Images in the slices of sets are ordinary images. Consequently Definition 3.2 for a set of concrete traces gives the usual set of satisfying assignments to the nodes of the chosen abstract path.

For graph structure maps that are arbitrary finite sets over X^2, the path object also records the choice of each concrete edge witness. The relational finite checker collapses parallel edges with the same endpoints and label. This collapse is sound and complete for existence of a finite bad path: any selected relational edge has a witness, and there are finitely many edges in the path. It changes the witness set whenever parallel evidence is present. This is a useful first distinction between the implemented existence test and the richer semantics.

4 Composition and towers of observations

Suppose $X \xrightarrow{q} A \xrightarrow{r} D$ are observations. We first compare their edgewise effects and then their effects on the space of concrete traces over a specified candidate.

Lemma 4.1 (Iterated images)**.** *For composable maps $Y \xrightarrow{f} Z \xrightarrow{g} W$, $\mathrm{im}(gf) \simeq \mathrm{im}(g|_{\mathrm{im}\, f})$ over W. In particular $r_!(q_! G) \simeq (rq)_! G$ as relational graphs with state object D.*

Proof. Write $Y \twoheadrightarrow \mathrm{im}\, f \hookrightarrow Z$. For any monomorphism $V \hookrightarrow W$, the composite $Y \to W$ factors through V exactly when $\mathrm{im}\, f \to W$ does. In the reverse implication composition suffices. For the forward implication, pull V back along $\mathrm{im}\, f \to W$. This subobject of $\mathrm{im}\, f$ contains the image of Y, which is all of $\mathrm{im}\, f$. The equality of the least subobjects follows. Apply it to the boundary maps and to the edge maps over $D \times D$. □

Consequently the associated path comparisons compose coherently. Equivalences such as $r_!(q_! G) \simeq (rq)_! G$ are used when identifying the codomain. An arbitrary chain of overapproximations can also be used when its comparison maps are specified. Leastness is needed only to identify the resulting graph with the direct may abstraction.

4.1 The dependent-sum law

Let $P \xrightarrow{p} Q \xrightarrow{r} W$ be maps of bad-path objects for a fixed word, or arbitrary maps in $\mathcal{E}$. For $b : U \to W$ let $H = U \times_W Q$ and $K = H \times_Q P$. Then $H \to U$ is the family of intermediate realizations. The fiber of $K \to H$ over an intermediate realization c is the family of finer realizations over c.

Theorem 4.2 (Fiber of a composite)**.** *The total object $K \to U$ is equivalent to the residual $F_b^{rp} \to U$. Internally, at a generalized point b,*

$$F_b^{rp} \simeq \sum_{c:F_b^r} F^p_{\pi_Q(c)}.$$

Its support is the relative image of the dependent sum.

Proof. The square defining K and the square defining H paste to the pullback square defining F_b^{rp}. Equivalently, if an element of the dependent sum is represented by $(y, \beta : r(y) = b, x, \alpha : p(x) = y)$, the forward map is $(x, \mathrm{ap}_r(\alpha) \cdot \beta)$. The inverse sends (x, γ) to $(p(x), \gamma, x, \mathsf{refl}_{p(x)})$. Path induction on α gives the nontrivial inverse law. The first law follows from the unit of path concatenation. Truncation in the slice gives the statement for support. □

Corollary 4.3 (Refuting a coarse path)**.** *For fixed b, a refutation of F_b^{rp} is equivalent to a dependent family*

$$\prod_{c:F_b^r} \left(F^p_{\pi_Q(c)} \to 0\right).$$

A refined path with a chosen finer lift supplies a chosen lift of the coarse path.

Proof. Use Theorem 4.2, and use the eliminator for a dependent sum to curry and uncurry the map to 0. A chosen pair is sent along the forward map of that theorem. □

The family in Corollary 4.3 includes intermediate paths that need not be globally given in a sheaf semantics. Proofs stated only for a list of global representatives require a justification that the list covers the intermediate family. At a groupoid point the same obligation includes compatibility with automorphisms.

4.2 Staging truncation

For each c, write $S(c) = \left\| F^p_{\pi_Q(c)} \right\|_{-1}$. The coarse support satisfies

$$\|F_b^{rp}\|_{-1} \simeq \left\| \sum_{c:F_b^r} S(c) \right\|_{-1}.$$

There is a map from left to right induced by the units in the inner fibers. In the other direction, the target is a proposition. It suffices to eliminate the outer truncation, then the inner truncation at each specified c, and send a lift to the corresponding pair. This construction retains the dependence on c. Replacing F_b^r by its support requires descent of the family S along the truncation unit, and there need not be such a descended family.

For a set example, take $F_b^r = \{c_0, c_1\}$, with finer fibers empty over c_0 and singleton over c_1. The coarse path has a lift, even though one intermediate path is impossible. If F_b^r is replaced by a singleton while asking a question about a designated descendant, that descendant cannot be recovered. At the level of propositions one can still compute the join over both original fibers.

4.3 Arbitrary finite chains

For a finite chain of composable maps, repeated use of Theorem 4.2 expresses the coarse residual as an iterated dependent sum. The sums can be rebracketed: both bracketings identify with the same total homotopy pullback. This gives a coherent statement of composition without requiring a strict choice of representatives of fibers. Concretely, the witness consists of a point at the finest level and a coherent sequence of paths at the intermediate levels.

A refinement tower of partitions of a fixed finite set is one example. If q' factors q through r, a fine bad path b' is a descendant of b when its image under r is b, with the necessary identification in the higher case. A refinement can eliminate all descendants of a chosen abstract path. It can also retain some abstract descendants while identifying new infeasibility obligations. The distinction is a quantifier over F_b^r.

4.4 Inverse systems

Consider a tower of residual objects $F_i \to U$ with specified transition maps. For example, F_i can be the relative Postnikov truncations $\tau^U_{\le i} F$ of a single residual. At each $i \ge -1$ the support of F_i equals that of F, since relative truncations compose. If the chosen tower converges in the slice, compatible sections give a section of F. Here compatibility means a point of the limit of the section spaces. Individual inhabitants of all section spaces need not supply it.

A discrete warning is the tower of sets $\{n, n+1, \dots\} \hookrightarrow \{n-1, n, \dots\}$. Each set is inhabited and the limit is empty. A finite compactness argument needs hypotheses on the inverse system, for example a nonempty finite tree with arbitrarily long branches in the classical setting. The finite CEGAR termination proof uses a descending measure on one concrete carrier and does not require a limit or any inverse-limit compactness principle.

5 Local witnesses and descent

A subobject in a topos can be a genuinely local truth value. For a fixed abstract path b over U, the support S_b determines precisely the subcontexts in which the residual is locally inhabited. A section of the residual over U solves a stronger problem.

Proposition 5.1 (Descent for chosen lifts)**.** *Let $v : V \to U$ be an effective epimorphism in $\mathcal{E}$, and let $V^{[n]}$ be its $(n+1)$-fold fiber product over U. The space of lifts over*

U is the limit of the cosimplicial diagram of spaces of lifts over $V^{[n]}$.

Proof. The augmented Čech nerve of v presents U as the colimit of its simplicial diagram. The presentation can be regarded in $\mathcal{E}/U$. Mapping out of that colimit to F_b gives the limit of the mapping spaces. A map $V^{[n]} \to F_b$ over U is a section of the pullback of F_b to $V^{[n]}$ by the pullback universal property. The result follows, including higher compatibility. □

For the image subobject, this reduces to ordinary descent of truth: a proposition is determined by its restrictions along an effective epimorphism. If $v^* S_b = V$, then $S_b = U$. Indeed, a monomorphism containing the image of an effective epimorphism is an equivalence. For the residual, the corresponding conclusion is that $F_b \to U$ is an effective epimorphism. Constructing a section still requires compatible data on the Čech nerve.

5.1 An equivariant obstruction

Let H be a nontrivial discrete group and use the ∞-topos $\mathrm{Fun}(BH, \mathcal{S})$ of H-spaces. Give $R = H$ the regular left action. The map $R \to 1$ is an effective epimorphism because it is surjective on components after evaluation at the unique object of BH. There is no map $1 \to R$. Since R is discrete, such a map would be a fixed point, and $h \cdot x = x$ for every h would imply H is trivial.

This exact example arises as a CEGAR obligation. Take $X = 1_s \amalg R \amalg 1_t$, observe the three summands, and give the concrete graph edges from s to r and from r to t for each $r \in R$. Each family of edges is itself an H-object isomorphic to R. The abstract graph has the two observed edges. The candidate s, m, t has residual R, because the middle states in its two edge witnesses must be equal. Its support is terminal. Over the atlas R a lift is given by the diagonal. On a pair (x, y) in $R \times R$, the two restrictions of this particular lift select x and y; they disagree whenever $x \neq y$. This makes the descent failure explicit.

If H is finite, this counterexample has finite data, including all edge orbits. Finiteness of a presentation does not change the internal distinction between a surjective map of H-sets and an equivariantly split surjection.

5.2 An open truth value

Let $\mathcal{E}$ be the ∞-topos of sheaves of spaces on $\mathbb{R}$, and $J = (0, \infty)$ the indicated subterminal object. Take the constant four-state graph of Example 2.2. Add the edge $u \to t$ only over J, by adding its subobject $J \to 1_{(u,t)}$ to the concrete edge relation. The may edges $S \to M \to T$ are global because $s \to u$ and $v \to t$ are global. The residual of this fixed candidate is J. A lift must use u as its middle state and use the new edge.

Subobjects of the terminal sheaf correspond to open sets. The Heyting negation of J is $\mathrm{int}(\mathbb{R} \setminus J) = (-\infty, 0)$. The join of J with its negation misses zero. The closed assertion that this trace lifts therefore has neither a global proof nor a global refutation. It is decidable over any open disjoint from zero, and a positive or negative decision can be made on each side. The context containing zero requires a semantics that permits an unresolved obligation.

5.3 What an abstract checker can return

A checker may return a section, a refutation, a proof that $S_b = U$, or an open decomposition on which distinct outcomes hold. These have different meanings. In the finite discrete implementation, positivity gives a chosen state tuple and negative means the witness set is empty. An internal higher checker supplied with an oracle for propositional support has a weaker positive interface. Its result can refute a universal safety proposition, but an external diagnostic requiring a chosen trace needs section data or effective descent.

For any $v : V \to U$, Theorem 3.3 makes every available result stable under reindexing. A local refutation over an effective epi $V \to U$ gives a global one, because the target is a proposition and local proofs are automatically compatible. An implementation that searches only for global sections could instead falsely classify an equivariant torsor as a spurious trace. The support condition prevents that error.

5.4 Internal and external points

One must specify the ambient context when reporting that a fiber is empty. In $\mathcal{S}$, a point can witness support of a space, and one can reason constructively with a specified representative. In a general $\mathcal{E}$, global points need not detect subobjects. For a concrete comparison $p : P \to Q$, testing fibers only over global $1 \to Q$ does not establish that p is an effective epimorphism. An internal assertion quantifies over generalized points, or equivalently uses the subobject $\mathrm{im}(p) \hookrightarrow Q$. This is the form needed for safety transfer in every slice.

6 Syntax, logic, and HoTT interpretation

We use the dependent type theory with a universe of small types, finite products and coproducts, dependent sums, dependent products, identity types, and propositional truncation [27, 16]. Judgments may be interpreted in a suitable split presentation of a Grothendieck ∞-topos with a univalent universe. The finite-limit statements from Section 3 do not depend on the split presentation. The strict interpretation of dependent type theory in a presentation uses the coherence results of [25, 19, 34]. Cubical type theory supplies a computational account of univalence and selected higher inductive types [10].

Computation of terms in a type theory and a decision procedure for inhabitedness of arbitrary semantic fibers are distinct problems.

6.1 Contexts and terms

A context Γ denotes an object. A type $\Gamma \vdash T\ \mathsf{type}$ denotes a family over Γ, and $\Gamma \vdash t : T$ denotes a section. Context extension $\Gamma.T$ is the total space of T. A substitution $\sigma : \Delta \to \Gamma$ reindexes the family and pulls back a section. In a split syntactic presentation the equations for identity and composition of substitutions hold at the chosen judgmental level. In the ∞-categorical semantics their content comes from canonical pullback comparisons.

If T is over Γ and V is over $\Gamma.T$, then $\Sigma_T V$ is obtained by composition of families. The type $\Pi_T V$ uses the right adjoint to reindexing along the projection. The equality type of two sections is the pullback of the fiberwise diagonal along their pair. Its points are paths, and its iterated equality types are higher paths. The identity eliminator J is interpreted by reduction to the reflexivity section of the based path family. This interpretation allows transports that act nontrivially, as in the circle calculation below.

6.2 Two meanings of existence

Write $\mathsf{isProp}(T) = \prod_{x,y:T}(x = y)$. For small proposition-valued families P, Q over a context, the internal connectives used here have the interpretations

formula	family
$\top, \bot$	$1,\ 0$
$P \wedge Q$	$P \times Q$
$P \vee Q$	$\lVert P + Q \rVert_{-1}$
$P \Rightarrow Q$	$P \to Q$
$\exists_{x:T} P(x)$	$\lVert \Sigma_{x:T} P(x) \rVert_{-1}$
$\forall_{x:T} P(x)$	$\Pi_{x:T} P(x)$

A dependent sum before truncation carries a selected witness and its evidence. A dependent product of propositions is a proposition with function extensionality, which is available in the stated HoTT setting. If equality of two states is intended as a formula in the same proposition-valued fragment, use $\lVert x = y \rVert_{-1}$. The raw identity type is appropriate when the path itself is an input to transport.

Existential quantification can also be read as a map on subobjects. Let $u : Y \to X$ be a family. For $M \hookrightarrow Y$, $\exists_u M$ is the image of $M \to X$. If $N \hookrightarrow X$, factorization through N gives

$$\exists_u M \le N \quad \Longleftrightarrow \quad M \le u^* N.$$

If a pullback square reindexes u, the image construction commutes with the square by Theorem 3.3. This is the Beck–Chevalley law for the existential. The universal quantifier has the corresponding Beck–Chevalley isomorphism by the pullback adjunctions in the locally cartesian closed setting [22, 18].

6.3 Path syntax

Fix an alphabet L. To interpret a word $w = \ell_1 \cdots \ell_m$, take state variables $x_0, \dots, x_m : X$, initial and bad witnesses, and edge witnesses. In an internal presentation with $I(x), B(x)$ and $T_\ell(x, y)$ as families, the proof-relevant type for a bad execution is

$$\sum_{x_0,\dots,x_m : X} I(x_0) \times \left(\prod_{j=1}^{m} T_{\ell_j}(x_{j-1}, x_j) \right) \times B(x_m). \quad (3)$$

Here the indexed product is finite. If the edges and boundary data are presented instead as total objects with maps to states, replace each family by the respective homotopy fiber. Expression (3) then agrees with the limit of the zigzag by the universal property of iterated dependent sums. The empty word gives $\Sigma_x I(x) \times B(x)$.

An observation map induces an interpretation of the same word in a may graph. Its action on a concrete witness uses the unit into each edge and boundary support. For an abstract term b of the path type, the residual syntax is the judgment

$$b : \mathsf{BadPath}(q_! G, w) \ \vdash \sum_{a : \mathsf{BadPath}(G,w)} (p_w(a) = b)\ \mathsf{type}.$$

The logic of feasibility is its propositional truncation. Substitution in the displayed context gives exactly Theorem 3.3. In particular, the existential formula is stable under substitution even though absolute truncation fails to preserve arbitrary homotopy pullbacks.

6.4 Identity in a residual

Write $F = \Sigma_{x:X}(p(x) = b)$ for a single residual over a point. For $u = (x, \alpha)$ and $v = (y, \beta)$ there is an equivalence

$$(u = v) \simeq \sum_{\delta : x = y} \big(\alpha = \mathrm{ap}_p(\delta) \cdot \beta\big).$$

Use the characterization of equality in a dependent sum. The equality between second coordinates after transport is equivalent to the displayed equation, by path induction on δ. Thus a path between two lifts includes a comparison of their abstract identifications. Iterating the same construction yields higher coherences. Treating an equality proof as irrelevant at the start would remove the comparison before its role is established.

Example 6.1 (Two different lifts of one observed point)**.** Let $p : 1 \to S^1$ pick its base point. The residual over that point is equivalent to ΩS^1. It contains $(*, \mathsf{refl})$ and $(*, \mathsf{loop})$, which are unequal because projection to the path coordinate would otherwise identify the loop with reflexivity. In the usual circle computation $\Omega S^1 \simeq \mathbb{Z}$. The support of the residual is terminal. This is realized by the empty word with both boundaries equal to the state object on each side, because p is effective epi in spaces.

6.5 Decidable fragments

If X is a finite set with an enumeration and decidable equality, each identity type has the set interpretation. If the atomic predicates are decidable, evaluation of a finite first-order formula follows its syntax recursively. Quantified variables are enumerated over X. The case of a logical disjunction uses the sum of two decidable propositions followed by truncation. To reflect a Boolean true into truth, take the displayed branch. To reflect a truncated disjunction into Boolean truth, eliminate the truncation into the proposition $\mathsf{true} = \mathsf{true}$ at the computed Boolean target. No elimination into a selected arbitrary branch is necessary.

These facts explain the division of verification between the kernels. The Lean 4 evaluator gives reflection for the finite relational language. The native HoTT development gives reflection for its explicitly specified propositional syntax, with disjunction interpreted by actual propositional truncation and with proof-relevant identity used elsewhere in the same development. The manuscript uses the internal language for the full categorical semantics. The claims about executable decisions apply to the decidable fragments just described.

7 Directed composition and Segal objects

Directed transitions need a composition law separate from the equivalences in the state space. A directed edge may have an inverse only in exceptional systems. Keeping the edge as an element of a separate object T_ℓ makes that distinction in a graph. In a category object it is built into the simplicial structure.

7.1 The compositional case

Let $C_\bullet : \Delta^{\mathrm{op}} \to \mathcal{E}$ satisfy the Segal condition. The state object is C_0, the arrow object is C_1, and for each $m \geq 2$ the spine map

$$C_m \longrightarrow C_1 \times_{C_0} \cdots \times_{C_0} C_1$$

is an equivalence. Complete Segal objects add a completeness condition identifying equivalences with the appropriate state identities [31, 32]. The path theorem below only requires the Segal condition.

For boundary maps $I, B \to C_0$, use the pullback of $C_m \to C_0 \times C_0$ along $I \times B$. If $C_1 \to L$ assigns an external discrete label to each arrow, restrict C_m along its map to L^m and the fixed word w. It is enough that labels are specified on the adjacent arrows. Assigning a label to composites requires extra structure, such as a monoid label functor.

Proposition 7.1 (Spines and bad paths)**.** *For a Segal object with boundary, the object of boundary-marked m-simplices is equivalent to the bad-path object of its directed graph for paths of length m. The equivalence is natural for maps of Segal objects preserving the boundary and any labels.*

Proof. For $m \geq 2$, pull the Segal equivalence back along the boundary maps and, when used, the word selection. Pullbacks preserve equivalences. For $m = 1$, the claim is the definition of the edge object with specified endpoints. For $m = 0$, it is the pullback $I \times_{C_0} B$. Naturality follows from the naturality of restriction to the spine and of the pullbacks. □

A map of category objects therefore induces the same residual over an abstract simplex as over its spine, provided the boundaries and labels are included in the comparison. The result compares actual simplicial objects. It does not supply a category structure on the edgewise may quotient of an arbitrary directed graph.

7.2 Why the may quotient is a graph

Take Example 2.2 as a directed category with identities added. There are no nonidentity composites beyond the two edges. The may graph contains a composable pair $S \to M \to T$ and no edge $S \to T$. Consequently its edge image cannot serve as the arrow relation of a category with those compositions. One can freely add formal composites. Such a construction gives a different abstract path representation and must retain a comparison from concrete executions. The formal composite of the two edges still need not lift to a concrete composite. A category quotient preserving all observed composites requires additional compatibility, such as a suitable lifting condition.

7.3 Laxity for relations

For a pair of spans $X \xleftarrow{s} R \xrightarrow{t} Y$ and $Y \xleftarrow{u} S \xrightarrow{v} Z$, composition uses $R \times_Y S$. Relational semantics takes the support of its map to $X \times Z$. To abstract through $X \to X'$, $Y \to Y'$, $Z \to Z'$, one can either compose concretely and take the image, or take the image of each span and compose in the observed carrier. There is a map from the support of the first to the support of the second. A concrete compatible pair maps to a compatible observed pair. The induced inclusion is associative in the sense that the maps obtained by bracketing a finite sequence factor the same image inclusion. Its failure to be invertible is the categorical form of the middle-witness problem.

The obstruction can have higher structure even when the image inclusion happens to be an equivalence. The original composite contains matching data in Y, which can include nontrivial paths; after taking support that data is invisible. Requiring both sides to have the same truth values is weaker than requiring a pullback square of the corresponding witness objects.

7.4 Simulation versus a lifting property

An ordinary forward simulation gives a map of execution witnesses, perhaps after packaging the simulation relation as an intermediate graph. It is sufficient for safety transfer. To conclude that every abstract path lifts over every context one needs a stronger hypothesis.

A useful sufficient hypothesis is a path lifting property with chosen initial lifts, relative to the given boundaries. For each abstract initial state choose a concrete initial representative. For each concrete representative already chosen and each next abstract edge out of its image, choose a concrete next edge and a compatible next representative. At the last step require that any representative reached over an abstract bad point is concretely bad, or choose an endpoint adjustment compatible with the last edge. Induction on the word gives a lift. In a slice one can state the same conditions as effective epimorphisms of the relevant extension objects and deduce local existence by iterated pullback stability and composition. A section-producing version needs specified choices or stronger lifting data. This distinction parallels the difference between the map of paths and its support.

7.5 Projection to temporal models

A deterministic finite monitor has a state m, an initial monitor state, and an update from a label and the current monitor state. Form the product graph with states (x, m). A transition is permitted when its system component is an edge with the label used by the monitor update. Mark bad monitor states as bad product states. Then a bad product path is equivalent to a system path whose word is rejected by the monitor, with the initial monitor state fixed. Proof is by induction along the deterministic updates.

This reduction supports regular safety specifications. It does not turn an arbitrary infinite-word liveness condition into finite bad-prefix reachability. For a finite-state system and a Büchi condition, a witness has a reachable accepting cycle, expressible by a lasso in the concrete graph. A projected lasso can fail to close at the same concrete state. Its residual must include the closing identification, and fairness conditions can require additional checks [2, 7]. The code supplied here checks finite bad walks with a bad predicate.

III. Algorithms and proof

Effective finite verification, certified finite logic, and conditional generalizations.

8 Exactification and finite termination

The refinement strategy in this section starts from a coarse partition and makes visited cells exact. It provides a simple completeness theorem. A more selective procedure can be cheaper on a large system.

8.1 A local exactness lemma

Let $q : X \to A$ be an observation in sets and let $b = (a_0, \dots, a_m)$ be a may bad path. If each $q^{-1}(a_i)$ contains at most one state, then b has a unique concrete state sequence. The initial fact, the edge facts, and the bad fact each give representatives in their respective cells. For the zero-length word, the initial and bad witnesses already lie in the same cell. For a positive-length word, each pair of adjacent witnesses has equal representatives at their shared cell. The same reasoning identifies the first edge witness with the initial representative and the last with the bad representative. With relational edges the path is determined by the state sequence. With proof-relevant edges, its endpoint sequence is determined and its edge witnesses can vary.

Theorem 8.1 (Eliminating a spurious path)**.** *Let X be a finite set and let b be a may bad path for the partition q, with empty residual. Refine q to q' by replacing every visited cell with its singletons. There is a coarsening r with $q = rq'$. No bad path in the may graph of q' projects to b, and the number of nonempty cells increases strictly.*

Proof. A hypothetical descendant b' lies entirely over visited cells of q. Each of its cells under q' is a singleton. By the local exactness lemma it lifts to a concrete bad path. Its projection is b, contradicting the empty residual. At least one of the original visited cells has two or more members, since otherwise the local exactness lemma would already give a lift of b. Each singleton split of a cell of size $d > 0$ increases the total cell count by $d - 1$, so the total increase is positive. □

Corollary 8.2 (Unbounded finite safety decision)**.** *On n concrete states with k initially nonempty observation cells, the loop using Theorem 8.1 makes at most $n - k$ refinements and calls the abstract checker at most $n-k+1$ times. It returns a valid safety proof or a concrete bad walk.*

Proof. If the abstract graph is safe, Proposition 2.1 transfers safety. A witness from the lift checker is a valid concrete bad walk. Any other case is a spurious abstract walk and strictly increases a quantity bounded by n. The discrete observation is exact on all realized cells, so that case cannot recur there. □

8.2 Sharpness of the round bound

For $m \geq 1$, take concrete states s, t, u_i, v_i for $1 \leq i \leq m$, with edges $s \to u_i$ and $v_i \to t$. Make s initial and t bad. The seed keeps s, t in singleton cells and pairs u_i, v_i in a middle cell M_i. Every bad path is abstract and has form s, M_i, t. The exactification of its visited cells splits M_i and has no effect on another middle cell. Thus any run splits exactly m cells before it proves safety. Here $n = 2m + 2$ and $k = m + 2$, so the bound $n - k + 1$ is attained independently of candidate selection. Part IV checks this family, including variants with one genuine bad branch among the infeasible candidates.

8.3 A representation with a simple recursion measure

The algorithm represents q by a seed map $s : X \to K$ and a list R of concrete states still assigned seed keys. Its carrier of keys is $X + K$. The two summands distinguish an isolated state from a seed block:

$$q_R(x) = \begin{cases} \mathsf{seed}(s(x)), & x \in R, \\ \mathsf{iso}(x), & x \notin R. \end{cases}$$

If b is spurious, define R' by removing every $x \in R$ whose current key is visited by b. With a duplicate-free initial list, this is exactly the singleton policy of Theorem 8.1, up to names for singleton cells. Complete enumerations with repeated elements are also accepted. Filtering removes all occurrences of a selected state, so the recursive measure is the length of R. A spurious walk must visit a key with at least one unisolated member. Otherwise every visited cell would already be a singleton and the lift checker would return a witness.

The factor map $r : X + K \to X + K$ sends $\mathsf{iso}(x)$ to $q_R(x)$ and keeps $\mathsf{seed}(s)$ unchanged. It satisfies $rq_{R'} = q_R$. A key outside the actual image carries no boundary evidence or edges. The sum of key enumerations is a convenient finite implementation even when many keys are unused in a particular round.

8.4 A relative higher version

For a monomorphism $j : V \hookrightarrow A$, write $X_V = X \times_A V$. Restrict edges to both endpoints in X_V and restrict the boundaries to X_V. Suppose $q_V : X_V \to V$ is a monomorphism. Its graph maps, after relative image in X_V and X_V^2, are exactly the restricted may maps over V and V^2, followed by the inclusions. This follows by stable image and Lemma 4.1.

Proposition 8.3 (Exactness on visited state subobjects)**.** *If a may bad path has all of its states in V and q_V is a monomorphism, its residual is locally inhabited in its context. If the restricted graph's boundary and edge maps are monomorphisms, the residual is terminal in that context.*

Proof. The abstract boundary and edge objects factor through the state subobject X_V at all their endpoints. Their path limit over V agrees with their path limit over X_V, since $X_V \times_V X_V \simeq X_V$. The abstract path therefore specifies a path in the relationalized restricted graph on X_V.

Over this concrete state sequence, each boundary or edge witness is locally inhabited by the image property. A finite product of such families has full support. Pullback stability and composition of effective epimorphisms prove this for two factors, and induction gives the finite product. This product describes the lift after the state identifications have been fixed. If the boundary and edge maps are monos, each factor is terminal over the context by the abstract witness. □

Corollary 8.4 (Higher elimination criterion)**.** *Let $q = rq'$ and let b be a coarse path refuted over U. Let $V \hookrightarrow A$ contain all its states. If the restriction of q' over r^*V is a monomorphism, then the object of fine may bad paths projecting to b is initial over U.*

Proof. Work over the object of proposed fine descendants, where there is a universal fine path and an identification of its coarsening with b. Its states lie in r^*V after transport along that identification. By Proposition 8.3 its concrete lifts form an effective epimorphism onto this descendant context. Each such lift projects to a lift of b, so the refutation restricts to a refutation of the effective epimorphism's domain. An effective epimorphism with initial domain has initial codomain by its image factorization. □

An available observation language may fail to express such a refinement. For example, 0-truncated keys cannot make an observation monic over every state of an object with nontrivial loops. The criterion applies when the chosen refinement can express the required monomorphism over the specified visited states.

9 Layered predicates and interpolation

Predicate abstraction can express the reason a path fails using fewer cells than bulk isolation. Fix an abstract word $b = (a_0, \dots, a_m)$ with labels ℓ_i. Put $A_i = q^{-1}(a_i)$ and let F_i, D_i be the exact prefix and suffix sets from Section 2. Take predicates $J_i \subseteq A_i$ satisfying

$$I \cap A_0 \subseteq J_0, \tag{4}$$

$$A_i \cap \mathrm{post}_{T_{\ell_i}}(J_{i-1}) \subseteq J_i \qquad (1 \le i \le m), \tag{5}$$

$$J_m \cap B = \varnothing. \tag{6}$$

These conditions can be checked with explicit finite sets or by a proof-producing decision procedure in a suitable assertion language.

Theorem 9.1 (Layered path blocking)**.** *A sequence satisfying (4)–(6) refutes b. Refining the observation by the vector of all bits 1_{J_i} eliminates every bad path projecting to b.*

Proof. The first assertion follows by induction along a concrete lift. For the second, the refined initial fact at index zero has a concrete initial witness in A_0. Its J_0 bit is true, and every state in the same refined cell has that bit. Suppose this holds at index $i-1$. The next may edge has witnesses $x \in A_{i-1}, y \in A_i$ with the designated finer keys. The first has J_{i-1}, so (5) gives $J_i(y)$. The next finer cell has its J_i bit true. The final bad fact would supply a concrete point in $J_m \cap B$, contradicting (6). □

If the observation is indexed by location, one can attach J_i at the occurrence of the i-th location in the selected word. If that location can be revisited, a location-only implementation can attach all the predicates needed there. A product with a word monitor gives the occurrences separate names. The global observation vector avoids assuming that the layers had distinct locations.

9.1 An interval of certificates

Proposition 9.2 (Extremal sequences)**.** *If b is spurious, the sequences*

$$J_i^{\min} = F_i, \qquad J_i^{\max} = A_i \setminus D_i$$

satisfy (4)–(6). Every satisfying sequence lies between them at each index.

Proof. The lower sequence satisfies the first two conditions by construction, and the final condition by the cut criterion. In the upper sequence the initial condition follows from $F_0 \cap D_0 = \varnothing$. An edge from a point outside D_{i-1} to a point in D_i would extend the suffix from the latter to the former, contradiction. The final condition follows from $D_m = A_m \cap B$.

The first two conditions for an arbitrary sequence imply $F_i \subseteq J_i$ by forward induction. To show $J_i \cap D_i = \varnothing$, start at m and reason backwards. A point in the intersection at $i-1$ has an edge to a point of D_i. By (5) the latter also lies in J_i, contradicting the induction hypothesis. □

The coordinatewise inclusions alone do not guarantee that an arbitrary choice of the J_i satisfies the step implications. For example, let s be initial, t bad, and allow

$s \to u$, $v \to v'$, and $w' \to t$ across three layers. Put u, v in the first middle layer and v', w' in the second. Both middle predicates can satisfy their interval conditions while choosing v at the first and omitting v' at the second. The edge $v \to v'$ then violates the step condition.

9.2 When one separator suffices

At a fixed cut index i, $F_i \subseteq K \subseteq A_i \setminus D_i$ says that K separates the exact feasible prefix endpoints from the exact feasible suffix endpoints. It suffices to block the join when each side is represented by its exact endpoint set. When prefixes or suffixes remain may approximations, witnesses can lose coherence away from the cut. The layered step conditions, or exactness on both sides, supply the needed guarantee.

Part V computes the exact sets and checks the refined may graph by a separate procedure. A certificate can be much smaller than the exact prefix sequence. A parity invariant common to a long prefix and a contradictory suffix can be expressed by one short formula reused at each layer. Exact forward sets give a finite construction of a certificate, and their description as a disjunction over states can be large.

9.3 Symbolic and internal interpolation

Let $A(\bar{x}, z)$ express feasibility of the prefix and $B(z, \bar{y})$ express feasibility of the suffix. If $A \wedge B$ is unsatisfiable and the relevant logic has interpolation, a Craig interpolant $K(z)$ satisfies $A \Rightarrow K$ and $K \wedge B \Rightarrow \bot$ [9, 28, 29]. Background symbols and theories are part of the interpolation contract. A sequence intended for refinement must satisfy the local conditions (4)–(6).

Internally, read each F_i as an existential image in the slice over the path context, and $A_i \setminus D_i$ as the relative Heyting negation of D_i. The two extreme predicates satisfy the same entailments for a refuted path. The proof uses composition and elimination into propositions. Forming a Boolean observation bit for a predicate requires its decidability. An internal entailment and its proposed Boolean implementation can have different domains of validity in a sheaf model.

10 Finite logical syntax and reflection

The finite relational syntax can be evaluated without running a refinement. Its 0-truncated HoTT meaning is given by the interpretation of Section 6.

10.1 Grammar and environments

For a state type X and context length n, a term is a variable $i : \mathsf{Fin}n$ or a named state $x : X$. An environment is $\rho : \mathsf{Fin}n \to X$. Quantification binds a variable at index zero. The old environment is carried along the successor injection. The formula constructors are

$$\top,\ \bot,\quad P_j(t),\quad s = t,\quad T(s,t),$$
$$\varphi \wedge \psi,\quad \varphi \vee \psi,\quad \varphi \Rightarrow \psi,\quad \exists x.\, \varphi,\quad \forall x.\, \varphi.$$

Each atom P_j and T has a Boolean implementation. The typing index enforces the scope of free variables. In the Lean 4 package equality uses an explicit `DecidableEq` instance. No default state is needed for an empty domain.

10.2 Reflection

Write $[\![\varphi]\!]_\rho$ for the relational semantics and $E(\varphi, \rho)$ for the Boolean evaluator. For the quantifiers,

$$E(\exists x.\varphi, \rho) = \mathrm{any}_{s \in \mathrm{enum}\, X}\, E(\varphi, s :: \rho),$$
$$E(\forall x.\varphi, \rho) = \mathrm{all}_{s \in \mathrm{enum}\, X}\, E(\varphi, s :: \rho).$$

An empty enumeration gives false for the existential and true for the universal. Completeness of the enumeration ensures the domain semantics. Duplicate elements leave both truth values invariant.

Theorem 10.1 (Finite reflection)**.** *For every well-scoped formula and environment,*

$$E(\varphi, \rho) = \mathsf{true} \quad \Longleftrightarrow \quad [\![\varphi]\!]_\rho.$$

Proof. Induct on the syntax. Atomic cases use the definition of the interpretation. Conjunction and disjunction use reflection of Boolean operations on their arguments. For implication split on the computed value of the antecedent. When false, the induction hypothesis refutes the antecedent, and when true it supplies it. Equality uses the decision instance and quantifiers use enumeration completeness. □

10.3 Renaming and reindexing

For a renaming $f : \mathsf{Fin}n \to \mathsf{Fin}m$, define $f^+(0) = 0$ and $f^+(i+1) = f(i) + 1$. Then

$$(s :: \rho) \circ f^+ = s :: (\rho \circ f).$$

Both environments agree at zero and at each successor by definition. Induction on formulas using this equality in the quantifier cases proves

$$[\![\mathrm{rename}_f(\varphi)]\!]_\rho \quad \Longleftrightarrow \quad [\![\varphi]\!]_{\rho \circ f}.$$

Reflection gives Boolean equality of the evaluations. This is a finite instance of substitution naturality. The native HoTT file proves a transport equation for reindexing an arbitrary family along a map.

10.4 Modal variance

For relational modalities, $\Diamond_\ell \varphi(x)$ asserts the existence of y with $xT_\ell y$ and $\varphi(y)$, while $\Box_\ell \varphi(x)$ asserts $\forall y,\ xT_\ell y \Rightarrow \varphi(y)$. A forward graph map preserves positive existential formulas built from forward-preserved atoms, $\wedge, \vee$, and diamonds. Induct on the formula and project the witness in the diamond case. Positive universal formulas built from backward-reflected atoms, $\wedge, \vee$, and boxes reflect truth from the abstract graph to the concrete one. A concrete successor projects to an abstract successor, and the induction hypothesis reflects truth there.

Negation and implication require the appropriate variance of the subformulas. Checking an existential property directly on the may graph can return a path with no concrete witness. Modal transition systems and three-valued abstractions record separate sufficient and possible conditions for related reasons [20, 17].

10.5 The finite reachability formula

For a given length and word, existence of a bad path is a closed existential formula with a variable for each node and conjuncts for edges and boundaries. To express feasibility of a fixed abstract path, also specify the observed key at each index. The evaluator and the lift checker decide the same proposition. The checker tabulates endpoints so it can share work across assignments and return a concrete witness.

A bounded formula also decides unbounded finite-state reachability at a sufficient bound. A bad walk has a simple bad subpath by removing repeated-state segments. Saturation obtains the answer without generating the bounded formula or choosing a path length by hand.

11 Kernel-checked algorithms

The companion includes a Lean 4 package pinned to release 4.19.0 [24], and three HoTT-mode files pinned to the archived Lean HoTT distribution [23]. The packages use their respective bundled libraries. The first verifier works with explicit finite enumerations; the second kernel has proof-relevant identity types. The source audit identifies which claims are checked in which kernel.

11.1 Proof-carrying finite reachability

In `Cegar.Core`, a graph has three Boolean functions. The inductive family `Walk G x` is indexed by its last state. A start constructor requires a proof that `G.initial x = true`. An extension requires a previous walk and a proof of the edge test. `BadWalk G` pairs a final state with a walk and a proof of its bad test. `Safe G` is the type of functions from that bad-walk type to false. Consequently a `Verdict G` value distinguishes a concrete witness from a proof of safety by type.

The worklist state has a list of known reachable states with their paths, and a list of remaining states. The invariant says every state is in one list or the other, and all known states are nonbad. The function `pick` either produces a value for one element or proves all tested candidates missing. In the found branch the remaining list strictly decreases. In the missing branch the known states include every initial state and are closed under successors. Induction on an arbitrary walk proves that its endpoint is known. The maintained bad-state condition then yields `Safe G`. This argument verifies all finite walk lengths at once.

The decision specification is `solve_safe_iff`. A positive bad verdict carries its own checked path. Projection to the command-line format erases the proofs and keeps the node sequence.

11.2 May abstraction and exact tabulation

The Lean definition `may` scans the concrete enumeration for boundary witnesses and scans pairs for edge witnesses. Three equivalences recover the existential semantics of those scans. `mapWalk` maps a walk along the observation using the three equivalences, and `safe_of_may_safe` transfers safety.

A term of `Lift G states q w x` follows one specific abstract walk w and ends at x. It has the same inductive shape as the walk and its constructors check observation and concrete edge facts. `Lift.nodes` proves the expected projection equation for the lists of nodes. `lift_of_nodes` proves the converse as propositional existence.

`tabulate` creates one row for every reachable endpoint of the given prefix that is discovered by the enumeration. Each row contains a witness. Its completeness property says that any actual lift ending at an endpoint has a row with that endpoint. In the successor case it suffices to retain one previous witness per endpoint, because the next test depends only on that endpoint and the proposed new state. Induction on the abstract walk proves the tabulation property. `checkBad` adds the final bad test and returns either a lift or a function ruling out every term of the lift type.

11.3 The refinement loop

`cut_refines` proves the factorization through the previous observation. `cut_shorter` proves the strict decrease needed for the well-founded definition `runFrom`. `cut_eliminates` formalizes the stronger claim that a refined bad walk cannot project to the rejected walk. Its proof uses uniqueness of states over visited keys, obtains a lift of the fine path, and uses the exact node projection equation to construct a lift of the old path. The resulting contradiction is independent of the search order.

`run_safe_iff` states that the loop returns a safety verdict exactly when the concrete graph is safe. In particular,

a fuel-exhaustion branch does not enter the result type. The mathematical bound by the number of realized cells in Corollary 8.2 is a separate proof from the executable recursion's decrease in list length.

11.4 Checking layered certificates

`Cegar.Layered` takes a finite sequence of proposed predicates and labelled transition tests. Its Boolean checker enumerates the initial, step, and bad obligations. `check_reflect` proves equivalence with the conjunction of those logical obligations. `check_sound` uses induction on a proof-relevant walk through the layers to refute a bad endpoint whenever the checker accepts the certificate. The theorem verifies a supplied predicate sequence. The scripts in Part V construct example sequences by forward and backward propagation.

11.5 Native HoTT declarations

`Residual.hlean` defines a residual using the kernel's identity type. It verifies the fiber of reindexing a family, transport along a base path, the dependent-sum form of a composite fiber, and the identity type between two lifts. Its circle example checks that a winding lift differs from the reflexive lift in the same residual.

`Logic.hlean` gives a deep syntax for Boolean atoms, truth, falsehood, conjunction, disjunction, and implication. `Holds` interprets these as types. In particular, disjunction uses the propositional truncation of a sum. `reflect` constructs both directions between Boolean truth and the interpretation. `check_rename` verifies evaluation after renaming atom symbols.

`FiniteLift.hlean` implements a witness decision for a fixed sequence of abstract Boolean keys in a Boolean concrete graph. At each step its semantic family is a dependent sum over predecessor states satisfying the previous semantic family and the new edge. The code reflects the Boolean recursion into that family. A true test constructs a lift and hence its support. A false test refutes the lift. Two small closed examples check both branches, including a spurious zero-edge candidate whose abstract boundary facts have different witnesses.

12 Generalizations and effectivity bounds

The categorical construction consumes a graph, an observation, a path, and the relevant limits and images. An effective CEGAR loop additionally needs a representation of all these inputs and a strategy with a computable progress bound. The following extensions make the additional hypotheses explicit.

12.1 Symbolic transition systems

Let states be valuations in a decidable background theory, edges relations presented by formulas, and a finite abstraction specified by Boolean predicate vectors. The may edge between two keys is a satisfiability query consisting of their defining predicates and the concrete transition formula. The initial and bad queries use only one endpoint. A selected path lifts exactly when its conjunction of indexed transition and boundary constraints is satisfiable. The finite image semantics is unchanged. The cost and evidence depend on the background solver [3, 12].

A verifier can require a concrete assignment in the satisfiable branch and a checked unsatisfiability proof in the negative branch. A proof-producing interpolant procedure can additionally generate the layered obligations. For an incomplete solver, a third result `unknown` is appropriate. Soundness of its positive and negative results can still be proved by the same semantic laws.

12.2 A finite predicate budget

Given a fixed finite pool of predicates, begin with a subset of that pool and add a new predicate on each refinement that can be justified by a certificate. The number of added predicates is bounded by the pool size. If no remaining predicate separates a spurious trace, the algorithm can report an unresolved obligation. A total safety decision from that budget requires an additional expressiveness theorem. For example, if the pool contains all concrete singletons and the algorithm may add every bit needed by exactification, Section 8 supplies the needed completeness.

A fixed pool can be effective and insufficient. On s, u, v, t from Example 2.2, a pool whose predicates all agree on u and v cannot separate their two occurrences in the path. Every observation from that pool identifies them and retains the same infeasible may path. The algorithm's witness and refutation rules remain sound.

12.3 Infinite systems and the halting boundary

There is a simple reduction for unbounded systems. Given a deterministic Turing machine and input, use its configurations as concrete states, its one-step relation as transitions, its initial configuration as the sole initial state, and its halting configurations as bad. Equality of encoded configurations, the transition test, and the bad predicate are all decidable. A finite bad path exists exactly when the machine halts. Thus a total safety decision for all such presentations would decide the halting problem [35]. A finite prefix checker can still be correct and total for each selected finite observation when its feasibility query is decidable. The unbounded verification task needs separate restrictions.

12.4 Proof interfaces for several analysis engines

A safety certificate may specify an inductive invariant J with $I \subseteq J$, $\mathrm{post}_T(J) \subseteq J$, and $J \cap B = \varnothing$. A counterexample certificate gives a concrete path. A refinement certificate gives an entailment sequence or a verified splitting criterion. These interfaces admit implementations based on graph search, symbolic fixed points, interpolation, or an inductive generalization of failed predecessor obligations [8, 5]. They share the semantics of the claims while using different discovery procedures.

One can lift the certificate descriptions to a topos by interpreting J as a subobject and all inclusions as morphisms in the subobject order. The proof of safety uses induction on a finite word. A higher residual may require additional witnesses that a purely relational invariant does not represent. The chosen interface determines whether existential truth or a selected path is required at its boundary.

12.5 Computational cost of the reference verifier

Let n be the number of concrete states and a the number of enumerated abstract keys, and assume constant-time concrete matrix tests and key equality. A single may-edge query checks at most n^2 pairs. The reference key computation can itself scan the list of unisolated states, giving a conservative $O(n^3)$ cost for a may-edge query without caching. The worklist can test at most a candidate targets against at most a seen states on each of at most a successful discoveries and one final scan. This gives $O(a^3 n^3)$ as a coarse per-round upper bound for that reference implementation. The JSON adapter uses lists for input edge membership, so its concrete edge query cost also depends on the size of that list.

For a specified path of length m, the tabulator makes $O(mn^2)$ concrete edge and observation work under the matrix assumption and the list-key cost just described: each of n target candidates can scan n previous rows and compute its key in $O(n)$. It retains witnesses rather than enumerating all concrete paths. Printing the witness can cost additional time for list construction. These estimates describe the reference functions, not an optimized model checker. Indexing transitions and caching image relations can improve the relevant costs without changing the proof interface.

13 Relational adjunctions and logical observations

The finite checker uses ordinary sets and Boolean observation bits. The following results give internal versions of two algebraic parts of the construction. The first describes state reindexing before path formation. The second gives a certificate-driven refinement using truth values in a general ∞-topos.

13.1 An adjunction of relational graphs

For a state object X, let

$$\mathsf{RelG}(X) = \mathrm{Sub}(X)^2 \times \prod_{\ell \in L} \mathrm{Sub}(X \times X).$$

The two first components encode the boundaries. For $q : X \to A$, the functor $q^* : \mathsf{RelG}(A) \to \mathsf{RelG}(X)$ pulls back each boundary along q and each edge along $q \times q$. The may image $q_!$ is its left adjoint. This holds componentwise: for a subobject $P \hookrightarrow X$ and $Q \hookrightarrow A$, $\mathrm{im}(P \to A) \le Q$ if and only if $P \le q^*Q$. The proof is the universal property of pullback and image. Use X^2, A^2 and q^2 for the edge component.

In a pullback square of state objects

$$\begin{array}{ccc} X' & \xrightarrow{g'} & X \\ {\scriptstyle q'}\downarrow & & \downarrow{\scriptstyle q} \\ A' & \xrightarrow[g]{} & A \end{array}$$

there is a Beck–Chevalley identification $g^* q_! \simeq q'_! (g')^*$ on relational graphs. It follows from pullback-stable images on boundaries. Taking products of the displayed square with itself gives a pullback square for the edges, so the same proof applies there. This compares entire state graphs after reindexing; Theorem 3.3 instead changes the context of an already specified path.

For subobjects $P \hookrightarrow X$ and $Q \hookrightarrow A$, Frobenius reciprocity reads

$$\exists_q (P \wedge q^* Q) \simeq (\exists_q P) \wedge Q.$$

Pull back the image factorization of $P \to A$ along $Q \to A$. The left domain becomes $P \times_A Q$, and its image over Q is $Q \times_A \exists_q P$. Composing this monomorphism with $Q \hookrightarrow A$ gives the claim in $\mathrm{Sub}(A)$. This is why an observation guard can be applied before or after a single existential image. It does not license pulling an image through a sequence of transition joins.

13.2 Reachability as a least subobject

Let $R \hookrightarrow X \times X$ be an internal relation, with source and target $s, t : R \to X$. Its forward operator on subobjects is $\mathrm{post}_R(U) = \exists_t s^* U$. A pullback functor between slices preserves colimits, since the ∞-topos is locally cartesian closed and pullback has a right adjoint. The induced subobject reindexing therefore preserves joins, using the image of the coproduct as the join. Existential image preserves joins by its left adjoint property. Thus post_R preserves joins.

Form $U_0 = I$, $U_{j+1} = U_j \vee \mathrm{post}_R(U_j)$ and $U_\omega = \bigvee_{j \in \mathbb{N}} U_j$. The latter is closed under the forward operator,

since $\mathrm{post}_R(U_\omega) = \bigvee_j \mathrm{post}_R(U_j) \le U_\omega$. Any invariant containing I contains each U_j by induction and contains their join. Equivalently, U_j is the image of the endpoint map from concrete paths of length at most j. This follows by induction, using the endpoint pullback and its image at the successor step. The image forgets path evidence while keeping reachability as an internal proposition.

Safety is $U_\omega \wedge B = 0$. Indeed, a point of $U_\omega \wedge B$ is locally the endpoint of a finite bad path. Conversely each concrete bad path factors its endpoint through that intersection. If the intersection is initial, every path is refuted. If every finite bad path is refuted, the union of their endpoint images is initial. This proves equivalence with the path-based safety definition in every slice.

13.3 Refinement by truth-value coordinates

Work internally in a slice $\mathcal{E}/U$. Write a chosen coarse path as the state sequence $a_0, \dots, a_m$, with its edge and boundary evidence. For each i, let $J_i : X \to \Omega$ be a predicate, where Ω denotes the object of truth values of the slice. Assume the following typed entailments:

$$\begin{aligned} I(x),\ q(x) = a_0 &\vdash J_0(x), \\ J_{i-1}(x),\ T_{\ell_i}(x,y),\ q(x) = a_{i-1},\ q(y) = a_i &\vdash J_i(y), \\ J_m(x),\ B(x),\ q(x) = a_m &\vdash 0. \end{aligned}$$

Here I, B, T are the witness families of the concrete graph. The equality hypotheses allow transport in the observed state object. No decidability hypothesis on J_i is required to state the entailments.

Theorem 13.1 (Internal layered blocking)**.** *Let the refined state map be*

$$q'(x) = (q(x), J_0(x), \dots, J_m(x)) \ \in A \times \Omega^{m+1}.$$

The object of fine may paths whose coarse projection is identified with the chosen b is initial over U.

Proof. Work in the context of a proposed fine path b' with state keys a'_i. The identification of its projection with b supplies identifications between the first coordinate of a'_i and a_i. The i-th truth coordinate of a'_i is itself a proposition. At the initial key it holds: eliminate the may initial witness into that proposition, use $q'(x) = a'_0$ to transport the first entailment to the displayed truth coordinate, and conclude.

Now suppose the $i-1$ coordinate holds at a'_{i-1}. The may edge is the support of concrete edge witnesses over the two fine keys. Eliminate that support into the i-th proposition at a'_i. A concrete edge witness transports the previous truth into $J_{i-1}(x)$. The equalities of the first coordinates and the specified coarse identification supply the hypotheses of the second entailment. Its conclusion transports to the required truth coordinate at a'_i. Inductively the m-th proposition holds at the final key. Eliminate the may bad witness into 0 and apply the last entailment. The resulting map from the proposed-descendant object to 0 makes that object initial. □

In sets, the truth coordinates are Boolean and this proves Theorem 9.1 after restricting each J_i to its observed cell. In a general slice an arbitrary subobject need not be complemented. The theorem uses truth values as observations. An effective Boolean implementation requires the relevant predicates to be decidable, which is a condition on the input presentation. A failed higher lifting problem also need not supply predicates satisfying the hypotheses: the obstruction can depend on higher equality or descent data that the specified observation language does not retain.

14 Branching obligations and tableau patterns

The original overview also identifies uses of CEGAR in modal tableaux [36]. The path residual extends to a fixed finite incidence diagram. This gives a common semantics for failed compatibility checks in linear paths and branching candidate countermodels. The extension does not assume the internal logic of the topos is Boolean.

14.1 Finite patterns

Let D be a finite directed graph of positions with an edge label $\ell(e)$ at each edge. Let $C_v \to X$ be the vertex constraint at a position v. An assignment of D in G is an element of

$$\sum_{(x_v) \in X^{D_0}} \left(\prod_{v \in D_0} C_v(x_v) \right) \times \left(\prod_{e \in D_1} T_{\ell(e)}(x_{s(e)}, x_{t(e)}) \right).$$

All families here are homotopy fibers of the indicated structure maps. This is a finite limit in the incidence diagram with one state at each position. A repeated position uses one state variable, regardless of how many edges are incident to it. For a path, choose $C_v = 1$ at interior positions and the boundary families at its endpoints.

Under an observation q, take the image of each vertex constraint and edge relation. Denote the comparison from the concrete assignment object to the abstract one by p_D. A chosen abstract assignment $d : U \to \mathrm{cod}(p_D)$ has residual $U \times_{\mathrm{cod}(p_D)} \mathrm{dom}(p_D)$. The proof of Theorem 3.3 uses only this map and image stability, so it applies to p_D. The dependent-sum law of Theorem 4.2 applies to its composites. Likewise, the proof of Proposition 8.3 works when the observation is monic over the states assigned to every position of D: the incidence identifications are taken within the same subobject, and the finite product of the local witnesses has full support.

14.2 A fork obstruction

Take a discrete graph with root candidates u, v. There is a transition from u to a state satisfying P and a transition from v to a different state satisfying Q. These are the only two outgoing edges. Put u, v in the same observed root cell, keeping both target states distinct. The coarse graph satisfies $\Diamond P \wedge \Diamond Q$ at the root cell. In the concrete graph the left diamond holds at u and the right at v. Neither concrete root satisfies their conjunction. The residual of the two-edge fork is empty because both incident edges must use the same concrete root. The two one-edge patterns each have a nonempty residual. This is a branching counterpart of Example 2.2.

14.3 Connection with modal tableau refinement

For propositional modal K, a finite open tableau can be read as local constraints on a finite Kripke tree. A node assigns consistent propositional literals. A diamond obligation demands a successor where its argument holds, and a box obligation imposes its argument at every chosen successor. An open branch yields a model of the original formula once those local constraints are all satisfied. The claim follows by induction on the formula: a literal uses its assigned truth value, conjunction uses both obligations, disjunction uses its chosen branch, and the modal clauses are exactly the stated successor constraints.

For a formula in negation normal form, the tree search can be bounded by its modal depth. At a node in a satisfying model, retain one successor for each required diamond among the current obligations. At each retained successor impose its corresponding diamond argument and the arguments of the boxes at the parent. The obligations still hold in the original model at the chosen successor. Their maximum modal depth drops by one, so recursion terminates. The converse is the same structural induction on the completed tree. If s bounds the number of subformulas and d is the modal depth, $1 + s + \cdots + s^d$ bounds this construction's number of vertices. One can decide the finite problem by enumerating the corresponding choices, even though practical procedures use symbolic constraints.

The graph of this fixed tree, together with selected local Boolean choices, gives one instance of D. If a coarse propositional abstraction allows separate local satisfying assignments, the residual asks whether those assignments extend to a single assignment to the entire tree. An empty residual justifies adding a constraint or refining the observation. The modal satisfiability algorithms of [21, 13] study how such refinements can be generated efficiently. The result here is a semantics for a supplied candidate pattern and its compatibility obligation. It also applies to finite patterns with sharing, where two incoming branches refer to the same designated concrete state.

15 Study design and comparison protocol

The computed cases provide explicit examples of the theorems and failure modes. The finite, labelled, and higher calculations use finite data small enough to retain the complete certificate tables. The companion contains the full machine-readable inputs and deterministic generators.

15.1 Finite runs

The finite cases specify concrete edges, boundaries, and seed observations. Each case has an analytic argument for its verdict, such as a modular invariant, a preserved channel, or an explicit path. A separate breadth-first traversal computes concrete distances and predecessors. The worklist simulation uses the same order as the Lean verifier: choose the first remaining key that is initial or has an incoming edge from a reached key, preferring the most recently inserted predecessor when there are several. Additional uninhabited abstract keys do not affect this order among discovered keys.

The published tables give the concrete successor rows, the distances from the initial predicate, and the rounds of the abstraction. For a rejected candidate, exact forward and backward recurrences record its infeasibility. The script also builds the new abstract graph and counts descendants of the old path, restricted by the factor map. For a safe round, the abstract invariant table lists all outgoing successors of each reached key. For a true violation, a concrete witness is reconstructed and checked against the original edge set.

The compiled Lean executable is run on the same finite cases. Its result is compared with the independently computed verdict and its bad path is validated against the concrete relation. A separate small exhaustive test exercises all directed graphs on three states, including self-loops, with state zero initial and state two bad, under all five partitions. These checks are regression evidence for the implementation and the data representation. The general correctness argument is the kernel-checked specification in Section 11.

15.2 Labelled certificates

The second collection fixes a word, a sequence of observed states, and explicit labelled transitions. It computes prefix and suffix tables and counts concrete lifts. A dynamic count sums the number of prefixes rather than just the number of endpoints. The same state sequence is also checked in the union of the label relations. This can change the result, as the guard and bypass cases illustrate.

For every infeasible word the script constructs the least and greatest certificates of Proposition 9.2, and an ad-

ditional certificate obtained by adding an allowed state and closing forward. The script verifies each local entailment and independently constructs the refined graph to confirm there are zero descendants. The case tables record how many distinct keys each choice generates. The smaller number of keys among these three choices is an observed comparison for that instance; it is not a minimality theorem over all possible predicates.

15.3 Higher calculations

The third collection computes group homomorphisms, transporters, double cosets, stabilizers, and equivariant sections for finite group actions. The groupoid equivalences that justify those calculations are proved using explicit action descriptions. The generators verify the relevant group operations, partitions into orbits, and stabilizer cardinalities. These cases exhibit differences that would be hidden by replacing a residual with its support or by forgetting descent. They do not approximate arbitrary higher homotopy types through a finite cutoff.

15.4 Verification results

All 109 manuscript graph cases agreed with the compiled Lean 4 checker, including the round count from the deterministic reference search. The run found 41 genuine bad walks and proved 68 graphs safe. The largest observed round count was 19, attained by the 18-cell decoy instance. The three-state census had 2,560 graph/partition pairs. Its 1,600 violations and 960 safety results agreed with breadth-first search. Independently, a three-state graph reaches state two from zero precisely when it contains the direct edge or both edges through state one. There are $256 + 64 = 320$ such directed graphs and five partitions per graph.

Of 32 prescribed labelled paths, 8 lifted and 24 were spurious. Five of the spurious paths would have a lift if the labels were ignored. The 54 groupoid and descent instances include examples whose supports are all true but whose residual automorphism groups or global section sets differ. The companion records the input and output of each calculation.

15.5 Conclusions

The same path-feasibility question appears as a relational join on sets, a homotopy fiber in an ∞-topos, and a dependent sum with identity evidence in the internal language. Its support gives a stable existential proposition. The fiber controls composition across observation levels and keeps the data needed for descent. For finite systems, explicit tabulation and a strict refinement measure turn those semantics into a complete decision procedure. For more general contexts, the semantic theorems identify precisely what additional decidability or lifting data a verification algorithm must supply.

IV. Finite verification cases

Explicit transition systems, quotient walks, refinements, and independently reconstructed certificates.

16 Lanes under changing offsets

At phase j, the transition $(j, x) \mapsto (j+1, x+j+o \bmod k)$ rotates the k lanes. The seed observes j. Independent edge witnesses preserve the phase structure and conceal the lane.

Tables enumerate concrete states by integer ID. The d column gives distance from an initial state, and p gives a predecessor attaining that distance. Columns I and B are encoded by the boundary field. Abstract keys b_i are seed blocks and s_x are isolated states.

16.1 F1.1 depth 4, width 3, offset 0, target 1

The lane at stage j is $j \cdot 0 + j(j-1)/2$ modulo 3. The terminal lane is 0 and the specified bad lane is 1.

The concrete carrier has 15 states and 12 edges. Its reachable set has 5 states. The reachable set is disjoint from the bad predicate.

ID	State	Seed	Bd.	d	p	Successors
0	(0, 0)	0	I	0	–	{3}
1	(0, 1)	0	–	∞	–	{4}
2	(0, 2)	0	–	∞	–	{5}
3	(1, 0)	1	–	1	0	{7}
4	(1, 1)	1	–	∞	–	{8}
5	(1, 2)	1	–	∞	–	{6}
6	(2, 0)	2	–	∞	–	{11}
7	(2, 1)	2	–	2	3	{9}
8	(2, 2)	2	–	∞	–	{10}
9	(3, 0)	3	–	3	7	{12}
10	(3, 1)	3	–	∞	–	{13}
11	(3, 2)	3	–	∞	–	{14}
12	(4, 0)	4	–	4	9	∅

ID	State	Seed	Bd.	d	p	Successors
13	(4, 1)	4	B	∞	–	∅
14	(4, 2)	4	–	∞	–	∅

Round 1. The partition has 5 nonempty cells, and the may relation has 4 edges.

The selected path is b_0, b_1, b_2, b_3, b_4. The exact number of concrete lifts is 0.

j	a_j	F_j	D_j	$\#\cap$
0	b_0	{0}	{1}	0
1	b_1	{3}	{4}	0
2	b_2	{7}	{8}	0
3	b_3	{9}	{10}	0
4	b_4	{12}	{13}	0

The isolated state IDs are $\{0, 1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12, 13, 14\}$. The refined graph has zero paths projecting to this candidate.

Round 2. The partition has 15 nonempty cells, and the may relation has 12 edges.

The abstract checker found the inductive invariant $\{s_0, s_3, s_7, s_9, s_{12}\}$. It contains $\{s_0\}$ and is disjoint from $\{s_{13}\}$.

Cell	All may successors
s_0	$\{s_3\}$
s_3	$\{s_7\}$
s_7	$\{s_9\}$
s_9	$\{s_{12}\}$
s_{12}	∅

16.2 F1.2 depth 5, width 2, offset 1, target 0

The lane at stage j is $j \cdot 1 + j(j-1)/2$ modulo 2. The terminal lane is 1 and the specified bad lane is 0.

The concrete carrier has 12 states and 10 edges. Its reachable set has 6 states. The reachable set is disjoint from the bad predicate.

ID	State	Seed	Bd.	d	p	Successors
0	(0, 0)	0	I	0	–	{3}
1	(0, 1)	0	–	∞	–	{2}
2	(1, 0)	1	–	∞	–	{4}
3	(1, 1)	1	–	1	0	{5}
4	(2, 0)	2	–	∞	–	{7}
5	(2, 1)	2	–	2	3	{6}
6	(3, 0)	3	–	3	5	{8}
7	(3, 1)	3	–	∞	–	{9}
8	(4, 0)	4	–	4	6	{11}
9	(4, 1)	4	–	∞	–	{10}
10	(5, 0)	5	B	∞	–	∅
11	(5, 1)	5	–	5	8	∅

Round 1. The partition has 6 nonempty cells, and the may relation has 5 edges.

The selected path is $b_0, b_1, b_2, b_3, b_4, b_5$. The exact number of concrete lifts is 0.

j	a_j	F_j	D_j	$\#\cap$
0	b_0	{0}	{1}	0
1	b_1	{3}	{2}	0
2	b_2	{5}	{4}	0
3	b_3	{6}	{7}	0
4	b_4	{8}	{9}	0
5	b_5	{11}	{10}	0

The isolated state IDs are $\{0, 1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11\}$. The refined graph has zero paths projecting to this candidate.

Round 2. The partition has 12 nonempty cells, and the may relation has 10 edges.

The abstract checker found the inductive invariant $\{s_0, s_3, s_5, s_6, s_8, s_{11}\}$. It contains $\{s_0\}$ and is disjoint from $\{s_{10}\}$.

Cell	All may successors
s_0	$\{s_3\}$
s_3	$\{s_5\}$
s_5	$\{s_6\}$
s_6	$\{s_8\}$
s_8	$\{s_{11}\}$
s_{11}	$\varnothing$

16.3 F1.3 depth 6, width 3, offset 2, target 0

The lane at stage j is $j \cdot 2 + j(j-1)/2$ modulo 3. The terminal lane is 0 and the specified bad lane is 0.

The concrete carrier has 21 states and 18 edges. Its reachable set has 7 states. The bad predicate is reached.

ID	State	Seed	Bd.	d	p	Successors
0	(0, 0)	0	I	0	–	{5}
1	(0, 1)	0	–	∞	–	{3}
2	(0, 2)	0	–	∞	–	{4}
3	(1, 0)	1	–	∞	–	{6}
4	(1, 1)	1	–	∞	–	{7}
5	(1, 2)	1	–	1	0	{8}
6	(2, 0)	2	–	∞	–	{10}
7	(2, 1)	2	–	∞	–	{11}
8	(2, 2)	2	–	2	5	{9}
9	(3, 0)	3	–	3	8	{14}
10	(3, 1)	3	–	∞	–	{12}
11	(3, 2)	3	–	∞	–	{13}
12	(4, 0)	4	–	∞	–	{15}

ID	State	Seed	Bd.	d	p	Successors
13	(4, 1)	4	–	∞	–	{16}
14	(4, 2)	4	–	4	9	{17}
15	(5, 0)	5	–	∞	–	{19}
16	(5, 1)	5	–	∞	–	{20}
17	(5, 2)	5	–	5	14	{18}
18	(6, 0)	6	B	6	17	$\varnothing$
19	(6, 1)	6	–	∞	–	$\varnothing$
20	(6, 2)	6	–	∞	–	$\varnothing$

Round 1. The partition has 7 nonempty cells, and the may relation has 6 edges.

The selected path is $b_0, b_1, b_2, b_3, b_4, b_5, b_6$. The exact number of concrete lifts is 1.

j	a_j	F_j	D_j	$\#\cap$
0	b_0	{0}	{0}	1
1	b_1	{5}	{5}	1
2	b_2	{8}	{8}	1
3	b_3	{9}	{9}	1
4	b_4	{14}	{14}	1
5	b_5	{17}	{17}	1
6	b_6	{18}	{18}	1

A concrete witness is $0, 5, 8, 9, 14, 17, 18$. Each adjacent pair occurs in the concrete successor table.

16.4 F1.4 depth 7, width 3, offset 1, target 0

The lane at stage j is $j \cdot 1 + j(j-1)/2$ modulo 3. The terminal lane is 1 and the specified bad lane is 0.

The concrete carrier has 24 states and 21 edges. Its reachable set has 8 states. The reachable set is disjoint from the bad predicate.

ID	State	Seed	Bd.	d	p	Successors
0	(0, 0)	0	I	0	–	{4}
1	(0, 1)	0	–	∞	–	{5}
2	(0, 2)	0	–	∞	–	{3}
3	(1, 0)	1	–	∞	–	{8}
4	(1, 1)	1	–	1	0	{6}
5	(1, 2)	1	–	∞	–	{7}
6	(2, 0)	2	–	2	4	{9}
7	(2, 1)	2	–	∞	–	{10}
8	(2, 2)	2	–	∞	–	{11}
9	(3, 0)	3	–	3	6	{13}
10	(3, 1)	3	–	∞	–	{14}
11	(3, 2)	3	–	∞	–	{12}
12	(4, 0)	4	–	∞	–	{17}

ID	State	Seed	Bd.	d	p	Successors
13	(4, 1)	4	–	4	9	{15}
14	(4, 2)	4	–	∞	–	{16}
15	(5, 0)	5	–	5	13	{18}
16	(5, 1)	5	–	∞	–	{19}
17	(5, 2)	5	–	∞	–	{20}
18	(6, 0)	6	–	6	15	{22}
19	(6, 1)	6	–	∞	–	{23}
20	(6, 2)	6	–	∞	–	{21}
21	(7, 0)	7	B	∞	–	$\varnothing$
22	(7, 1)	7	–	7	18	$\varnothing$
23	(7, 2)	7	–	∞	–	$\varnothing$

Round 1. The partition has 8 nonempty cells, and the may relation has 7 edges.

The selected path is $b_0, b_1, b_2, b_3, b_4, b_5, b_6, b_7$. The exact number of concrete lifts is 0.

j	a_j	F_j	D_j	$\#\cap$
0	b_0	$\{0\}$	$\{2\}$	0
1	b_1	$\{4\}$	$\{3\}$	0
2	b_2	$\{6\}$	$\{8\}$	0
3	b_3	$\{9\}$	$\{11\}$	0
4	b_4	$\{13\}$	$\{12\}$	0
5	b_5	$\{15\}$	$\{17\}$	0
6	b_6	$\{18\}$	$\{20\}$	0
7	b_7	$\{22\}$	$\{21\}$	0

The isolated state IDs are $\{0, 1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12, 13, 14, 15, 16, 17, 18, 19, 20, 21, 22, 23\}$. The refined graph has zero paths projecting to this candidate.

Round 2. The partition has 24 nonempty cells, and the may relation has 21 edges.

The abstract checker found the inductive invariant $\{s_0, s_4, s_6, s_9, s_{13}, s_{15}, s_{18}, s_{22}\}$. It contains $\{s_0\}$ and is disjoint from $\{s_{21}\}$.

Cell	All may successors
s_0	$\{s_4\}$
s_4	$\{s_6\}$
s_6	$\{s_9\}$
s_9	$\{s_{13}\}$
s_{13}	$\{s_{15}\}$
s_{15}	$\{s_{18}\}$
s_{18}	$\{s_{22}\}$
s_{22}	$\varnothing$

16.5 F1.5 depth 8, width 2, offset 2, target 1

The lane at stage j is $j \cdot 2 + j(j-1)/2$ modulo 2. The terminal lane is 0 and the specified bad lane is 1.

The concrete carrier has 18 states and 16 edges. Its reachable set has 9 states. The reachable set is disjoint from the bad predicate.

ID	State	Seed	Bd.	d	p	Successors
0	(0, 0)	0	I	0	–	{2}
1	(0, 1)	0	–	∞	–	{3}
2	(1, 0)	1	–	1	0	{5}
3	(1, 1)	1	–	∞	–	{4}
4	(2, 0)	2	–	∞	–	{6}
5	(2, 1)	2	–	2	2	{7}
6	(3, 0)	3	–	∞	–	{9}
7	(3, 1)	3	–	3	5	{8}
8	(4, 0)	4	–	4	7	{10}
9	(4, 1)	4	–	∞	–	{11}
10	(5, 0)	5	–	5	8	{13}
11	(5, 1)	5	–	∞	–	{12}
12	(6, 0)	6	–	∞	–	{14}
13	(6, 1)	6	–	6	10	{15}
14	(7, 0)	7	–	∞	–	{17}
15	(7, 1)	7	–	7	13	{16}
16	(8, 0)	8	–	8	15	∅
17	(8, 1)	8	B	∞	–	∅

Round 1. The partition has 9 nonempty cells, and the may relation has 8 edges.

The selected path is $b_0, b_1, b_2, b_3, b_4, b_5, b_6, b_7, b_8$. The exact number of concrete lifts is 0.

j	a_j	F_j	D_j	$\#\cap$
0	b_0	$\{0\}$	$\{1\}$	0
1	b_1	$\{2\}$	$\{3\}$	0
2	b_2	$\{5\}$	$\{4\}$	0
3	b_3	$\{7\}$	$\{6\}$	0
4	b_4	$\{8\}$	$\{9\}$	0
5	b_5	$\{10\}$	$\{11\}$	0
6	b_6	$\{13\}$	$\{12\}$	0
7	b_7	$\{15\}$	$\{14\}$	0
8	b_8	$\{16\}$	$\{17\}$	0

The isolated state IDs are $\{0, 1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12, 13, 14, 15, 16, 17\}$. The refined graph has zero paths projecting to this candidate.

Round 2. The partition has 18 nonempty cells, and the may relation has 16 edges.

The abstract checker found the inductive invariant $\{s_0, s_2, s_5, s_7, s_8, s_{10}, s_{13}, s_{15}, s_{16}\}$. It contains $\{s_0\}$ and is disjoint from $\{s_{17}\}$.

Cell	All may successors
s_0	$\{s_2\}$
s_2	$\{s_5\}$
s_5	$\{s_7\}$
s_7	$\{s_8\}$
s_8	$\{s_{10}\}$
s_{10}	$\{s_{13}\}$
s_{13}	$\{s_{15}\}$
s_{15}	$\{s_{16}\}$
s_{16}	$\varnothing$

16.6 F1.6 depth 9, width 3, offset 0, target 1

The lane at stage j is $j \cdot 0 + j(j-1)/2$ modulo 3. The terminal lane is 0 and the specified bad lane is 1.

The concrete carrier has 30 states and 27 edges. Its reachable set has 10 states. The reachable set is disjoint from the bad predicate.

ID	State	Seed	Bd.	d	p	Successors
0	(0, 0)	0	I	0	–	{3}
1	(0, 1)	0	–	∞	–	{4}
2	(0, 2)	0	–	∞	–	{5}
3	(1, 0)	1	–	1	0	{7}
4	(1, 1)	1	–	∞	–	{8}
5	(1, 2)	1	–	∞	–	{6}
6	(2, 0)	2	–	∞	–	{11}
7	(2, 1)	2	–	2	3	{9}
8	(2, 2)	2	–	∞	–	{10}
9	(3, 0)	3	–	3	7	{12}
10	(3, 1)	3	–	∞	–	{13}
11	(3, 2)	3	–	∞	–	{14}
12	(4, 0)	4	–	4	9	{16}

ID	State	Seed	Bd.	d	p	Successors
13	(4, 1)	4	–	∞	–	{17}
14	(4, 2)	4	–	∞	–	{15}
15	(5, 0)	5	–	∞	–	{20}
16	(5, 1)	5	–	5	12	{18}
17	(5, 2)	5	–	∞	–	{19}
18	(6, 0)	6	–	6	16	{21}
19	(6, 1)	6	–	∞	–	{22}
20	(6, 2)	6	–	∞	–	{23}
21	(7, 0)	7	–	7	18	{25}
22	(7, 1)	7	–	∞	–	{26}
23	(7, 2)	7	–	∞	–	{24}
24	(8, 0)	8	–	∞	–	{29}
25	(8, 1)	8	–	8	21	{27}

ID	State	Seed	Bd.	d	p	Successors
26	(8, 2)	8	–	∞	–	{28}
27	(9, 0)	9	–	9	25	$\varnothing$
28	(9, 1)	9	B	∞	–	$\varnothing$
29	(9, 2)	9	–	∞	–	$\varnothing$

Round 1. The partition has 10 nonempty cells, and the may relation has 9 edges.

The selected path is $b_0, b_1, b_2, b_3, b_4, b_5, b_6, b_7, b_8, b_9$. The exact number of concrete lifts is 0.

j	a_j	F_j	D_j	$\#\cap$
0	b_0	{0}	{1}	0
1	b_1	{3}	{4}	0
2	b_2	{7}	{8}	0
3	b_3	{9}	{10}	0
4	b_4	{12}	{13}	0
5	b_5	{16}	{17}	0
6	b_6	{18}	{19}	0
7	b_7	{21}	{22}	0
8	b_8	{25}	{26}	0
9	b_9	{27}	{28}	0

The isolated state IDs are $\{0, 1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12, 13, 14, 15, 16, 17, 18, 19, 20, 21, 22, 23, 24, 25, 26, 27, 28, 29\}$. The refined graph has zero paths projecting to this candidate.

Round 2. The partition has 30 nonempty cells, and the may relation has 27 edges.

The abstract checker found the inductive invariant $\{s_0, s_3, s_7, s_9, s_{12}, s_{16}, s_{18}, s_{21}, s_{25}, s_{27}\}$. It contains $\{s_0\}$ and is disjoint from $\{s_{28}\}$.

Cell	All may successors
s_0	$\{s_3\}$
s_3	$\{s_7\}$
s_7	$\{s_9\}$
s_9	$\{s_{12}\}$
s_{12}	$\{s_{16}\}$
s_{16}	$\{s_{18}\}$
s_{18}	$\{s_{21}\}$
s_{21}	$\{s_{25}\}$
s_{25}	$\{s_{27}\}$
s_{27}	$\varnothing$

16.7 F1.7 depth 10, width 3, offset 1, target 1

The lane at stage j is $j \cdot 1 + j(j-1)/2$ modulo 3. The terminal lane is 1 and the specified bad lane is 1.

The concrete carrier has 33 states and 30 edges. Its reachable set has 11 states. The bad predicate is reached.

ID	State	Seed	Bd.	d	p	Successors
0	(0, 0)	0	I	0	–	{4}
1	(0, 1)	0	–	∞	–	{5}
2	(0, 2)	0	–	∞	–	{3}
3	(1, 0)	1	–	∞	–	{8}
4	(1, 1)	1	–	1	0	{6}
5	(1, 2)	1	–	∞	–	{7}
6	(2, 0)	2	–	2	4	{9}
7	(2, 1)	2	–	∞	–	{10}
8	(2, 2)	2	–	∞	–	{11}
9	(3, 0)	3	–	3	6	{13}
10	(3, 1)	3	–	∞	–	{14}
11	(3, 2)	3	–	∞	–	{12}
12	(4, 0)	4	–	∞	–	{17}

ID	State	Seed	Bd.	d	p	Successors
13	(4, 1)	4	–	4	9	{15}
14	(4, 2)	4	–	∞	–	{16}
15	(5, 0)	5	–	5	13	{18}
16	(5, 1)	5	–	∞	–	{19}
17	(5, 2)	5	–	∞	–	{20}
18	(6, 0)	6	–	6	15	{22}
19	(6, 1)	6	–	∞	–	{23}
20	(6, 2)	6	–	∞	–	{21}
21	(7, 0)	7	–	∞	–	{26}
22	(7, 1)	7	–	7	18	{24}
23	(7, 2)	7	–	∞	–	{25}
24	(8, 0)	8	–	8	22	{27}
25	(8, 1)	8	–	∞	–	{28}

ID	State	Seed	Bd.	d	p	Successors
26	(8, 2)	8	–	∞	–	{29}
27	(9, 0)	9	–	9	24	{31}
28	(9, 1)	9	–	∞	–	{32}
29	(9, 2)	9	–	∞	–	{30}
30	(10, 0)	10	–	∞	–	$\varnothing$
31	(10, 1)	10	B	10	27	$\varnothing$
32	(10, 2)	10	–	∞	–	$\varnothing$

Round 1. The partition has 11 nonempty cells, and the may relation has 10 edges.

The selected path is $b_0, b_1, b_2, b_3, b_4, b_5, b_6, b_7, b_8, b_9, b_{10}$. The exact number of concrete lifts is 1.

j	a_j	F_j	D_j	$\#\cap$
0	b_0	{0}	{0}	1
1	b_1	{4}	{4}	1
2	b_2	{6}	{6}	1
3	b_3	{9}	{9}	1
4	b_4	{13}	{13}	1
5	b_5	{15}	{15}	1
6	b_6	{18}	{18}	1
7	b_7	{22}	{22}	1
8	b_8	{24}	{24}	1
9	b_9	{27}	{27}	1
10	b_{10}	{31}	{31}	1

A concrete witness is $0, 4, 6, 9, 13, 15, 18, 22, 24, 27, 31$. Each adjacent pair occurs in the concrete successor table.

16.8 F1.8 depth 11, width 2, offset 0, target 0

The lane at stage j is $j \cdot 0 + j(j-1)/2$ modulo 2. The terminal lane is 1 and the specified bad lane is 0.

The concrete carrier has 24 states and 22 edges. Its reachable set has 12 states. The reachable set is disjoint from the bad predicate.

ID	State	Seed	Bd.	d	p	Successors
0	(0, 0)	0	I	0	–	{2}
1	(0, 1)	0	–	∞	–	{3}
2	(1, 0)	1	–	1	0	{5}
3	(1, 1)	1	–	∞	–	{4}
4	(2, 0)	2	–	∞	–	{6}
5	(2, 1)	2	–	2	2	{7}
6	(3, 0)	3	–	∞	–	{9}
7	(3, 1)	3	–	3	5	{8}
8	(4, 0)	4	–	4	7	{10}
9	(4, 1)	4	–	∞	–	{11}
10	(5, 0)	5	–	5	8	{13}
11	(5, 1)	5	–	∞	–	{12}
12	(6, 0)	6	–	∞	–	{14}
13	(6, 1)	6	–	6	10	{15}
14	(7, 0)	7	–	∞	–	{17}
15	(7, 1)	7	–	7	13	{16}
16	(8, 0)	8	–	8	15	{18}
17	(8, 1)	8	–	∞	–	{19}
18	(9, 0)	9	–	9	16	{21}
19	(9, 1)	9	–	∞	–	{20}
20	(10, 0)	10	–	∞	–	{22}
21	(10, 1)	10	–	10	18	{23}
22	(11, 0)	11	B	∞	–	$\varnothing$
23	(11, 1)	11	–	11	21	$\varnothing$

Round 1. The partition has 12 nonempty cells, and the may relation has 11 edges.

The selected path is $b_0, b_1, b_2, b_3, b_4, b_5, b_6, b_7, b_8, b_9, b_{10}, b_{11}$. The exact number of concrete lifts is 0.

j	a_j	F_j	D_j	$\#\cap$
0	b_0	{0}	{1}	0
1	b_1	{2}	{3}	0
2	b_2	{5}	{4}	0
3	b_3	{7}	{6}	0
4	b_4	{8}	{9}	0
5	b_5	{10}	{11}	0
6	b_6	{13}	{12}	0
7	b_7	{15}	{14}	0
8	b_8	{16}	{17}	0
9	b_9	{18}	{19}	0
10	b_{10}	{21}	{20}	0
11	b_{11}	{23}	{22}	0

The isolated state IDs are $\{0, 1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12, 13, 14, 15, 16, 17, 18, 19, 20, 21, 22, 23\}$. The refined graph has zero paths projecting to this candidate.

Round 2. The partition has 24 nonempty cells, and the may relation has 22 edges.

The abstract checker found the inductive invariant $\{s_0, s_2, s_5, s_7, s_8, s_{10}, s_{13}, s_{15}, s_{16}, s_{18}, s_{21}, s_{23}\}$. It contains $\{s_0\}$ and is disjoint from $\{s_{22}\}$.

Cell	All may successors
s_0	$\{s_2\}$
s_2	$\{s_5\}$
s_5	$\{s_7\}$
s_7	$\{s_8\}$
s_8	$\{s_{10}\}$
s_{10}	$\{s_{13}\}$
s_{13}	$\{s_{15}\}$
s_{15}	$\{s_{16}\}$
s_{16}	$\{s_{18}\}$
s_{18}	$\{s_{21}\}$
s_{21}	$\{s_{23}\}$
s_{23}	$\varnothing$

16.9 F1.9 depth 12, width 3, offset 1, target 1

The lane at stage j is $j \cdot 1 + j(j-1)/2$ modulo 3. The terminal lane is 0 and the specified bad lane is 1.

The concrete carrier has 39 states and 36 edges. Its reachable set has 13 states. The reachable set is disjoint from the bad predicate.

ID	State	Seed	Bd.	d	p	Successors
0	(0, 0)	0	I	0	–	{4}
1	(0, 1)	0	–	∞	–	{5}
2	(0, 2)	0	–	∞	–	{3}
3	(1, 0)	1	–	∞	–	{8}
4	(1, 1)	1	–	1	0	{6}
5	(1, 2)	1	–	∞	–	{7}
6	(2, 0)	2	–	2	4	{9}
7	(2, 1)	2	–	∞	–	{10}
8	(2, 2)	2	–	∞	–	{11}
9	(3, 0)	3	–	3	6	{13}
10	(3, 1)	3	–	∞	–	{14}
11	(3, 2)	3	–	∞	–	{12}
12	(4, 0)	4	–	∞	–	{17}

ID	State	Seed	Bd.	d	p	Successors
13	(4, 1)	4	–	4	9	{15}
14	(4, 2)	4	–	∞	–	{16}
15	(5, 0)	5	–	5	13	{18}
16	(5, 1)	5	–	∞	–	{19}
17	(5, 2)	5	–	∞	–	{20}
18	(6, 0)	6	–	6	15	{22}
19	(6, 1)	6	–	∞	–	{23}
20	(6, 2)	6	–	∞	–	{21}
21	(7, 0)	7	–	∞	–	{26}
22	(7, 1)	7	–	7	18	{24}
23	(7, 2)	7	–	∞	–	{25}
24	(8, 0)	8	–	8	22	{27}
25	(8, 1)	8	–	∞	–	{28}

ID	State	Seed	Bd.	d	p	Successors
26	(8, 2)	8	–	∞	–	{29}
27	(9, 0)	9	–	9	24	{31}
28	(9, 1)	9	–	∞	–	{32}
29	(9, 2)	9	–	∞	–	{30}
30	(10, 0)	10	–	∞	–	{35}
31	(10, 1)	10	–	10	27	{33}
32	(10, 2)	10	–	∞	–	{34}
33	(11, 0)	11	–	11	31	{36}
34	(11, 1)	11	–	∞	–	{37}
35	(11, 2)	11	–	∞	–	{38}
36	(12, 0)	12	–	12	33	∅
37	(12, 1)	12	B	∞	–	∅
38	(12, 2)	12	–	∞	–	∅

Round 1. The partition has 13 nonempty cells, and the may relation has 12 edges.

The selected path is $b_0, b_1, b_2, b_3, b_4, b_5, b_6, b_7, b_8, b_9, b_{10}, b_{11}, b_{12}$. The exact number of concrete lifts is 0.

j	a_j	F_j	D_j	$\#\cap$
0	b_0	{0}	{1}	0
1	b_1	{4}	{5}	0
2	b_2	{6}	{7}	0
3	b_3	{9}	{10}	0
4	b_4	{13}	{14}	0
5	b_5	{15}	{16}	0
6	b_6	{18}	{19}	0
7	b_7	{22}	{23}	0
8	b_8	{24}	{25}	0
9	b_9	{27}	{28}	0
10	b_{10}	{31}	{32}	0
11	b_{11}	{33}	{34}	0
12	b_{12}	{36}	{37}	0

The isolated state IDs are $\{0, 1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12, 13, 14, 15, 16, 17, 18, 19, 20, 21, 22, 23, 24, 25, 26, 27, 28, 29, 30, 31, 32, 33, 34, 35, 36, 37, 38\}$. The refined graph has zero paths projecting to this candidate.

Round 2. The partition has 39 nonempty cells, and the may relation has 36 edges.

The abstract checker found the inductive invariant $\{s_0, s_4, s_6, s_9, s_{13}, s_{15}, s_{18}, s_{22}, s_{24}, s_{27}, s_{31}, s_{33}, s_{36}\}$. It contains $\{s_0\}$ and is disjoint from $\{s_{37}\}$.

Cell	All may successors
s_0	$\{s_4\}$
s_4	$\{s_6\}$
s_6	$\{s_9\}$
s_9	$\{s_{13}\}$
s_{13}	$\{s_{15}\}$
s_{15}	$\{s_{18}\}$
s_{18}	$\{s_{22}\}$
s_{22}	$\{s_{24}\}$
s_{24}	$\{s_{27}\}$
s_{27}	$\{s_{31}\}$
s_{31}	$\{s_{33}\}$
s_{33}	$\{s_{36}\}$
s_{36}	∅

17 Additive counters

Rows of the concrete graph are generated by addition in a finite cyclic group. The observation x mod k uses the chosen integer representatives and need not be a group homomorphism.

Tables enumerate concrete states by integer ID. The d column gives distance from an initial state, and p gives a predecessor attaining that distance. Columns I and B are encoded by the boundary field. Abstract keys b_i are seed blocks and s_x are isolated states.

17.1 F2.1 order 12, jumps (4, 8), target 5, observation modulus 3

The additive submonoid generated by the jumps in $\mathbb{Z}/12$ is a subgroup, since each jump has finite order and hence

has its inverse in the submonoid. The subgroup consists of multiples of 4.

The concrete carrier has 12 states and 24 edges. Its reachable set has 3 states. The reachable set is disjoint from the bad predicate.

ID	State	Seed	Bd.	d	p	Successors
0	0	0	I	0	–	$\{4, 8\}$
1	1	1	–	∞	–	$\{5, 9\}$
2	2	2	–	∞	–	$\{6, 10\}$
3	3	0	–	∞	–	$\{7, 11\}$
4	4	1	–	1	0	$\{0, 8\}$
5	5	2	B	∞	–	$\{1, 9\}$
6	6	0	–	∞	–	$\{2, 10\}$
7	7	1	–	∞	–	$\{3, 11\}$
8	8	2	–	1	0	$\{0, 4\}$
9	9	0	–	∞	–	$\{1, 5\}$
10	10	1	–	∞	–	$\{2, 6\}$
11	11	2	–	∞	–	$\{3, 7\}$

Round 1. The partition has 3 nonempty cells, and the may relation has 6 edges.

The selected path is b_0, b_1, b_2. The exact number of concrete lifts is 0.

j	a_j	F_j	D_j	$\#\cap$
0	b_0	$\{0\}$	$\{9\}$	0
1	b_1	$\{4\}$	$\{1\}$	0
2	b_2	$\{8\}$	$\{5\}$	0

The isolated state IDs are $\{0, 1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11\}$. The refined graph has zero paths projecting to this candidate.

Round 2. The partition has 12 nonempty cells, and the may relation has 24 edges.

The abstract checker found the inductive invariant $\{s_0, s_4, s_8\}$. It contains $\{s_0\}$ and is disjoint from $\{s_5\}$.

Cell	All may successors
s_0	$\{s_4, s_8\}$
s_4	$\{s_0, s_8\}$
s_8	$\{s_0, s_4\}$

17.2 F2.2 order 18, jumps (6, 9), target 12, observation modulus 4

The additive submonoid generated by the jumps in $\mathbb{Z}/18$ is a subgroup, since each jump has finite order and hence has its inverse in the submonoid. The subgroup consists of multiples of 3.

The concrete carrier has 18 states and 36 edges. Its reachable set has 6 states. The bad predicate is reached.

ID	State	Seed	Bd.	d	p	Successors
0	0	0	I	0	–	$\{6, 9\}$
1	1	1	–	∞	–	$\{7, 10\}$
2	2	2	–	∞	–	$\{8, 11\}$
3	3	3	–	3	12	$\{9, 12\}$
4	4	0	–	∞	–	$\{10, 13\}$
5	5	1	–	∞	–	$\{11, 14\}$
6	6	2	–	1	0	$\{12, 15\}$
7	7	3	–	∞	–	$\{13, 16\}$
8	8	0	–	∞	–	$\{14, 17\}$
9	9	1	–	1	0	$\{0, 15\}$
10	10	2	–	∞	–	$\{1, 16\}$
11	11	3	–	∞	–	$\{2, 17\}$
12	12	0	B	2	6	$\{0, 3\}$
13	13	1	–	∞	–	$\{1, 4\}$
14	14	2	–	∞	–	$\{2, 5\}$
15	15	3	–	2	6	$\{3, 6\}$
16	16	0	–	∞	–	$\{4, 7\}$
17	17	1	–	∞	–	$\{5, 8\}$

Round 1. The partition has 4 nonempty cells, and the may relation has 16 edges.

The selected path is b_0. The exact number of concrete lifts is 0.

j	a_j	F_j	D_j	$\#\cap$
0	b_0	$\{0\}$	$\{12\}$	0

The isolated state IDs are $\{0, 4, 8, 12, 16\}$. The refined graph has zero paths projecting to this candidate.

Round 2. The partition has 8 nonempty cells, and the may relation has 27 edges.

The selected path is $s_0, b_1, s_8, b_2, s_{12}$. The exact number of concrete lifts is 0.

j	a_j	F_j	D_j	$\#\cap$
0	s_0	$\{0\}$	$\varnothing$	0
1	b_1	$\{9\}$	$\varnothing$	0
2	s_8	$\varnothing$	$\varnothing$	0
3	b_2	$\varnothing$	$\{6\}$	0
4	s_{12}	$\varnothing$	$\{12\}$	0

The isolated state IDs are $\{1, 2, 5, 6, 9, 10, 13, 14, 17\}$. The refined graph has zero paths projecting to this candidate.

Round 3. The partition has 15 nonempty cells, and the may relation has 36 edges.

The selected path is s_0, s_6, s_{12}. The exact number of concrete lifts is 1.

j	a_j	F_j	D_j	$\#\cap$
0	s_0	$\{0\}$	$\{0\}$	1
1	s_6	$\{6\}$	$\{6\}$	1
2	s_{12}	$\{12\}$	$\{12\}$	1

A concrete witness is $0, 6, 12$. Each adjacent pair occurs in the concrete successor table.

17.3 F2.3 order 20, jumps (4, 10), target 7, observation modulus 3

The additive submonoid generated by the jumps in $\mathbb{Z}/20$ is a subgroup, since each jump has finite order and hence has its inverse in the submonoid. The subgroup consists of multiples of 2.

The concrete carrier has 20 states and 40 edges. Its reachable set has 10 states. The reachable set is disjoint from the bad predicate.

ID	State	Seed	Bd.	d	p	Successors
0	0	0	I	0	–	$\{4, 10\}$
1	1	1	–	∞	–	$\{5, 11\}$
2	2	2	–	4	12	$\{6, 12\}$
3	3	0	–	∞	–	$\{7, 13\}$
4	4	1	–	1	0	$\{8, 14\}$
5	5	2	–	∞	–	$\{9, 15\}$
6	6	0	–	5	2	$\{10, 16\}$
7	7	1	B	∞	–	$\{11, 17\}$
8	8	2	–	2	4	$\{12, 18\}$
9	9	0	–	∞	–	$\{13, 19\}$
10	10	1	–	1	0	$\{0, 14\}$
11	11	2	–	∞	–	$\{1, 15\}$
12	12	0	–	3	8	$\{2, 16\}$

ID	State	Seed	Bd.	d	p	Successors
13	13	1	–	∞	–	$\{3, 17\}$
14	14	2	–	2	4	$\{4, 18\}$
15	15	0	–	∞	–	$\{5, 19\}$
16	16	1	–	4	12	$\{0, 6\}$
17	17	2	–	∞	–	$\{1, 7\}$
18	18	0	–	3	8	$\{2, 8\}$
19	19	1	–	∞	–	$\{3, 9\}$

Round 1. The partition has 3 nonempty cells, and the may relation has 6 edges.

The selected path is b_0, b_1. The exact number of concrete lifts is 0.

j	a_j	F_j	D_j	$\#\cap$
0	b_0	$\{0\}$	$\{3\}$	0
1	b_1	$\{4, 10\}$	$\{7\}$	0

The isolated state IDs are $\{0, 1, 3, 4, 6, 7, 9, 10, 12, 13, 15, 16, 18, 19\}$. The refined graph has zero paths projecting to this candidate.

Round 2. The partition has 15 nonempty cells, and the may relation has 32 edges.

The selected path is s_0, s_{10}, b_2, s_7. The exact number of concrete lifts is 0.

j	a_j	F_j	D_j	$\#\cap$
0	s_0	$\{0\}$	$\varnothing$	0
1	s_{10}	$\{10\}$	$\varnothing$	0
2	b_2	$\{14\}$	$\{17\}$	0
3	s_7	$\varnothing$	$\{7\}$	0

The isolated state IDs are $\{2, 5, 8, 11, 14, 17\}$. The refined graph has zero paths projecting to this candidate.

Round 3. The partition has 20 nonempty cells, and the may relation has 40 edges.

The abstract checker found the inductive invariant $\{s_0, s_2, s_4, s_6, s_8, s_{10}, s_{12}, s_{14}, s_{16}, s_{18}\}$. It contains $\{s_0\}$ and is disjoint from $\{s_7\}$.

Cell	All may successors
s_0	$\{s_4, s_{10}\}$
s_2	$\{s_6, s_{12}\}$
s_4	$\{s_8, s_{14}\}$
s_6	$\{s_{10}, s_{16}\}$
s_8	$\{s_{12}, s_{18}\}$
s_{10}	$\{s_0, s_{14}\}$
s_{12}	$\{s_2, s_{16}\}$
s_{14}	$\{s_4, s_{18}\}$
s_{16}	$\{s_0, s_6\}$
s_{18}	$\{s_2, s_8\}$

17.4 F2.4 order 24, jumps (6, 9), target 19, observation modulus 5

The additive submonoid generated by the jumps in $\mathbb{Z}/24$ is a subgroup, since each jump has finite order and hence has its inverse in the submonoid. The subgroup consists of multiples of 3.

The concrete carrier has 24 states and 48 edges. Its reachable set has 8 states. The reachable set is disjoint from the bad predicate.

ID	State	Seed	Bd.	d	p	Successors
0	0	0	I	0	–	$\{6, 9\}$
1	1	1	–	∞	–	$\{7, 10\}$
2	2	2	–	∞	–	$\{8, 11\}$
3	3	3	–	3	18	$\{9, 12\}$
4	4	4	–	∞	–	$\{10, 13\}$
5	5	0	–	∞	–	$\{11, 14\}$
6	6	1	–	1	0	$\{12, 15\}$
7	7	2	–	∞	–	$\{13, 16\}$
8	8	3	–	∞	–	$\{14, 17\}$
9	9	4	–	1	0	$\{15, 18\}$
10	10	0	–	∞	–	$\{16, 19\}$
11	11	1	–	∞	–	$\{17, 20\}$
12	12	2	–	2	6	$\{18, 21\}$

ID	State	Seed	Bd.	d	p	Successors
13	13	3	–	∞	–	$\{19, 22\}$
14	14	4	–	∞	–	$\{20, 23\}$
15	15	0	–	2	6	$\{0, 21\}$
16	16	1	–	∞	–	$\{1, 22\}$
17	17	2	–	∞	–	$\{2, 23\}$
18	18	3	–	2	9	$\{0, 3\}$
19	19	4	B	∞	–	$\{1, 4\}$
20	20	0	–	∞	–	$\{2, 5\}$
21	21	1	–	3	12	$\{3, 6\}$
22	22	2	–	∞	–	$\{4, 7\}$
23	23	3	–	∞	–	$\{5, 8\}$

Round 1. The partition has 5 nonempty cells, and the may relation has 20 edges.

The selected path is b_0, b_1, b_2, b_3, b_4. The exact number of concrete lifts is 0.

j	a_j	F_j	D_j	$\#\cap$
0	b_0	$\{0\}$	$\varnothing$	0
1	b_1	$\{6\}$	$\{1\}$	0
2	b_2	$\{12\}$	$\{7\}$	0
3	b_3	$\{18\}$	$\{13\}$	0
4	b_4	$\varnothing$	$\{19\}$	0

The isolated state IDs are $\{0, 1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12, 13, 14, 15, 16, 17, 18, 19, 20, 21, 22, 23\}$. The refined graph has zero paths projecting to this candidate.

Round 2. The partition has 24 nonempty cells, and the may relation has 48 edges.

The abstract checker found the inductive invariant $\{s_0, s_3, s_6, s_9, s_{12}, s_{15}, s_{18}, s_{21}\}$. It contains $\{s_0\}$ and is disjoint from $\{s_{19}\}$.

Cell	All may successors
s_0	$\{s_6, s_9\}$
s_3	$\{s_9, s_{12}\}$
s_6	$\{s_{12}, s_{15}\}$
s_9	$\{s_{15}, s_{18}\}$
s_{12}	$\{s_{18}, s_{21}\}$
s_{15}	$\{s_0, s_{21}\}$
s_{18}	$\{s_0, s_3\}$
s_{21}	$\{s_3, s_6\}$

17.5 F2.5 order 21, jumps (6, 12), target 15, observation modulus 4

The additive submonoid generated by the jumps in $\mathbb{Z}/21$ is a subgroup, since each jump has finite order and hence has its inverse in the submonoid. The subgroup consists of multiples of 3.

The concrete carrier has 21 states and 42 edges. Its reachable set has 7 states. The bad predicate is reached.

ID	State	Seed	Bd.	d	p	Successors
0	0	0	I	0	–	$\{6, 12\}$
1	1	1	–	∞	–	$\{7, 13\}$
2	2	2	–	∞	–	$\{8, 14\}$
3	3	3	–	2	12	$\{9, 15\}$
4	4	0	–	∞	–	$\{10, 16\}$
5	5	1	–	∞	–	$\{11, 17\}$
6	6	2	–	1	0	$\{12, 18\}$
7	7	3	–	∞	–	$\{13, 19\}$
8	8	0	–	∞	–	$\{14, 20\}$
9	9	1	–	3	18	$\{0, 15\}$
10	10	2	–	∞	–	$\{1, 16\}$
11	11	3	–	∞	–	$\{2, 17\}$
12	12	0	–	1	0	$\{3, 18\}$

ID	State	Seed	Bd.	d	p	Successors
13	13	1	–	∞	–	$\{4, 19\}$
14	14	2	–	∞	–	$\{5, 20\}$
15	15	3	B	3	3	$\{0, 6\}$
16	16	0	–	∞	–	$\{1, 7\}$
17	17	1	–	∞	–	$\{2, 8\}$
18	18	2	–	2	6	$\{3, 9\}$
19	19	3	–	∞	–	$\{4, 10\}$
20	20	0	–	∞	–	$\{5, 11\}$

Round 1. The partition has 4 nonempty cells, and the may relation has 16 edges.

The selected path is b_0, b_1, b_2, b_3. The exact number of concrete lifts is 0.

j	a_j	F_j	D_j	$\#\cap$
0	b_0	$\{0\}$	$\varnothing$	0
1	b_1	$\varnothing$	$\varnothing$	0
2	b_2	$\varnothing$	$\varnothing$	0
3	b_3	$\varnothing$	$\{15\}$	0

The isolated state IDs are $\{0, 1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12, 13, 14, 15, 16, 17, 18, 19, 20\}$. The refined graph has zero paths projecting to this candidate.

Round 2. The partition has 21 nonempty cells, and the may relation has 42 edges.

The selected path is $s_0, s_6, s_{12}, s_3, s_9, s_{15}$. The exact number of concrete lifts is 1.

j	a_j	F_j	D_j	$\#\cap$
0	s_0	$\{0\}$	$\{0\}$	1
1	s_6	$\{6\}$	$\{6\}$	1
2	s_{12}	$\{12\}$	$\{12\}$	1
3	s_3	$\{3\}$	$\{3\}$	1
4	s_9	$\{9\}$	$\{9\}$	1
5	s_{15}	$\{15\}$	$\{15\}$	1

A concrete witness is $0, 6, 12, 3, 9, 15$. Each adjacent pair occurs in the concrete successor table.

17.6 F2.6 order 25, jumps (5, 10), target 17, observation modulus 4

The additive submonoid generated by the jumps in $\mathbb{Z}/25$ is a subgroup, since each jump has finite order and hence has its inverse in the submonoid. The subgroup consists of multiples of 5.

The concrete carrier has 25 states and 50 edges. Its reachable set has 5 states. The reachable set is disjoint from the bad predicate.

ID	State	Seed	Bd.	d	p	Successors
0	0	0	I	0	–	$\{5, 10\}$
1	1	1	–	∞	–	$\{6, 11\}$
2	2	2	–	∞	–	$\{7, 12\}$
3	3	3	–	∞	–	$\{8, 13\}$
4	4	0	–	∞	–	$\{9, 14\}$
5	5	1	–	1	0	$\{10, 15\}$
6	6	2	–	∞	–	$\{11, 16\}$
7	7	3	–	∞	–	$\{12, 17\}$
8	8	0	–	∞	–	$\{13, 18\}$
9	9	1	–	∞	–	$\{14, 19\}$
10	10	2	–	1	0	$\{15, 20\}$
11	11	3	–	∞	–	$\{16, 21\}$
12	12	0	–	∞	–	$\{17, 22\}$

ID	State	Seed	Bd.	d	p	Successors
13	13	1	–	∞	–	$\{18, 23\}$
14	14	2	–	∞	–	$\{19, 24\}$
15	15	3	–	2	5	$\{0, 20\}$
16	16	0	–	∞	–	$\{1, 21\}$
17	17	1	B	∞	–	$\{2, 22\}$
18	18	2	–	∞	–	$\{3, 23\}$
19	19	3	–	∞	–	$\{4, 24\}$
20	20	0	–	2	10	$\{0, 5\}$
21	21	1	–	∞	–	$\{1, 6\}$
22	22	2	–	∞	–	$\{2, 7\}$
23	23	3	–	∞	–	$\{3, 8\}$
24	24	0	–	∞	–	$\{4, 9\}$

Round 1. The partition has 4 nonempty cells, and the may relation has 12 edges.

The selected path is b_0, b_1. The exact number of concrete lifts is 0.

j	a_j	F_j	D_j	$\#\cap$
0	b_0	$\{0\}$	$\{12\}$	0
1	b_1	$\{5\}$	$\{17\}$	0

The isolated state IDs are $\{0, 1, 4, 5, 8, 9, 12, 13, 16, 17, 20, 21, 24\}$. The refined graph has zero paths projecting to this candidate.

Round 2. The partition has 15 nonempty cells, and the may relation has 42 edges.

The selected path is $s_0, s_5, b_2, s_{12}, s_{17}$. The exact number of concrete lifts is 0.

j	a_j	F_j	D_j	$\#\cap$
0	s_0	$\{0\}$	$\varnothing$	0
1	s_5	$\{5\}$	$\varnothing$	0
2	b_2	$\{10\}$	$\{2\}$	0
3	s_{12}	$\varnothing$	$\{12\}$	0
4	s_{17}	$\varnothing$	$\{17\}$	0

The isolated state IDs are $\{2, 6, 10, 14, 18, 22\}$. The refined graph has zero paths projecting to this candidate.

Round 3. The partition has 20 nonempty cells, and the may relation has 48 edges.

The selected path is $s_0, s_5, s_{10}, b_3, s_{12}, s_{17}$. The exact number of concrete lifts is 0.

j	a_j	F_j	D_j	$\#\cap$
0	s_0	$\{0\}$	$\varnothing$	0
1	s_5	$\{5\}$	$\varnothing$	0
2	s_{10}	$\{10\}$	$\varnothing$	0
3	b_3	$\{15\}$	$\{7\}$	0
4	s_{12}	$\varnothing$	$\{12\}$	0
5	s_{17}	$\varnothing$	$\{17\}$	0

The isolated state IDs are $\{3, 7, 11, 15, 19, 23\}$. The refined graph has zero paths projecting to this candidate.

Round 4. The partition has 25 nonempty cells, and the may relation has 50 edges.

The abstract checker found the inductive invariant $\{s_0, s_5, s_{10}, s_{15}, s_{20}\}$. It contains $\{s_0\}$ and is disjoint from $\{s_{17}\}$.

Cell	All may successors
s_0	$\{s_5, s_{10}\}$
s_5	$\{s_{10}, s_{15}\}$
s_{10}	$\{s_{15}, s_{20}\}$
s_{15}	$\{s_0, s_{20}\}$
s_{20}	$\{s_0, s_5\}$

17.7 F2.7 order 28, jumps (8, 14), target 23, observation modulus 3

The additive submonoid generated by the jumps in $\mathbb{Z}/28$ is a subgroup, since each jump has finite order and hence has its inverse in the submonoid. The subgroup consists of multiples of 2.

The concrete carrier has 28 states and 56 edges. Its reachable set has 14 states. The reachable set is disjoint from the bad predicate.

ID	State	Seed	Bd.	d	p	Successors
0	0	0	I	0	–	$\{8, 14\}$
1	1	1	–	∞	–	$\{9, 15\}$
2	2	2	–	3	16	$\{10, 16\}$
3	3	0	–	∞	–	$\{11, 17\}$
4	4	1	–	4	24	$\{12, 18\}$
5	5	2	–	∞	–	$\{13, 19\}$
6	6	0	–	7	26	$\{14, 20\}$
7	7	1	–	∞	–	$\{15, 21\}$
8	8	2	–	1	0	$\{16, 22\}$
9	9	0	–	∞	–	$\{17, 23\}$
10	10	1	–	4	2	$\{18, 24\}$
11	11	2	–	∞	–	$\{19, 25\}$
12	12	0	–	5	4	$\{20, 26\}$

ID	State	Seed	Bd.	d	p	Successors
13	13	1	–	∞	–	$\{21, 27\}$
14	14	2	–	1	0	$\{0, 22\}$
15	15	0	–	∞	–	$\{1, 23\}$
16	16	1	–	2	8	$\{2, 24\}$
17	17	2	–	∞	–	$\{3, 25\}$
18	18	0	–	5	10	$\{4, 26\}$
19	19	1	–	∞	–	$\{5, 27\}$
20	20	2	–	6	12	$\{0, 6\}$
21	21	0	–	∞	–	$\{1, 7\}$
22	22	1	–	2	8	$\{2, 8\}$
23	23	2	B	∞	–	$\{3, 9\}$
24	24	0	–	3	16	$\{4, 10\}$
25	25	1	–	∞	–	$\{5, 11\}$

ID	State	Seed	Bd.	d	p	Successors
26	26	2	–	6	18	$\{6, 12\}$
27	27	0	–	∞	–	$\{7, 13\}$

Round 1. The partition has 3 nonempty cells, and the may relation has 6 edges.

The selected path is b_0, b_1, b_2. The exact number of concrete lifts is 0.

j	a_j	F_j	D_j	$\#\cap$
0	b_0	$\{0\}$	$\varnothing$	0
1	b_1	$\varnothing$	$\varnothing$	0
2	b_2	$\varnothing$	$\{23\}$	0

The isolated state IDs are $\{0, 1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12, 13, 14, 15, 16, 17, 18, 19, 20, 21, 22, 23, 24, 25, 26, 27\}$. The refined graph has zero paths projecting to this candidate.

Round 2. The partition has 28 nonempty cells, and the may relation has 56 edges.

The abstract checker found the inductive invariant $\{s_0, s_2, s_4, s_6, s_8, s_{10}, s_{12}, s_{14}, s_{16}, s_{18}, s_{20}, s_{22}, s_{24}, s_{26}\}$. It contains $\{s_0\}$ and is disjoint from $\{s_{23}\}$.

Cell	All may successors
s_0	$\{s_8, s_{14}\}$
s_2	$\{s_{10}, s_{16}\}$
s_4	$\{s_{12}, s_{18}\}$
s_6	$\{s_{14}, s_{20}\}$
s_8	$\{s_{16}, s_{22}\}$
s_{10}	$\{s_{18}, s_{24}\}$
s_{12}	$\{s_{20}, s_{26}\}$
s_{14}	$\{s_0, s_{22}\}$
s_{16}	$\{s_2, s_{24}\}$
s_{18}	$\{s_4, s_{26}\}$
s_{20}	$\{s_0, s_6\}$
s_{22}	$\{s_2, s_8\}$
s_{24}	$\{s_4, s_{10}\}$

Cell	All may successors
s_{26}	$\{s_6, s_{12}\}$

17.8 F2.8 order 30, jumps (6, 15), target 26, observation modulus 4

The additive submonoid generated by the jumps in $\mathbb{Z}/30$ is a subgroup, since each jump has finite order and hence has its inverse in the submonoid. The subgroup consists of multiples of 3.

The concrete carrier has 30 states and 60 edges. Its reachable set has 10 states. The reachable set is disjoint from the bad predicate.

ID	State	Seed	Bd.	d	p	Successors
0	0	0	I	0	–	$\{6, 15\}$
1	1	1	–	∞	–	$\{7, 16\}$
2	2	2	–	∞	–	$\{8, 17\}$
3	3	3	–	4	18	$\{9, 18\}$
4	4	0	–	∞	–	$\{10, 19\}$
5	5	1	–	∞	–	$\{11, 20\}$
6	6	2	–	1	0	$\{12, 21\}$
7	7	3	–	∞	–	$\{13, 22\}$
8	8	0	–	∞	–	$\{14, 23\}$
9	9	1	–	5	3	$\{15, 24\}$
10	10	2	–	∞	–	$\{16, 25\}$
11	11	3	–	∞	–	$\{17, 26\}$
12	12	0	–	2	6	$\{18, 27\}$

ID	State	Seed	Bd.	d	p	Successors
13	13	1	–	∞	–	$\{19, 28\}$
14	14	2	–	∞	–	$\{20, 29\}$
15	15	3	–	1	0	$\{0, 21\}$
16	16	0	–	∞	–	$\{1, 22\}$
17	17	1	–	∞	–	$\{2, 23\}$
18	18	2	–	3	12	$\{3, 24\}$
19	19	3	–	∞	–	$\{4, 25\}$
20	20	0	–	∞	–	$\{5, 26\}$
21	21	1	–	2	6	$\{6, 27\}$
22	22	2	–	∞	–	$\{7, 28\}$
23	23	3	–	∞	–	$\{8, 29\}$
24	24	0	–	4	18	$\{0, 9\}$
25	25	1	–	∞	–	$\{1, 10\}$

ID	State	Seed	Bd.	d	p	Successors
26	26	2	B	∞	–	$\{2, 11\}$
27	27	3	–	3	12	$\{3, 12\}$
28	28	0	–	∞	–	$\{4, 13\}$
29	29	1	–	∞	–	$\{5, 14\}$

Round 1. The partition has 4 nonempty cells, and the may relation has 16 edges.

The selected path is b_0, b_1, b_2. The exact number of concrete lifts is 0.

j	a_j	F_j	D_j	$\#\cap$
0	b_0	$\{0\}$	$\varnothing$	0
1	b_1	$\varnothing$	$\varnothing$	0
2	b_2	$\varnothing$	$\{26\}$	0

The isolated state IDs are $\{0, 1, 2, 4, 5, 6, 8, 9, 10, 12, 13, 14, 16, 17, 18, 20, 21, 22, 24, 25, 26, 28, 29\}$. The refined graph has zero paths projecting to this candidate.

Round 2. The partition has 24 nonempty cells, and the may relation has 60 edges.

The selected path is $s_0, s_6, s_{12}, s_{18}, s_{24}, s_9, b_3, s_8, s_{14}, s_{20}, s_{26}$. The exact number of concrete lifts is 0.

j	a_j	F_j	D_j	$\#\cap$
0	s_0	$\{0\}$	$\varnothing$	0
1	s_6	$\{6\}$	$\varnothing$	0
2	s_{12}	$\{12\}$	$\varnothing$	0
3	s_{18}	$\{18\}$	$\varnothing$	0
4	s_{24}	$\{24\}$	$\varnothing$	0
5	s_9	$\{9\}$	$\varnothing$	0
6	b_3	$\{15\}$	$\{23\}$	0
7	s_8	$\varnothing$	$\{8\}$	0
8	s_{14}	$\varnothing$	$\{14\}$	0
9	s_{20}	$\varnothing$	$\{20\}$	0
10	s_{26}	$\varnothing$	$\{26\}$	0

The isolated state IDs are $\{3, 7, 11, 15, 19, 23, 27\}$. The refined graph has zero paths projecting to this candidate.

Round 3. The partition has 30 nonempty cells, and the may relation has 60 edges.

The abstract checker found the inductive invariant $\{s_0, s_3, s_6, s_9, s_{12}, s_{15}, s_{18}, s_{21}, s_{24}, s_{27}\}$. It contains $\{s_0\}$ and is disjoint from $\{s_{26}\}$.

Cell	All may successors
s_0	$\{s_6, s_{15}\}$
s_3	$\{s_9, s_{18}\}$
s_6	$\{s_{12}, s_{21}\}$
s_9	$\{s_{15}, s_{24}\}$
s_{12}	$\{s_{18}, s_{27}\}$
s_{15}	$\{s_0, s_{21}\}$
s_{18}	$\{s_3, s_{24}\}$
s_{21}	$\{s_6, s_{27}\}$
s_{24}	$\{s_0, s_9\}$
s_{27}	$\{s_3, s_{12}\}$

17.9 F2.9 order 32, jumps (8, 12), target 28, observation modulus 5

The additive submonoid generated by the jumps in $\mathbb{Z}/32$ is a subgroup, since each jump has finite order and hence has its inverse in the submonoid. The subgroup consists of multiples of 4.

The concrete carrier has 32 states and 64 edges. Its reachable set has 8 states. The bad predicate is reached.

ID	State	Seed	Bd.	d	p	Successors
0	0	0	I	0	–	$\{8, 12\}$
1	1	1	–	∞	–	$\{9, 13\}$
2	2	2	–	∞	–	$\{10, 14\}$
3	3	3	–	∞	–	$\{11, 15\}$
4	4	4	–	3	24	$\{12, 16\}$
5	5	0	–	∞	–	$\{13, 17\}$
6	6	1	–	∞	–	$\{14, 18\}$
7	7	2	–	∞	–	$\{15, 19\}$
8	8	3	–	1	0	$\{16, 20\}$
9	9	4	–	∞	–	$\{17, 21\}$
10	10	0	–	∞	–	$\{18, 22\}$
11	11	1	–	∞	–	$\{19, 23\}$
12	12	2	–	1	0	$\{20, 24\}$
13	13	3	–	∞	–	$\{21, 25\}$
14	14	4	–	∞	–	$\{22, 26\}$
15	15	0	–	∞	–	$\{23, 27\}$
16	16	1	–	2	8	$\{24, 28\}$
17	17	2	–	∞	–	$\{25, 29\}$
18	18	3	–	∞	–	$\{26, 30\}$
19	19	4	–	∞	–	$\{27, 31\}$
20	20	0	–	2	8	$\{0, 28\}$
21	21	1	–	∞	–	$\{1, 29\}$
22	22	2	–	∞	–	$\{2, 30\}$
23	23	3	–	∞	–	$\{3, 31\}$
24	24	4	–	2	12	$\{0, 4\}$
25	25	0	–	∞	–	$\{1, 5\}$
26	26	1	–	∞	–	$\{2, 6\}$
27	27	2	–	∞	–	$\{3, 7\}$
28	28	3	B	3	16	$\{4, 8\}$
29	29	4	–	∞	–	$\{5, 9\}$
30	30	0	–	∞	–	$\{6, 10\}$
31	31	1	–	∞	–	$\{7, 11\}$

Round 1. The partition has 5 nonempty cells, and the may relation has 20 edges.

The selected path is b_0, b_1, b_2, b_3. The exact number of concrete lifts is 0.

j	a_j	F_j	D_j	$\#\cap$
0	b_0	$\{0\}$	$\varnothing$	0
1	b_1	$\varnothing$	$\varnothing$	0
2	b_2	$\varnothing$	$\varnothing$	0
3	b_3	$\varnothing$	$\{28\}$	0

The isolated state IDs are $\{0, 1, 2, 3, 5, 6, 7, 8, 10, 11, 12, 13, 15, 16, 17, 18, 20, 21, 22, 23, 25, 26, 27, 28, 30, 31\}$. The refined graph has zero paths projecting to this candidate.

Round 2. The partition has 27 nonempty cells, and the may relation has 63 edges.

The selected path is $s_0, s_{12}, s_{20}, s_{28}$. The exact number of concrete lifts is 1.

j	a_j	F_j	D_j	$\#\cap$
0	s_0	$\{0\}$	$\{0\}$	1
1	s_{12}	$\{12\}$	$\{12\}$	1
2	s_{20}	$\{20\}$	$\{20\}$	1
3	s_{28}	$\{28\}$	$\{28\}$	1

A concrete witness is $0, 12, 20, 28$. Each adjacent pair occurs in the concrete successor table.

18 Lockstep clocks

The state records two clocks in $\mathbb{Z}/n$. The ordinary edge advances both. In the skew variant a state whose first clock is zero has another edge advancing only that clock.

Tables enumerate concrete states by integer ID. The d column gives distance from an initial state, and p gives a predecessor attaining that distance. Columns I and B are encoded by the boundary field. Abstract keys b_i are seed blocks and s_x are isolated states.

18.1 F3.1 order 3, target (2, 0), skew edge 0

Synchronous steps preserve $y - x$ modulo 3. Only the diagonal can be reached from the initial state.

The concrete carrier has 9 states and 9 edges. Its reachable set has 3 states. The reachable set is disjoint from the bad predicate.

ID	State	Seed	Bd.	d	p	Successors
0	(0, 0)	0	I	0	–	{4}
1	(0, 1)	0	–	∞	–	{5}
2	(0, 2)	0	–	∞	–	{3}
3	(1, 0)	1	–	∞	–	{7}
4	(1, 1)	1	–	1	0	{8}
5	(1, 2)	1	–	∞	–	{6}
6	(2, 0)	2	B	∞	–	{1}
7	(2, 1)	2	–	∞	–	{2}
8	(2, 2)	2	–	2	4	{0}

Round 1. The partition has 3 nonempty cells, and the may relation has 3 edges.

The selected path is b_0, b_1, b_2. The exact number of concrete lifts is 0.

j	a_j	F_j	D_j	$\#\cap$
0	b_0	$\{0\}$	$\{1\}$	0
1	b_1	$\{4\}$	$\{5\}$	0
2	b_2	$\{8\}$	$\{6\}$	0

The isolated state IDs are $\{0, 1, 2, 3, 4, 5, 6, 7, 8\}$. The refined graph has zero paths projecting to this candidate.

Round 2. The partition has 9 nonempty cells, and the may relation has 9 edges.

The abstract checker found the inductive invariant $\{s_0, s_4, s_8\}$. It contains $\{s_0\}$ and is disjoint from $\{s_6\}$.

Cell	All may successors
s_0	$\{s_4\}$
s_4	$\{s_8\}$
s_8	$\{s_0\}$

18.2 F3.2 order 3, target (2, 0), skew edge 1

At first coordinate zero the skew edge adds -1 to $y - x$ modulo 3. A complete trip returns the first coordinate to zero, so repetition reaches every offset.

The concrete carrier has 9 states and 12 edges. Its reachable set has 9 states. The bad predicate is reached.

ID	State	Seed	Bd.	d	p	Successors
0	(0, 0)	0	I	0	–	{3, 4}
1	(0, 1)	0	–	6	6	{4, 5}
2	(0, 2)	0	–	3	7	{3, 5}
3	(1, 0)	1	–	1	0	{7}
4	(1, 1)	1	–	1	0	{8}
5	(1, 2)	1	–	4	2	{6}
6	(2, 0)	2	B	5	5	{1}
7	(2, 1)	2	–	2	3	{2}
8	(2, 2)	2	–	2	4	{0}

Round 1. The partition has 3 nonempty cells, and the may relation has 3 edges.

The selected path is b_0, b_1, b_2. The exact number of concrete lifts is 0.

j	a_j	F_j	D_j	$\#\cap$
0	b_0	$\{0\}$	$\{1, 2\}$	0
1	b_1	$\{3, 4\}$	$\{5\}$	0
2	b_2	$\{7, 8\}$	$\{6\}$	0

The isolated state IDs are $\{0, 1, 2, 3, 4, 5, 6, 7, 8\}$. The refined graph has zero paths projecting to this candidate.

Round 2. The partition has 9 nonempty cells, and the may relation has 12 edges.

The selected path is $s_0, s_3, s_7, s_2, s_5, s_6$. The exact number of concrete lifts is 1.

j	a_j	F_j	D_j	$\#\cap$
0	s_0	$\{0\}$	$\{0\}$	1
1	s_3	$\{3\}$	$\{3\}$	1
2	s_7	$\{7\}$	$\{7\}$	1
3	s_2	$\{2\}$	$\{2\}$	1
4	s_5	$\{5\}$	$\{5\}$	1
5	s_6	$\{6\}$	$\{6\}$	1

A concrete witness is $0, 3, 7, 2, 5, 6$. Each adjacent pair occurs in the concrete successor table.

18.3 F3.3 order 4, target (2, 2), skew edge 0

Synchronous steps preserve $y - x$ modulo 4. Only the diagonal can be reached from the initial state.

The concrete carrier has 16 states and 16 edges. Its reachable set has 4 states. The bad predicate is reached.

ID	State	Seed	Bd.	d	p	Successors
0	(0, 0)	0	I	0	–	{5}
1	(0, 1)	0	–	∞	–	{6}
2	(0, 2)	0	–	∞	–	{7}
3	(0, 3)	0	–	∞	–	{4}
4	(1, 0)	1	–	∞	–	{9}
5	(1, 1)	1	–	1	0	{10}
6	(1, 2)	1	–	∞	–	{11}
7	(1, 3)	1	–	∞	–	{8}
8	(2, 0)	2	–	∞	–	{13}
9	(2, 1)	2	–	∞	–	{14}
10	(2, 2)	2	B	2	5	{15}
11	(2, 3)	2	–	∞	–	{12}
12	(3, 0)	3	–	∞	–	{1}

ID	State	Seed	Bd.	d	p	Successors
13	(3, 1)	3	–	∞	–	{2}
14	(3, 2)	3	–	∞	–	{3}
15	(3, 3)	3	–	3	10	{0}

Round 1. The partition has 4 nonempty cells, and the may relation has 4 edges.

The selected path is b_0, b_1, b_2. The exact number of concrete lifts is 1.

j	a_j	F_j	D_j	$\#\cap$
0	b_0	{0}	{0}	1
1	b_1	{5}	{5}	1
2	b_2	{10}	{10}	1

A concrete witness is $0, 5, 10$. Each adjacent pair occurs in the concrete successor table.

18.4 F3.4 order 4, target (3, 0), skew edge 0

Synchronous steps preserve $y - x$ modulo 4. Only the diagonal can be reached from the initial state.

The concrete carrier has 16 states and 16 edges. Its reachable set has 4 states. The reachable set is disjoint from the bad predicate.

ID	State	Seed	Bd.	d	p	Successors
0	(0, 0)	0	I	0	–	{5}
1	(0, 1)	0	–	∞	–	{6}
2	(0, 2)	0	–	∞	–	{7}
3	(0, 3)	0	–	∞	–	{4}
4	(1, 0)	1	–	∞	–	{9}
5	(1, 1)	1	–	1	0	{10}
6	(1, 2)	1	–	∞	–	{11}
7	(1, 3)	1	–	∞	–	{8}
8	(2, 0)	2	–	∞	–	{13}
9	(2, 1)	2	–	∞	–	{14}
10	(2, 2)	2	–	2	5	{15}
11	(2, 3)	2	–	∞	–	{12}
12	(3, 0)	3	B	∞	–	{1}

ID	State	Seed	Bd.	d	p	Successors
13	(3, 1)	3	–	∞	–	{2}
14	(3, 2)	3	–	∞	–	{3}
15	(3, 3)	3	–	3	10	{0}

Round 1. The partition has 4 nonempty cells, and the may relation has 4 edges.

The selected path is b_0, b_1, b_2, b_3. The exact number of concrete lifts is 0.

j	a_j	F_j	D_j	$\#\cap$
0	b_0	{0}	{1}	0
1	b_1	{5}	{6}	0
2	b_2	{10}	{11}	0
3	b_3	{15}	{12}	0

The isolated state IDs are $\{0, 1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12, 13, 14, 15\}$. The refined graph has zero paths projecting to this candidate.

Round 2. The partition has 16 nonempty cells, and the may relation has 16 edges.

The abstract checker found the inductive invariant $\{s_0, s_5, s_{10}, s_{15}\}$. It contains $\{s_0\}$ and is disjoint from $\{s_{12}\}$.

Cell	All may successors
s_0	$\{s_5\}$
s_5	$\{s_{10}\}$
s_{10}	$\{s_{15}\}$
s_{15}	$\{s_0\}$

18.5 F3.5 order 5, target (4, 3), skew edge 1

At first coordinate zero the skew edge adds -1 to $y - x$ modulo 5. A complete trip returns the first coordinate to zero, so repetition reaches every offset.

The concrete carrier has 25 states and 30 edges. Its reachable set has 25 states. The bad predicate is reached.

ID	State	Seed	Bd.	d	p	Successors
0	(0, 0)	0	I	0	–	$\{5, 6\}$
1	(0, 1)	0	–	20	20	$\{6, 7\}$
2	(0, 2)	0	–	15	21	$\{7, 8\}$
3	(0, 3)	0	–	10	22	$\{8, 9\}$
4	(0, 4)	0	–	5	23	$\{5, 9\}$
5	(1, 0)	1	–	1	0	$\{11\}$
6	(1, 1)	1	–	1	0	$\{12\}$
7	(1, 2)	1	–	16	2	$\{13\}$
8	(1, 3)	1	–	11	3	$\{14\}$
9	(1, 4)	1	–	6	4	$\{10\}$
10	(2, 0)	2	–	7	9	$\{16\}$
11	(2, 1)	2	–	2	5	$\{17\}$
12	(2, 2)	2	–	2	6	$\{18\}$
13	(2, 3)	2	–	17	7	$\{19\}$
14	(2, 4)	2	–	12	8	$\{15\}$
15	(3, 0)	3	–	13	14	$\{21\}$
16	(3, 1)	3	–	8	10	$\{22\}$
17	(3, 2)	3	–	3	11	$\{23\}$
18	(3, 3)	3	–	3	12	$\{24\}$
19	(3, 4)	3	–	18	13	$\{20\}$
20	(4, 0)	4	–	19	19	$\{1\}$
21	(4, 1)	4	–	14	15	$\{2\}$
22	(4, 2)	4	–	9	16	$\{3\}$
23	(4, 3)	4	B	4	17	$\{4\}$
24	(4, 4)	4	–	4	18	$\{0\}$

Round 1. The partition has 5 nonempty cells, and the may relation has 5 edges.

The selected path is b_0, b_1, b_2, b_3, b_4. The exact number of concrete lifts is 1.

j	a_j	F_j	D_j	$\#\cap$
0	b_0	$\{0\}$	$\{0, 4\}$	1
1	b_1	$\{5, 6\}$	$\{5\}$	1
2	b_2	$\{11, 12\}$	$\{11\}$	1
3	b_3	$\{17, 18\}$	$\{17\}$	1
4	b_4	$\{23, 24\}$	$\{23\}$	1

A concrete witness is $0, 5, 11, 17, 23$. Each adjacent pair occurs in the concrete successor table.

18.6 F3.6 order 5, target (0, 2), skew edge 0

Synchronous steps preserve $y - x$ modulo 5. Only the diagonal can be reached from the initial state.

The concrete carrier has 25 states and 25 edges. Its reachable set has 5 states. The reachable set is disjoint from the bad predicate.

ID	State	Seed	Bd.	d	p	Successors
0	(0, 0)	0	I	0	–	$\{6\}$
1	(0, 1)	0	–	∞	–	$\{7\}$
2	(0, 2)	0	B	∞	–	$\{8\}$
3	(0, 3)	0	–	∞	–	$\{9\}$
4	(0, 4)	0	–	∞	–	$\{5\}$
5	(1, 0)	1	–	∞	–	$\{11\}$
6	(1, 1)	1	–	1	0	$\{12\}$
7	(1, 2)	1	–	∞	–	$\{13\}$
8	(1, 3)	1	–	∞	–	$\{14\}$
9	(1, 4)	1	–	∞	–	$\{10\}$
10	(2, 0)	2	–	∞	–	$\{16\}$
11	(2, 1)	2	–	∞	–	$\{17\}$
12	(2, 2)	2	–	2	6	$\{18\}$
13	(2, 3)	2	–	∞	–	$\{19\}$
14	(2, 4)	2	–	∞	–	$\{15\}$
15	(3, 0)	3	–	∞	–	$\{21\}$
16	(3, 1)	3	–	∞	–	$\{22\}$
17	(3, 2)	3	–	∞	–	$\{23\}$
18	(3, 3)	3	–	3	12	$\{24\}$
19	(3, 4)	3	–	∞	–	$\{20\}$
20	(4, 0)	4	–	∞	–	$\{1\}$
21	(4, 1)	4	–	∞	–	$\{2\}$
22	(4, 2)	4	–	∞	–	$\{3\}$
23	(4, 3)	4	–	∞	–	$\{4\}$
24	(4, 4)	4	–	4	18	$\{0\}$

Round 1. The partition has 5 nonempty cells, and the may relation has 5 edges.

The selected path is b_0. The exact number of concrete lifts is 0.

j	a_j	F_j	D_j	$\#\cap$
0	b_0	$\{0\}$	$\{2\}$	0

The isolated state IDs are $\{0, 1, 2, 3, 4\}$. The refined graph has zero paths projecting to this candidate.

Round 2. The partition has 9 nonempty cells, and the may relation has 13 edges.

The selected path is $s_0, b_1, b_2, b_3, b_4, s_2$. The exact number of concrete lifts is 0.

j	a_j	F_j	D_j	$\#\cap$
0	s_0	$\{0\}$	$\varnothing$	0
1	b_1	$\{6\}$	$\{8\}$	0
2	b_2	$\{12\}$	$\{14\}$	0
3	b_3	$\{18\}$	$\{15\}$	0
4	b_4	$\{24\}$	$\{21\}$	0
5	s_2	$\varnothing$	$\{2\}$	0

The isolated state IDs are $\{5, 6, 7, 8, 9, 10, 11, 12, 13, 14, 15, 16, 17, 18, 19, 20, 21, 22, 23, 24\}$. The refined graph has zero paths projecting to this candidate.

Round 3. The partition has 25 nonempty cells, and the may relation has 25 edges.

The abstract checker found the inductive invariant $\{s_0, s_6, s_{12}, s_{18}, s_{24}\}$. It contains $\{s_0\}$ and is disjoint from $\{s_2\}$.

Cell	All may successors
s_0	$\{s_6\}$
s_6	$\{s_{12}\}$
s_{12}	$\{s_{18}\}$
s_{18}	$\{s_{24}\}$
s_{24}	$\{s_0\}$

18.7 F3.7 order 6, target (4, 3), skew edge 0

Synchronous steps preserve $y - x$ modulo 6. Only the diagonal can be reached from the initial state.

The concrete carrier has 36 states and 36 edges. Its reachable set has 6 states. The reachable set is disjoint from the bad predicate.

ID	State	Seed	Bd.	d	p	Successors
0	(0, 0)	0	I	0	–	{7}
1	(0, 1)	0	–	∞	–	{8}
2	(0, 2)	0	–	∞	–	{9}
3	(0, 3)	0	–	∞	–	{10}
4	(0, 4)	0	–	∞	–	{11}
5	(0, 5)	0	–	∞	–	{6}
6	(1, 0)	1	–	∞	–	{13}
7	(1, 1)	1	–	1	0	{14}
8	(1, 2)	1	–	∞	–	{15}
9	(1, 3)	1	–	∞	–	{16}
10	(1, 4)	1	–	∞	–	{17}
11	(1, 5)	1	–	∞	–	{12}
12	(2, 0)	2	–	∞	–	{19}
13	(2, 1)	2	–	∞	–	{20}
14	(2, 2)	2	–	2	7	{21}
15	(2, 3)	2	–	∞	–	{22}
16	(2, 4)	2	–	∞	–	{23}
17	(2, 5)	2	–	∞	–	{18}
18	(3, 0)	3	–	∞	–	{25}
19	(3, 1)	3	–	∞	–	{26}
20	(3, 2)	3	–	∞	–	{27}
21	(3, 3)	3	–	3	14	{28}
22	(3, 4)	3	–	∞	–	{29}
23	(3, 5)	3	–	∞	–	{24}
24	(4, 0)	4	–	∞	–	{31}
25	(4, 1)	4	–	∞	–	{32}
26	(4, 2)	4	–	∞	–	{33}
27	(4, 3)	4	B	∞	–	{34}
28	(4, 4)	4	–	4	21	{35}
29	(4, 5)	4	–	∞	–	{30}
30	(5, 0)	5	–	∞	–	{1}
31	(5, 1)	5	–	∞	–	{2}
32	(5, 2)	5	–	∞	–	{3}
33	(5, 3)	5	–	∞	–	{4}
34	(5, 4)	5	–	∞	–	{5}
35	(5, 5)	5	–	5	28	{0}

Round 1. The partition has 6 nonempty cells, and the may relation has 6 edges.

The selected path is b_0, b_1, b_2, b_3, b_4. The exact number of concrete lifts is 0.

j	a_j	F_j	D_j	$\#\cap$
0	b_0	{0}	{5}	0
1	b_1	{7}	{6}	0
2	b_2	{14}	{13}	0
3	b_3	{21}	{20}	0
4	b_4	{28}	{27}	0

The isolated state IDs are $\{0, 1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12, 13, 14, 15, 16, 17, 18, 19, 20, 21, 22, 23, 24, 25, 26, 27, 28, 29\}$. The refined graph has zero paths projecting to this candidate.

Round 2. The partition has 31 nonempty cells, and the may relation has 36 edges.

The selected path is $s_0, s_7, s_{14}, s_{21}, s_{28}, b_5, s_5, s_6, s_{13}, s_{20}, s_{27}$. The exact number of concrete lifts is 0.

j	a_j	F_j	D_j	$\#\cap$
0	s_0	{0}	$\varnothing$	0
1	s_7	{7}	$\varnothing$	0
2	s_{14}	{14}	$\varnothing$	0
3	s_{21}	{21}	$\varnothing$	0
4	s_{28}	{28}	$\varnothing$	0
5	b_5	{35}	{34}	0
6	s_5	$\varnothing$	{5}	0
7	s_6	$\varnothing$	{6}	0
8	s_{13}	$\varnothing$	{13}	0
9	s_{20}	$\varnothing$	{20}	0
10	s_{27}	$\varnothing$	{27}	0

The isolated state IDs are $\{30, 31, 32, 33, 34, 35\}$. The refined graph has zero paths projecting to this candidate.

Round 3. The partition has 36 nonempty cells, and the may relation has 36 edges.

The abstract checker found the inductive invariant $\{s_0, s_7, s_{14}, s_{21}, s_{28}, s_{35}\}$. It contains $\{s_0\}$ and is disjoint from $\{s_{27}\}$.

Cell	All may successors
s_0	$\{s_7\}$
s_7	$\{s_{14}\}$
s_{14}	$\{s_{21}\}$
s_{21}	$\{s_{28}\}$
s_{28}	$\{s_{35}\}$
s_{35}	$\{s_0\}$

18.8 F3.8 order 6, target (5, 1), skew edge 1

At first coordinate zero the skew edge adds -1 to $y - x$ modulo 6. A complete trip returns the first coordinate to zero, so repetition reaches every offset.

The concrete carrier has 36 states and 42 edges. Its reachable set has 36 states. The bad predicate is reached.

ID	State	Seed	Bd.	d	p	Successors
0	(0, 0)	0	I	0	–	{6, 7}
1	(0, 1)	0	–	30	30	{7, 8}
2	(0, 2)	0	–	24	31	{8, 9}
3	(0, 3)	0	–	18	32	{9, 10}
4	(0, 4)	0	–	12	33	{10, 11}
5	(0, 5)	0	–	6	34	{6, 11}
6	(1, 0)	1	–	1	0	{13}
7	(1, 1)	1	–	1	0	{14}
8	(1, 2)	1	–	25	2	{15}
9	(1, 3)	1	–	19	3	{16}
10	(1, 4)	1	–	13	4	{17}
11	(1, 5)	1	–	7	5	{12}
12	(2, 0)	2	–	8	11	{19}

ID	State	Seed	Bd.	d	p	Successors
13	(2, 1)	2	–	2	6	{20}
14	(2, 2)	2	–	2	7	{21}
15	(2, 3)	2	–	26	8	{22}
16	(2, 4)	2	–	20	9	{23}
17	(2, 5)	2	–	14	10	{18}
18	(3, 0)	3	–	15	17	{25}
19	(3, 1)	3	–	9	12	{26}
20	(3, 2)	3	–	3	13	{27}
21	(3, 3)	3	–	3	14	{28}
22	(3, 4)	3	–	27	15	{29}
23	(3, 5)	3	–	21	16	{24}
24	(4, 0)	4	–	22	23	{31}
25	(4, 1)	4	–	16	18	{32}

ID	State	Seed	Bd.	d	p	Successors
26	(4, 2)	4	–	10	19	{33}
27	(4, 3)	4	–	4	20	{34}
28	(4, 4)	4	–	4	21	{35}
29	(4, 5)	4	–	28	22	{30}
30	(5, 0)	5	–	29	29	{1}
31	(5, 1)	5	B	23	24	{2}
32	(5, 2)	5	–	17	25	{3}
33	(5, 3)	5	–	11	26	{4}
34	(5, 4)	5	–	5	27	{5}
35	(5, 5)	5	–	5	28	{0}

Round 1. The partition has 6 nonempty cells, and the may relation has 6 edges.

The selected path is $b_0, b_1, b_2, b_3, b_4, b_5$. The exact number of concrete lifts is 0.

j	a_j	F_j	D_j	$\#\cap$
0	b_0	$\{0\}$	$\{2, 3\}$	0
1	b_1	$\{6, 7\}$	$\{9\}$	0
2	b_2	$\{13, 14\}$	$\{16\}$	0
3	b_3	$\{20, 21\}$	$\{23\}$	0
4	b_4	$\{27, 28\}$	$\{24\}$	0
5	b_5	$\{34, 35\}$	$\{31\}$	0

The isolated state IDs are $\{0, 1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12, 13, 14, 15, 16, 17, 18, 19, 20, 21, 22, 23, 24, 25, 26, 27, 28, 29, 30, 31, 32, 33, 34, 35\}$. The refined graph has zero paths projecting to this candidate.

Round 2. The partition has 36 nonempty cells, and the may relation has 42 edges.

The selected path is $s_0, s_6, s_{13}, s_{20}, s_{27}, s_{34}, s_5, s_{11}, s_{12}, s_{19}, s_{26}, s_{33}, s_4, s_{10}, s_{17}, s_{18}, s_{25}, s_{32}, s_3, s_9, s_{16}, s_{23}, s_{24}, s_{31}$. The exact number of concrete lifts is 1.

j	a_j	F_j	D_j	$\#\cap$
0	s_0	$\{0\}$	$\{0\}$	1
1	s_6	$\{6\}$	$\{6\}$	1
2	s_{13}	$\{13\}$	$\{13\}$	1
3	s_{20}	$\{20\}$	$\{20\}$	1
4	s_{27}	$\{27\}$	$\{27\}$	1
5	s_{34}	$\{34\}$	$\{34\}$	1
6	s_5	$\{5\}$	$\{5\}$	1
7	s_{11}	$\{11\}$	$\{11\}$	1
8	s_{12}	$\{12\}$	$\{12\}$	1
9	s_{19}	$\{19\}$	$\{19\}$	1
10	s_{26}	$\{26\}$	$\{26\}$	1
11	s_{33}	$\{33\}$	$\{33\}$	1
12	s_4	$\{4\}$	$\{4\}$	1

j	a_j	F_j	D_j	$\#\cap$
13	s_{10}	$\{10\}$	$\{10\}$	1
14	s_{17}	$\{17\}$	$\{17\}$	1
15	s_{18}	$\{18\}$	$\{18\}$	1
16	s_{25}	$\{25\}$	$\{25\}$	1
17	s_{32}	$\{32\}$	$\{32\}$	1
18	s_3	$\{3\}$	$\{3\}$	1
19	s_9	$\{9\}$	$\{9\}$	1
20	s_{16}	$\{16\}$	$\{16\}$	1
21	s_{23}	$\{23\}$	$\{23\}$	1
22	s_{24}	$\{24\}$	$\{24\}$	1
23	s_{31}	$\{31\}$	$\{31\}$	1

A concrete witness is $0, 6, 13, 20, 27, 34, 5, 11, 12, 19, 26, 33, 4, 10, 17, 18, 25, 32, 3, 9, 16, 23, 24, 31$. Each adjacent pair occurs in the concrete successor table.

18.9 F3.9 order 6, target (2, 4), skew edge 0

Synchronous steps preserve $y - x$ modulo 6. Only the diagonal can be reached from the initial state.

The concrete carrier has 36 states and 36 edges. Its reachable set has 6 states. The reachable set is disjoint from the bad predicate.

ID	State	Seed	Bd.	d	p	Successors
0	(0, 0)	0	I	0	–	{7}
1	(0, 1)	0	–	∞	–	{8}
2	(0, 2)	0	–	∞	–	{9}
3	(0, 3)	0	–	∞	–	{10}
4	(0, 4)	0	–	∞	–	{11}
5	(0, 5)	0	–	∞	–	{6}
6	(1, 0)	1	–	∞	–	{13}
7	(1, 1)	1	–	1	0	{14}
8	(1, 2)	1	–	∞	–	{15}
9	(1, 3)	1	–	∞	–	{16}
10	(1, 4)	1	–	∞	–	{17}
11	(1, 5)	1	–	∞	–	{12}
12	(2, 0)	2	–	∞	–	{19}

ID	State	Seed	Bd.	d	p	Successors
13	(2, 1)	2	–	∞	–	{20}
14	(2, 2)	2	–	2	7	{21}
15	(2, 3)	2	–	∞	–	{22}
16	(2, 4)	2	B	∞	–	{23}
17	(2, 5)	2	–	∞	–	{18}
18	(3, 0)	3	–	∞	–	{25}
19	(3, 1)	3	–	∞	–	{26}
20	(3, 2)	3	–	∞	–	{27}
21	(3, 3)	3	–	3	14	{28}
22	(3, 4)	3	–	∞	–	{29}
23	(3, 5)	3	–	∞	–	{24}
24	(4, 0)	4	–	∞	–	{31}
25	(4, 1)	4	–	∞	–	{32}

ID	State	Seed	Bd.	d	p	Successors
26	(4, 2)	4	–	∞	–	{33}
27	(4, 3)	4	–	∞	–	{34}
28	(4, 4)	4	–	4	21	{35}
29	(4, 5)	4	–	∞	–	{30}
30	(5, 0)	5	–	∞	–	{1}
31	(5, 1)	5	–	∞	–	{2}
32	(5, 2)	5	–	∞	–	{3}
33	(5, 3)	5	–	∞	–	{4}
34	(5, 4)	5	–	∞	–	{5}
35	(5, 5)	5	–	5	28	{0}

Round 1. The partition has 6 nonempty cells, and the may relation has 6 edges.

The selected path is b_0, b_1, b_2. The exact number of concrete lifts is 0.

j	a_j	F_j	D_j	$\#\cap$
0	b_0	{0}	{2}	0
1	b_1	{7}	{9}	0
2	b_2	{14}	{16}	0

The isolated state IDs are $\{0, 1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12, 13, 14, 15, 16, 17\}$. The refined graph has zero paths projecting to this candidate.

Round 2. The partition has 21 nonempty cells, and the may relation has 26 edges.

The selected path is $s_0, s_7, s_{14}, b_3, b_4, b_5, s_2, s_9, s_{16}$. The exact number of concrete lifts is 0.

j	a_j	F_j	D_j	$\#\cap$
0	s_0	{0}	$\varnothing$	0
1	s_7	{7}	$\varnothing$	0
2	s_{14}	{14}	$\varnothing$	0
3	b_3	{21}	{23}	0
4	b_4	{28}	{24}	0
5	b_5	{35}	{31}	0
6	s_2	$\varnothing$	{2}	0
7	s_9	$\varnothing$	{9}	0
8	s_{16}	$\varnothing$	{16}	0

The isolated state IDs are $\{18, 19, 20, 21, 22, 23, 24, 25, 26, 27, 28, 29, 30, 31, 32, 33, 34, 35\}$. The refined graph has zero paths projecting to this candidate.

Round 3. The partition has 36 nonempty cells, and the may relation has 36 edges.

The abstract checker found the inductive invariant $\{s_0, s_7, s_{14}, s_{21}, s_{28}, s_{35}\}$. It contains $\{s_0\}$ and is disjoint from $\{s_{16}\}$.

Cell	All may successors
s_0	$\{s_7\}$
s_7	$\{s_{14}\}$
s_{14}	$\{s_{21}\}$
s_{21}	$\{s_{28}\}$
s_{28}	$\{s_{35}\}$
s_{35}	$\{s_0\}$

19 Queues with a checksum

Ordinary queue operations change the occupancy by one and complement its stored parity. The faulty edge is a pop leaving the parity unchanged. The observation is occupancy or occupancy modulo three.

Tables enumerate concrete states by integer ID. The d column gives distance from an initial state, and p gives a predecessor attaining that distance. Columns I and B are encoded by the boundary field. Abstract keys b_i are seed blocks and s_x are isolated states.

19.1 F4.1 capacity 4, faulty pop None, target (3, 0), observation mode 0

Each ordinary push or pop changes occupancy and parity together. The invariant is $p = n \bmod 2$.

The concrete carrier has 10 states and 16 edges. Its reachable set has 5 states. The reachable set is disjoint from the bad predicate.

ID	State	Seed	Bd.	d	p	Successors
0	(0, 0)	0	I	0	–	$\{3\}$
1	(0, 1)	0	–	∞	–	$\{2\}$
2	(1, 0)	1	–	∞	–	$\{1, 5\}$
3	(1, 1)	1	–	1	0	$\{0, 4\}$
4	(2, 0)	2	–	2	3	$\{3, 7\}$
5	(2, 1)	2	–	∞	–	$\{2, 6\}$
6	(3, 0)	3	B	∞	–	$\{5, 9\}$
7	(3, 1)	3	–	3	4	$\{4, 8\}$
8	(4, 0)	4	–	4	7	$\{7\}$
9	(4, 1)	4	–	∞	–	$\{6\}$

Round 1. The partition has 5 nonempty cells, and the may relation has 8 edges.

The selected path is b_0, b_1, b_2, b_3. The exact number of concrete lifts is 0.

j	a_j	F_j	D_j	$\#\cap$
0	b_0	$\{0\}$	$\{1\}$	0
1	b_1	$\{3\}$	$\{2\}$	0
2	b_2	$\{4\}$	$\{5\}$	0
3	b_3	$\{7\}$	$\{6\}$	0

The isolated state IDs are $\{0, 1, 2, 3, 4, 5, 6, 7\}$. The refined graph has zero paths projecting to this candidate.

Round 2. The partition has 9 nonempty cells, and the may relation has 16 edges.

The selected path is $s_0, s_3, s_4, s_7, b_4, s_6$. The exact number of concrete lifts is 0.

j	a_j	F_j	D_j	$\#\cap$
0	s_0	$\{0\}$	$\varnothing$	0
1	s_3	$\{3\}$	$\varnothing$	0
2	s_4	$\{4\}$	$\varnothing$	0
3	s_7	$\{7\}$	$\varnothing$	0
4	b_4	$\{8\}$	$\{9\}$	0
5	s_6	$\varnothing$	$\{6\}$	0

The isolated state IDs are $\{8, 9\}$. The refined graph has zero paths projecting to this candidate.

Round 3. The partition has 10 nonempty cells, and the may relation has 16 edges.

The abstract checker found the inductive invariant $\{s_0, s_3, s_4, s_7, s_8\}$. It contains $\{s_0\}$ and is disjoint from $\{s_6\}$.

Cell	All may successors
s_0	$\{s_3\}$
s_3	$\{s_0, s_4\}$
s_4	$\{s_3, s_7\}$
s_7	$\{s_4, s_8\}$
s_8	$\{s_7\}$

19.2 F4.2 capacity 5, faulty pop 3, target (4, 1), observation mode 0

The faulty pop at occupancy 3 changes n without changing parity. After it is used, ordinary pushes and pops can reach every occupancy with the opposite parity.

The concrete carrier has 12 states and 22 edges. Its reachable set has 12 states. The bad predicate is reached.

ID	State	Seed	Bd.	d	p	Successors
0	(0, 0)	0	I	0	–	$\{3\}$
1	(0, 1)	0	–	6	2	$\{2\}$
2	(1, 0)	1	–	5	5	$\{1, 5\}$
3	(1, 1)	1	–	1	0	$\{0, 4\}$
4	(2, 0)	2	–	2	3	$\{3, 7\}$
5	(2, 1)	2	–	4	7	$\{2, 6\}$
6	(3, 0)	3	–	5	5	$\{4, 5, 9\}$
7	(3, 1)	3	–	3	4	$\{4, 5, 8\}$
8	(4, 0)	4	–	4	7	$\{7, 11\}$
9	(4, 1)	4	B	6	6	$\{6, 10\}$
10	(5, 0)	5	–	7	9	$\{9\}$
11	(5, 1)	5	–	5	8	$\{8\}$

Round 1. The partition has 6 nonempty cells, and the may relation has 10 edges.

The selected path is b_0, b_1, b_2, b_3, b_4. The exact number of concrete lifts is 0.

j	a_j	F_j	D_j	$\#\cap$
0	b_0	$\{0\}$	$\{1\}$	0
1	b_1	$\{3\}$	$\{2\}$	0
2	b_2	$\{4\}$	$\{5\}$	0
3	b_3	$\{7\}$	$\{6\}$	0
4	b_4	$\{8\}$	$\{9\}$	0

The isolated state IDs are $\{0, 1, 2, 3, 4, 5, 6, 7, 8, 9\}$. The refined graph has zero paths projecting to this candidate.

Round 2. The partition has 11 nonempty cells, and the may relation has 22 edges.

The selected path is $s_0, s_3, s_4, s_7, s_5, s_6, s_9$. The exact number of concrete lifts is 1.

j	a_j	F_j	D_j	$\#\cap$
0	s_0	$\{0\}$	$\{0\}$	1
1	s_3	$\{3\}$	$\{3\}$	1
2	s_4	$\{4\}$	$\{4\}$	1
3	s_7	$\{7\}$	$\{7\}$	1
4	s_5	$\{5\}$	$\{5\}$	1
5	s_6	$\{6\}$	$\{6\}$	1
6	s_9	$\{9\}$	$\{9\}$	1

A concrete witness is $0, 3, 4, 7, 5, 6, 9$. Each adjacent pair occurs in the concrete successor table.

19.3 F4.3 capacity 6, faulty pop None, target (6, 1), observation mode 1

Each ordinary push or pop changes occupancy and parity together. The invariant is $p = n \bmod 2$.

The concrete carrier has 14 states and 24 edges. Its reachable set has 7 states. The reachable set is disjoint from the bad predicate.

ID	State	Seed	Bd.	d	p	Successors
0	(0, 0)	0	I	0	–	$\{3\}$
1	(0, 1)	0	–	∞	–	$\{2\}$
2	(1, 0)	1	–	∞	–	$\{1, 5\}$
3	(1, 1)	1	–	1	0	$\{0, 4\}$
4	(2, 0)	2	–	2	3	$\{3, 7\}$
5	(2, 1)	2	–	∞	–	$\{2, 6\}$
6	(3, 0)	0	–	∞	–	$\{5, 9\}$
7	(3, 1)	0	–	3	4	$\{4, 8\}$
8	(4, 0)	1	–	4	7	$\{7, 11\}$
9	(4, 1)	1	–	∞	–	$\{6, 10\}$
10	(5, 0)	2	–	∞	–	$\{9, 13\}$
11	(5, 1)	2	–	5	8	$\{8, 12\}$
12	(6, 0)	0	–	6	11	$\{11\}$

ID	State	Seed	Bd.	d	p	Successors
13	(6, 1)	0	B	∞	–	$\{10\}$

Round 1. The partition has 3 nonempty cells, and the may relation has 6 edges.

The selected path is b_0. The exact number of concrete lifts is 0.

j	a_j	F_j	D_j	$\#\cap$
0	b_0	$\{0\}$	$\{13\}$	0

The isolated state IDs are $\{0, 1, 6, 7, 12, 13\}$. The refined graph has zero paths projecting to this candidate.

Round 2. The partition has 8 nonempty cells, and the may relation has 18 edges.

The selected path is $s_0, b_1, s_7, b_2, s_{13}$. The exact number of concrete lifts is 0.

j	a_j	F_j	D_j	$\#\cap$
0	s_0	$\{0\}$	$\varnothing$	0
1	b_1	$\{3\}$	$\varnothing$	0
2	s_7	$\varnothing$	$\varnothing$	0
3	b_2	$\varnothing$	$\{10\}$	0
4	s_{13}	$\varnothing$	$\{13\}$	0

The isolated state IDs are $\{2, 3, 4, 5, 8, 9, 10, 11\}$. The refined graph has zero paths projecting to this candidate.

Round 3. The partition has 14 nonempty cells, and the may relation has 24 edges.

The abstract checker found the inductive invariant $\{s_0, s_3, s_4, s_7, s_8, s_{11}, s_{12}\}$. It contains $\{s_0\}$ and is disjoint from $\{s_{13}\}$.

Cell	All may successors
s_0	$\{s_3\}$
s_3	$\{s_0, s_4\}$
s_4	$\{s_3, s_7\}$
s_7	$\{s_4, s_8\}$
s_8	$\{s_7, s_{11}\}$
s_{11}	$\{s_8, s_{12}\}$
s_{12}	$\{s_{11}\}$

19.4 F4.4 capacity 7, faulty pop None, target (5, 0), observation mode 1

Each ordinary push or pop changes occupancy and parity together. The invariant is $p = n \bmod 2$.

The concrete carrier has 16 states and 28 edges. Its reachable set has 8 states. The reachable set is disjoint from the bad predicate.

ID	State	Seed	Bd.	d	p	Successors
0	(0, 0)	0	I	0	–	$\{3\}$
1	(0, 1)	0	–	∞	–	$\{2\}$
2	(1, 0)	1	–	∞	–	$\{1, 5\}$
3	(1, 1)	1	–	1	0	$\{0, 4\}$
4	(2, 0)	2	–	2	3	$\{3, 7\}$
5	(2, 1)	2	–	∞	–	$\{2, 6\}$
6	(3, 0)	0	–	∞	–	$\{5, 9\}$
7	(3, 1)	0	–	3	4	$\{4, 8\}$
8	(4, 0)	1	–	4	7	$\{7, 11\}$
9	(4, 1)	1	–	∞	–	$\{6, 10\}$
10	(5, 0)	2	B	∞	–	$\{9, 13\}$
11	(5, 1)	2	–	5	8	$\{8, 12\}$
12	(6, 0)	0	–	6	11	$\{11, 15\}$

ID	State	Seed	Bd.	d	p	Successors
13	(6, 1)	0	–	∞	–	$\{10, 14\}$
14	(7, 0)	1	–	∞	–	$\{13\}$
15	(7, 1)	1	–	7	12	$\{12\}$

Round 1. The partition has 3 nonempty cells, and the may relation has 6 edges.

The selected path is b_0, b_1, b_2. The exact number of concrete lifts is 0.

j	a_j	F_j	D_j	$\#\cap$
0	b_0	$\{0\}$	$\{6\}$	0
1	b_1	$\{3\}$	$\{9\}$	0
2	b_2	$\{4\}$	$\{10\}$	0

The isolated state IDs are $\{0, 1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12, 13, 14, 15\}$. The refined graph has zero paths projecting to this candidate.

Round 2. The partition has 16 nonempty cells, and the may relation has 28 edges.

The abstract checker found the inductive invariant $\{s_0, s_3, s_4, s_7, s_8, s_{11}, s_{12}, s_{15}\}$. It contains $\{s_0\}$ and is disjoint from $\{s_{10}\}$.

Cell	All may successors
s_0	$\{s_3\}$
s_3	$\{s_0, s_4\}$
s_4	$\{s_3, s_7\}$
s_7	$\{s_4, s_8\}$
s_8	$\{s_7, s_{11}\}$
s_{11}	$\{s_8, s_{12}\}$
s_{12}	$\{s_{11}, s_{15}\}$
s_{15}	$\{s_{12}\}$

19.5 F4.5 capacity 8, faulty pop 8, target (1, 0), observation mode 1

The faulty pop at occupancy 8 changes n without changing parity. After it is used, ordinary pushes and pops can reach every occupancy with the opposite parity.

The concrete carrier has 18 states and 34 edges. Its reachable set has 18 states. The bad predicate is reached.

ID	State	Seed	Bd.	d	p	Successors
0	(0, 0)	0	I	0	–	{3}
1	(0, 1)	0	–	16	2	{2}
2	(1, 0)	1	B	15	5	{1, 5}
3	(1, 1)	1	–	1	0	{0, 4}
4	(2, 0)	2	–	2	3	{3, 7}
5	(2, 1)	2	–	14	6	{2, 6}
6	(3, 0)	0	–	13	9	{5, 9}
7	(3, 1)	0	–	3	4	{4, 8}
8	(4, 0)	1	–	4	7	{7, 11}
9	(4, 1)	1	–	12	10	{6, 10}
10	(5, 0)	2	–	11	13	{9, 13}
11	(5, 1)	2	–	5	8	{8, 12}
12	(6, 0)	0	–	6	11	{11, 15}

ID	State	Seed	Bd.	d	p	Successors
13	(6, 1)	0	–	10	14	{10, 14}
14	(7, 0)	1	–	9	16	{13, 17}
15	(7, 1)	1	–	7	12	{12, 16}
16	(8, 0)	2	–	8	15	{14, 15}
17	(8, 1)	2	–	10	14	{14, 15}

Round 1. The partition has 3 nonempty cells, and the may relation has 6 edges.

The selected path is b_0, b_1. The exact number of concrete lifts is 0.

j	a_j	F_j	D_j	$\#\cap$
0	b_0	{0}	{1}	0
1	b_1	{3}	{2}	0

The isolated state IDs are $\{0, 1, 2, 3, 6, 7, 8, 9, 12, 13, 14, 15\}$. The refined graph has zero paths projecting to this candidate.

Round 2. The partition has 13 nonempty cells, and the may relation has 32 edges.

The selected path is s_0, s_3, b_2, s_2. The exact number of concrete lifts is 0.

j	a_j	F_j	D_j	$\#\cap$
0	s_0	{0}	∅	0
1	s_3	{3}	∅	0
2	b_2	{4}	{5}	0
3	s_2	∅	{2}	0

The isolated state IDs are $\{4, 5, 10, 11, 16, 17\}$. The refined graph has zero paths projecting to this candidate.

Round 3. The partition has 18 nonempty cells, and the may relation has 34 edges.

The selected path is $s_0, s_3, s_4, s_7, s_8, s_{11}, s_{12}, s_{15}, s_{16}, s_{14}, s_{13}, s_{10}, s_9, s_6, s_5, s_2$. The exact number of concrete lifts is 1.

j	a_j	F_j	D_j	$\#\cap$
0	s_0	{0}	{0}	1
1	s_3	{3}	{3}	1
2	s_4	{4}	{4}	1
3	s_7	{7}	{7}	1
4	s_8	{8}	{8}	1
5	s_{11}	{11}	{11}	1
6	s_{12}	{12}	{12}	1
7	s_{15}	{15}	{15}	1
8	s_{16}	{16}	{16}	1
9	s_{14}	{14}	{14}	1
10	s_{13}	{13}	{13}	1
11	s_{10}	{10}	{10}	1
12	s_9	{9}	{9}	1

j	a_j	F_j	D_j	$\#\cap$
13	s_6	{6}	{6}	1
14	s_5	{5}	{5}	1
15	s_2	{2}	{2}	1

A concrete witness is $0, 3, 4, 7, 8, 11, 12, 15, 16, 14, 13, 10, 9, 6, 5, 2$. Each adjacent pair occurs in the concrete successor table.

19.6 F4.6 capacity 9, faulty pop None, target (6, 0), observation mode 1

Each ordinary push or pop changes occupancy and parity together. The invariant is $p = n \bmod 2$.

The concrete carrier has 20 states and 36 edges. Its reachable set has 10 states. The bad predicate is reached.

ID	State	Seed	Bd.	d	p	Successors
0	(0, 0)	0	I	0	–	{3}
1	(0, 1)	0	–	∞	–	{2}
2	(1, 0)	1	–	∞	–	{1, 5}
3	(1, 1)	1	–	1	0	{0, 4}
4	(2, 0)	2	–	2	3	{3, 7}
5	(2, 1)	2	–	∞	–	{2, 6}
6	(3, 0)	0	–	∞	–	{5, 9}
7	(3, 1)	0	–	3	4	{4, 8}
8	(4, 0)	1	–	4	7	{7, 11}
9	(4, 1)	1	–	∞	–	{6, 10}
10	(5, 0)	2	–	∞	–	{9, 13}
11	(5, 1)	2	–	5	8	{8, 12}
12	(6, 0)	0	B	6	11	{11, 15}

ID	State	Seed	Bd.	d	p	Successors
13	(6, 1)	0	–	∞	–	{10, 14}
14	(7, 0)	1	–	∞	–	{13, 17}
15	(7, 1)	1	–	7	12	{12, 16}
16	(8, 0)	2	–	8	15	{15, 19}
17	(8, 1)	2	–	∞	–	{14, 18}
18	(9, 0)	0	–	∞	–	{17}
19	(9, 1)	0	–	9	16	{16}

Round 1. The partition has 3 nonempty cells, and the may relation has 6 edges.

The selected path is b_0. The exact number of concrete lifts is 0.

j	a_j	F_j	D_j	#∩
0	b_0	{0}	{12}	0

The isolated state IDs are $\{0, 1, 6, 7, 12, 13, 18, 19\}$. The refined graph has zero paths projecting to this candidate.

Round 2. The partition has 10 nonempty cells, and the may relation has 26 edges.

The selected path is s_0, b_1, s_{12}. The exact number of concrete lifts is 0.

j	a_j	F_j	D_j	#∩
0	s_0	{0}	∅	0
1	b_1	{3}	{15}	0
2	s_{12}	∅	{12}	0

The isolated state IDs are $\{2, 3, 8, 9, 14, 15\}$. The refined graph has zero paths projecting to this candidate.

Round 3. The partition has 15 nonempty cells, and the may relation has 36 edges.

The selected path is s_0, s_3, b_2, s_{12}. The exact number of concrete lifts is 0.

j	a_j	F_j	D_j	#∩
0	s_0	{0}	∅	0
1	s_3	{3}	∅	0
2	b_2	{4}	{11}	0
3	s_{12}	∅	{12}	0

The isolated state IDs are $\{4, 5, 10, 11, 16, 17\}$. The refined graph has zero paths projecting to this candidate.

Round 4. The partition has 20 nonempty cells, and the may relation has 36 edges.

The selected path is $s_0, s_3, s_4, s_7, s_8, s_{11}, s_{12}$. The exact number of concrete lifts is 1.

j	a_j	F_j	D_j	#∩
0	s_0	{0}	{0}	1
1	s_3	{3}	{3}	1
2	s_4	{4}	{4}	1
3	s_7	{7}	{7}	1
4	s_8	{8}	{8}	1
5	s_{11}	{11}	{11}	1
6	s_{12}	{12}	{12}	1

A concrete witness is $0, 3, 4, 7, 8, 11, 12$. Each adjacent pair occurs in the concrete successor table.

19.7 F4.7 capacity 10, faulty pop None, target (10, 1), observation mode 0

Each ordinary push or pop changes occupancy and parity together. The invariant is $p = n \bmod 2$.

The concrete carrier has 22 states and 40 edges. Its reachable set has 11 states. The reachable set is disjoint from the bad predicate.

ID	State	Seed	Bd.	d	p	Successors
0	(0, 0)	0	I	0	–	{3}
1	(0, 1)	0	–	∞	–	{2}
2	(1, 0)	1	–	∞	–	{1, 5}
3	(1, 1)	1	–	1	0	{0, 4}
4	(2, 0)	2	–	2	3	{3, 7}
5	(2, 1)	2	–	∞	–	{2, 6}
6	(3, 0)	3	–	∞	–	{5, 9}
7	(3, 1)	3	–	3	4	{4, 8}
8	(4, 0)	4	–	4	7	{7, 11}
9	(4, 1)	4	–	∞	–	{6, 10}
10	(5, 0)	5	–	∞	–	{9, 13}
11	(5, 1)	5	–	5	8	{8, 12}
12	(6, 0)	6	–	6	11	{11, 15}

ID	State	Seed	Bd.	d	p	Successors
13	(6, 1)	6	–	∞	–	{10, 14}
14	(7, 0)	7	–	∞	–	{13, 17}
15	(7, 1)	7	–	7	12	{12, 16}
16	(8, 0)	8	–	8	15	{15, 19}
17	(8, 1)	8	–	∞	–	{14, 18}
18	(9, 0)	9	–	∞	–	{17, 21}
19	(9, 1)	9	–	9	16	{16, 20}
20	(10, 0)	10	–	10	19	{19}
21	(10, 1)	10	B	∞	–	{18}

Round 1. The partition has 11 nonempty cells, and the may relation has 20 edges.

The selected path is $b_0, b_1, b_2, b_3, b_4, b_5, b_6, b_7, b_8, b_9, b_{10}$. The exact number of concrete lifts is 0.

j	a_j	F_j	D_j	$\#\cap$
0	b_0	$\{0\}$	$\{1\}$	0
1	b_1	$\{3\}$	$\{2\}$	0
2	b_2	$\{4\}$	$\{5\}$	0
3	b_3	$\{7\}$	$\{6\}$	0
4	b_4	$\{8\}$	$\{9\}$	0
5	b_5	$\{11\}$	$\{10\}$	0
6	b_6	$\{12\}$	$\{13\}$	0
7	b_7	$\{15\}$	$\{14\}$	0
8	b_8	$\{16\}$	$\{17\}$	0
9	b_9	$\{19\}$	$\{18\}$	0
10	b_{10}	$\{20\}$	$\{21\}$	0

The isolated state IDs are $\{0, 1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12, 13, 14, 15, 16, 17, 18, 19, 20, 21\}$. The refined graph has zero paths projecting to this candidate.

Round 2. The partition has 22 nonempty cells, and the may relation has 40 edges.

The abstract checker found the inductive invariant $\{s_0, s_3, s_4, s_7, s_8, s_{11}, s_{12}, s_{15}, s_{16}, s_{19}, s_{20}\}$. It contains $\{s_0\}$ and is disjoint from $\{s_{21}\}$.

Cell	All may successors
s_0	$\{s_3\}$
s_3	$\{s_0, s_4\}$
s_4	$\{s_3, s_7\}$
s_7	$\{s_4, s_8\}$
s_8	$\{s_7, s_{11}\}$
s_{11}	$\{s_8, s_{12}\}$
s_{12}	$\{s_{11}, s_{15}\}$
s_{15}	$\{s_{12}, s_{16}\}$
s_{16}	$\{s_{15}, s_{19}\}$
s_{19}	$\{s_{16}, s_{20}\}$
s_{20}	$\{s_{19}\}$

19.8 F4.8 capacity 11, faulty pop 7, target (3, 0), observation mode 0

The faulty pop at occupancy 7 changes n without changing parity. After it is used, ordinary pushes and pops can reach every occupancy with the opposite parity.

The concrete carrier has 24 states and 46 edges. Its reachable set has 24 states. The bad predicate is reached.

ID	State	Seed	Bd.	d	p	Successors
0	(0, 0)	0	I	0	–	$\{3\}$
1	(0, 1)	0	–	14	2	$\{2\}$
2	(1, 0)	1	–	13	5	$\{1, 5\}$
3	(1, 1)	1	–	1	0	$\{0, 4\}$
4	(2, 0)	2	–	2	3	$\{3, 7\}$
5	(2, 1)	2	–	12	6	$\{2, 6\}$
6	(3, 0)	3	B	11	9	$\{5, 9\}$
7	(3, 1)	3	–	3	4	$\{4, 8\}$
8	(4, 0)	4	–	4	7	$\{7, 11\}$
9	(4, 1)	4	–	10	10	$\{6, 10\}$
10	(5, 0)	5	–	9	13	$\{9, 13\}$
11	(5, 1)	5	–	5	8	$\{8, 12\}$
12	(6, 0)	6	–	6	11	$\{11, 15\}$
13	(6, 1)	6	–	8	15	$\{10, 14\}$
14	(7, 0)	7	–	9	13	$\{12, 13, 17\}$
15	(7, 1)	7	–	7	12	$\{12, 13, 16\}$
16	(8, 0)	8	–	8	15	$\{15, 19\}$
17	(8, 1)	8	–	10	14	$\{14, 18\}$
18	(9, 0)	9	–	11	17	$\{17, 21\}$
19	(9, 1)	9	–	9	16	$\{16, 20\}$
20	(10, 0)	10	–	10	19	$\{19, 23\}$
21	(10, 1)	10	–	12	18	$\{18, 22\}$
22	(11, 0)	11	–	13	21	$\{21\}$
23	(11, 1)	11	–	11	20	$\{20\}$

Round 1. The partition has 12 nonempty cells, and the may relation has 22 edges.

The selected path is b_0, b_1, b_2, b_3. The exact number of concrete lifts is 0.

j	a_j	F_j	D_j	$\#\cap$
0	b_0	$\{0\}$	$\{1\}$	0
1	b_1	$\{3\}$	$\{2\}$	0
2	b_2	$\{4\}$	$\{5\}$	0
3	b_3	$\{7\}$	$\{6\}$	0

The isolated state IDs are $\{0, 1, 2, 3, 4, 5, 6, 7\}$. The refined graph has zero paths projecting to this candidate.

Round 2. The partition has 16 nonempty cells, and the may relation has 30 edges.

The selected path is $s_0, s_3, s_4, s_7, b_4, s_6$. The exact number of concrete lifts is 0.

j	a_j	F_j	D_j	$\#\cap$
0	s_0	$\{0\}$	$\varnothing$	0
1	s_3	$\{3\}$	$\varnothing$	0
2	s_4	$\{4\}$	$\varnothing$	0
3	s_7	$\{7\}$	$\varnothing$	0
4	b_4	$\{8\}$	$\{9\}$	0
5	s_6	$\varnothing$	$\{6\}$	0

The isolated state IDs are $\{8, 9\}$. The refined graph has zero paths projecting to this candidate.

Round 3. The partition has 17 nonempty cells, and the may relation has 32 edges.

The selected path is $s_0, s_3, s_4, s_7, s_8, b_5, s_9, s_6$. The exact number of concrete lifts is 0.

j	a_j	F_j	D_j	$\#\cap$
0	s_0	$\{0\}$	$\varnothing$	0
1	s_3	$\{3\}$	$\varnothing$	0
2	s_4	$\{4\}$	$\varnothing$	0
3	s_7	$\{7\}$	$\varnothing$	0
4	s_8	$\{8\}$	$\varnothing$	0
5	b_5	$\{11\}$	$\{10\}$	0
6	s_9	$\varnothing$	$\{9\}$	0
7	s_6	$\varnothing$	$\{6\}$	0

The isolated state IDs are $\{10, 11\}$. The refined graph has zero paths projecting to this candidate.

Round 4. The partition has 18 nonempty cells, and the may relation has 34 edges.

The selected path is $s_0, s_3, s_4, s_7, s_8, s_{11}, b_6, s_{10}, s_9, s_6$. The exact number of concrete lifts is 0.

j	a_j	F_j	D_j	$\#\cap$
0	s_0	$\{0\}$	$\varnothing$	0
1	s_3	$\{3\}$	$\varnothing$	0
2	s_4	$\{4\}$	$\varnothing$	0
3	s_7	$\{7\}$	$\varnothing$	0
4	s_8	$\{8\}$	$\varnothing$	0
5	s_{11}	$\{11\}$	$\varnothing$	0
6	b_6	$\{12\}$	$\{13\}$	0
7	s_{10}	$\varnothing$	$\{10\}$	0
8	s_9	$\varnothing$	$\{9\}$	0
9	s_6	$\varnothing$	$\{6\}$	0

The isolated state IDs are $\{12, 13\}$. The refined graph has zero paths projecting to this candidate.

Round 5. The partition has 19 nonempty cells, and the may relation has 36 edges.

The selected path is $s_0, s_3, s_4, s_7, s_8, s_{11}, s_{12}, b_7, s_{13}, s_{10}, s_9, s_6$. The exact number of concrete lifts is 1.

j	a_j	F_j	D_j	$\#\cap$
0	s_0	$\{0\}$	$\{0\}$	1
1	s_3	$\{3\}$	$\{3\}$	1
2	s_4	$\{4\}$	$\{4\}$	1
3	s_7	$\{7\}$	$\{7\}$	1
4	s_8	$\{8\}$	$\{8\}$	1
5	s_{11}	$\{11\}$	$\{11\}$	1
6	s_{12}	$\{12\}$	$\{12\}$	1
7	b_7	$\{15\}$	$\{14, 15\}$	1
8	s_{13}	$\{13\}$	$\{13\}$	1
9	s_{10}	$\{10\}$	$\{10\}$	1
10	s_9	$\{9\}$	$\{9\}$	1
11	s_6	$\{6\}$	$\{6\}$	1

A concrete witness is $0, 3, 4, 7, 8, 11, 12, 15, 13, 10, 9, 6$. Each adjacent pair occurs in the concrete successor table.

19.9 F4.9 capacity 12, faulty pop None, target (11, 0), observation mode 0

Each ordinary push or pop changes occupancy and parity together. The invariant is $p = n \bmod 2$.

The concrete carrier has 26 states and 48 edges. Its reachable set has 13 states. The reachable set is disjoint from the bad predicate.

ID	State	Seed	Bd.	d	p	Successors
0	(0, 0)	0	I	0	–	$\{3\}$
1	(0, 1)	0	–	∞	–	$\{2\}$
2	(1, 0)	1	–	∞	–	$\{1, 5\}$
3	(1, 1)	1	–	1	0	$\{0, 4\}$
4	(2, 0)	2	–	2	3	$\{3, 7\}$
5	(2, 1)	2	–	∞	–	$\{2, 6\}$
6	(3, 0)	3	–	∞	–	$\{5, 9\}$
7	(3, 1)	3	–	3	4	$\{4, 8\}$
8	(4, 0)	4	–	4	7	$\{7, 11\}$
9	(4, 1)	4	–	∞	–	$\{6, 10\}$
10	(5, 0)	5	–	∞	–	$\{9, 13\}$
11	(5, 1)	5	–	5	8	$\{8, 12\}$
12	(6, 0)	6	–	6	11	$\{11, 15\}$

ID	State	Seed	Bd.	d	p	Successors
13	(6, 1)	6	–	∞	–	$\{10, 14\}$
14	(7, 0)	7	–	∞	–	$\{13, 17\}$
15	(7, 1)	7	–	7	12	$\{12, 16\}$
16	(8, 0)	8	–	8	15	$\{15, 19\}$
17	(8, 1)	8	–	∞	–	$\{14, 18\}$
18	(9, 0)	9	–	∞	–	$\{17, 21\}$
19	(9, 1)	9	–	9	16	$\{16, 20\}$
20	(10, 0)	10	–	10	19	$\{19, 23\}$
21	(10, 1)	10	–	∞	–	$\{18, 22\}$
22	(11, 0)	11	B	∞	–	$\{21, 25\}$
23	(11, 1)	11	–	11	20	$\{20, 24\}$
24	(12, 0)	12	–	12	23	$\{23\}$
25	(12, 1)	12	–	∞	–	$\{22\}$

Round 1. The partition has 13 nonempty cells, and the may relation has 24 edges.

The selected path is $b_0, b_1, b_2, b_3, b_4, b_5, b_6, b_7, b_8, b_9, b_{10}, b_{11}$. The exact number of concrete lifts is 0.

j	a_j	F_j	D_j	$\#\cap$
0	b_0	$\{0\}$	$\{1\}$	0
1	b_1	$\{3\}$	$\{2\}$	0
2	b_2	$\{4\}$	$\{5\}$	0
3	b_3	$\{7\}$	$\{6\}$	0
4	b_4	$\{8\}$	$\{9\}$	0
5	b_5	$\{11\}$	$\{10\}$	0
6	b_6	$\{12\}$	$\{13\}$	0
7	b_7	$\{15\}$	$\{14\}$	0
8	b_8	$\{16\}$	$\{17\}$	0
9	b_9	$\{19\}$	$\{18\}$	0
10	b_{10}	$\{20\}$	$\{21\}$	0
11	b_{11}	$\{23\}$	$\{22\}$	0

The isolated state IDs are $\{0, 1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12, 13, 14, 15, 16, 17, 18, 19, 20, 21, 22, 23\}$. The refined graph has zero paths projecting to this candidate.

Round 2. The partition has 25 nonempty cells, and the may relation has 48 edges.

The selected path is $s_0, s_3, s_4, s_7, s_8, s_{11}, s_{12}, s_{15}, s_{16}, s_{19}, s_{20}, s_{23}, b_{12}, s_{22}$. The exact number of concrete lifts is 0.

j	a_j	F_j	D_j	$\#\cap$
0	s_0	$\{0\}$	$\varnothing$	0
1	s_3	$\{3\}$	$\varnothing$	0
2	s_4	$\{4\}$	$\varnothing$	0
3	s_7	$\{7\}$	$\varnothing$	0
4	s_8	$\{8\}$	$\varnothing$	0
5	s_{11}	$\{11\}$	$\varnothing$	0
6	s_{12}	$\{12\}$	$\varnothing$	0
7	s_{15}	$\{15\}$	$\varnothing$	0
8	s_{16}	$\{16\}$	$\varnothing$	0
9	s_{19}	$\{19\}$	$\varnothing$	0
10	s_{20}	$\{20\}$	$\varnothing$	0
11	s_{23}	$\{23\}$	$\varnothing$	0
12	b_{12}	$\{24\}$	$\{25\}$	0
13	s_{22}	$\varnothing$	$\{22\}$	0

The isolated state IDs are $\{24, 25\}$. The refined graph has zero paths projecting to this candidate.

Round 3. The partition has 26 nonempty cells, and the may relation has 48 edges.

The abstract checker found the inductive invariant $\{s_0, s_3, s_4, s_7, s_8, s_{11}, s_{12}, s_{15}, s_{16}, s_{19}, s_{20}, s_{23}, s_{24}\}$. It contains $\{s_0\}$ and is disjoint from $\{s_{22}\}$.

Cell	All may successors
s_0	$\{s_3\}$
s_3	$\{s_0, s_4\}$
s_4	$\{s_3, s_7\}$
s_7	$\{s_4, s_8\}$
s_8	$\{s_7, s_{11}\}$
s_{11}	$\{s_8, s_{12}\}$
s_{12}	$\{s_{11}, s_{15}\}$
s_{15}	$\{s_{12}, s_{16}\}$
s_{16}	$\{s_{15}, s_{19}\}$
s_{19}	$\{s_{16}, s_{20}\}$
s_{20}	$\{s_{19}, s_{23}\}$
s_{23}	$\{s_{20}, s_{24}\}$
s_{24}	$\{s_{23}\}$

20 Leases and handoff

A lease state has a phase and an owner. Advancing wraps the phase. An optional handoff from the final phase starts the next owner at phase zero. Some additional restart edges preserve ownership.

Tables enumerate concrete states by integer ID. The d column gives distance from an initial state, and p gives a predecessor attaining that distance. Columns I and B are encoded by the boundary field. Abstract keys b_i are seed blocks and s_x are isolated states.

20.1 F5.1 owners 2, phases 4, transfer 0, target owner 1

Advancing and restarting preserve the owner. Owner zero is an invariant and its entire phase cycle is reachable.

The concrete carrier has 8 states and 10 edges. Its reachable set has 4 states. The reachable set is disjoint from the bad predicate.

ID	State	Seed	Bd.	d	p	Successors
0	(0, 0)	0	I	0	–	$\{2\}$
1	(0, 1)	0	–	∞	–	$\{3\}$
2	(1, 0)	1	–	1	0	$\{4\}$
3	(1, 1)	1	–	∞	–	$\{1, 5\}$
4	(2, 0)	2	–	2	2	$\{0, 6\}$
5	(2, 1)	2	–	∞	–	$\{7\}$
6	(3, 0)	3	–	3	4	$\{0\}$
7	(3, 1)	3	B	∞	–	$\{1\}$

Round 1. The partition has 4 nonempty cells, and the may relation has 6 edges.

The selected path is b_0, b_1, b_2, b_3. The exact number of concrete lifts is 0.

j	a_j	F_j	D_j	$\#\cap$
0	b_0	$\{0\}$	$\{1\}$	0
1	b_1	$\{2\}$	$\{3\}$	0
2	b_2	$\{4\}$	$\{5\}$	0
3	b_3	$\{6\}$	$\{7\}$	0

The isolated state IDs are $\{0, 1, 2, 3, 4, 5, 6, 7\}$. The refined graph has zero paths projecting to this candidate.

Round 2. The partition has 8 nonempty cells, and the may relation has 10 edges.

The abstract checker found the inductive invariant $\{s_0, s_2, s_4, s_6\}$. It contains $\{s_0\}$ and is disjoint from $\{s_7\}$.

Cell	All may successors
s_0	$\{s_2\}$
s_2	$\{s_4\}$
s_4	$\{s_0, s_6\}$
s_6	$\{s_0\}$

20.2 F5.2 owners 2, phases 5, transfer 1, target owner 1

Advancing and restarting preserve the owner. A transfer at the final phase advances the owner by one, after which all phases of that owner are reachable.

The concrete carrier has 10 states and 14 edges. Its reachable set has 10 states. The bad predicate is reached.

ID	State	Seed	Bd.	d	p	Successors
0	(0, 0)	0	I	0	–	{2}
1	(0, 1)	0	–	5	8	{3}
2	(1, 0)	1	–	1	0	{4}
3	(1, 1)	1	–	6	1	{1, 5}
4	(2, 0)	2	–	2	2	{0, 6}
5	(2, 1)	2	–	7	3	{7}
6	(3, 0)	3	–	3	4	{8}
7	(3, 1)	3	–	8	5	{1, 9}
8	(4, 0)	4	–	4	6	{0, 1}
9	(4, 1)	4	B	9	7	{1}

Round 1. The partition has 5 nonempty cells, and the may relation has 8 edges.

The selected path is b_0, b_1, b_2, b_3, b_4. The exact number of concrete lifts is 0.

j	a_j	F_j	D_j	$\#\cap$
0	b_0	{0}	{1}	0
1	b_1	{2}	{3}	0
2	b_2	{4}	{5}	0
3	b_3	{6}	{7}	0
4	b_4	{8}	{9}	0

The isolated state IDs are $\{0, 1, 2, 3, 4, 5, 6, 7, 8, 9\}$. The refined graph has zero paths projecting to this candidate.

Round 2. The partition has 10 nonempty cells, and the may relation has 14 edges.

The selected path is $s_0, s_2, s_4, s_6, s_8, s_1, s_3, s_5, s_7, s_9$. The exact number of concrete lifts is 1.

j	a_j	F_j	D_j	$\#\cap$
0	s_0	{0}	{0}	1
1	s_2	{2}	{2}	1
2	s_4	{4}	{4}	1
3	s_6	{6}	{6}	1
4	s_8	{8}	{8}	1
5	s_1	{1}	{1}	1
6	s_3	{3}	{3}	1
7	s_5	{5}	{5}	1
8	s_7	{7}	{7}	1
9	s_9	{9}	{9}	1

A concrete witness is $0, 2, 4, 6, 8, 1, 3, 5, 7, 9$. Each adjacent pair occurs in the concrete successor table.

20.3 F5.3 owners 3, phases 5, transfer 0, target owner 2

Advancing and restarting preserve the owner. Owner zero is an invariant and its entire phase cycle is reachable.

The concrete carrier has 15 states and 19 edges. Its reachable set has 5 states. The reachable set is disjoint from the bad predicate.

ID	State	Seed	Bd.	d	p	Successors
0	(0, 0)	0	I	0	–	{3}
1	(0, 1)	0	–	∞	–	{4}
2	(0, 2)	0	–	∞	–	{5}
3	(1, 0)	1	–	1	0	{6}
4	(1, 1)	1	–	∞	–	{1, 7}
5	(1, 2)	1	–	∞	–	{8}
6	(2, 0)	2	–	2	3	{0, 9}
7	(2, 1)	2	–	∞	–	{10}
8	(2, 2)	2	–	∞	–	{2, 11}
9	(3, 0)	3	–	3	6	{12}
10	(3, 1)	3	–	∞	–	{1, 13}
11	(3, 2)	3	–	∞	–	{14}
12	(4, 0)	4	–	4	9	{0}
13	(4, 1)	4	–	∞	–	{1}
14	(4, 2)	4	B	∞	–	{2}

Round 1. The partition has 5 nonempty cells, and the may relation has 8 edges.

The selected path is b_0, b_1, b_2, b_3, b_4. The exact number of concrete lifts is 0.

j	a_j	F_j	D_j	$\#\cap$
0	b_0	{0}	{2}	0
1	b_1	{3}	{5}	0
2	b_2	{6}	{8}	0
3	b_3	{9}	{11}	0
4	b_4	{12}	{14}	0

The isolated state IDs are $\{0, 1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12, 13, 14\}$. The refined graph has zero paths projecting to this candidate.

Round 2. The partition has 15 nonempty cells, and the may relation has 19 edges.

The abstract checker found the inductive invariant $\{s_0, s_3, s_6, s_9, s_{12}\}$. It contains $\{s_0\}$ and is disjoint from $\{s_{14}\}$.

Cell	All may successors
s_0	$\{s_3\}$
s_3	$\{s_6\}$
s_6	$\{s_0, s_9\}$
s_9	$\{s_{12}\}$
s_{12}	$\{s_0\}$

20.4 F5.4 owners 4, phases 4, transfer 0, target owner 1

Advancing and restarting preserve the owner. Owner zero is an invariant and its entire phase cycle is reachable.

The concrete carrier has 16 states and 20 edges. Its reachable set has 4 states. The reachable set is disjoint from the bad predicate.

ID	State	Seed	Bd.	d	p	Successors
0	(0, 0)	0	I	0	–	{4}
1	(0, 1)	0	–	∞	–	{5}
2	(0, 2)	0	–	∞	–	{6}
3	(0, 3)	0	–	∞	–	{7}
4	(1, 0)	1	–	1	0	{8}
5	(1, 1)	1	–	∞	–	{1, 9}
6	(1, 2)	1	–	∞	–	{10}
7	(1, 3)	1	–	∞	–	{3, 11}
8	(2, 0)	2	–	2	4	{0, 12}
9	(2, 1)	2	–	∞	–	{13}
10	(2, 2)	2	–	∞	–	{2, 14}
11	(2, 3)	2	–	∞	–	{15}
12	(3, 0)	3	–	3	8	{0}
13	(3, 1)	3	B	∞	–	{1}
14	(3, 2)	3	–	∞	–	{2}
15	(3, 3)	3	–	∞	–	{3}

Round 1. The partition has 4 nonempty cells, and the may relation has 6 edges.

The selected path is b_0, b_1, b_2, b_3. The exact number of concrete lifts is 0.

j	a_j	F_j	D_j	$\#\cap$
0	b_0	{0}	{1}	0
1	b_1	{4}	{5}	0
2	b_2	{8}	{9}	0
3	b_3	{12}	{13}	0

The isolated state IDs are $\{0, 1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12, 13, 14, 15\}$. The refined graph has zero paths projecting to this candidate.

Round 2. The partition has 16 nonempty cells, and the may relation has 20 edges.

The abstract checker found the inductive invariant $\{s_0, s_4, s_8, s_{12}\}$. It contains $\{s_0\}$ and is disjoint from $\{s_{13}\}$.

Cell	All may successors
s_0	$\{s_4\}$
s_4	$\{s_8\}$
s_8	$\{s_0, s_{12}\}$
s_{12}	$\{s_0\}$

20.5 F5.5 owners 4, phases 5, transfer 1, target owner 3

Advancing and restarting preserve the owner. A transfer at the final phase advances the owner by one, after which all phases of that owner are reachable.

The concrete carrier has 20 states and 29 edges. Its reachable set has 20 states. The bad predicate is reached.

ID	State	Seed	Bd.	d	p	Successors
0	(0, 0)	0	I	0	–	{4}
1	(0, 1)	0	–	5	16	{5}
2	(0, 2)	0	–	10	17	{6}
3	(0, 3)	0	–	15	18	{7}
4	(1, 0)	1	–	1	0	{8}
5	(1, 1)	1	–	6	1	{1, 9}
6	(1, 2)	1	–	11	2	{10}
7	(1, 3)	1	–	16	3	{3, 11}
8	(2, 0)	2	–	2	4	{0, 12}
9	(2, 1)	2	–	7	5	{13}
10	(2, 2)	2	–	12	6	{2, 14}
11	(2, 3)	2	–	17	7	{15}
12	(3, 0)	3	–	3	8	{16}
13	(3, 1)	3	–	8	9	{1, 17}
14	(3, 2)	3	–	13	10	{18}
15	(3, 3)	3	–	18	11	{3, 19}
16	(4, 0)	4	–	4	12	{0, 1}
17	(4, 1)	4	–	9	13	{1, 2}
18	(4, 2)	4	–	14	14	{2, 3}
19	(4, 3)	4	B	19	15	{3}

Round 1. The partition has 5 nonempty cells, and the may relation has 8 edges.

The selected path is b_0, b_1, b_2, b_3, b_4. The exact number of concrete lifts is 0.

j	a_j	F_j	D_j	$\#\cap$
0	b_0	{0}	{3}	0
1	b_1	{4}	{7}	0
2	b_2	{8}	{11}	0
3	b_3	{12}	{15}	0
4	b_4	{16}	{19}	0

The isolated state IDs are $\{0, 1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12, 13, 14, 15, 16, 17, 18, 19\}$. The refined graph has zero paths projecting to this candidate.

Round 2. The partition has 20 nonempty cells, and the may relation has 29 edges.

The selected path is $s_0, s_4, s_8, s_{12}, s_{16}, s_1, s_5, s_9, s_{13}, s_{17}, s_2, s_6, s_{10}, s_{14}, s_{18}, s_3, s_7, s_{11}, s_{15}, s_{19}$. The exact number of concrete lifts is 1.

j	a_j	F_j	D_j	$\#\cap$
0	s_0	$\{0\}$	$\{0\}$	1
1	s_4	$\{4\}$	$\{4\}$	1
2	s_8	$\{8\}$	$\{8\}$	1
3	s_{12}	$\{12\}$	$\{12\}$	1
4	s_{16}	$\{16\}$	$\{16\}$	1
5	s_1	$\{1\}$	$\{1\}$	1
6	s_5	$\{5\}$	$\{5\}$	1
7	s_9	$\{9\}$	$\{9\}$	1
8	s_{13}	$\{13\}$	$\{13\}$	1
9	s_{17}	$\{17\}$	$\{17\}$	1
10	s_2	$\{2\}$	$\{2\}$	1
11	s_6	$\{6\}$	$\{6\}$	1
12	s_{10}	$\{10\}$	$\{10\}$	1

j	a_j	F_j	D_j	$\#\cap$
13	s_{14}	$\{14\}$	$\{14\}$	1
14	s_{18}	$\{18\}$	$\{18\}$	1
15	s_3	$\{3\}$	$\{3\}$	1
16	s_7	$\{7\}$	$\{7\}$	1
17	s_{11}	$\{11\}$	$\{11\}$	1
18	s_{15}	$\{15\}$	$\{15\}$	1
19	s_{19}	$\{19\}$	$\{19\}$	1

A concrete witness is $0, 4, 8, 12, 16, 1, 5, 9, 13, 17, 2, 6, 10, 14, 18, 3, 7, 11, 15, 19$. Each adjacent pair occurs in the concrete successor table.

20.6 F5.6 owners 5, phases 5, transfer 0, target owner 4

Advancing and restarting preserve the owner. Owner zero is an invariant and its entire phase cycle is reachable.

The concrete carrier has 25 states and 32 edges. Its reachable set has 5 states. The reachable set is disjoint from the bad predicate.

ID	State	Seed	Bd.	d	p	Successors
0	(0, 0)	0	I	0	–	$\{5\}$
1	(0, 1)	0	–	∞	–	$\{6\}$
2	(0, 2)	0	–	∞	–	$\{7\}$
3	(0, 3)	0	–	∞	–	$\{8\}$
4	(0, 4)	0	–	∞	–	$\{9\}$
5	(1, 0)	1	–	1	0	$\{10\}$
6	(1, 1)	1	–	∞	–	$\{1, 11\}$
7	(1, 2)	1	–	∞	–	$\{12\}$
8	(1, 3)	1	–	∞	–	$\{3, 13\}$
9	(1, 4)	1	–	∞	–	$\{14\}$
10	(2, 0)	2	–	2	5	$\{0, 15\}$
11	(2, 1)	2	–	∞	–	$\{16\}$
12	(2, 2)	2	–	∞	–	$\{2, 17\}$
13	(2, 3)	2	–	∞	–	$\{18\}$
14	(2, 4)	2	–	∞	–	$\{4, 19\}$
15	(3, 0)	3	–	3	10	$\{20\}$
16	(3, 1)	3	–	∞	–	$\{1, 21\}$
17	(3, 2)	3	–	∞	–	$\{22\}$
18	(3, 3)	3	–	∞	–	$\{3, 23\}$
19	(3, 4)	3	–	∞	–	$\{24\}$
20	(4, 0)	4	–	4	15	$\{0\}$
21	(4, 1)	4	–	∞	–	$\{1\}$
22	(4, 2)	4	–	∞	–	$\{2\}$
23	(4, 3)	4	–	∞	–	$\{3\}$
24	(4, 4)	4	B	∞	–	$\{4\}$

Round 1. The partition has 5 nonempty cells, and the may relation has 8 edges.

The selected path is b_0, b_1, b_2, b_3, b_4. The exact number of concrete lifts is 0.

j	a_j	F_j	D_j	$\#\cap$
0	b_0	$\{0\}$	$\{4\}$	0
1	b_1	$\{5\}$	$\{9\}$	0
2	b_2	$\{10\}$	$\{14\}$	0
3	b_3	$\{15\}$	$\{19\}$	0
4	b_4	$\{20\}$	$\{24\}$	0

The isolated state IDs are $\{0, 1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12, 13, 14, 15, 16, 17, 18, 19, 20, 21, 22, 23, 24\}$. The refined graph has zero paths projecting to this candidate.

Round 2. The partition has 25 nonempty cells, and the may relation has 32 edges.

The abstract checker found the inductive invariant $\{s_0, s_5, s_{10}, s_{15}, s_{20}\}$. It contains $\{s_0\}$ and is disjoint from $\{s_{24}\}$.

Cell	All may successors
s_0	$\{s_5\}$
s_5	$\{s_{10}\}$
s_{10}	$\{s_0, s_{15}\}$
s_{15}	$\{s_{20}\}$
s_{20}	$\{s_0\}$

20.7 F5.7 owners 6, phases 4, transfer 1, target owner 5

Advancing and restarting preserve the owner. A transfer at the final phase advances the owner by one, after which all phases of that owner are reachable.

The concrete carrier has 24 states and 35 edges. Its reachable set has 24 states. The bad predicate is reached.

ID	State	Seed	Bd.	d	p	Successors
0	(0, 0)	0	I	0	–	{6}
1	(0, 1)	0	–	4	18	{7}
2	(0, 2)	0	–	8	19	{8}
3	(0, 3)	0	–	12	20	{9}
4	(0, 4)	0	–	16	21	{10}
5	(0, 5)	0	–	20	22	{11}
6	(1, 0)	1	–	1	0	{12}
7	(1, 1)	1	–	5	1	{1, 13}
8	(1, 2)	1	–	9	2	{14}
9	(1, 3)	1	–	13	3	{3, 15}
10	(1, 4)	1	–	17	4	{16}
11	(1, 5)	1	–	21	5	{5, 17}
12	(2, 0)	2	–	2	6	{0, 18}

ID	State	Seed	Bd.	d	p	Successors
13	(2, 1)	2	–	6	7	{19}
14	(2, 2)	2	–	10	8	{2, 20}
15	(2, 3)	2	–	14	9	{21}
16	(2, 4)	2	–	18	10	{4, 22}
17	(2, 5)	2	–	22	11	{23}
18	(3, 0)	3	–	3	12	{0, 1}
19	(3, 1)	3	–	7	13	{1, 2}
20	(3, 2)	3	–	11	14	{2, 3}
21	(3, 3)	3	–	15	15	{3, 4}
22	(3, 4)	3	–	19	16	{4, 5}
23	(3, 5)	3	B	23	17	{5}

Round 1. The partition has 4 nonempty cells, and the may relation has 6 edges.

The selected path is b_0, b_1, b_2, b_3. The exact number of concrete lifts is 0.

j	a_j	F_j	D_j	$\#\cap$
0	b_0	{0}	{5}	0
1	b_1	{6}	{11}	0
2	b_2	{12}	{17}	0
3	b_3	{18}	{23}	0

The isolated state IDs are $\{0, 1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12, 13, 14, 15, 16, 17, 18, 19, 20, 21, 22, 23\}$. The refined graph has zero paths projecting to this candidate.

Round 2. The partition has 24 nonempty cells, and the may relation has 35 edges.

The selected path is $s_0, s_6, s_{12}, s_{18}, s_1, s_7, s_{13}, s_{19}, s_2, s_8, s_{14}, s_{20}, s_3, s_9, s_{15}, s_{21}, s_4, s_{10}, s_{16}, s_{22}, s_5, s_{11}, s_{17}, s_{23}$. The exact number of concrete lifts is 1.

j	a_j	F_j	D_j	$\#\cap$
0	s_0	{0}	{0}	1
1	s_6	{6}	{6}	1
2	s_{12}	{12}	{12}	1
3	s_{18}	{18}	{18}	1
4	s_1	{1}	{1}	1
5	s_7	{7}	{7}	1
6	s_{13}	{13}	{13}	1
7	s_{19}	{19}	{19}	1
8	s_2	{2}	{2}	1
9	s_8	{8}	{8}	1
10	s_{14}	{14}	{14}	1
11	s_{20}	{20}	{20}	1
12	s_3	{3}	{3}	1

j	a_j	F_j	D_j	$\#\cap$
13	s_9	{9}	{9}	1
14	s_{15}	{15}	{15}	1
15	s_{21}	{21}	{21}	1
16	s_4	{4}	{4}	1
17	s_{10}	{10}	{10}	1
18	s_{16}	{16}	{16}	1
19	s_{22}	{22}	{22}	1
20	s_5	{5}	{5}	1
21	s_{11}	{11}	{11}	1
22	s_{17}	{17}	{17}	1
23	s_{23}	{23}	{23}	1

A concrete witness is $0, 6, 12, 18, 1, 7, 13, 19, 2, 8, 14, 20, 3, 9, 15, 21, 4, 10, 16, 22, 5, 11, 17, 23$. Each adjacent pair occurs in the concrete successor table.

20.8 F5.8 owners 6, phases 5, transfer 0, target owner 2

Advancing and restarting preserve the owner. Owner zero is an invariant and its entire phase cycle is reachable.

The concrete carrier has 30 states and 39 edges. Its reachable set has 5 states. The reachable set is disjoint from the bad predicate.

ID	State	Seed	Bd.	d	p	Successors
0	(0, 0)	0	I	0	–	{6}
1	(0, 1)	0	–	∞	–	{7}
2	(0, 2)	0	–	∞	–	{8}
3	(0, 3)	0	–	∞	–	{9}
4	(0, 4)	0	–	∞	–	{10}
5	(0, 5)	0	–	∞	–	{11}
6	(1, 0)	1	–	1	0	{12}
7	(1, 1)	1	–	∞	–	{1, 13}
8	(1, 2)	1	–	∞	–	{14}
9	(1, 3)	1	–	∞	–	{3, 15}
10	(1, 4)	1	–	∞	–	{16}
11	(1, 5)	1	–	∞	–	{5, 17}
12	(2, 0)	2	–	2	6	{0, 18}

ID	State	Seed	Bd.	d	p	Successors
13	(2, 1)	2	–	∞	–	$\{19\}$
14	(2, 2)	2	–	∞	–	$\{2, 20\}$
15	(2, 3)	2	–	∞	–	$\{21\}$
16	(2, 4)	2	–	∞	–	$\{4, 22\}$
17	(2, 5)	2	–	∞	–	$\{23\}$
18	(3, 0)	3	–	3	12	$\{24\}$
19	(3, 1)	3	–	∞	–	$\{1, 25\}$
20	(3, 2)	3	–	∞	–	$\{26\}$
21	(3, 3)	3	–	∞	–	$\{3, 27\}$
22	(3, 4)	3	–	∞	–	$\{28\}$
23	(3, 5)	3	–	∞	–	$\{5, 29\}$
24	(4, 0)	4	–	4	18	$\{0\}$
25	(4, 1)	4	–	∞	–	$\{1\}$

ID	State	Seed	Bd.	d	p	Successors
26	(4, 2)	4	B	∞	–	$\{2\}$
27	(4, 3)	4	–	∞	–	$\{3\}$
28	(4, 4)	4	–	∞	–	$\{4\}$
29	(4, 5)	4	–	∞	–	$\{5\}$

Round 1. The partition has 5 nonempty cells, and the may relation has 8 edges.

The selected path is b_0, b_1, b_2, b_3, b_4. The exact number of concrete lifts is 0.

j	a_j	F_j	D_j	$\#\cap$
0	b_0	$\{0\}$	$\{2\}$	0
1	b_1	$\{6\}$	$\{8\}$	0
2	b_2	$\{12\}$	$\{14\}$	0
3	b_3	$\{18\}$	$\{20\}$	0
4	b_4	$\{24\}$	$\{26\}$	0

The isolated state IDs are $\{0, 1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12, 13, 14, 15, 16, 17, 18, 19, 20, 21, 22, 23, 24, 25, 26, 27, 28, 29\}$. The refined graph has zero paths projecting to this candidate.

Round 2. The partition has 30 nonempty cells, and the may relation has 39 edges.

The abstract checker found the inductive invariant $\{s_0, s_6, s_{12}, s_{18}, s_{24}\}$. It contains $\{s_0\}$ and is disjoint from $\{s_{26}\}$.

Cell	All may successors
s_0	$\{s_6\}$
s_6	$\{s_{12}\}$
s_{12}	$\{s_0, s_{18}\}$
s_{18}	$\{s_{24}\}$
s_{24}	$\{s_0\}$

20.9 F5.9 owners 5, phases 6, transfer 0, target owner 0

Advancing and restarting preserve the owner. Owner zero is an invariant and its entire phase cycle is reachable.

The concrete carrier has 30 states and 40 edges. Its reachable set has 6 states. The bad predicate is reached.

ID	State	Seed	Bd.	d	p	Successors
0	(0, 0)	0	I	0	–	$\{5\}$
1	(0, 1)	0	–	∞	–	$\{6\}$
2	(0, 2)	0	–	∞	–	$\{7\}$
3	(0, 3)	0	–	∞	–	$\{8\}$
4	(0, 4)	0	–	∞	–	$\{9\}$
5	(1, 0)	1	–	1	0	$\{10\}$
6	(1, 1)	1	–	∞	–	$\{1, 11\}$
7	(1, 2)	1	–	∞	–	$\{12\}$
8	(1, 3)	1	–	∞	–	$\{3, 13\}$
9	(1, 4)	1	–	∞	–	$\{14\}$
10	(2, 0)	2	–	2	5	$\{0, 15\}$
11	(2, 1)	2	–	∞	–	$\{16\}$
12	(2, 2)	2	–	∞	–	$\{2, 17\}$

ID	State	Seed	Bd.	d	p	Successors
13	(2, 3)	2	–	∞	–	$\{18\}$
14	(2, 4)	2	–	∞	–	$\{4, 19\}$
15	(3, 0)	3	–	3	10	$\{20\}$
16	(3, 1)	3	–	∞	–	$\{1, 21\}$
17	(3, 2)	3	–	∞	–	$\{22\}$
18	(3, 3)	3	–	∞	–	$\{3, 23\}$
19	(3, 4)	3	–	∞	–	$\{24\}$
20	(4, 0)	4	–	4	15	$\{0, 25\}$
21	(4, 1)	4	–	∞	–	$\{26\}$
22	(4, 2)	4	–	∞	–	$\{2, 27\}$
23	(4, 3)	4	–	∞	–	$\{28\}$
24	(4, 4)	4	–	∞	–	$\{4, 29\}$
25	(5, 0)	5	B	5	20	$\{0\}$

ID	State	Seed	Bd.	d	p	Successors
26	(5, 1)	5	–	∞	–	$\{1\}$
27	(5, 2)	5	–	∞	–	$\{2\}$
28	(5, 3)	5	–	∞	–	$\{3\}$
29	(5, 4)	5	–	∞	–	$\{4\}$

Round 1. The partition has 6 nonempty cells, and the may relation has 10 edges.

The selected path is $b_0, b_1, b_2, b_3, b_4, b_5$. The exact number of concrete lifts is 1.

j	a_j	F_j	D_j	$\#\cap$
0	b_0	$\{0\}$	$\{0\}$	1
1	b_1	$\{5\}$	$\{5\}$	1
2	b_2	$\{10\}$	$\{10\}$	1
3	b_3	$\{15\}$	$\{15\}$	1
4	b_4	$\{20\}$	$\{20\}$	1
5	b_5	$\{25\}$	$\{25\}$	1

A concrete witness is $0, 5, 10, 15, 20, 25$. Each adjacent pair occurs in the concrete successor table.

21 Two cyclic rails

Position advances by the indicated stride on either rail. A return link is available at positions divisible by three.

Only the optional directed bridge permits entering the second rail.

Tables enumerate concrete states by integer ID. The d column gives distance from an initial state, and p gives a predecessor attaining that distance. Columns I and B are encoded by the boundary field. Abstract keys b_i are seed blocks and s_x are isolated states.

21.1 F6.1 length 8, stride 2, bridge None, target 6

Every edge preserves the position residue modulo 2. Return links leave the second rail. The specified bridge is the only way to enter that rail. Its source and the bad position must both lie in the initial residue class.

The concrete carrier has 16 states and 19 edges. Its reachable set has 4 states. The reachable set is disjoint from the bad predicate.

ID	State	Seed	Bd.	d	p	Successors
0	(0, 0)	0	I	0	–	{2}
1	(0, 1)	1	–	∞	–	{3}
2	(0, 2)	2	–	1	0	{4}
3	(0, 3)	3	–	∞	–	{5}
4	(0, 4)	0	–	2	2	{6}
5	(0, 5)	1	–	∞	–	{7}
6	(0, 6)	2	–	3	4	{0}
7	(0, 7)	3	–	∞	–	{1}
8	(1, 0)	0	–	∞	–	{0, 10}
9	(1, 1)	1	–	∞	–	{11}
10	(1, 2)	2	–	∞	–	{12}
11	(1, 3)	3	–	∞	–	{3, 13}
12	(1, 4)	0	–	∞	–	{14}

ID	State	Seed	Bd.	d	p	Successors
13	(1, 5)	1	–	∞	–	{15}
14	(1, 6)	2	B	∞	–	{6, 8}
15	(1, 7)	3	–	∞	–	{9}

Round 1. The partition has 4 nonempty cells, and the may relation has 7 edges.

The selected path is b_0, b_2. The exact number of concrete lifts is 0.

j	a_j	F_j	D_j	#∩
0	b_0	{0}	{12}	0
1	b_2	{2}	{14}	0

The isolated state IDs are $\{0, 2, 4, 6, 8, 10, 12, 14\}$. The refined graph has zero paths projecting to this candidate.

Round 2. The partition has 10 nonempty cells, and the may relation has 13 edges.

The abstract checker found the inductive invariant $\{s_0, s_2, s_4, s_6\}$. It contains $\{s_0\}$ and is disjoint from $\{s_{14}\}$.

Cell	All may successors
s_0	$\{s_2\}$
s_2	$\{s_4\}$
s_4	$\{s_6\}$
s_6	$\{s_0\}$

21.2 F6.2 length 10, stride 2, bridge 5, target 8

Every edge preserves the position residue modulo 2. Return links leave the second rail. The specified bridge is the only way to enter that rail. Its source and the bad position must both lie in the initial residue class.

The concrete carrier has 20 states and 25 edges. Its reachable set has 5 states. The reachable set is disjoint from the bad predicate.

ID	State	Seed	Bd.	d	p	Successors
0	(0, 0)	0	I	0	–	{2}
1	(0, 1)	1	–	∞	–	{3}
2	(0, 2)	2	–	1	0	{4}
3	(0, 3)	3	–	∞	–	{5}
4	(0, 4)	0	–	2	2	{6}
5	(0, 5)	1	–	∞	–	{7, 15}
6	(0, 6)	2	–	3	4	{8}
7	(0, 7)	3	–	∞	–	{9}
8	(0, 8)	0	–	4	6	{0}
9	(0, 9)	1	–	∞	–	{1}
10	(1, 0)	0	–	∞	–	{0, 12}
11	(1, 1)	1	–	∞	–	{13}
12	(1, 2)	2	–	∞	–	{14}

ID	State	Seed	Bd.	d	p	Successors
13	(1, 3)	3	–	∞	–	{3, 15}
14	(1, 4)	0	–	∞	–	{16}
15	(1, 5)	1	–	∞	–	{17}
16	(1, 6)	2	–	∞	–	{6, 18}
17	(1, 7)	3	–	∞	–	{19}
18	(1, 8)	0	B	∞	–	{10}
19	(1, 9)	1	–	∞	–	{9, 11}

Round 1. The partition has 4 nonempty cells, and the may relation has 8 edges.

The selected path is b_0. The exact number of concrete lifts is 0.

j	a_j	F_j	D_j	#∩
0	b_0	{0}	{18}	0

The isolated state IDs are $\{0, 4, 8, 10, 14, 18\}$. The refined graph has zero paths projecting to this candidate.

Round 2. The partition has 9 nonempty cells, and the may relation has 16 edges.

The selected path is s_0, b_2, s_{18}. The exact number of concrete lifts is 0.

j	a_j	F_j	D_j	$\#\cap$
0	s_0	$\{0\}$	$\varnothing$	0
1	b_2	$\{2\}$	$\{16\}$	0
2	s_{18}	$\varnothing$	$\{18\}$	0

The isolated state IDs are $\{2, 6, 12, 16\}$. The refined graph has zero paths projecting to this candidate.

Round 3. The partition has 12 nonempty cells, and the may relation has 16 edges.

The abstract checker found the inductive invariant $\{s_0, s_2, s_4, s_6, s_8\}$. It contains $\{s_0\}$ and is disjoint from $\{s_{18}\}$.

Cell	All may successors
s_0	$\{s_2\}$
s_2	$\{s_4\}$
s_4	$\{s_6\}$
s_6	$\{s_8\}$
s_8	$\{s_0\}$

21.3 F6.3 length 12, stride 3, bridge 6, target 9

Every edge preserves the position residue modulo 3. Return links leave the second rail. The specified bridge is the only way to enter that rail. Its source and the bad position must both lie in the initial residue class.

The concrete carrier has 24 states and 29 edges. Its reachable set has 8 states. The bad predicate is reached.

ID	State	Seed	Bd.	d	p	Successors
0	(0, 0)	0	I	0	–	{3}
1	(0, 1)	1	–	∞	–	{4}
2	(0, 2)	2	–	∞	–	{5}
3	(0, 3)	3	–	1	0	{6}
4	(0, 4)	0	–	∞	–	{7}
5	(0, 5)	1	–	∞	–	{8}
6	(0, 6)	2	–	2	3	{9, 18}
7	(0, 7)	3	–	∞	–	{10}
8	(0, 8)	0	–	∞	–	{11}
9	(0, 9)	1	–	3	6	{0}
10	(0, 10)	2	–	∞	–	{1}
11	(0, 11)	3	–	∞	–	{2}
12	(1, 0)	0	–	5	21	{0, 15}
13	(1, 1)	1	–	∞	–	{16}
14	(1, 2)	2	–	∞	–	{17}
15	(1, 3)	3	–	6	12	{3, 18}
16	(1, 4)	0	–	∞	–	{19}
17	(1, 5)	1	–	∞	–	{20}
18	(1, 6)	2	–	3	6	{6, 21}
19	(1, 7)	3	–	∞	–	{22}
20	(1, 8)	0	–	∞	–	{23}
21	(1, 9)	1	B	4	18	{9, 12}
22	(1, 10)	2	–	∞	–	{13}
23	(1, 11)	3	–	∞	–	{14}

Round 1. The partition has 4 nonempty cells, and the may relation has 8 edges.

The selected path is b_0, b_3, b_2, b_1. The exact number of concrete lifts is 0.

j	a_j	F_j	D_j	$\#\cap$
0	b_0	$\{0\}$	$\{12\}$	0
1	b_3	$\{3\}$	$\{15\}$	0
2	b_2	$\{6\}$	$\{18\}$	0
3	b_1	$\{9\}$	$\{21\}$	0

The isolated state IDs are $\{0, 1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12, 13, 14, 15, 16, 17, 18, 19, 20, 21, 22, 23\}$. The refined graph has zero paths projecting to this candidate.

Round 2. The partition has 24 nonempty cells, and the may relation has 29 edges.

The selected path is $s_0, s_3, s_6, s_{18}, s_{21}$. The exact number of concrete lifts is 1.

j	a_j	F_j	D_j	$\#\cap$
0	s_0	$\{0\}$	$\{0\}$	1
1	s_3	$\{3\}$	$\{3\}$	1
2	s_6	$\{6\}$	$\{6\}$	1
3	s_{18}	$\{18\}$	$\{18\}$	1
4	s_{21}	$\{21\}$	$\{21\}$	1

A concrete witness is $0, 3, 6, 18, 21$. Each adjacent pair occurs in the concrete successor table.

21.4 F6.4 length 14, stride 2, bridge 4, target 9

Every edge preserves the position residue modulo 2. Return links leave the second rail. The specified bridge is the only way to enter that rail. Its source and the bad position must both lie in the initial residue class.

The concrete carrier has 28 states and 34 edges. Its reachable set has 14 states. The reachable set is disjoint from the bad predicate.

ID	State	Seed	Bd.	d	p	Successors
0	(0, 0)	0	I	0	–	{2}
1	(0, 1)	1	–	∞	–	{3}
2	(0, 2)	2	–	1	0	{4}
3	(0, 3)	3	–	∞	–	{5}
4	(0, 4)	0	–	2	2	{6, 18}
5	(0, 5)	1	–	∞	–	{7}
6	(0, 6)	2	–	3	4	{8}
7	(0, 7)	3	–	∞	–	{9}
8	(0, 8)	0	–	4	6	{10}
9	(0, 9)	1	–	∞	–	{11}
10	(0, 10)	2	–	5	8	{12}
11	(0, 11)	3	–	∞	–	{13}
12	(0, 12)	0	–	6	10	{0}

ID	State	Seed	Bd.	d	p	Successors
13	(0, 13)	1	–	∞	–	$\{1\}$
14	(1, 0)	0	–	8	26	$\{0, 16\}$
15	(1, 1)	1	–	∞	–	$\{17\}$
16	(1, 2)	2	–	9	14	$\{18\}$
17	(1, 3)	3	–	∞	–	$\{3, 19\}$
18	(1, 4)	0	–	3	4	$\{20\}$
19	(1, 5)	1	–	∞	–	$\{21\}$
20	(1, 6)	2	–	4	18	$\{6, 22\}$
21	(1, 7)	3	–	∞	–	$\{23\}$
22	(1, 8)	0	–	5	20	$\{24\}$
23	(1, 9)	1	B	∞	–	$\{9, 25\}$
24	(1, 10)	2	–	6	22	$\{26\}$
25	(1, 11)	3	–	∞	–	$\{27\}$
26	(1, 12)	0	–	7	24	$\{12, 14\}$
27	(1, 13)	1	–	∞	–	$\{15\}$

Round 1. The partition has 4 nonempty cells, and the may relation has 8 edges.

The abstract checker found the inductive invariant $\{b_0, b_2\}$. It contains $\{b_0\}$ and is disjoint from $\{b_1\}$.

Cell	All may successors
b_0	$\{b_0, b_2\}$
b_2	$\{b_0, b_2\}$

21.5 F6.5 length 15, stride 5, bridge 10, target 5

Every edge preserves the position residue modulo 5. Return links leave the second rail. The specified bridge is the only way to enter that rail. Its source and the bad position must both lie in the initial residue class.

The concrete carrier has 30 states and 36 edges. Its reachable set has 6 states. The bad predicate is reached.

ID	State	Seed	Bd.	d	p	Successors
0	(0, 0)	0	I	0	–	$\{5\}$
1	(0, 1)	1	–	∞	–	$\{6\}$
2	(0, 2)	2	–	∞	–	$\{7\}$
3	(0, 3)	3	–	∞	–	$\{8\}$
4	(0, 4)	0	–	∞	–	$\{9\}$
5	(0, 5)	1	–	1	0	$\{10\}$
6	(0, 6)	2	–	∞	–	$\{11\}$
7	(0, 7)	3	–	∞	–	$\{12\}$
8	(0, 8)	0	–	∞	–	$\{13\}$
9	(0, 9)	1	–	∞	–	$\{14\}$
10	(0, 10)	2	–	2	5	$\{0, 25\}$
11	(0, 11)	3	–	∞	–	$\{1\}$
12	(0, 12)	0	–	∞	–	$\{2\}$
13	(0, 13)	1	–	∞	–	$\{3\}$
14	(0, 14)	2	–	∞	–	$\{4\}$
15	(1, 0)	0	–	4	25	$\{0, 20\}$
16	(1, 1)	1	–	∞	–	$\{21\}$
17	(1, 2)	2	–	∞	–	$\{22\}$
18	(1, 3)	3	–	∞	–	$\{3, 23\}$
19	(1, 4)	0	–	∞	–	$\{24\}$
20	(1, 5)	1	B	5	15	$\{25\}$
21	(1, 6)	2	–	∞	–	$\{6, 26\}$
22	(1, 7)	3	–	∞	–	$\{27\}$
23	(1, 8)	0	–	∞	–	$\{28\}$
24	(1, 9)	1	–	∞	–	$\{9, 29\}$
25	(1, 10)	2	–	3	10	$\{15\}$
26	(1, 11)	3	–	∞	–	$\{16\}$
27	(1, 12)	0	–	∞	–	$\{12, 17\}$
28	(1, 13)	1	–	∞	–	$\{18\}$
29	(1, 14)	2	–	∞	–	$\{19\}$

Round 1. The partition has 4 nonempty cells, and the may relation has 12 edges.

The selected path is b_0, b_1. The exact number of concrete lifts is 0.

j	a_j	F_j	D_j	$\#\cap$
0	b_0	$\{0\}$	$\{15\}$	0
1	b_1	$\{5\}$	$\{20\}$	0

The isolated state IDs are $\{0, 1, 4, 5, 8, 9, 12, 13, 15, 16, 19, 20, 23, 24, 27, 28\}$. The refined graph has zero paths projecting to this candidate.

Round 2. The partition has 18 nonempty cells, and the may relation has 32 edges.

The selected path is $s_0, s_5, b_2, s_{15}, s_{20}$. The exact number of concrete lifts is 0.

j	a_j	F_j	D_j	$\#\cap$
0	s_0	$\{0\}$	$\varnothing$	0
1	s_5	$\{5\}$	$\varnothing$	0
2	b_2	$\{10\}$	$\{25\}$	0
3	s_{15}	$\varnothing$	$\{15\}$	0
4	s_{20}	$\varnothing$	$\{20\}$	0

The isolated state IDs are $\{2, 6, 10, 14, 17, 21, 25, 29\}$. The refined graph has zero paths projecting to this candidate.

Round 3. The partition has 25 nonempty cells, and the may relation has 36 edges.

The selected path is $s_0, s_5, s_{10}, s_{25}, s_{15}, s_{20}$. The exact number of concrete lifts is 1.

j	a_j	F_j	D_j	$\#\cap$
0	s_0	$\{0\}$	$\{0\}$	1
1	s_5	$\{5\}$	$\{5\}$	1
2	s_{10}	$\{10\}$	$\{10\}$	1
3	s_{25}	$\{25\}$	$\{25\}$	1
4	s_{15}	$\{15\}$	$\{15\}$	1
5	s_{20}	$\{20\}$	$\{20\}$	1

A concrete witness is $0, 5, 10, 25, 15, 20$. Each adjacent pair occurs in the concrete successor table.

21.6 F6.6 length 16, stride 4, bridge None, target 12

Every edge preserves the position residue modulo 4. Return links leave the second rail. The specified bridge is the only way to enter that rail. Its source and the bad position must both lie in the initial residue class.

The concrete carrier has 32 states and 38 edges. Its reachable set has 4 states. The reachable set is disjoint from the bad predicate.

ID	State	Seed	Bd.	d	p	Successors
0	(0, 0)	0	I	0	–	{4}
1	(0, 1)	1	–	∞	–	{5}
2	(0, 2)	2	–	∞	–	{6}
3	(0, 3)	3	–	∞	–	{7}
4	(0, 4)	0	–	1	0	{8}
5	(0, 5)	1	–	∞	–	{9}
6	(0, 6)	2	–	∞	–	{10}
7	(0, 7)	3	–	∞	–	{11}
8	(0, 8)	0	–	2	4	{12}
9	(0, 9)	1	–	∞	–	{13}
10	(0, 10)	2	–	∞	–	{14}
11	(0, 11)	3	–	∞	–	{15}
12	(0, 12)	0	–	3	8	{0}

ID	State	Seed	Bd.	d	p	Successors
13	(0, 13)	1	–	∞	–	{1}
14	(0, 14)	2	–	∞	–	{2}
15	(0, 15)	3	–	∞	–	{3}
16	(1, 0)	0	–	∞	–	{0, 20}
17	(1, 1)	1	–	∞	–	{21}
18	(1, 2)	2	–	∞	–	{22}
19	(1, 3)	3	–	∞	–	{3, 23}
20	(1, 4)	0	–	∞	–	{24}
21	(1, 5)	1	–	∞	–	{25}
22	(1, 6)	2	–	∞	–	{6, 26}
23	(1, 7)	3	–	∞	–	{27}
24	(1, 8)	0	–	∞	–	{28}
25	(1, 9)	1	–	∞	–	{9, 29}

ID	State	Seed	Bd.	d	p	Successors
26	(1, 10)	2	–	∞	–	{30}
27	(1, 11)	3	–	∞	–	{31}
28	(1, 12)	0	B	∞	–	{12, 16}
29	(1, 13)	1	–	∞	–	{17}
30	(1, 14)	2	–	∞	–	{18}
31	(1, 15)	3	–	∞	–	{15, 19}

Round 1. The partition has 4 nonempty cells, and the may relation has 4 edges.

The selected path is b_0. The exact number of concrete lifts is 0.

j	a_j	F_j	D_j	$\#\cap$
0	b_0	$\{0\}$	$\{28\}$	0

The isolated state IDs are $\{0, 4, 8, 12, 16, 20, 24, 28\}$. The refined graph has zero paths projecting to this candidate.

Round 2. The partition has 11 nonempty cells, and the may relation has 13 edges.

The abstract checker found the inductive invariant $\{s_0, s_4, s_8, s_{12}\}$. It contains $\{s_0\}$ and is disjoint from $\{s_{28}\}$.

Cell	All may successors
s_0	$\{s_4\}$
s_4	$\{s_8\}$
s_8	$\{s_{12}\}$
s_{12}	$\{s_0\}$

21.7 F6.7 length 17, stride 3, bridge 16, target 8

Every edge preserves the position residue modulo 1. Return links leave the second rail. The specified bridge is the only way to enter that rail. Its source and the bad position must both lie in the initial residue class.

The concrete carrier has 34 states and 41 edges. Its reachable set has 34 states. The bad predicate is reached.

ID	State	Seed	Bd.	d	p	Successors
0	(0, 0)	0	I	0	–	{3}
1	(0, 1)	1	–	6	15	{4}
2	(0, 2)	2	–	12	16	{5}
3	(0, 3)	3	–	1	0	{6}
4	(0, 4)	0	–	7	1	{7}
5	(0, 5)	1	–	13	2	{8}
6	(0, 6)	2	–	2	3	{9}
7	(0, 7)	3	–	8	4	{10}
8	(0, 8)	0	–	14	5	{11}
9	(0, 9)	1	–	3	6	{12}
10	(0, 10)	2	–	9	7	{13}
11	(0, 11)	3	–	15	8	{14}
12	(0, 12)	0	–	4	9	{15}

ID	State	Seed	Bd.	d	p	Successors
13	(0, 13)	1	–	10	10	{16}
14	(0, 14)	2	–	16	11	{0}
15	(0, 15)	3	–	5	12	{1}
16	(0, 16)	0	–	11	13	{2, 33}
17	(1, 0)	0	–	18	31	{0, 20}
18	(1, 1)	1	–	24	32	{21}
19	(1, 2)	2	–	13	33	{22}
20	(1, 3)	3	–	19	17	{3, 23}
21	(1, 4)	0	–	25	18	{24}
22	(1, 5)	1	–	14	19	{25}
23	(1, 6)	2	–	20	20	{6, 26}
24	(1, 7)	3	–	26	21	{27}
25	(1, 8)	0	B	15	22	{28}

ID	State	Seed	Bd.	d	p	Successors
26	(1, 9)	1	–	21	23	{9, 29}
27	(1, 10)	2	–	27	24	{30}
28	(1, 11)	3	–	16	25	{31}
29	(1, 12)	0	–	22	26	{12, 32}
30	(1, 13)	1	–	28	27	{33}
31	(1, 14)	2	–	17	28	{17}
32	(1, 15)	3	–	23	29	{15, 18}
33	(1, 16)	0	–	12	16	{19}

Round 1. The partition has 4 nonempty cells, and the may relation has 11 edges.

The selected path is b_0. The exact number of concrete lifts is 0.

j	a_j	F_j	D_j	$\#\cap$
0	b_0	{0}	{25}	0

The isolated state IDs are $\{0, 4, 8, 12, 16, 17, 21, 25, 29, 33\}$. The refined graph has zero paths projecting to this candidate.

Round 2. The partition has 13 nonempty cells, and the may relation has 29 edges.

The selected path is s_0, b_3, b_1, s_{25}. The exact number of concrete lifts is 0.

j	a_j	F_j	D_j	$\#\cap$
0	s_0	{0}	$\varnothing$	0
1	b_3	{3}	$\varnothing$	0
2	b_1	$\varnothing$	{22}	0
3	s_{25}	$\varnothing$	{25}	0

The isolated state IDs are $\{1, 3, 5, 7, 9, 11, 13, 15, 18, 20, 22, 24, 26, 28, 30, 32\}$. The refined graph has zero paths projecting to this candidate.

Round 3. The partition has 27 nonempty cells, and the may relation has 41 edges.

The selected path is $s_0, s_3, b_2, s_{22}, s_{25}$. The exact number of concrete lifts is 0.

j	a_j	F_j	D_j	$\#\cap$
0	s_0	{0}	$\varnothing$	0
1	s_3	{3}	$\varnothing$	0
2	b_2	{6}	{19}	0
3	s_{22}	$\varnothing$	{22}	0
4	s_{25}	$\varnothing$	{25}	0

The isolated state IDs are $\{2, 6, 10, 14, 19, 23, 27, 31\}$. The refined graph has zero paths projecting to this candidate.

Round 4. The partition has 34 nonempty cells, and the may relation has 41 edges.

The selected path is $s_0, s_3, s_6, s_9, s_{12}, s_{15}, s_1, s_4, s_7, s_{10}, s_{13}, s_{16}, s_{33}, s_{19}, s_{22}, s_{25}$. The exact number of concrete lifts is 1.

j	a_j	F_j	D_j	$\#\cap$
0	s_0	{0}	{0}	1
1	s_3	{3}	{3}	1
2	s_6	{6}	{6}	1
3	s_9	{9}	{9}	1
4	s_{12}	{12}	{12}	1
5	s_{15}	{15}	{15}	1
6	s_1	{1}	{1}	1
7	s_4	{4}	{4}	1
8	s_7	{7}	{7}	1
9	s_{10}	{10}	{10}	1
10	s_{13}	{13}	{13}	1
11	s_{16}	{16}	{16}	1
12	s_{33}	{33}	{33}	1

j	a_j	F_j	D_j	$\#\cap$
13	s_{19}	{19}	{19}	1
14	s_{22}	{22}	{22}	1
15	s_{25}	{25}	{25}	1

A concrete witness is $0, 3, 6, 9, 12, 15, 1, 4, 7, 10, 13, 16, 33, 19, 22, 25$. Each adjacent pair occurs in the concrete successor table.

21.8 F6.8 length 18, stride 6, bridge 12, target 15

Every edge preserves the position residue modulo 6. Return links leave the second rail. The specified bridge is the only way to enter that rail. Its source and the bad position must both lie in the initial residue class.

The concrete carrier has 36 states and 43 edges. Its reachable set has 6 states. The reachable set is disjoint from the bad predicate.

ID	State	Seed	Bd.	d	p	Successors
0	(0, 0)	0	I	0	–	{6}
1	(0, 1)	1	–	∞	–	{7}
2	(0, 2)	2	–	∞	–	{8}
3	(0, 3)	3	–	∞	–	{9}
4	(0, 4)	0	–	∞	–	{10}
5	(0, 5)	1	–	∞	–	{11}
6	(0, 6)	2	–	1	0	{12}
7	(0, 7)	3	–	∞	–	{13}
8	(0, 8)	0	–	∞	–	{14}
9	(0, 9)	1	–	∞	–	{15}
10	(0, 10)	2	–	∞	–	{16}
11	(0, 11)	3	–	∞	–	{17}
12	(0, 12)	0	–	2	6	{0, 30}

ID	State	Seed	Bd.	d	p	Successors
13	(0, 13)	1	–	∞	–	{1}
14	(0, 14)	2	–	∞	–	{2}
15	(0, 15)	3	–	∞	–	{3}
16	(0, 16)	0	–	∞	–	{4}
17	(0, 17)	1	–	∞	–	{5}
18	(1, 0)	0	–	4	30	{0, 24}
19	(1, 1)	1	–	∞	–	{25}
20	(1, 2)	2	–	∞	–	{26}
21	(1, 3)	3	–	∞	–	{3, 27}
22	(1, 4)	0	–	∞	–	{28}
23	(1, 5)	1	–	∞	–	{29}
24	(1, 6)	2	–	5	18	{6, 30}
25	(1, 7)	3	–	∞	–	{31}

ID	State	Seed	Bd.	d	p	Successors
26	(1, 8)	0	–	∞	–	{32}
27	(1, 9)	1	–	∞	–	{9, 33}
28	(1, 10)	2	–	∞	–	{34}
29	(1, 11)	3	–	∞	–	{35}
30	(1, 12)	0	–	3	12	{12, 18}
31	(1, 13)	1	–	∞	–	{19}
32	(1, 14)	2	–	∞	–	{20}
33	(1, 15)	3	B	∞	–	{15, 21}
34	(1, 16)	0	–	∞	–	{22}
35	(1, 17)	1	–	∞	–	{23}

Round 1. The partition has 4 nonempty cells, and the may relation has 8 edges.

The abstract checker found the inductive invariant $\{b_0, b_2\}$. It contains $\{b_0\}$ and is disjoint from $\{b_3\}$.

Cell	All may successors
b_0	$\{b_0, b_2\}$
b_2	$\{b_0, b_2\}$

21.9 F6.9 length 20, stride 4, bridge 12, target 16

Every edge preserves the position residue modulo 4. Return links leave the second rail. The specified bridge is the only way to enter that rail. Its source and the bad position must both lie in the initial residue class.

The concrete carrier has 40 states and 48 edges. Its reachable set has 10 states. The bad predicate is reached.

ID	State	Seed	Bd.	d	p	Successors
0	(0, 0)	0	I	0	–	{4}
1	(0, 1)	1	–	∞	–	{5}
2	(0, 2)	2	–	∞	–	{6}
3	(0, 3)	3	–	∞	–	{7}
4	(0, 4)	0	–	1	0	{8}
5	(0, 5)	1	–	∞	–	{9}
6	(0, 6)	2	–	∞	–	{10}
7	(0, 7)	3	–	∞	–	{11}
8	(0, 8)	0	–	2	4	{12}
9	(0, 9)	1	–	∞	–	{13}
10	(0, 10)	2	–	∞	–	{14}
11	(0, 11)	3	–	∞	–	{15}
12	(0, 12)	0	–	3	8	{16, 32}

ID	State	Seed	Bd.	d	p	Successors
13	(0, 13)	1	–	∞	–	{17}
14	(0, 14)	2	–	∞	–	{18}
15	(0, 15)	3	–	∞	–	{19}
16	(0, 16)	0	–	4	12	{0}
17	(0, 17)	1	–	∞	–	{1}
18	(0, 18)	2	–	∞	–	{2}
19	(0, 19)	3	–	∞	–	{3}
20	(1, 0)	0	–	6	36	{0, 24}
21	(1, 1)	1	–	∞	–	{25}
22	(1, 2)	2	–	∞	–	{26}
23	(1, 3)	3	–	∞	–	{3, 27}
24	(1, 4)	0	–	7	20	{28}
25	(1, 5)	1	–	∞	–	{29}

ID	State	Seed	Bd.	d	p	Successors
26	(1, 6)	2	–	∞	–	{6, 30}
27	(1, 7)	3	–	∞	–	{31}
28	(1, 8)	0	–	8	24	{32}
29	(1, 9)	1	–	∞	–	{9, 33}
30	(1, 10)	2	–	∞	–	{34}
31	(1, 11)	3	–	∞	–	{35}
32	(1, 12)	0	–	4	12	{12, 36}
33	(1, 13)	1	–	∞	–	{37}
34	(1, 14)	2	–	∞	–	{38}
35	(1, 15)	3	–	∞	–	{15, 39}
36	(1, 16)	0	B	5	32	{20}
37	(1, 17)	1	–	∞	–	{21}
38	(1, 18)	2	–	∞	–	{18, 22}

ID	State	Seed	Bd.	d	p	Successors
39	(1, 19)	3	–	∞	–	{23}

Round 1. The partition has 4 nonempty cells, and the may relation has 4 edges.

The selected path is b_0. The exact number of concrete lifts is 0.

j	a_j	F_j	D_j	$\#\cap$
0	b_0	{0}	{36}	0

The isolated state IDs are $\{0, 4, 8, 12, 16, 20, 24, 28, 32, 36\}$. The refined graph has zero paths projecting to this candidate.

Round 2. The partition has 13 nonempty cells, and the may relation has 16 edges.

The selected path is $s_0, s_4, s_8, s_{12}, s_{32}, s_{36}$. The exact number of concrete lifts is 1.

j	a_j	F_j	D_j	$\#\cap$
0	s_0	$\{0\}$	$\{0\}$	1
1	s_4	$\{4\}$	$\{4\}$	1
2	s_8	$\{8\}$	$\{8\}$	1
3	s_{12}	$\{12\}$	$\{12\}$	1
4	s_{32}	$\{32\}$	$\{32\}$	1
5	s_{36}	$\{36\}$	$\{36\}$	1

A concrete witness is $0, 4, 8, 12, 32, 36$. Each adjacent pair occurs in the concrete successor table.

22 Grid channels

East and north edges move through a finite rectangle. East complements a separate channel bit. The seed is the sum of the coordinates. The optional skew edge moves east while preserving the bit.

Tables enumerate concrete states by integer ID. The d column gives distance from an initial state, and p gives a predecessor attaining that distance. Columns I and B are encoded by the boundary field. Abstract keys b_i are seed blocks and s_x are isolated states.

22.1 F7.1 rectangle 2 by 3, skew None, target (2, 3, 1)

East changes the parity channel and north preserves it. The ordinary channel is determined by the east coordinate. The exceptional channel can reach precisely the northeast rectangle above the destination of the skew edge, if that edge exists.

The concrete carrier has 24 states and 34 edges. Its reachable set has 12 states. The reachable set is disjoint from the bad predicate.

ID	State	Seed	Bd.	d	p	Successors
0	(0, 0, 0)	0	I	0	–	$\{2, 9\}$
1	(0, 0, 1)	0	–	∞	–	$\{3, 8\}$
2	(0, 1, 0)	1	–	1	0	$\{4, 11\}$
3	(0, 1, 1)	1	–	∞	–	$\{5, 10\}$
4	(0, 2, 0)	2	–	2	2	$\{6, 13\}$
5	(0, 2, 1)	2	–	∞	–	$\{7, 12\}$
6	(0, 3, 0)	3	–	3	4	$\{15\}$
7	(0, 3, 1)	3	–	∞	–	$\{14\}$
8	(1, 0, 0)	1	–	∞	–	$\{10, 17\}$
9	(1, 0, 1)	1	–	1	0	$\{11, 16\}$
10	(1, 1, 0)	2	–	∞	–	$\{12, 19\}$
11	(1, 1, 1)	2	–	2	2	$\{13, 18\}$
12	(1, 2, 0)	3	–	∞	–	$\{14, 21\}$
13	(1, 2, 1)	3	–	3	4	$\{15, 20\}$
14	(1, 3, 0)	4	–	∞	–	$\{23\}$
15	(1, 3, 1)	4	–	4	6	$\{22\}$
16	(2, 0, 0)	2	–	2	9	$\{18\}$
17	(2, 0, 1)	2	–	∞	–	$\{19\}$
18	(2, 1, 0)	3	–	3	11	$\{20\}$
19	(2, 1, 1)	3	–	∞	–	$\{21\}$
20	(2, 2, 0)	4	–	4	13	$\{22\}$
21	(2, 2, 1)	4	–	∞	–	$\{23\}$
22	(2, 3, 0)	5	–	5	15	$\varnothing$
23	(2, 3, 1)	5	B	∞	–	$\varnothing$

Round 1. The partition has 6 nonempty cells, and the may relation has 5 edges.

The selected path is $b_0, b_1, b_2, b_3, b_4, b_5$. The exact number of concrete lifts is 0.

j	a_j	F_j	D_j	$\#\cap$
0	b_0	$\{0\}$	$\{1\}$	0
1	b_1	$\{2, 9\}$	$\{3, 8\}$	0
2	b_2	$\{4, 11, 16\}$	$\{5, 10, 17\}$	0
3	b_3	$\{6, 13, 18\}$	$\{7, 12, 19\}$	0
4	b_4	$\{15, 20\}$	$\{14, 21\}$	0
5	b_5	$\{22\}$	$\{23\}$	0

The isolated state IDs are $\{0, 1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12, 13, 14, 15, 16, 17, 18, 19, 20, 21, 22, 23\}$. The refined graph has zero paths projecting to this candidate.

Round 2. The partition has 24 nonempty cells, and the may relation has 34 edges.

The abstract checker found the inductive invariant $\{s_0, s_2, s_4, s_6, s_9, s_{11}, s_{13}, s_{15}, s_{16}, s_{18}, s_{20}, s_{22}\}$. It contains $\{s_0\}$ and is disjoint from $\{s_{23}\}$.

Cell	All may successors
s_0	$\{s_2, s_9\}$
s_2	$\{s_4, s_{11}\}$
s_4	$\{s_6, s_{13}\}$
s_6	$\{s_{15}\}$
s_9	$\{s_{11}, s_{16}\}$
s_{11}	$\{s_{13}, s_{18}\}$
s_{13}	$\{s_{15}, s_{20}\}$
s_{15}	$\{s_{22}\}$
s_{16}	$\{s_{18}\}$
s_{18}	$\{s_{20}\}$
s_{20}	$\{s_{22}\}$
s_{22}	$\varnothing$

22.2 F7.2 rectangle 3 by 2, skew (1, 1), target (3, 2, 0)

East changes the parity channel and north preserves it. The ordinary channel is determined by the east coordinate. The exceptional channel can reach precisely the northeast rectangle above the destination of the skew edge, if that edge exists.

The concrete carrier has 24 states and 35 edges. Its reachable set has 16 states. The bad predicate is reached.

ID	State	Seed	Bd.	d	p	Successors
0	(0, 0, 0)	0	I	0	–	$\{2, 7\}$
1	(0, 0, 1)	0	–	∞	–	$\{3, 6\}$
2	(0, 1, 0)	1	–	1	0	$\{4, 9\}$
3	(0, 1, 1)	1	–	∞	–	$\{5, 8\}$
4	(0, 2, 0)	2	–	2	2	$\{11\}$
5	(0, 2, 1)	2	–	∞	–	$\{10\}$
6	(1, 0, 0)	1	–	∞	–	$\{8, 13\}$
7	(1, 0, 1)	1	–	1	0	$\{9, 12\}$
8	(1, 1, 0)	2	–	∞	–	$\{10, 15\}$
9	(1, 1, 1)	2	–	2	2	$\{11, 14, 15\}$
10	(1, 2, 0)	3	–	∞	–	$\{17\}$
11	(1, 2, 1)	3	–	3	4	$\{16\}$
12	(2, 0, 0)	2	–	2	7	$\{14, 19\}$

ID	State	Seed	Bd.	d	p	Successors
13	(2, 0, 1)	2	–	∞	–	$\{15, 18\}$
14	(2, 1, 0)	3	–	3	9	$\{16, 21\}$
15	(2, 1, 1)	3	–	3	9	$\{17, 20\}$
16	(2, 2, 0)	4	–	4	11	$\{23\}$
17	(2, 2, 1)	4	–	4	15	$\{22\}$
18	(3, 0, 0)	3	–	∞	–	$\{20\}$
19	(3, 0, 1)	3	–	3	12	$\{21\}$
20	(3, 1, 0)	4	–	4	15	$\{22\}$
21	(3, 1, 1)	4	–	4	14	$\{23\}$
22	(3, 2, 0)	5	B	5	17	$\varnothing$
23	(3, 2, 1)	5	–	5	16	$\varnothing$

Round 1. The partition has 6 nonempty cells, and the may relation has 5 edges.

The selected path is $b_0, b_1, b_2, b_3, b_4, b_5$. The exact number of concrete lifts is 4.

j	a_j	F_j	D_j	$\#\cap$
0	b_0	$\{0\}$	$\{0, 1\}$	1
1	b_1	$\{2, 7\}$	$\{2, 3, 6, 7\}$	2
2	b_2	$\{4, 9, 12\}$	$\{5, 8, 9, 13\}$	1
3	b_3	$\{11, 14, 15, 19\}$	$\{10, 15, 18\}$	1
4	b_4	$\{16, 17, 20, 21\}$	$\{17, 20\}$	2
5	b_5	$\{22, 23\}$	$\{22\}$	1

A concrete witness is $0, 2, 9, 15, 17, 22$. Each adjacent pair occurs in the concrete successor table.

22.3 F7.3 rectangle 3 by 3, skew (2, 2), target (2, 3, 1)

East changes the parity channel and north preserves it. The ordinary channel is determined by the east coordinate. The exceptional channel can reach precisely the northeast rectangle above the destination of the skew edge, if that edge exists.

The concrete carrier has 32 states and 49 edges. Its reachable set has 18 states. The reachable set is disjoint from the bad predicate.

ID	State	Seed	Bd.	d	p	Successors
0	(0, 0, 0)	0	I	0	–	$\{2, 9\}$
1	(0, 0, 1)	0	–	∞	–	$\{3, 8\}$
2	(0, 1, 0)	1	–	1	0	$\{4, 11\}$
3	(0, 1, 1)	1	–	∞	–	$\{5, 10\}$
4	(0, 2, 0)	2	–	2	2	$\{6, 13\}$
5	(0, 2, 1)	2	–	∞	–	$\{7, 12\}$
6	(0, 3, 0)	3	–	3	4	$\{15\}$
7	(0, 3, 1)	3	–	∞	–	$\{14\}$
8	(1, 0, 0)	1	–	∞	–	$\{10, 17\}$
9	(1, 0, 1)	1	–	1	0	$\{11, 16\}$
10	(1, 1, 0)	2	–	∞	–	$\{12, 19\}$
11	(1, 1, 1)	2	–	2	2	$\{13, 18\}$
12	(1, 2, 0)	3	–	∞	–	$\{14, 21\}$

ID	State	Seed	Bd.	d	p	Successors
13	(1, 2, 1)	3	–	3	4	$\{15, 20\}$
14	(1, 3, 0)	4	–	∞	–	$\{23\}$
15	(1, 3, 1)	4	–	4	6	$\{22\}$
16	(2, 0, 0)	2	–	2	9	$\{18, 25\}$
17	(2, 0, 1)	2	–	∞	–	$\{19, 24\}$
18	(2, 1, 0)	3	–	3	11	$\{20, 27\}$
19	(2, 1, 1)	3	–	∞	–	$\{21, 26\}$
20	(2, 2, 0)	4	–	4	13	$\{22, 28, 29\}$
21	(2, 2, 1)	4	–	∞	–	$\{23, 28\}$
22	(2, 3, 0)	5	–	5	15	$\{31\}$
23	(2, 3, 1)	5	B	∞	–	$\{30\}$
24	(3, 0, 0)	3	–	∞	–	$\{26\}$
25	(3, 0, 1)	3	–	3	16	$\{27\}$

ID	State	Seed	Bd.	d	p	Successors
26	(3, 1, 0)	4	–	∞	–	$\{28\}$
27	(3, 1, 1)	4	–	4	18	$\{29\}$
28	(3, 2, 0)	5	–	5	20	$\{30\}$
29	(3, 2, 1)	5	–	5	20	$\{31\}$
30	(3, 3, 0)	6	–	6	28	$\varnothing$
31	(3, 3, 1)	6	–	6	22	$\varnothing$

Round 1. The partition has 7 nonempty cells, and the may relation has 6 edges.

The selected path is $b_0, b_1, b_2, b_3, b_4, b_5$. The exact number of concrete lifts is 0.

j	a_j	F_j	D_j	$\#\cap$
0	b_0	$\{0\}$	$\{1\}$	0
1	b_1	$\{2, 9\}$	$\{3, 8\}$	0
2	b_2	$\{4, 11, 16\}$	$\{5, 10, 17\}$	0
3	b_3	$\{6, 13, 18, 25\}$	$\{7, 12, 19\}$	0
4	b_4	$\{15, 20, 27\}$	$\{14, 21\}$	0
5	b_5	$\{22, 28, 29\}$	$\{23\}$	0

The isolated state IDs are $\{0, 1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12, 13, 14, 15, 16, 17, 18, 19, 20, 21, 22, 23, 24, 25, 26, 27, 28, 29\}$. The refined graph has zero paths projecting to this candidate.

Round 2. The partition has 31 nonempty cells, and the may relation has 49 edges.

The abstract checker found the inductive invariant $\{s_0, s_2, s_4, s_6, s_9, s_{11}, s_{13}, s_{15}, s_{16}, s_{18}, s_{20}, s_{22}, s_{25}, s_{27}, s_{28}, s_{29}, b_6\}$. It contains $\{s_0\}$ and is disjoint from $\{s_{23}\}$.

Cell	All may successors
s_0	$\{s_2, s_9\}$
s_2	$\{s_4, s_{11}\}$
s_4	$\{s_6, s_{13}\}$
s_6	$\{s_{15}\}$
s_9	$\{s_{11}, s_{16}\}$
s_{11}	$\{s_{13}, s_{18}\}$
s_{13}	$\{s_{15}, s_{20}\}$
s_{15}	$\{s_{22}\}$
s_{16}	$\{s_{18}, s_{25}\}$
s_{18}	$\{s_{20}, s_{27}\}$
s_{20}	$\{s_{22}, s_{28}, s_{29}\}$
s_{22}	$\{b_6\}$
s_{25}	$\{s_{27}\}$

Cell	All may successors
s_{27}	$\{s_{29}\}$
s_{28}	$\{b_6\}$
s_{29}	$\{b_6\}$
b_6	$\varnothing$

22.4 F7.4 rectangle 4 by 2, skew None, target (4, 2, 0)

East changes the parity channel and north preserves it. The ordinary channel is determined by the east coordinate. The exceptional channel can reach precisely the northeast rectangle above the destination of the skew edge, if that edge exists.

The concrete carrier has 30 states and 44 edges. Its reachable set has 15 states. The bad predicate is reached.

ID	State	Seed	Bd.	d	p	Successors
0	(0, 0, 0)	0	I	0	–	$\{2, 7\}$
1	(0, 0, 1)	0	–	∞	–	$\{3, 6\}$
2	(0, 1, 0)	1	–	1	0	$\{4, 9\}$
3	(0, 1, 1)	1	–	∞	–	$\{5, 8\}$
4	(0, 2, 0)	2	–	2	2	$\{11\}$
5	(0, 2, 1)	2	–	∞	–	$\{10\}$
6	(1, 0, 0)	1	–	∞	–	$\{8, 13\}$
7	(1, 0, 1)	1	–	1	0	$\{9, 12\}$
8	(1, 1, 0)	2	–	∞	–	$\{10, 15\}$
9	(1, 1, 1)	2	–	2	2	$\{11, 14\}$
10	(1, 2, 0)	3	–	∞	–	$\{17\}$
11	(1, 2, 1)	3	–	3	4	$\{16\}$
12	(2, 0, 0)	2	–	2	7	$\{14, 19\}$
13	(2, 0, 1)	2	–	∞	–	$\{15, 18\}$
14	(2, 1, 0)	3	–	3	9	$\{16, 21\}$
15	(2, 1, 1)	3	–	∞	–	$\{17, 20\}$
16	(2, 2, 0)	4	–	4	11	$\{23\}$
17	(2, 2, 1)	4	–	∞	–	$\{22\}$
18	(3, 0, 0)	3	–	∞	–	$\{20, 25\}$
19	(3, 0, 1)	3	–	3	12	$\{21, 24\}$
20	(3, 1, 0)	4	–	∞	–	$\{22, 27\}$
21	(3, 1, 1)	4	–	4	14	$\{23, 26\}$
22	(3, 2, 0)	5	–	∞	–	$\{29\}$
23	(3, 2, 1)	5	–	5	16	$\{28\}$
24	(4, 0, 0)	4	–	4	19	$\{26\}$
25	(4, 0, 1)	4	–	∞	–	$\{27\}$
26	(4, 1, 0)	5	–	5	21	$\{28\}$
27	(4, 1, 1)	5	–	∞	–	$\{29\}$
28	(4, 2, 0)	6	B	6	23	$\varnothing$
29	(4, 2, 1)	6	–	∞	–	$\varnothing$

Round 1. The partition has 7 nonempty cells, and the may relation has 6 edges.

The selected path is $b_0, b_1, b_2, b_3, b_4, b_5, b_6$. The exact number of concrete lifts is 15.

j	a_j	F_j	D_j	$\#\cap$
0	b_0	$\{0\}$	$\{0\}$	1
1	b_1	$\{2, 7\}$	$\{2, 7\}$	2
2	b_2	$\{4, 9, 12\}$	$\{4, 9, 12\}$	3
3	b_3	$\{11, 14, 19\}$	$\{11, 14, 19\}$	3
4	b_4	$\{16, 21, 24\}$	$\{16, 21, 24\}$	3
5	b_5	$\{23, 26\}$	$\{23, 26\}$	2
6	b_6	$\{28\}$	$\{28\}$	1

A concrete witness is $0, 2, 4, 11, 16, 23, 28$. Each adjacent pair occurs in the concrete successor table.

22.5 F7.5 rectangle 4 by 3, skew (1, 1), target (4, 3, 1)

East changes the parity channel and north preserves it. The ordinary channel is determined by the east coordinate. The exceptional channel can reach precisely the northeast rectangle above the destination of the skew edge, if that edge exists.

The concrete carrier has 40 states and 63 edges. Its reachable set has 29 states. The bad predicate is reached.

ID	State	Seed	Bd.	d	p	Successors
0	(0, 0, 0)	0	I	0	–	$\{2, 9\}$
1	(0, 0, 1)	0	–	∞	–	$\{3, 8\}$
2	(0, 1, 0)	1	–	1	0	$\{4, 11\}$
3	(0, 1, 1)	1	–	∞	–	$\{5, 10\}$
4	(0, 2, 0)	2	–	2	2	$\{6, 13\}$
5	(0, 2, 1)	2	–	∞	–	$\{7, 12\}$
6	(0, 3, 0)	3	–	3	4	$\{15\}$
7	(0, 3, 1)	3	–	∞	–	$\{14\}$
8	(1, 0, 0)	1	–	∞	–	$\{10, 17\}$
9	(1, 0, 1)	1	–	1	0	$\{11, 16\}$
10	(1, 1, 0)	2	–	∞	–	$\{12, 19\}$
11	(1, 1, 1)	2	–	2	2	$\{13, 18, 19\}$
12	(1, 2, 0)	3	–	∞	–	$\{14, 21\}$

ID	State	Seed	Bd.	d	p	Successors
13	(1, 2, 1)	3	–	3	4	$\{15, 20\}$
14	(1, 3, 0)	4	–	∞	–	$\{23\}$
15	(1, 3, 1)	4	–	4	6	$\{22\}$
16	(2, 0, 0)	2	–	2	9	$\{18, 25\}$
17	(2, 0, 1)	2	–	∞	–	$\{19, 24\}$
18	(2, 1, 0)	3	–	3	11	$\{20, 27\}$
19	(2, 1, 1)	3	–	3	11	$\{21, 26\}$
20	(2, 2, 0)	4	–	4	13	$\{22, 29\}$
21	(2, 2, 1)	4	–	4	19	$\{23, 28\}$
22	(2, 3, 0)	5	–	5	15	$\{31\}$
23	(2, 3, 1)	5	–	5	21	$\{30\}$
24	(3, 0, 0)	3	–	∞	–	$\{26, 33\}$
25	(3, 0, 1)	3	–	3	16	$\{27, 32\}$

ID	State	Seed	Bd.	d	p	Successors
26	(3, 1, 0)	4	–	4	19	$\{28, 35\}$
27	(3, 1, 1)	4	–	4	18	$\{29, 34\}$
28	(3, 2, 0)	5	–	5	21	$\{30, 37\}$
29	(3, 2, 1)	5	–	5	20	$\{31, 36\}$
30	(3, 3, 0)	6	–	6	23	$\{39\}$
31	(3, 3, 1)	6	–	6	22	$\{38\}$
32	(4, 0, 0)	4	–	4	25	$\{34\}$
33	(4, 0, 1)	4	–	∞	–	$\{35\}$
34	(4, 1, 0)	5	–	5	27	$\{36\}$
35	(4, 1, 1)	5	–	5	26	$\{37\}$
36	(4, 2, 0)	6	–	6	29	$\{38\}$
37	(4, 2, 1)	6	–	6	28	$\{39\}$
38	(4, 3, 0)	7	–	7	31	$\varnothing$

ID	State	Seed	Bd.	d	p	Successors
39	(4, 3, 1)	7	B	7	30	$\varnothing$

Round 1. The partition has 8 nonempty cells, and the may relation has 7 edges.

The selected path is $b_0, b_1, b_2, b_3, b_4, b_5, b_6, b_7$. The exact number of concrete lifts is 12.

j	a_j	F_j	D_j	$\#\cap$
0	b_0	$\{0\}$	$\{0, 1\}$	1
1	b_1	$\{2, 9\}$	$\{2, 3, 8, 9\}$	2
2	b_2	$\{4, 11, 16\}$	$\{5, 10, 11, 17\}$	1
3	b_3	$\{6, 13, 18, 19, 25\}$	$\{7, 12, 19, 24\}$	1
4	b_4	$\{15, 20, 21, 26, 27, 32\}$	$\{14, 21, 26, 33\}$	2
5	b_5	$\{22, 23, 28, 29, 34, 35\}$	$\{23, 28, 35\}$	3
6	b_6	$\{30, 31, 36, 37\}$	$\{30, 37\}$	2
7	b_7	$\{38, 39\}$	$\{39\}$	1

A concrete witness is $0, 2, 11, 19, 21, 23, 30, 39$. Each adjacent pair occurs in the concrete successor table.

22.6 F7.6 rectangle 4 by 4, skew None, target (4, 4, 1)

East changes the parity channel and north preserves it. The ordinary channel is determined by the east coordinate. The exceptional channel can reach precisely the northeast rectangle above the destination of the skew edge, if that edge exists.

The concrete carrier has 50 states and 80 edges. Its reachable set has 25 states. The reachable set is disjoint from the bad predicate.

ID	State	Seed	Bd.	d	p	Successors
0	(0, 0, 0)	0	I	0	–	$\{2, 11\}$
1	(0, 0, 1)	0	–	∞	–	$\{3, 10\}$
2	(0, 1, 0)	1	–	1	0	$\{4, 13\}$
3	(0, 1, 1)	1	–	∞	–	$\{5, 12\}$
4	(0, 2, 0)	2	–	2	2	$\{6, 15\}$
5	(0, 2, 1)	2	–	∞	–	$\{7, 14\}$
6	(0, 3, 0)	3	–	3	4	$\{8, 17\}$
7	(0, 3, 1)	3	–	∞	–	$\{9, 16\}$
8	(0, 4, 0)	4	–	4	6	$\{19\}$
9	(0, 4, 1)	4	–	∞	–	$\{18\}$
10	(1, 0, 0)	1	–	∞	–	$\{12, 21\}$
11	(1, 0, 1)	1	–	1	0	$\{13, 20\}$
12	(1, 1, 0)	2	–	∞	–	$\{14, 23\}$

ID	State	Seed	Bd.	d	p	Successors
13	(1, 1, 1)	2	–	2	2	$\{15, 22\}$
14	(1, 2, 0)	3	–	∞	–	$\{16, 25\}$
15	(1, 2, 1)	3	–	3	4	$\{17, 24\}$
16	(1, 3, 0)	4	–	∞	–	$\{18, 27\}$
17	(1, 3, 1)	4	–	4	6	$\{19, 26\}$
18	(1, 4, 0)	5	–	∞	–	$\{29\}$
19	(1, 4, 1)	5	–	5	8	$\{28\}$
20	(2, 0, 0)	2	–	2	11	$\{22, 31\}$
21	(2, 0, 1)	2	–	∞	–	$\{23, 30\}$
22	(2, 1, 0)	3	–	3	13	$\{24, 33\}$
23	(2, 1, 1)	3	–	∞	–	$\{25, 32\}$
24	(2, 2, 0)	4	–	4	15	$\{26, 35\}$
25	(2, 2, 1)	4	–	∞	–	$\{27, 34\}$

ID	State	Seed	Bd.	d	p	Successors
26	(2, 3, 0)	5	–	5	17	$\{28, 37\}$
27	(2, 3, 1)	5	–	∞	–	$\{29, 36\}$
28	(2, 4, 0)	6	–	6	19	$\{39\}$
29	(2, 4, 1)	6	–	∞	–	$\{38\}$
30	(3, 0, 0)	3	–	∞	–	$\{32, 41\}$
31	(3, 0, 1)	3	–	3	20	$\{33, 40\}$
32	(3, 1, 0)	4	–	∞	–	$\{34, 43\}$
33	(3, 1, 1)	4	–	4	22	$\{35, 42\}$
34	(3, 2, 0)	5	–	∞	–	$\{36, 45\}$
35	(3, 2, 1)	5	–	5	24	$\{37, 44\}$
36	(3, 3, 0)	6	–	∞	–	$\{38, 47\}$
37	(3, 3, 1)	6	–	6	26	$\{39, 46\}$
38	(3, 4, 0)	7	–	∞	–	$\{49\}$
39	(3, 4, 1)	7	–	7	28	$\{48\}$
40	(4, 0, 0)	4	–	4	31	$\{42\}$
41	(4, 0, 1)	4	–	∞	–	$\{43\}$
42	(4, 1, 0)	5	–	5	33	$\{44\}$
43	(4, 1, 1)	5	–	∞	–	$\{45\}$
44	(4, 2, 0)	6	–	6	35	$\{46\}$
45	(4, 2, 1)	6	–	∞	–	$\{47\}$
46	(4, 3, 0)	7	–	7	37	$\{48\}$
47	(4, 3, 1)	7	–	∞	–	$\{49\}$
48	(4, 4, 0)	8	–	8	39	$\varnothing$
49	(4, 4, 1)	8	B	∞	–	$\varnothing$

Round 1. The partition has 9 nonempty cells, and the may relation has 8 edges.

The selected path is $b_0, b_1, b_2, b_3, b_4, b_5, b_6, b_7, b_8$. The exact number of concrete lifts is 0.

j	a_j	F_j	D_j	$\#\cap$
0	b_0	$\{0\}$	$\{1\}$	0
1	b_1	$\{2, 11\}$	$\{3, 10\}$	0
2	b_2	$\{4, 13, 20\}$	$\{5, 12, 21\}$	0
3	b_3	$\{6, 15, 22, 31\}$	$\{7, 14, 23, 30\}$	0
4	b_4	$\{8, 17, 24, 33, 40\}$	$\{9, 16, 25, 32, 41\}$	0
5	b_5	$\{19, 26, 35, 42\}$	$\{18, 27, 34, 43\}$	0
6	b_6	$\{28, 37, 44\}$	$\{29, 36, 45\}$	0
7	b_7	$\{39, 46\}$	$\{38, 47\}$	0
8	b_8	$\{48\}$	$\{49\}$	0

The isolated state IDs are $\{0, 1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12, 13, 14, 15, 16, 17, 18, 19, 20, 21, 22, 23, 24, 25, 26, 27, 28, 29, 30, 31, 32, 33, 34, 35, 36, 37, 38, 39, 40, 41, 42, 43, 44, 45, 46, 47, 48, 49\}$. The refined graph has zero paths projecting to this candidate.

Round 2. The partition has 50 nonempty cells, and the may relation has 80 edges.

The abstract checker found the inductive invariant $\{s_0, s_2, s_4, s_6, s_8, s_{11}, s_{13}, s_{15}, s_{17}, s_{19}, s_{20}, s_{22}, s_{24}, s_{26}, s_{28}, s_{31}, s_{33}, s_{35}, s_{37}, s_{39}, s_{40}, s_{42}, s_{44}, s_{46}, s_{48}\}$. It contains $\{s_0\}$ and is disjoint from $\{s_{49}\}$.

Cell	All may successors
s_0	$\{s_2, s_{11}\}$
s_2	$\{s_4, s_{13}\}$
s_4	$\{s_6, s_{15}\}$
s_6	$\{s_8, s_{17}\}$
s_8	$\{s_{19}\}$
s_{11}	$\{s_{13}, s_{20}\}$
s_{13}	$\{s_{15}, s_{22}\}$
s_{15}	$\{s_{17}, s_{24}\}$
s_{17}	$\{s_{19}, s_{26}\}$
s_{19}	$\{s_{28}\}$
s_{20}	$\{s_{22}, s_{31}\}$
s_{22}	$\{s_{24}, s_{33}\}$
s_{24}	$\{s_{26}, s_{35}\}$
s_{26}	$\{s_{28}, s_{37}\}$
s_{28}	$\{s_{39}\}$
s_{31}	$\{s_{33}, s_{40}\}$
s_{33}	$\{s_{35}, s_{42}\}$
s_{35}	$\{s_{37}, s_{44}\}$
s_{37}	$\{s_{39}, s_{46}\}$
s_{39}	$\{s_{48}\}$
s_{40}	$\{s_{42}\}$
s_{42}	$\{s_{44}\}$
s_{44}	$\{s_{46}\}$
s_{46}	$\{s_{48}\}$
s_{48}	$\varnothing$

22.7 F7.7 rectangle 5 by 3, skew (3, 2), target (3, 3, 0)

East changes the parity channel and north preserves it. The ordinary channel is determined by the east coordinate. The exceptional channel can reach precisely the northeast rectangle above the destination of the skew edge, if that edge exists.

The concrete carrier has 48 states and 77 edges. Its reachable set has 28 states. The reachable set is disjoint from the bad predicate.

ID	State	Seed	Bd.	d	p	Successors
0	(0, 0, 0)	0	I	0	–	$\{2, 9\}$
1	(0, 0, 1)	0	–	∞	–	$\{3, 8\}$
2	(0, 1, 0)	1	–	1	0	$\{4, 11\}$
3	(0, 1, 1)	1	–	∞	–	$\{5, 10\}$
4	(0, 2, 0)	2	–	2	2	$\{6, 13\}$
5	(0, 2, 1)	2	–	∞	–	$\{7, 12\}$
6	(0, 3, 0)	3	–	3	4	$\{15\}$
7	(0, 3, 1)	3	–	∞	–	$\{14\}$
8	(1, 0, 0)	1	–	∞	–	$\{10, 17\}$
9	(1, 0, 1)	1	–	1	0	$\{11, 16\}$
10	(1, 1, 0)	2	–	∞	–	$\{12, 19\}$
11	(1, 1, 1)	2	–	2	2	$\{13, 18\}$
12	(1, 2, 0)	3	–	∞	–	$\{14, 21\}$

ID	State	Seed	Bd.	d	p	Successors
13	(1, 2, 1)	3	–	3	4	$\{15, 20\}$
14	(1, 3, 0)	4	–	∞	–	$\{23\}$
15	(1, 3, 1)	4	–	4	6	$\{22\}$
16	(2, 0, 0)	2	–	2	9	$\{18, 25\}$
17	(2, 0, 1)	2	–	∞	–	$\{19, 24\}$
18	(2, 1, 0)	3	–	3	11	$\{20, 27\}$
19	(2, 1, 1)	3	–	∞	–	$\{21, 26\}$
20	(2, 2, 0)	4	–	4	13	$\{22, 29\}$
21	(2, 2, 1)	4	–	∞	–	$\{23, 28\}$
22	(2, 3, 0)	5	–	5	15	$\{31\}$
23	(2, 3, 1)	5	–	∞	–	$\{30\}$
24	(3, 0, 0)	3	–	∞	–	$\{26, 33\}$
25	(3, 0, 1)	3	–	3	16	$\{27, 32\}$

ID	State	Seed	Bd.	d	p	Successors
26	(3, 1, 0)	4	–	∞	–	$\{28, 35\}$
27	(3, 1, 1)	4	–	4	18	$\{29, 34\}$
28	(3, 2, 0)	5	–	∞	–	$\{30, 37\}$
29	(3, 2, 1)	5	–	5	20	$\{31, 36, 37\}$
30	(3, 3, 0)	6	B	∞	–	$\{39\}$
31	(3, 3, 1)	6	–	6	22	$\{38\}$
32	(4, 0, 0)	4	–	4	25	$\{34, 41\}$
33	(4, 0, 1)	4	–	∞	–	$\{35, 40\}$
34	(4, 1, 0)	5	–	5	27	$\{36, 43\}$
35	(4, 1, 1)	5	–	∞	–	$\{37, 42\}$
36	(4, 2, 0)	6	–	6	29	$\{38, 45\}$
37	(4, 2, 1)	6	–	6	29	$\{39, 44\}$
38	(4, 3, 0)	7	–	7	31	$\{47\}$

ID	State	Seed	Bd.	d	p	Successors
39	(4, 3, 1)	7	–	7	37	$\{46\}$
40	(5, 0, 0)	5	–	∞	–	$\{42\}$
41	(5, 0, 1)	5	–	5	32	$\{43\}$
42	(5, 1, 0)	6	–	∞	–	$\{44\}$
43	(5, 1, 1)	6	–	6	34	$\{45\}$
44	(5, 2, 0)	7	–	7	37	$\{46\}$
45	(5, 2, 1)	7	–	7	36	$\{47\}$
46	(5, 3, 0)	8	–	8	39	$\varnothing$
47	(5, 3, 1)	8	–	8	38	$\varnothing$

Round 1. The partition has 9 nonempty cells, and the may relation has 8 edges.

The selected path is $b_0, b_1, b_2, b_3, b_4, b_5, b_6$. The exact number of concrete lifts is 0.

j	a_j	F_j	D_j	$\#\cap$
0	b_0	$\{0\}$	$\{1\}$	0
1	b_1	$\{2, 9\}$	$\{3, 8\}$	0
2	b_2	$\{4, 11, 16\}$	$\{5, 10, 17\}$	0
3	b_3	$\{6, 13, 18, 25\}$	$\{7, 12, 19, 24\}$	0
4	b_4	$\{15, 20, 27, 32\}$	$\{14, 21, 26\}$	0
5	b_5	$\{22, 29, 34, 41\}$	$\{23, 28\}$	0
6	b_6	$\{31, 36, 37, 43\}$	$\{30\}$	0

The isolated state IDs are $\{0, 1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12, 13, 14, 15, 16, 17, 18, 19, 20, 21, 22, 23, 24, 25, 26, 27, 28, 29, 30, 31, 32, 33, 34, 35, 36, 37, 40, 41, 42, 43\}$. The refined graph has zero paths projecting to this candidate.

Round 2. The partition has 44 nonempty cells, and the may relation has 72 edges.

The abstract checker found the inductive invariant $\{s_0, s_2, s_4, s_6, s_9, s_{11}, s_{13}, s_{15}, s_{16}, s_{18}, s_{20}, s_{22}, s_{25}, s_{27}, s_{29}, s_{31}, s_{32}, s_{34}, s_{36}, s_{37}, s_{41}, s_{43}, b_7, b_8\}$. It contains $\{s_0\}$ and is disjoint from $\{s_{30}\}$.

Cell	All may successors
s_0	$\{s_2, s_9\}$
s_2	$\{s_4, s_{11}\}$
s_4	$\{s_6, s_{13}\}$
s_6	$\{s_{15}\}$
s_9	$\{s_{11}, s_{16}\}$
s_{11}	$\{s_{13}, s_{18}\}$
s_{13}	$\{s_{15}, s_{20}\}$
s_{15}	$\{s_{22}\}$
s_{16}	$\{s_{18}, s_{25}\}$
s_{18}	$\{s_{20}, s_{27}\}$
s_{20}	$\{s_{22}, s_{29}\}$
s_{22}	$\{s_{31}\}$
s_{25}	$\{s_{27}, s_{32}\}$

Cell	All may successors
s_{27}	$\{s_{29}, s_{34}\}$
s_{29}	$\{s_{31}, s_{36}, s_{37}\}$
s_{31}	$\{b_7\}$
s_{32}	$\{s_{34}, s_{41}\}$
s_{34}	$\{s_{36}, s_{43}\}$
s_{36}	$\{b_7\}$
s_{37}	$\{b_7\}$
s_{41}	$\{s_{43}\}$
s_{43}	$\{b_7\}$
b_7	$\{b_8\}$
b_8	$\varnothing$

22.8 F7.8 rectangle 5 by 4, skew (2, 2), target (5, 4, 0)

East changes the parity channel and north preserves it. The ordinary channel is determined by the east coordinate. The exceptional channel can reach precisely the northeast rectangle above the destination of the skew edge, if that edge exists.

The concrete carrier has 60 states and 99 edges. Its reachable set has 39 states. The bad predicate is reached.

ID	State	Seed	Bd.	d	p	Successors
0	(0, 0, 0)	0	I	0	–	$\{2, 11\}$
1	(0, 0, 1)	0	–	∞	–	$\{3, 10\}$
2	(0, 1, 0)	1	–	1	0	$\{4, 13\}$
3	(0, 1, 1)	1	–	∞	–	$\{5, 12\}$
4	(0, 2, 0)	2	–	2	2	$\{6, 15\}$
5	(0, 2, 1)	2	–	∞	–	$\{7, 14\}$
6	(0, 3, 0)	3	–	3	4	$\{8, 17\}$
7	(0, 3, 1)	3	–	∞	–	$\{9, 16\}$
8	(0, 4, 0)	4	–	4	6	$\{19\}$
9	(0, 4, 1)	4	–	∞	–	$\{18\}$
10	(1, 0, 0)	1	–	∞	–	$\{12, 21\}$
11	(1, 0, 1)	1	–	1	0	$\{13, 20\}$
12	(1, 1, 0)	2	–	∞	–	$\{14, 23\}$

ID	State	Seed	Bd.	d	p	Successors
13	(1, 1, 1)	2	–	2	2	$\{15, 22\}$
14	(1, 2, 0)	3	–	∞	–	$\{16, 25\}$
15	(1, 2, 1)	3	–	3	4	$\{17, 24\}$
16	(1, 3, 0)	4	–	∞	–	$\{18, 27\}$
17	(1, 3, 1)	4	–	4	6	$\{19, 26\}$
18	(1, 4, 0)	5	–	∞	–	$\{29\}$
19	(1, 4, 1)	5	–	5	8	$\{28\}$
20	(2, 0, 0)	2	–	2	11	$\{22, 31\}$
21	(2, 0, 1)	2	–	∞	–	$\{23, 30\}$
22	(2, 1, 0)	3	–	3	13	$\{24, 33\}$
23	(2, 1, 1)	3	–	∞	–	$\{25, 32\}$
24	(2, 2, 0)	4	–	4	15	$\{26, 34, 35\}$
25	(2, 2, 1)	4	–	∞	–	$\{27, 34\}$

ID	State	Seed	Bd.	d	p	Successors
26	(2, 3, 0)	5	–	5	17	$\{28, 37\}$
27	(2, 3, 1)	5	–	∞	–	$\{29, 36\}$
28	(2, 4, 0)	6	–	6	19	$\{39\}$
29	(2, 4, 1)	6	–	∞	–	$\{38\}$
30	(3, 0, 0)	3	–	∞	–	$\{32, 41\}$
31	(3, 0, 1)	3	–	3	20	$\{33, 40\}$
32	(3, 1, 0)	4	–	∞	–	$\{34, 43\}$
33	(3, 1, 1)	4	–	4	22	$\{35, 42\}$
34	(3, 2, 0)	5	–	5	24	$\{36, 45\}$
35	(3, 2, 1)	5	–	5	24	$\{37, 44\}$
36	(3, 3, 0)	6	–	6	34	$\{38, 47\}$
37	(3, 3, 1)	6	–	6	26	$\{39, 46\}$
38	(3, 4, 0)	7	–	7	36	$\{49\}$

ID	State	Seed	Bd.	d	p	Successors
39	(3, 4, 1)	7	–	7	28	$\{48\}$
40	(4, 0, 0)	4	–	4	31	$\{42, 51\}$
41	(4, 0, 1)	4	–	∞	–	$\{43, 50\}$
42	(4, 1, 0)	5	–	5	33	$\{44, 53\}$
43	(4, 1, 1)	5	–	∞	–	$\{45, 52\}$
44	(4, 2, 0)	6	–	6	35	$\{46, 55\}$
45	(4, 2, 1)	6	–	6	34	$\{47, 54\}$
46	(4, 3, 0)	7	–	7	37	$\{48, 57\}$
47	(4, 3, 1)	7	–	7	36	$\{49, 56\}$
48	(4, 4, 0)	8	–	8	39	$\{59\}$
49	(4, 4, 1)	8	–	8	38	$\{58\}$
50	(5, 0, 0)	5	–	∞	–	$\{52\}$
51	(5, 0, 1)	5	–	5	40	$\{53\}$

ID	State	Seed	Bd.	d	p	Successors
52	(5, 1, 0)	6	–	∞	–	$\{54\}$
53	(5, 1, 1)	6	–	6	42	$\{55\}$
54	(5, 2, 0)	7	–	7	45	$\{56\}$
55	(5, 2, 1)	7	–	7	44	$\{57\}$
56	(5, 3, 0)	8	–	8	47	$\{58\}$
57	(5, 3, 1)	8	–	8	46	$\{59\}$
58	(5, 4, 0)	9	B	9	49	$\varnothing$
59	(5, 4, 1)	9	–	9	48	$\varnothing$

Round 1. The partition has 10 nonempty cells, and the may relation has 9 edges.

The selected path is $b_0, b_1, b_2, b_3, b_4, b_5, b_6, b_7, b_8, b_9$. The exact number of concrete lifts is 36.

j	a_j	F_j	D_j	$\#\cap$
0	b_0	$\{0\}$	$\{0, 1\}$	1
1	b_1	$\{2, 11\}$	$\{2, 3, 10, 11\}$	2
2	b_2	$\{4, 13, 20\}$	$\{4, 5, 12, 13, 20, 21\}$	3
3	b_3	$\{6, 15, 22, 31\}$	$\{7, 14, 15, 22, 23, 30\}$	2
4	b_4	$\{8, 17, 24, 33, 40\}$	$\{9, 16, 24, 25, 32, 41\}$	1
5	b_5	$\{19, 26, 34, 35, 42, 51\}$	$\{18, 27, 34, 43, 50\}$	1
6	b_6	$\{28, 36, 37, 44, 45, 53\}$	$\{29, 36, 45, 52\}$	2
7	b_7	$\{38, 39, 46, 47, 54, 55\}$	$\{38, 47, 54\}$	3
8	b_8	$\{48, 49, 56, 57\}$	$\{49, 56\}$	2
9	b_9	$\{58, 59\}$	$\{58\}$	1

A concrete witness is $0, 2, 4, 15, 24, 34, 36, 38, 49, 58$. Each adjacent pair occurs in the concrete successor table.

22.9 F7.9 rectangle 5 by 5, skew None, target (5, 5, 0)

East changes the parity channel and north preserves it. The ordinary channel is determined by the east coordinate. The exceptional channel can reach precisely the northeast rectangle above the destination of the skew edge, if that edge exists.

The concrete carrier has 72 states and 120 edges. Its reachable set has 36 states. The reachable set is disjoint from the bad predicate.

ID	State	Seed	Bd.	d	p	Successors
0	(0, 0, 0)	0	I	0	–	$\{2, 13\}$
1	(0, 0, 1)	0	–	∞	–	$\{3, 12\}$
2	(0, 1, 0)	1	–	1	0	$\{4, 15\}$
3	(0, 1, 1)	1	–	∞	–	$\{5, 14\}$
4	(0, 2, 0)	2	–	2	2	$\{6, 17\}$
5	(0, 2, 1)	2	–	∞	–	$\{7, 16\}$
6	(0, 3, 0)	3	–	3	4	$\{8, 19\}$
7	(0, 3, 1)	3	–	∞	–	$\{9, 18\}$
8	(0, 4, 0)	4	–	4	6	$\{10, 21\}$
9	(0, 4, 1)	4	–	∞	–	$\{11, 20\}$
10	(0, 5, 0)	5	–	5	8	$\{23\}$
11	(0, 5, 1)	5	–	∞	–	$\{22\}$
12	(1, 0, 0)	1	–	∞	–	$\{14, 25\}$

ID	State	Seed	Bd.	d	p	Successors
13	(1, 0, 1)	1	–	1	0	$\{15, 24\}$
14	(1, 1, 0)	2	–	∞	–	$\{16, 27\}$
15	(1, 1, 1)	2	–	2	2	$\{17, 26\}$
16	(1, 2, 0)	3	–	∞	–	$\{18, 29\}$
17	(1, 2, 1)	3	–	3	4	$\{19, 28\}$
18	(1, 3, 0)	4	–	∞	–	$\{20, 31\}$
19	(1, 3, 1)	4	–	4	6	$\{21, 30\}$
20	(1, 4, 0)	5	–	∞	–	$\{22, 33\}$
21	(1, 4, 1)	5	–	5	8	$\{23, 32\}$
22	(1, 5, 0)	6	–	∞	–	$\{35\}$
23	(1, 5, 1)	6	–	6	10	$\{34\}$
24	(2, 0, 0)	2	–	2	13	$\{26, 37\}$
25	(2, 0, 1)	2	–	∞	–	$\{27, 36\}$

ID	State	Seed	Bd.	d	p	Successors
26	(2, 1, 0)	3	–	3	15	$\{28, 39\}$
27	(2, 1, 1)	3	–	∞	–	$\{29, 38\}$
28	(2, 2, 0)	4	–	4	17	$\{30, 41\}$
29	(2, 2, 1)	4	–	∞	–	$\{31, 40\}$
30	(2, 3, 0)	5	–	5	19	$\{32, 43\}$
31	(2, 3, 1)	5	–	∞	–	$\{33, 42\}$
32	(2, 4, 0)	6	–	6	21	$\{34, 45\}$
33	(2, 4, 1)	6	–	∞	–	$\{35, 44\}$
34	(2, 5, 0)	7	–	7	23	$\{47\}$
35	(2, 5, 1)	7	–	∞	–	$\{46\}$
36	(3, 0, 0)	3	–	∞	–	$\{38, 49\}$
37	(3, 0, 1)	3	–	3	24	$\{39, 48\}$
38	(3, 1, 0)	4	–	∞	–	$\{40, 51\}$

ID	State	Seed	Bd.	d	p	Successors
39	(3, 1, 1)	4	–	4	26	$\{41, 50\}$
40	(3, 2, 0)	5	–	∞	–	$\{42, 53\}$
41	(3, 2, 1)	5	–	5	28	$\{43, 52\}$
42	(3, 3, 0)	6	–	∞	–	$\{44, 55\}$
43	(3, 3, 1)	6	–	6	30	$\{45, 54\}$
44	(3, 4, 0)	7	–	∞	–	$\{46, 57\}$
45	(3, 4, 1)	7	–	7	32	$\{47, 56\}$
46	(3, 5, 0)	8	–	∞	–	$\{59\}$
47	(3, 5, 1)	8	–	8	34	$\{58\}$
48	(4, 0, 0)	4	–	4	37	$\{50, 61\}$
49	(4, 0, 1)	4	–	∞	–	$\{51, 60\}$
50	(4, 1, 0)	5	–	5	39	$\{52, 63\}$
51	(4, 1, 1)	5	–	∞	–	$\{53, 62\}$

ID	State	Seed	Bd.	d	p	Successors
52	(4, 2, 0)	6	–	6	41	$\{54, 65\}$
53	(4, 2, 1)	6	–	∞	–	$\{55, 64\}$
54	(4, 3, 0)	7	–	7	43	$\{56, 67\}$
55	(4, 3, 1)	7	–	∞	–	$\{57, 66\}$
56	(4, 4, 0)	8	–	8	45	$\{58, 69\}$
57	(4, 4, 1)	8	–	∞	–	$\{59, 68\}$
58	(4, 5, 0)	9	–	9	47	$\{71\}$
59	(4, 5, 1)	9	–	∞	–	$\{70\}$
60	(5, 0, 0)	5	–	∞	–	$\{62\}$
61	(5, 0, 1)	5	–	5	48	$\{63\}$
62	(5, 1, 0)	6	–	∞	–	$\{64\}$
63	(5, 1, 1)	6	–	6	50	$\{65\}$
64	(5, 2, 0)	7	–	∞	–	$\{66\}$

ID	State	Seed	Bd.	d	p	Successors
65	(5, 2, 1)	7	–	7	52	$\{67\}$
66	(5, 3, 0)	8	–	∞	–	$\{68\}$
67	(5, 3, 1)	8	–	8	54	$\{69\}$
68	(5, 4, 0)	9	–	∞	–	$\{70\}$
69	(5, 4, 1)	9	–	9	56	$\{71\}$
70	(5, 5, 0)	10	B	∞	–	$\varnothing$
71	(5, 5, 1)	10	–	10	58	$\varnothing$

Round 1. The partition has 11 nonempty cells, and the may relation has 10 edges.

The selected path is $b_0, b_1, b_2, b_3, b_4, b_5, b_6, b_7, b_8, b_9, b_{10}$. The exact number of concrete lifts is 0.

j	a_j	F_j	D_j	$\#\cap$
0	b_0	$\{0\}$	$\{1\}$	0
1	b_1	$\{2, 13\}$	$\{3, 12\}$	0
2	b_2	$\{4, 15, 24\}$	$\{5, 14, 25\}$	0
3	b_3	$\{6, 17, 26, 37\}$	$\{7, 16, 27, 36\}$	0
4	b_4	$\{8, 19, 28, 39, 48\}$	$\{9, 18, 29, 38, 49\}$	0
5	b_5	$\{10, 21, 30, 41, 50, 61\}$	$\{11, 20, 31, 40, 51, 60\}$	0
6	b_6	$\{23, 32, 43, 52, 63\}$	$\{22, 33, 42, 53, 62\}$	0
7	b_7	$\{34, 45, 54, 65\}$	$\{35, 44, 55, 64\}$	0
8	b_8	$\{47, 56, 67\}$	$\{46, 57, 66\}$	0
9	b_9	$\{58, 69\}$	$\{59, 68\}$	0
10	b_{10}	$\{71\}$	$\{70\}$	0

The isolated state IDs are $\{0, 1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12, 13, 14, 15, 16, 17, 18, 19, 20, 21, 22, 23, 24, 25, 26, 27, 28, 29, 30, 31, 32, 33, 34, 35, 36, 37, 38, 39, 40, 41, 42, 43, 44, 45, 46, 47, 48, 49, 50, 51, 52, 53, 54, 55, 56, 57, 58, 59, 60, 61, 62, 63, 64, 65, 66, 67, 68, 69, 70, 71\}$. The refined graph has zero paths projecting to this candidate.

Round 2. The partition has 72 nonempty cells, and the may relation has 120 edges.

The abstract checker found the inductive invariant $\{s_0, s_2, s_4, s_6, s_8, s_{10}, s_{13}, s_{15}, s_{17}, s_{19}, s_{21}, s_{23}, s_{24}, s_{26}, s_{28}, s_{30}, s_{32}, s_{34}, s_{37}, s_{39}, s_{41}, s_{43}, s_{45}, s_{47}, s_{48}, s_{50}, s_{52},$

$s_{54}, s_{56}, s_{58}, s_{61}, s_{63}, s_{65}, s_{67}, s_{69}, s_{71}\}$. It contains $\{s_0\}$ and is disjoint from $\{s_{70}\}$.

Cell	All may successors
s_0	$\{s_2, s_{13}\}$
s_2	$\{s_4, s_{15}\}$
s_4	$\{s_6, s_{17}\}$
s_6	$\{s_8, s_{19}\}$
s_8	$\{s_{10}, s_{21}\}$
s_{10}	$\{s_{23}\}$
s_{13}	$\{s_{15}, s_{24}\}$
s_{15}	$\{s_{17}, s_{26}\}$
s_{17}	$\{s_{19}, s_{28}\}$
s_{19}	$\{s_{21}, s_{30}\}$
s_{21}	$\{s_{23}, s_{32}\}$
s_{23}	$\{s_{34}\}$
s_{24}	$\{s_{26}, s_{37}\}$

Cell	All may successors
s_{26}	$\{s_{28}, s_{39}\}$
s_{28}	$\{s_{30}, s_{41}\}$
s_{30}	$\{s_{32}, s_{43}\}$
s_{32}	$\{s_{34}, s_{45}\}$
s_{34}	$\{s_{47}\}$
s_{37}	$\{s_{39}, s_{48}\}$
s_{39}	$\{s_{41}, s_{50}\}$
s_{41}	$\{s_{43}, s_{52}\}$
s_{43}	$\{s_{45}, s_{54}\}$
s_{45}	$\{s_{47}, s_{56}\}$
s_{47}	$\{s_{58}\}$
s_{48}	$\{s_{50}, s_{61}\}$
s_{50}	$\{s_{52}, s_{63}\}$

Cell	All may successors
s_{52}	$\{s_{54}, s_{65}\}$
s_{54}	$\{s_{56}, s_{67}\}$
s_{56}	$\{s_{58}, s_{69}\}$
s_{58}	$\{s_{71}\}$
s_{61}	$\{s_{63}\}$
s_{63}	$\{s_{65}\}$
s_{65}	$\{s_{67}\}$
s_{67}	$\{s_{69}\}$
s_{69}	$\{s_{71}\}$
s_{71}	$\varnothing$

23 Boundary behavior

These cases vary the empty carrier, missing boundary predicates, zero-length bad paths, and long paths in an observation with one realized cell.

Tables enumerate concrete states by integer ID. The d column gives distance from an initial state, and p gives a predecessor attaining that distance. Columns I and B are encoded by the boundary field. Abstract keys b_i are seed blocks and s_x are isolated states.

23.1 F8.1 carrier 0, mode 0

Self-loops alone preserve the state. In successor mode the ordered chain reaches every state from zero. A bad initial state has a witness of length zero, and an empty initial or bad predicate gives immediate safety.

The concrete carrier has 0 states and 0 edges. Its reachable set has 0 states. The reachable set is disjoint from the bad predicate.

The empty concrete carrier has no rows, boundaries, or transitions.

Round 1. The partition has 0 nonempty cells, and the may relation has 0 edges.

The abstract checker found the inductive invariant $\varnothing$. It contains $\varnothing$ and is disjoint from $\varnothing$.

23.2 F8.2 carrier 1, mode 0

Self-loops alone preserve the state. In successor mode the ordered chain reaches every state from zero. A bad initial state has a witness of length zero, and an empty initial or bad predicate gives immediate safety.

The concrete carrier has 1 states and 1 edges. Its reachable set has 1 states. The reachable set is disjoint from the bad predicate.

ID	State	Seed	Bd.	d	p	Successors
0	0	0	I	0	–	{0}

Round 1. The partition has 1 nonempty cells, and the may relation has 1 edges.

The abstract checker found the inductive invariant $\{b_0\}$. It contains $\{b_0\}$ and is disjoint from $\varnothing$.

Cell	All may successors
b_0	$\{b_0\}$

23.3 F8.3 carrier 1, mode 1

Self-loops alone preserve the state. In successor mode the ordered chain reaches every state from zero. A bad initial state has a witness of length zero, and an empty initial or bad predicate gives immediate safety.

The concrete carrier has 1 states and 1 edges. Its reachable set has 1 states. The bad predicate is reached.

ID	State	Seed	Bd.	d	p	Successors
0	0	0	IB	0	–	{0}

Round 1. The partition has 1 nonempty cells, and the may relation has 1 edges.

The selected path is b_0. The exact number of concrete lifts is 1.

j	a_j	F_j	D_j	$\#\cap$
0	b_0	$\{0\}$	$\{0\}$	1

A concrete witness is 0. Each adjacent pair occurs in the concrete successor table.

23.4 F8.4 carrier 2, mode 2

Self-loops alone preserve the state. In successor mode the ordered chain reaches every state from zero. A bad initial state has a witness of length zero, and an empty initial or bad predicate gives immediate safety.

The concrete carrier has 2 states and 2 edges. Its reachable set has 1 states. The reachable set is disjoint from the bad predicate.

ID	State	Seed	Bd.	d	p	Successors
0	0	0	I	0	–	$\{0\}$
1	1	0	B	∞	–	$\{1\}$

Round 1. The partition has 1 nonempty cells, and the may relation has 1 edges.

The selected path is b_0. The exact number of concrete lifts is 0.

j	a_j	F_j	D_j	$\#\cap$
0	b_0	$\{0\}$	$\{1\}$	0

The isolated state IDs are $\{0, 1\}$. The refined graph has zero paths projecting to this candidate.

Round 2. The partition has 2 nonempty cells, and the may relation has 2 edges.

The abstract checker found the inductive invariant $\{s_0\}$. It contains $\{s_0\}$ and is disjoint from $\{s_1\}$.

Cell	All may successors
s_0	$\{s_0\}$

23.5 F8.5 carrier 5, mode 2

Self-loops alone preserve the state. In successor mode the ordered chain reaches every state from zero. A bad initial state has a witness of length zero, and an empty initial or bad predicate gives immediate safety.

The concrete carrier has 5 states and 5 edges. Its reachable set has 1 states. The reachable set is disjoint from the bad predicate.

ID	State	Seed	Bd.	d	p	Successors
0	0	0	I	0	–	$\{0\}$
1	1	0	–	∞	–	$\{1\}$
2	2	0	–	∞	–	$\{2\}$
3	3	0	–	∞	–	$\{3\}$
4	4	0	B	∞	–	$\{4\}$

Round 1. The partition has 1 nonempty cells, and the may relation has 1 edges.

The selected path is b_0. The exact number of concrete lifts is 0.

j	a_j	F_j	D_j	$\#\cap$
0	b_0	$\{0\}$	$\{4\}$	0

The isolated state IDs are $\{0, 1, 2, 3, 4\}$. The refined graph has zero paths projecting to this candidate.

Round 2. The partition has 5 nonempty cells, and the may relation has 5 edges.

The abstract checker found the inductive invariant $\{s_0\}$. It contains $\{s_0\}$ and is disjoint from $\{s_4\}$.

Cell	All may successors
s_0	$\{s_0\}$

23.6 F8.6 carrier 7, mode 3

Self-loops alone preserve the state. In successor mode the ordered chain reaches every state from zero. A bad initial state has a witness of length zero, and an empty initial or bad predicate gives immediate safety.

The concrete carrier has 7 states and 13 edges. Its reachable set has 7 states. The bad predicate is reached.

ID	State	Seed	Bd.	d	p	Successors
0	0	0	I	0	–	$\{0, 1\}$
1	1	0	–	1	0	$\{1, 2\}$
2	2	0	–	2	1	$\{2, 3\}$
3	3	0	–	3	2	$\{3, 4\}$
4	4	0	–	4	3	$\{4, 5\}$
5	5	0	–	5	4	$\{5, 6\}$
6	6	0	B	6	5	$\{6\}$

Round 1. The partition has 1 nonempty cells, and the may relation has 1 edges.

The selected path is b_0. The exact number of concrete lifts is 0.

j	a_j	F_j	D_j	$\#\cap$
0	b_0	$\{0\}$	$\{6\}$	0

The isolated state IDs are $\{0, 1, 2, 3, 4, 5, 6\}$. The refined graph has zero paths projecting to this candidate.

Round 2. The partition has 7 nonempty cells, and the may relation has 13 edges.

The selected path is $s_0, s_1, s_2, s_3, s_4, s_5, s_6$. The exact number of concrete lifts is 1.

j	a_j	F_j	D_j	$\#\cap$
0	s_0	{0}	{0}	1
1	s_1	{1}	{1}	1
2	s_2	{2}	{2}	1
3	s_3	{3}	{3}	1
4	s_4	{4}	{4}	1
5	s_5	{5}	{5}	1
6	s_6	{6}	{6}	1

A concrete witness is $0, 1, 2, 3, 4, 5, 6$. Each adjacent pair occurs in the concrete successor table.

23.7 F8.7 carrier 10, mode 4

Self-loops alone preserve the state. In successor mode the ordered chain reaches every state from zero. A bad initial state has a witness of length zero, and an empty initial or bad predicate gives immediate safety.

The concrete carrier has 10 states and 10 edges. Its reachable set has 0 states. The reachable set is disjoint from the bad predicate.

ID	State	Seed	Bd.	d	p	Successors
0	0	0	–	∞	–	{0}
1	1	0	–	∞	–	{1}
2	2	0	–	∞	–	{2}
3	3	0	–	∞	–	{3}
4	4	0	–	∞	–	{4}
5	5	0	–	∞	–	{5}
6	6	0	–	∞	–	{6}
7	7	0	–	∞	–	{7}
8	8	0	–	∞	–	{8}
9	9	0	B	∞	–	{9}

Round 1. The partition has 1 nonempty cells, and the may relation has 1 edges.

The abstract checker found the inductive invariant $\varnothing$. It contains $\varnothing$ and is disjoint from $\{b_0\}$.

23.8 F8.8 carrier 12, mode 2

Self-loops alone preserve the state. In successor mode the ordered chain reaches every state from zero. A bad initial state has a witness of length zero, and an empty initial or bad predicate gives immediate safety.

The concrete carrier has 12 states and 12 edges. Its reachable set has 1 states. The reachable set is disjoint from the bad predicate.

ID	State	Seed	Bd.	d	p	Successors
0	0	0	I	0	–	{0}
1	1	0	–	∞	–	{1}
2	2	0	–	∞	–	{2}
3	3	0	–	∞	–	{3}
4	4	0	–	∞	–	{4}
5	5	0	–	∞	–	{5}
6	6	0	–	∞	–	{6}
7	7	0	–	∞	–	{7}
8	8	0	–	∞	–	{8}
9	9	0	–	∞	–	{9}
10	10	0	–	∞	–	{10}
11	11	0	B	∞	–	{11}

Round 1. The partition has 1 nonempty cells, and the may relation has 1 edges.

The selected path is b_0. The exact number of concrete lifts is 0.

j	a_j	F_j	D_j	$\#\cap$
0	b_0	{0}	{11}	0

The isolated state IDs are $\{0, 1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11\}$. The refined graph has zero paths projecting to this candidate.

Round 2. The partition has 12 nonempty cells, and the may relation has 12 edges.

The abstract checker found the inductive invariant $\{s_0\}$. It contains $\{s_0\}$ and is disjoint from $\{s_{11}\}$.

Cell	All may successors
s_0	$\{s_0\}$

23.9 F8.9 carrier 18, mode 3

Self-loops alone preserve the state. In successor mode the ordered chain reaches every state from zero. A bad initial state has a witness of length zero, and an empty initial or bad predicate gives immediate safety.

The concrete carrier has 18 states and 35 edges. Its reachable set has 18 states. The bad predicate is reached.

ID	State	Seed	Bd.	d	p	Successors
0	0	0	I	0	–	$\{0, 1\}$
1	1	0	–	1	0	$\{1, 2\}$
2	2	0	–	2	1	$\{2, 3\}$
3	3	0	–	3	2	$\{3, 4\}$
4	4	0	–	4	3	$\{4, 5\}$
5	5	0	–	5	4	$\{5, 6\}$
6	6	0	–	6	5	$\{6, 7\}$
7	7	0	–	7	6	$\{7, 8\}$
8	8	0	–	8	7	$\{8, 9\}$
9	9	0	–	9	8	$\{9, 10\}$
10	10	0	–	10	9	$\{10, 11\}$
11	11	0	–	11	10	$\{11, 12\}$
12	12	0	–	12	11	$\{12, 13\}$

ID	State	Seed	Bd.	d	p	Successors
13	13	0	–	13	12	$\{13, 14\}$
14	14	0	–	14	13	$\{14, 15\}$
15	15	0	–	15	14	$\{15, 16\}$
16	16	0	–	16	15	$\{16, 17\}$
17	17	0	B	17	16	$\{17\}$

Round 1. The partition has 1 nonempty cells, and the may relation has 1 edges.

The selected path is b_0. The exact number of concrete lifts is 0.

j	a_j	F_j	D_j	$\#\cap$
0	b_0	$\{0\}$	$\{17\}$	0

The isolated state IDs are $\{0, 1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12, 13, 14, 15, 16, 17\}$. The refined graph has zero paths projecting to this candidate.

Round 2. The partition has 18 nonempty cells, and the may relation has 35 edges.

The selected path is $s_0, s_1, s_2, s_3, s_4, s_5, s_6, s_7, s_8, s_9, s_{10}, s_{11}, s_{12}, s_{13}, s_{14}, s_{15}, s_{16}, s_{17}$. The exact number of concrete lifts is 1.

j	a_j	F_j	D_j	$\#\cap$
0	s_0	$\{0\}$	$\{0\}$	1
1	s_1	$\{1\}$	$\{1\}$	1
2	s_2	$\{2\}$	$\{2\}$	1
3	s_3	$\{3\}$	$\{3\}$	1
4	s_4	$\{4\}$	$\{4\}$	1
5	s_5	$\{5\}$	$\{5\}$	1
6	s_6	$\{6\}$	$\{6\}$	1
7	s_7	$\{7\}$	$\{7\}$	1
8	s_8	$\{8\}$	$\{8\}$	1
9	s_9	$\{9\}$	$\{9\}$	1
10	s_{10}	$\{10\}$	$\{10\}$	1
11	s_{11}	$\{11\}$	$\{11\}$	1
12	s_{12}	$\{12\}$	$\{12\}$	1

j	a_j	F_j	D_j	$\#\cap$
13	s_{13}	$\{13\}$	$\{13\}$	1
14	s_{14}	$\{14\}$	$\{14\}$	1
15	s_{15}	$\{15\}$	$\{15\}$	1
16	s_{16}	$\{16\}$	$\{16\}$	1
17	s_{17}	$\{17\}$	$\{17\}$	1

A concrete witness is $0, 1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12, 13, 14, 15, 16, 17$. Each adjacent pair occurs in the concrete successor table.

24 Sharpness of the round bound

Each unrefined middle cell witnesses an abstract two-edge bad path, and exactifying it does not remove the other coarse middle cells. These cases separate the bound on one particular path's descendants from the number of other candidates that can remain.

Tables enumerate concrete states by integer ID. The d column gives distance from an initial state, and p gives a predecessor attaining that distance. Columns I and B are encoded by the boundary field. Abstract keys b_i are seed blocks and s_x are isolated states.

24.1 F9.1 4 middle cells, concrete branch None

Every intermediate cell has an entry point reachable from the root and a separate exit point. Each failed round isolates precisely one as-yet coarse intermediate cell. The $2 \cdot 4 + 2$ concrete states give $4 + 2$ initial cells, so this case attains the bound of $4 + 1$ abstract rounds.

The concrete carrier has 10 states and 8 edges. Its reachable set has 5 states. The reachable set is disjoint from the bad predicate.

ID	State	Seed	Bd.	d	p	Successors
0	start	0	I	0	–	$\{1, 2, 3, 4\}$
1	(0, 'entry')	1	–	1	0	$\varnothing$
2	(1, 'entry')	2	–	1	0	$\varnothing$
3	(2, 'entry')	3	–	1	0	$\varnothing$
4	(3, 'entry')	4	–	1	0	$\varnothing$
5	(0, 'exit')	1	–	∞	–	$\{9\}$
6	(1, 'exit')	2	–	∞	–	$\{9\}$
7	(2, 'exit')	3	–	∞	–	$\{9\}$
8	(3, 'exit')	4	–	∞	–	$\{9\}$
9	bad	5	B	∞	–	$\varnothing$

Round 1. The partition has 6 nonempty cells, and the may relation has 8 edges.

The selected path is b_0, b_4, b_5. The exact number of concrete lifts is 0.

j	a_j	F_j	D_j	$\#\cap$
0	b_0	$\{0\}$	$\varnothing$	0
1	b_4	$\{4\}$	$\{8\}$	0
2	b_5	$\varnothing$	$\{9\}$	0

The isolated state IDs are $\{0, 4, 8, 9\}$. The refined graph has zero paths projecting to this candidate.

Round 2. The partition has 7 nonempty cells, and the may relation has 8 edges.

The selected path is s_0, b_1, s_9. The exact number of concrete lifts is 0.

j	a_j	F_j	D_j	$\#\cap$
0	s_0	$\{0\}$	$\varnothing$	0
1	b_1	$\{1\}$	$\{5\}$	0
2	s_9	$\varnothing$	$\{9\}$	0

The isolated state IDs are $\{1, 5\}$. The refined graph has zero paths projecting to this candidate.

Round 3. The partition has 8 nonempty cells, and the may relation has 8 edges.

The selected path is s_0, b_2, s_9. The exact number of concrete lifts is 0.

j	a_j	F_j	D_j	$\#\cap$
0	s_0	$\{0\}$	$\varnothing$	0
1	b_2	$\{2\}$	$\{6\}$	0
2	s_9	$\varnothing$	$\{9\}$	0

The isolated state IDs are $\{2, 6\}$. The refined graph has zero paths projecting to this candidate.

Round 4. The partition has 9 nonempty cells, and the may relation has 8 edges.

The selected path is s_0, b_3, s_9. The exact number of concrete lifts is 0.

j	a_j	F_j	D_j	$\#\cap$
0	s_0	$\{0\}$	$\varnothing$	0
1	b_3	$\{3\}$	$\{7\}$	0
2	s_9	$\varnothing$	$\{9\}$	0

The isolated state IDs are $\{3, 7\}$. The refined graph has zero paths projecting to this candidate.

Round 5. The partition has 10 nonempty cells, and the may relation has 8 edges.

The abstract checker found the inductive invariant $\{s_0, s_1, s_2, s_3, s_4\}$. It contains $\{s_0\}$ and is disjoint from $\{s_9\}$.

Cell	All may successors
s_0	$\{s_1, s_2, s_3, s_4\}$
s_1	$\varnothing$
s_2	$\varnothing$
s_3	$\varnothing$
s_4	$\varnothing$

24.2 F9.2 5 middle cells, concrete branch None

Every intermediate cell has an entry point reachable from the root and a separate exit point. Each failed round isolates precisely one as-yet coarse intermediate cell. The $2 \cdot 5 + 2$ concrete states give $5 + 2$ initial cells, so this case attains the bound of $5 + 1$ abstract rounds.

The concrete carrier has 12 states and 10 edges. Its reachable set has 6 states. The reachable set is disjoint from the bad predicate.

ID	State	Seed	Bd.	d	p	Successors
0	start	0	I	0	–	$\{1, 2, 3, 4, 5\}$
1	(0, 'entry')	1	–	1	0	$\varnothing$
2	(1, 'entry')	2	–	1	0	$\varnothing$
3	(2, 'entry')	3	–	1	0	$\varnothing$
4	(3, 'entry')	4	–	1	0	$\varnothing$
5	(4, 'entry')	5	–	1	0	$\varnothing$
6	(0, 'exit')	1	–	∞	–	$\{11\}$
7	(1, 'exit')	2	–	∞	–	$\{11\}$
8	(2, 'exit')	3	–	∞	–	$\{11\}$
9	(3, 'exit')	4	–	∞	–	$\{11\}$
10	(4, 'exit')	5	–	∞	–	$\{11\}$
11	bad	6	B	∞	–	$\varnothing$

Round 1. The partition has 7 nonempty cells, and the may relation has 10 edges.

The selected path is b_0, b_5, b_6. The exact number of concrete lifts is 0.

j	a_j	F_j	D_j	$\#\cap$
0	b_0	$\{0\}$	$\varnothing$	0
1	b_5	$\{5\}$	$\{10\}$	0
2	b_6	$\varnothing$	$\{11\}$	0

The isolated state IDs are $\{0, 5, 10, 11\}$. The refined graph has zero paths projecting to this candidate.

Round 2. The partition has 8 nonempty cells, and the may relation has 10 edges.

The selected path is s_0, b_1, s_{11}. The exact number of concrete lifts is 0.

j	a_j	F_j	D_j	$\#\cap$
0	s_0	$\{0\}$	$\varnothing$	0
1	b_1	$\{1\}$	$\{6\}$	0
2	s_{11}	$\varnothing$	$\{11\}$	0

The isolated state IDs are $\{1, 6\}$. The refined graph has zero paths projecting to this candidate.

Round 3. The partition has 9 nonempty cells, and the may relation has 10 edges.

The selected path is s_0, b_2, s_{11}. The exact number of concrete lifts is 0.

j	a_j	F_j	D_j	$\#\cap$
0	s_0	$\{0\}$	$\varnothing$	0
1	b_2	$\{2\}$	$\{7\}$	0
2	s_{11}	$\varnothing$	$\{11\}$	0

The isolated state IDs are $\{2, 7\}$. The refined graph has zero paths projecting to this candidate.

Round 4. The partition has 10 nonempty cells, and the may relation has 10 edges.

The selected path is s_0, b_3, s_{11}. The exact number of concrete lifts is 0.

j	a_j	F_j	D_j	$\#\cap$
0	s_0	$\{0\}$	$\varnothing$	0
1	b_3	$\{3\}$	$\{8\}$	0
2	s_{11}	$\varnothing$	$\{11\}$	0

The isolated state IDs are $\{3, 8\}$. The refined graph has zero paths projecting to this candidate.

Round 5. The partition has 11 nonempty cells, and the may relation has 10 edges.

The selected path is s_0, b_4, s_{11}. The exact number of concrete lifts is 0.

j	a_j	F_j	D_j	$\#\cap$
0	s_0	$\{0\}$	$\varnothing$	0
1	b_4	$\{4\}$	$\{9\}$	0
2	s_{11}	$\varnothing$	$\{11\}$	0

The isolated state IDs are $\{4, 9\}$. The refined graph has zero paths projecting to this candidate.

Round 6. The partition has 12 nonempty cells, and the may relation has 10 edges.

The abstract checker found the inductive invariant $\{s_0, s_1, s_2, s_3, s_4, s_5\}$. It contains $\{s_0\}$ and is disjoint from $\{s_{11}\}$.

Cell	All may successors
s_0	$\{s_1, s_2, s_3, s_4, s_5\}$
s_1	$\varnothing$
s_2	$\varnothing$
s_3	$\varnothing$
s_4	$\varnothing$
s_5	$\varnothing$

24.3 F9.3 6 middle cells, concrete branch None

Every intermediate cell has an entry point reachable from the root and a separate exit point. Each failed round isolates precisely one as-yet coarse intermediate cell. The $2 \cdot 6 + 2$ concrete states give $6 + 2$ initial cells, so this case attains the bound of $6 + 1$ abstract rounds.

The concrete carrier has 14 states and 12 edges. Its reachable set has 7 states. The reachable set is disjoint from the bad predicate.

ID	State	Seed	Bd.	d	p	Successors
0	start	0	I	0	–	$\{1, 2, 3, 4, 5, 6\}$
1	(0, 'entry')	1	–	1	0	$\varnothing$
2	(1, 'entry')	2	–	1	0	$\varnothing$
3	(2, 'entry')	3	–	1	0	$\varnothing$
4	(3, 'entry')	4	–	1	0	$\varnothing$
5	(4, 'entry')	5	–	1	0	$\varnothing$
6	(5, 'entry')	6	–	1	0	$\varnothing$
7	(0, 'exit')	1	–	∞	–	$\{13\}$
8	(1, 'exit')	2	–	∞	–	$\{13\}$
9	(2, 'exit')	3	–	∞	–	$\{13\}$
10	(3, 'exit')	4	–	∞	–	$\{13\}$
11	(4, 'exit')	5	–	∞	–	$\{13\}$
12	(5, 'exit')	6	–	∞	–	$\{13\}$

ID	State	Seed	Bd.	d	p	Successors
13	bad	7	B	∞	–	$\varnothing$

Round 1. The partition has 8 nonempty cells, and the may relation has 12 edges.

The selected path is b_0, b_6, b_7. The exact number of concrete lifts is 0.

j	a_j	F_j	D_j	$\#\cap$
0	b_0	$\{0\}$	$\varnothing$	0
1	b_6	$\{6\}$	$\{12\}$	0
2	b_7	$\varnothing$	$\{13\}$	0

The isolated state IDs are $\{0, 6, 12, 13\}$. The refined graph has zero paths projecting to this candidate.

Round 2. The partition has 9 nonempty cells, and the may relation has 12 edges.

The selected path is s_0, b_1, s_{13}. The exact number of concrete lifts is 0.

j	a_j	F_j	D_j	$\#\cap$
0	s_0	$\{0\}$	$\varnothing$	0
1	b_1	$\{1\}$	$\{7\}$	0
2	s_{13}	$\varnothing$	$\{13\}$	0

The isolated state IDs are $\{1, 7\}$. The refined graph has zero paths projecting to this candidate.

Round 3. The partition has 10 nonempty cells, and the may relation has 12 edges.

The selected path is s_0, b_2, s_{13}. The exact number of concrete lifts is 0.

j	a_j	F_j	D_j	$\#\cap$
0	s_0	$\{0\}$	$\varnothing$	0
1	b_2	$\{2\}$	$\{8\}$	0
2	s_{13}	$\varnothing$	$\{13\}$	0

The isolated state IDs are $\{2, 8\}$. The refined graph has zero paths projecting to this candidate.

Round 4. The partition has 11 nonempty cells, and the may relation has 12 edges.

The selected path is s_0, b_3, s_{13}. The exact number of concrete lifts is 0.

j	a_j	F_j	D_j	$\#\cap$
0	s_0	$\{0\}$	$\varnothing$	0
1	b_3	$\{3\}$	$\{9\}$	0
2	s_{13}	$\varnothing$	$\{13\}$	0

The isolated state IDs are $\{3, 9\}$. The refined graph has zero paths projecting to this candidate.

Round 5. The partition has 12 nonempty cells, and the may relation has 12 edges.

The selected path is s_0, b_4, s_{13}. The exact number of concrete lifts is 0.

j	a_j	F_j	D_j	$\#\cap$
0	s_0	$\{0\}$	$\varnothing$	0
1	b_4	$\{4\}$	$\{10\}$	0
2	s_{13}	$\varnothing$	$\{13\}$	0

The isolated state IDs are $\{4, 10\}$. The refined graph has zero paths projecting to this candidate.

Round 6. The partition has 13 nonempty cells, and the may relation has 12 edges.

The selected path is s_0, b_5, s_{13}. The exact number of concrete lifts is 0.

j	a_j	F_j	D_j	$\#\cap$
0	s_0	$\{0\}$	$\varnothing$	0
1	b_5	$\{5\}$	$\{11\}$	0
2	s_{13}	$\varnothing$	$\{13\}$	0

The isolated state IDs are $\{5, 11\}$. The refined graph has zero paths projecting to this candidate.

Round 7. The partition has 14 nonempty cells, and the may relation has 12 edges.

The abstract checker found the inductive invariant $\{s_0, s_1, s_2, s_3, s_4, s_5, s_6\}$. It contains $\{s_0\}$ and is disjoint from $\{s_{13}\}$.

Cell	All may successors
s_0	$\{s_1, s_2, s_3, s_4, s_5, s_6\}$
s_1	$\varnothing$
s_2	$\varnothing$
s_3	$\varnothing$
s_4	$\varnothing$
s_5	$\varnothing$
s_6	$\varnothing$

24.4 F9.4 7 middle cells, concrete branch None

Every intermediate cell has an entry point reachable from the root and a separate exit point. Each failed round isolates precisely one as-yet coarse intermediate cell. The $2 \cdot 7 + 2$ concrete states give $7 + 2$ initial cells, so this case attains the bound of $7 + 1$ abstract rounds.

The concrete carrier has 16 states and 14 edges. Its reachable set has 8 states. The reachable set is disjoint from the bad predicate.

ID	State	Seed	Bd.	d	p	Successors
0	start	0	I	0	–	$\{1, 2, 3, 4, 5, 6, 7\}$
1	(0, 'entry')	1	–	1	0	$\varnothing$
2	(1, 'entry')	2	–	1	0	$\varnothing$
3	(2, 'entry')	3	–	1	0	$\varnothing$
4	(3, 'entry')	4	–	1	0	$\varnothing$
5	(4, 'entry')	5	–	1	0	$\varnothing$
6	(5, 'entry')	6	–	1	0	$\varnothing$
7	(6, 'entry')	7	–	1	0	$\varnothing$
8	(0, 'exit')	1	–	∞	–	$\{15\}$
9	(1, 'exit')	2	–	∞	–	$\{15\}$
10	(2, 'exit')	3	–	∞	–	$\{15\}$
11	(3, 'exit')	4	–	∞	–	$\{15\}$
12	(4, 'exit')	5	–	∞	–	$\{15\}$
13	(5, 'exit')	6	–	∞	–	$\{15\}$
14	(6, 'exit')	7	–	∞	–	$\{15\}$
15	bad	8	B	∞	–	$\varnothing$

Round 1. The partition has 9 nonempty cells, and the may relation has 14 edges.

The selected path is b_0, b_7, b_8. The exact number of concrete lifts is 0.

j	a_j	F_j	D_j	$\#\cap$
0	b_0	$\{0\}$	$\varnothing$	0
1	b_7	$\{7\}$	$\{14\}$	0
2	b_8	$\varnothing$	$\{15\}$	0

The isolated state IDs are $\{0, 7, 14, 15\}$. The refined graph has zero paths projecting to this candidate.

Round 2. The partition has 10 nonempty cells, and the may relation has 14 edges.

The selected path is s_0, b_1, s_{15}. The exact number of concrete lifts is 0.

j	a_j	F_j	D_j	$\#\cap$
0	s_0	$\{0\}$	$\varnothing$	0
1	b_1	$\{1\}$	$\{8\}$	0
2	s_{15}	$\varnothing$	$\{15\}$	0

The isolated state IDs are $\{1, 8\}$. The refined graph has zero paths projecting to this candidate.

Round 3. The partition has 11 nonempty cells, and the may relation has 14 edges.

The selected path is s_0, b_2, s_{15}. The exact number of concrete lifts is 0.

j	a_j	F_j	D_j	$\#\cap$
0	s_0	$\{0\}$	$\varnothing$	0
1	b_2	$\{2\}$	$\{9\}$	0
2	s_{15}	$\varnothing$	$\{15\}$	0

The isolated state IDs are $\{2, 9\}$. The refined graph has zero paths projecting to this candidate.

Round 4. The partition has 12 nonempty cells, and the may relation has 14 edges.

The selected path is s_0, b_3, s_{15}. The exact number of concrete lifts is 0.

j	a_j	F_j	D_j	$\#\cap$
0	s_0	$\{0\}$	$\varnothing$	0
1	b_3	$\{3\}$	$\{10\}$	0
2	s_{15}	$\varnothing$	$\{15\}$	0

The isolated state IDs are $\{3, 10\}$. The refined graph has zero paths projecting to this candidate.

Round 5. The partition has 13 nonempty cells, and the may relation has 14 edges.

The selected path is s_0, b_4, s_{15}. The exact number of concrete lifts is 0.

j	a_j	F_j	D_j	$\#\cap$
0	s_0	$\{0\}$	$\varnothing$	0
1	b_4	$\{4\}$	$\{11\}$	0
2	s_{15}	$\varnothing$	$\{15\}$	0

The isolated state IDs are $\{4, 11\}$. The refined graph has zero paths projecting to this candidate.

Round 6. The partition has 14 nonempty cells, and the may relation has 14 edges.

The selected path is s_0, b_5, s_{15}. The exact number of concrete lifts is 0.

j	a_j	F_j	D_j	$\#\cap$
0	s_0	$\{0\}$	$\varnothing$	0
1	b_5	$\{5\}$	$\{12\}$	0
2	s_{15}	$\varnothing$	$\{15\}$	0

The isolated state IDs are $\{5, 12\}$. The refined graph has zero paths projecting to this candidate.

Round 7. The partition has 15 nonempty cells, and the may relation has 14 edges.

The selected path is s_0, b_6, s_{15}. The exact number of concrete lifts is 0.

j	a_j	F_j	D_j	$\#\cap$
0	s_0	$\{0\}$	$\varnothing$	0
1	b_6	$\{6\}$	$\{13\}$	0
2	s_{15}	$\varnothing$	$\{15\}$	0

The isolated state IDs are $\{6, 13\}$. The refined graph has zero paths projecting to this candidate.

Round 8. The partition has 16 nonempty cells, and the may relation has 14 edges.

The abstract checker found the inductive invariant $\{s_0, s_1, s_2, s_3, s_4, s_5, s_6, s_7\}$. It contains $\{s_0\}$ and is disjoint from $\{s_{15}\}$.

Cell	All may successors
s_0	$\{s_1, s_2, s_3, s_4, s_5, s_6, s_7\}$
s_1	$\varnothing$
s_2	$\varnothing$
s_3	$\varnothing$
s_4	$\varnothing$
s_5	$\varnothing$
s_6	$\varnothing$
s_7	$\varnothing$

24.5 F9.5 8 middle cells, concrete branch None

Every intermediate cell has an entry point reachable from the root and a separate exit point. Each failed round isolates precisely one as-yet coarse intermediate cell. The $2 \cdot 8 + 2$ concrete states give $8 + 2$ initial cells, so this case attains the bound of $8 + 1$ abstract rounds.

The concrete carrier has 18 states and 16 edges. Its reachable set has 9 states. The reachable set is disjoint from the bad predicate.

ID	State	Seed	Bd.	d	p	Successors
0	start	0	I	0	–	$\{1, 2, 3, 4, 5, 6, 7, 8\}$
1	(0, 'entry')	1	–	1	0	$\varnothing$
2	(1, 'entry')	2	–	1	0	$\varnothing$
3	(2, 'entry')	3	–	1	0	$\varnothing$
4	(3, 'entry')	4	–	1	0	$\varnothing$
5	(4, 'entry')	5	–	1	0	$\varnothing$
6	(5, 'entry')	6	–	1	0	$\varnothing$
7	(6, 'entry')	7	–	1	0	$\varnothing$
8	(7, 'entry')	8	–	1	0	$\varnothing$
9	(0, 'exit')	1	–	∞	–	$\{17\}$
10	(1, 'exit')	2	–	∞	–	$\{17\}$
11	(2, 'exit')	3	–	∞	–	$\{17\}$
12	(3, 'exit')	4	–	∞	–	$\{17\}$
13	(4, 'exit')	5	–	∞	–	$\{17\}$
14	(5, 'exit')	6	–	∞	–	$\{17\}$
15	(6, 'exit')	7	–	∞	–	$\{17\}$
16	(7, 'exit')	8	–	∞	–	$\{17\}$
17	bad	9	B	∞	–	$\varnothing$

Round 1. The partition has 10 nonempty cells, and the may relation has 16 edges.

The selected path is b_0, b_8, b_9. The exact number of concrete lifts is 0.

j	a_j	F_j	D_j	$\#\cap$
0	b_0	$\{0\}$	$\varnothing$	0
1	b_8	$\{8\}$	$\{16\}$	0
2	b_9	$\varnothing$	$\{17\}$	0

The isolated state IDs are $\{0, 8, 16, 17\}$. The refined graph has zero paths projecting to this candidate.

Round 2. The partition has 11 nonempty cells, and the may relation has 16 edges.

The selected path is s_0, b_1, s_{17}. The exact number of concrete lifts is 0.

j	a_j	F_j	D_j	$\#\cap$
0	s_0	$\{0\}$	$\varnothing$	0
1	b_1	$\{1\}$	$\{9\}$	0
2	s_{17}	$\varnothing$	$\{17\}$	0

The isolated state IDs are $\{1, 9\}$. The refined graph has zero paths projecting to this candidate.

Round 3. The partition has 12 nonempty cells, and the may relation has 16 edges.

The selected path is s_0, b_2, s_{17}. The exact number of concrete lifts is 0.

j	a_j	F_j	D_j	$\#\cap$
0	s_0	$\{0\}$	$\varnothing$	0
1	b_2	$\{2\}$	$\{10\}$	0
2	s_{17}	$\varnothing$	$\{17\}$	0

The isolated state IDs are $\{2, 10\}$. The refined graph has zero paths projecting to this candidate.

Round 4. The partition has 13 nonempty cells, and the may relation has 16 edges.

The selected path is s_0, b_3, s_{17}. The exact number of concrete lifts is 0.

j	a_j	F_j	D_j	$\#\cap$
0	s_0	$\{0\}$	$\varnothing$	0
1	b_3	$\{3\}$	$\{11\}$	0
2	s_{17}	$\varnothing$	$\{17\}$	0

The isolated state IDs are $\{3, 11\}$. The refined graph has zero paths projecting to this candidate.

Round 5. The partition has 14 nonempty cells, and the may relation has 16 edges.

The selected path is s_0, b_4, s_{17}. The exact number of concrete lifts is 0.

j	a_j	F_j	D_j	$\#\cap$
0	s_0	$\{0\}$	$\varnothing$	0
1	b_4	$\{4\}$	$\{12\}$	0
2	s_{17}	$\varnothing$	$\{17\}$	0

The isolated state IDs are $\{4, 12\}$. The refined graph has zero paths projecting to this candidate.

Round 6. The partition has 15 nonempty cells, and the may relation has 16 edges.

The selected path is s_0, b_5, s_{17}. The exact number of concrete lifts is 0.

j	a_j	F_j	D_j	$\#\cap$
0	s_0	$\{0\}$	$\varnothing$	0
1	b_5	$\{5\}$	$\{13\}$	0
2	s_{17}	$\varnothing$	$\{17\}$	0

The isolated state IDs are $\{5, 13\}$. The refined graph has zero paths projecting to this candidate.

Round 7. The partition has 16 nonempty cells, and the may relation has 16 edges.

The selected path is s_0, b_6, s_{17}. The exact number of concrete lifts is 0.

j	a_j	F_j	D_j	$\#\cap$
0	s_0	$\{0\}$	$\varnothing$	0
1	b_6	$\{6\}$	$\{14\}$	0
2	s_{17}	$\varnothing$	$\{17\}$	0

The isolated state IDs are $\{6, 14\}$. The refined graph has zero paths projecting to this candidate.

Round 8. The partition has 17 nonempty cells, and the may relation has 16 edges.

The selected path is s_0, b_7, s_{17}. The exact number of concrete lifts is 0.

j	a_j	F_j	D_j	$\#\cap$
0	s_0	$\{0\}$	$\varnothing$	0
1	b_7	$\{7\}$	$\{15\}$	0
2	s_{17}	$\varnothing$	$\{17\}$	0

The isolated state IDs are $\{7, 15\}$. The refined graph has zero paths projecting to this candidate.

Round 9. The partition has 18 nonempty cells, and the may relation has 16 edges.

The abstract checker found the inductive invariant $\{s_0, s_1, s_2, s_3, s_4, s_5, s_6, s_7, s_8\}$. It contains $\{s_0\}$ and is disjoint from $\{s_{17}\}$.

Cell	All may successors
s_0	$\{s_1, s_2, s_3, s_4, s_5, s_6, s_7, s_8\}$
s_1	$\varnothing$
s_2	$\varnothing$
s_3	$\varnothing$
s_4	$\varnothing$
s_5	$\varnothing$
s_6	$\varnothing$
s_7	$\varnothing$
s_8	$\varnothing$

24.6 F9.6 10 middle cells, concrete branch None

Every intermediate cell has an entry point reachable from the root and a separate exit point. Each failed round isolates precisely one as-yet coarse intermediate cell. The $2 \cdot 10 + 2$ concrete states give $10 + 2$ initial cells, so this case attains the bound of $10 + 1$ abstract rounds.

The concrete carrier has 22 states and 20 edges. Its reachable set has 11 states. The reachable set is disjoint from the bad predicate.

ID	State	Seed	Bd.	d	p	Successors
0	start	0	I	0	–	{1, 2, 3, 4, 5, 6, 7, 8, 9, 10}
1	(0, 'entry')	1	–	1	0	∅
2	(1, 'entry')	2	–	1	0	∅
3	(2, 'entry')	3	–	1	0	∅
4	(3, 'entry')	4	–	1	0	∅
5	(4, 'entry')	5	–	1	0	∅
6	(5, 'entry')	6	–	1	0	∅
7	(6, 'entry')	7	–	1	0	∅
8	(7, 'entry')	8	–	1	0	∅
9	(8, 'entry')	9	–	1	0	∅
10	(9, 'entry')	10	–	1	0	∅
11	(0, 'exit')	1	–	∞	–	{21}
12	(1, 'exit')	2	–	∞	–	{21}
13	(2, 'exit')	3	–	∞	–	{21}
14	(3, 'exit')	4	–	∞	–	{21}
15	(4, 'exit')	5	–	∞	–	{21}
16	(5, 'exit')	6	–	∞	–	{21}
17	(6, 'exit')	7	–	∞	–	{21}
18	(7, 'exit')	8	–	∞	–	{21}
19	(8, 'exit')	9	–	∞	–	{21}
20	(9, 'exit')	10	–	∞	–	{21}
21	bad	11	B	∞	–	∅

Round 1. The partition has 12 nonempty cells, and the may relation has 20 edges.

The selected path is b_0, b_{10}, b_{11}. The exact number of concrete lifts is 0.

j	a_j	F_j	D_j	$\#\cap$
0	b_0	{0}	∅	0
1	b_{10}	{10}	{20}	0
2	b_{11}	∅	{21}	0

The isolated state IDs are $\{0, 10, 20, 21\}$. The refined graph has zero paths projecting to this candidate.

Round 2. The partition has 13 nonempty cells, and the may relation has 20 edges.

The selected path is s_0, b_1, s_{21}. The exact number of concrete lifts is 0.

j	a_j	F_j	D_j	$\#\cap$
0	s_0	{0}	∅	0
1	b_1	{1}	{11}	0
2	s_{21}	∅	{21}	0

The isolated state IDs are $\{1, 11\}$. The refined graph has zero paths projecting to this candidate.

Round 3. The partition has 14 nonempty cells, and the may relation has 20 edges.

The selected path is s_0, b_2, s_{21}. The exact number of concrete lifts is 0.

j	a_j	F_j	D_j	$\#\cap$
0	s_0	{0}	∅	0
1	b_2	{2}	{12}	0
2	s_{21}	∅	{21}	0

The isolated state IDs are $\{2, 12\}$. The refined graph has zero paths projecting to this candidate.

Round 4. The partition has 15 nonempty cells, and the may relation has 20 edges.

The selected path is s_0, b_3, s_{21}. The exact number of concrete lifts is 0.

j	a_j	F_j	D_j	$\#\cap$
0	s_0	{0}	∅	0
1	b_3	{3}	{13}	0
2	s_{21}	∅	{21}	0

The isolated state IDs are $\{3, 13\}$. The refined graph has zero paths projecting to this candidate.

Round 5. The partition has 16 nonempty cells, and the may relation has 20 edges.

The selected path is s_0, b_4, s_{21}. The exact number of concrete lifts is 0.

j	a_j	F_j	D_j	$\#\cap$
0	s_0	{0}	∅	0
1	b_4	{4}	{14}	0
2	s_{21}	∅	{21}	0

The isolated state IDs are $\{4, 14\}$. The refined graph has zero paths projecting to this candidate.

Round 6. The partition has 17 nonempty cells, and the may relation has 20 edges.

The selected path is s_0, b_5, s_{21}. The exact number of concrete lifts is 0.

j	a_j	F_j	D_j	$\#\cap$
0	s_0	$\{0\}$	$\varnothing$	0
1	b_5	$\{5\}$	$\{15\}$	0
2	s_{21}	$\varnothing$	$\{21\}$	0

The isolated state IDs are $\{5, 15\}$. The refined graph has zero paths projecting to this candidate.

Round 7. The partition has 18 nonempty cells, and the may relation has 20 edges.

The selected path is s_0, b_6, s_{21}. The exact number of concrete lifts is 0.

j	a_j	F_j	D_j	$\#\cap$
0	s_0	$\{0\}$	$\varnothing$	0
1	b_6	$\{6\}$	$\{16\}$	0
2	s_{21}	$\varnothing$	$\{21\}$	0

The isolated state IDs are $\{6, 16\}$. The refined graph has zero paths projecting to this candidate.

Round 8. The partition has 19 nonempty cells, and the may relation has 20 edges.

The selected path is s_0, b_7, s_{21}. The exact number of concrete lifts is 0.

j	a_j	F_j	D_j	$\#\cap$
0	s_0	$\{0\}$	$\varnothing$	0
1	b_7	$\{7\}$	$\{17\}$	0
2	s_{21}	$\varnothing$	$\{21\}$	0

The isolated state IDs are $\{7, 17\}$. The refined graph has zero paths projecting to this candidate.

Round 9. The partition has 20 nonempty cells, and the may relation has 20 edges.

The selected path is s_0, b_8, s_{21}. The exact number of concrete lifts is 0.

j	a_j	F_j	D_j	$\#\cap$
0	s_0	$\{0\}$	$\varnothing$	0
1	b_8	$\{8\}$	$\{18\}$	0
2	s_{21}	$\varnothing$	$\{21\}$	0

The isolated state IDs are $\{8, 18\}$. The refined graph has zero paths projecting to this candidate.

Round 10. The partition has 21 nonempty cells, and the may relation has 20 edges.

The selected path is s_0, b_9, s_{21}. The exact number of concrete lifts is 0.

j	a_j	F_j	D_j	$\#\cap$
0	s_0	$\{0\}$	$\varnothing$	0
1	b_9	$\{9\}$	$\{19\}$	0
2	s_{21}	$\varnothing$	$\{21\}$	0

The isolated state IDs are $\{9, 19\}$. The refined graph has zero paths projecting to this candidate.

Round 11. The partition has 22 nonempty cells, and the may relation has 20 edges.

The abstract checker found the inductive invariant $\{s_0, s_1, s_2, s_3, s_4, s_5, s_6, s_7, s_8, s_9, s_{10}\}$. It contains $\{s_0\}$ and is disjoint from $\{s_{21}\}$.

Cell	All may successors
s_0	$\{s_1, s_2, s_3, s_4, s_5, s_6, s_7, s_8, s_9, s_{10}\}$
s_1	$\varnothing$
s_2	$\varnothing$
s_3	$\varnothing$
s_4	$\varnothing$
s_5	$\varnothing$
s_6	$\varnothing$
s_7	$\varnothing$
s_8	$\varnothing$
s_9	$\varnothing$
s_{10}	$\varnothing$

24.7 F9.7 12 middle cells, concrete branch None

Every intermediate cell has an entry point reachable from the root and a separate exit point. Each failed round isolates precisely one as-yet coarse intermediate cell. The $2 \cdot 12 + 2$ concrete states give $12 + 2$ initial cells, so this case attains the bound of $12 + 1$ abstract rounds.

The concrete carrier has 26 states and 24 edges. Its reachable set has 13 states. The reachable set is disjoint from the bad predicate.

ID	State	Seed	Bd.	d	p	Successors
0	start	0	I	0	–	$\{1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12\}$
1	(0, 'entry')	1	–	1	0	$\varnothing$
2	(1, 'entry')	2	–	1	0	$\varnothing$
3	(2, 'entry')	3	–	1	0	$\varnothing$
4	(3, 'entry')	4	–	1	0	$\varnothing$
5	(4, 'entry')	5	–	1	0	$\varnothing$
6	(5, 'entry')	6	–	1	0	$\varnothing$
7	(6, 'entry')	7	–	1	0	$\varnothing$
8	(7, 'entry')	8	–	1	0	$\varnothing$
9	(8, 'entry')	9	–	1	0	$\varnothing$
10	(9, 'entry')	10	–	1	0	$\varnothing$
11	(10, 'entry')	11	–	1	0	$\varnothing$
12	(11, 'entry')	12	–	1	0	$\varnothing$

ID	State	Seed	Bd.	d	p	Successors
13	(0, 'exit')	1	–	∞	–	{25}
14	(1, 'exit')	2	–	∞	–	{25}
15	(2, 'exit')	3	–	∞	–	{25}
16	(3, 'exit')	4	–	∞	–	{25}
17	(4, 'exit')	5	–	∞	–	{25}
18	(5, 'exit')	6	–	∞	–	{25}
19	(6, 'exit')	7	–	∞	–	{25}
20	(7, 'exit')	8	–	∞	–	{25}
21	(8, 'exit')	9	–	∞	–	{25}
22	(9, 'exit')	10	–	∞	–	{25}
23	(10, 'exit')	11	–	∞	–	{25}
24	(11, 'exit')	12	–	∞	–	{25}
25	bad	13	B	∞	–	$\varnothing$

Round 1. The partition has 14 nonempty cells, and the may relation has 24 edges.

The selected path is b_0, b_{12}, b_{13}. The exact number of concrete lifts is 0.

j	a_j	F_j	D_j	$\#\cap$
0	b_0	{0}	$\varnothing$	0
1	b_{12}	{12}	{24}	0
2	b_{13}	$\varnothing$	{25}	0

The isolated state IDs are $\{0, 12, 24, 25\}$. The refined graph has zero paths projecting to this candidate.

Round 2. The partition has 15 nonempty cells, and the may relation has 24 edges.

The selected path is s_0, b_1, s_{25}. The exact number of concrete lifts is 0.

j	a_j	F_j	D_j	$\#\cap$
0	s_0	{0}	$\varnothing$	0
1	b_1	{1}	{13}	0
2	s_{25}	$\varnothing$	{25}	0

The isolated state IDs are $\{1, 13\}$. The refined graph has zero paths projecting to this candidate.

Round 3. The partition has 16 nonempty cells, and the may relation has 24 edges.

The selected path is s_0, b_2, s_{25}. The exact number of concrete lifts is 0.

j	a_j	F_j	D_j	$\#\cap$
0	s_0	{0}	$\varnothing$	0
1	b_2	{2}	{14}	0
2	s_{25}	$\varnothing$	{25}	0

The isolated state IDs are $\{2, 14\}$. The refined graph has zero paths projecting to this candidate.

Round 4. The partition has 17 nonempty cells, and the may relation has 24 edges.

The selected path is s_0, b_3, s_{25}. The exact number of concrete lifts is 0.

j	a_j	F_j	D_j	$\#\cap$
0	s_0	{0}	$\varnothing$	0
1	b_3	{3}	{15}	0
2	s_{25}	$\varnothing$	{25}	0

The isolated state IDs are $\{3, 15\}$. The refined graph has zero paths projecting to this candidate.

Round 5. The partition has 18 nonempty cells, and the may relation has 24 edges.

The selected path is s_0, b_4, s_{25}. The exact number of concrete lifts is 0.

j	a_j	F_j	D_j	$\#\cap$
0	s_0	{0}	$\varnothing$	0
1	b_4	{4}	{16}	0
2	s_{25}	$\varnothing$	{25}	0

The isolated state IDs are $\{4, 16\}$. The refined graph has zero paths projecting to this candidate.

Round 6. The partition has 19 nonempty cells, and the may relation has 24 edges.

The selected path is s_0, b_5, s_{25}. The exact number of concrete lifts is 0.

j	a_j	F_j	D_j	$\#\cap$
0	s_0	{0}	$\varnothing$	0
1	b_5	{5}	{17}	0
2	s_{25}	$\varnothing$	{25}	0

The isolated state IDs are $\{5, 17\}$. The refined graph has zero paths projecting to this candidate.

Round 7. The partition has 20 nonempty cells, and the may relation has 24 edges.

The selected path is s_0, b_6, s_{25}. The exact number of concrete lifts is 0.

j	a_j	F_j	D_j	$\#\cap$
0	s_0	{0}	$\varnothing$	0
1	b_6	{6}	{18}	0
2	s_{25}	$\varnothing$	{25}	0

The isolated state IDs are $\{6, 18\}$. The refined graph has zero paths projecting to this candidate.

Round 8. The partition has 21 nonempty cells, and the may relation has 24 edges.

The selected path is s_0, b_7, s_{25}. The exact number of concrete lifts is 0.

j	a_j	F_j	D_j	$\#\cap$
0	s_0	{0}	$\varnothing$	0
1	b_7	{7}	{19}	0
2	s_{25}	$\varnothing$	{25}	0

The isolated state IDs are $\{7, 19\}$. The refined graph has zero paths projecting to this candidate.

Round 9. The partition has 22 nonempty cells, and the may relation has 24 edges.

The selected path is s_0, b_8, s_{25}. The exact number of concrete lifts is 0.

j	a_j	F_j	D_j	$\#\cap$
0	s_0	$\{0\}$	$\varnothing$	0
1	b_8	$\{8\}$	$\{20\}$	0
2	s_{25}	$\varnothing$	$\{25\}$	0

The isolated state IDs are $\{8, 20\}$. The refined graph has zero paths projecting to this candidate.

Round 10. The partition has 23 nonempty cells, and the may relation has 24 edges.

The selected path is s_0, b_9, s_{25}. The exact number of concrete lifts is 0.

j	a_j	F_j	D_j	$\#\cap$
0	s_0	$\{0\}$	$\varnothing$	0
1	b_9	$\{9\}$	$\{21\}$	0
2	s_{25}	$\varnothing$	$\{25\}$	0

The isolated state IDs are $\{9, 21\}$. The refined graph has zero paths projecting to this candidate.

Round 11. The partition has 24 nonempty cells, and the may relation has 24 edges.

The selected path is s_0, b_{10}, s_{25}. The exact number of concrete lifts is 0.

j	a_j	F_j	D_j	$\#\cap$
0	s_0	$\{0\}$	$\varnothing$	0
1	b_{10}	$\{10\}$	$\{22\}$	0
2	s_{25}	$\varnothing$	$\{25\}$	0

The isolated state IDs are $\{10, 22\}$. The refined graph has zero paths projecting to this candidate.

Round 12. The partition has 25 nonempty cells, and the may relation has 24 edges.

The selected path is s_0, b_{11}, s_{25}. The exact number of concrete lifts is 0.

j	a_j	F_j	D_j	$\#\cap$
0	s_0	$\{0\}$	$\varnothing$	0
1	b_{11}	$\{11\}$	$\{23\}$	0
2	s_{25}	$\varnothing$	$\{25\}$	0

The isolated state IDs are $\{11, 23\}$. The refined graph has zero paths projecting to this candidate.

Round 13. The partition has 26 nonempty cells, and the may relation has 24 edges.

The abstract checker found the inductive invariant $\{s_0, s_1, s_2, s_3, s_4, s_5, s_6, s_7, s_8, s_9, s_{10}, s_{11}, s_{12}\}$. It contains $\{s_0\}$ and is disjoint from $\{s_{25}\}$.

Cell	All may successors
s_0	$\{s_1, s_2, s_3, s_4, s_5, s_6, s_7, s_8, s_9, s_{10}, s_{11}, s_{12}\}$
s_1	$\varnothing$
s_2	$\varnothing$
s_3	$\varnothing$
s_4	$\varnothing$
s_5	$\varnothing$
s_6	$\varnothing$
s_7	$\varnothing$
s_8	$\varnothing$
s_9	$\varnothing$
s_{10}	$\varnothing$
s_{11}	$\varnothing$
s_{12}	$\varnothing$

24.8 F9.8 14 middle cells, concrete branch None

Every intermediate cell has an entry point reachable from the root and a separate exit point. Each failed round isolates precisely one as-yet coarse intermediate cell. The $2 \cdot 14 + 2$ concrete states give $14 + 2$ initial cells, so this case attains the bound of $14 + 1$ abstract rounds.

The concrete carrier has 30 states and 28 edges. Its reachable set has 15 states. The reachable set is disjoint from the bad predicate.

ID	State	Seed	Bd.	d	p	Successors
0	start	0	I	0	–	{1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12, 13, 14}
1	(0, 'entry')	1	–	1	0	$\varnothing$
2	(1, 'entry')	2	–	1	0	$\varnothing$
3	(2, 'entry')	3	–	1	0	$\varnothing$
4	(3, 'entry')	4	–	1	0	$\varnothing$
5	(4, 'entry')	5	–	1	0	$\varnothing$
6	(5, 'entry')	6	–	1	0	$\varnothing$
7	(6, 'entry')	7	–	1	0	$\varnothing$
8	(7, 'entry')	8	–	1	0	$\varnothing$
9	(8, 'entry')	9	–	1	0	$\varnothing$
10	(9, 'entry')	10	–	1	0	$\varnothing$
11	(10, 'entry')	11	–	1	0	$\varnothing$
12	(11, 'entry')	12	–	1	0	$\varnothing$
13	(12, 'entry')	13	–	1	0	$\varnothing$
14	(13, 'entry')	14	–	1	0	$\varnothing$
15	(0, 'exit')	1	–	∞	–	{29}
16	(1, 'exit')	2	–	∞	–	{29}
17	(2, 'exit')	3	–	∞	–	{29}
18	(3, 'exit')	4	–	∞	–	{29}
19	(4, 'exit')	5	–	∞	–	{29}
20	(5, 'exit')	6	–	∞	–	{29}
21	(6, 'exit')	7	–	∞	–	{29}
22	(7, 'exit')	8	–	∞	–	{29}
23	(8, 'exit')	9	–	∞	–	{29}
24	(9, 'exit')	10	–	∞	–	{29}
25	(10, 'exit')	11	–	∞	–	{29}

ID	State	Seed	Bd.	d	p	Successors
26	(11, 'exit')	12	–	∞	–	{29}
27	(12, 'exit')	13	–	∞	–	{29}
28	(13, 'exit')	14	–	∞	–	{29}
29	bad	15	B	∞	–	$\varnothing$

Round 1. The partition has 16 nonempty cells, and the may relation has 28 edges.

The selected path is b_0, b_{14}, b_{15}. The exact number of concrete lifts is 0.

j	a_j	F_j	D_j	$\#\cap$
0	b_0	{0}	$\varnothing$	0
1	b_{14}	{14}	{28}	0
2	b_{15}	$\varnothing$	{29}	0

The isolated state IDs are $\{0, 14, 28, 29\}$. The refined graph has zero paths projecting to this candidate.

Round 2. The partition has 17 nonempty cells, and the may relation has 28 edges.

The selected path is s_0, b_1, s_{29}. The exact number of concrete lifts is 0.

j	a_j	F_j	D_j	$\#\cap$
0	s_0	{0}	$\varnothing$	0
1	b_1	{1}	{15}	0
2	s_{29}	$\varnothing$	{29}	0

The isolated state IDs are $\{1, 15\}$. The refined graph has zero paths projecting to this candidate.

Round 3. The partition has 18 nonempty cells, and the may relation has 28 edges.

The selected path is s_0, b_2, s_{29}. The exact number of concrete lifts is 0.

j	a_j	F_j	D_j	$\#\cap$
0	s_0	{0}	$\varnothing$	0
1	b_2	{2}	{16}	0
2	s_{29}	$\varnothing$	{29}	0

The isolated state IDs are $\{2, 16\}$. The refined graph has zero paths projecting to this candidate.

Round 4. The partition has 19 nonempty cells, and the may relation has 28 edges.

The selected path is s_0, b_3, s_{29}. The exact number of concrete lifts is 0.

j	a_j	F_j	D_j	$\#\cap$
0	s_0	{0}	$\varnothing$	0
1	b_3	{3}	{17}	0
2	s_{29}	$\varnothing$	{29}	0

The isolated state IDs are $\{3, 17\}$. The refined graph has zero paths projecting to this candidate.

Round 5. The partition has 20 nonempty cells, and the may relation has 28 edges.

The selected path is s_0, b_4, s_{29}. The exact number of concrete lifts is 0.

j	a_j	F_j	D_j	$\#\cap$
0	s_0	{0}	$\varnothing$	0
1	b_4	{4}	{18}	0
2	s_{29}	$\varnothing$	{29}	0

The isolated state IDs are $\{4, 18\}$. The refined graph has zero paths projecting to this candidate.

Round 6. The partition has 21 nonempty cells, and the may relation has 28 edges.

The selected path is s_0, b_5, s_{29}. The exact number of concrete lifts is 0.

j	a_j	F_j	D_j	$\#\cap$
0	s_0	{0}	$\varnothing$	0
1	b_5	{5}	{19}	0
2	s_{29}	$\varnothing$	{29}	0

The isolated state IDs are $\{5, 19\}$. The refined graph has zero paths projecting to this candidate.

Round 7. The partition has 22 nonempty cells, and the may relation has 28 edges.

The selected path is s_0, b_6, s_{29}. The exact number of concrete lifts is 0.

j	a_j	F_j	D_j	$\#\cap$
0	s_0	{0}	$\varnothing$	0
1	b_6	{6}	{20}	0
2	s_{29}	$\varnothing$	{29}	0

The isolated state IDs are $\{6, 20\}$. The refined graph has zero paths projecting to this candidate.

Round 8. The partition has 23 nonempty cells, and the may relation has 28 edges.

The selected path is s_0, b_7, s_{29}. The exact number of concrete lifts is 0.

j	a_j	F_j	D_j	$\#\cap$
0	s_0	{0}	$\varnothing$	0
1	b_7	{7}	{21}	0
2	s_{29}	$\varnothing$	{29}	0

The isolated state IDs are $\{7, 21\}$. The refined graph has zero paths projecting to this candidate.

Round 9. The partition has 24 nonempty cells, and the may relation has 28 edges.

The selected path is s_0, b_8, s_{29}. The exact number of concrete lifts is 0.

j	a_j	F_j	D_j	$\#\cap$
0	s_0	$\{0\}$	$\varnothing$	0
1	b_8	$\{8\}$	$\{22\}$	0
2	s_{29}	$\varnothing$	$\{29\}$	0

The isolated state IDs are $\{8, 22\}$. The refined graph has zero paths projecting to this candidate.

Round 10. The partition has 25 nonempty cells, and the may relation has 28 edges.

The selected path is s_0, b_9, s_{29}. The exact number of concrete lifts is 0.

j	a_j	F_j	D_j	$\#\cap$
0	s_0	$\{0\}$	$\varnothing$	0
1	b_9	$\{9\}$	$\{23\}$	0
2	s_{29}	$\varnothing$	$\{29\}$	0

The isolated state IDs are $\{9, 23\}$. The refined graph has zero paths projecting to this candidate.

Round 11. The partition has 26 nonempty cells, and the may relation has 28 edges.

The selected path is s_0, b_{10}, s_{29}. The exact number of concrete lifts is 0.

j	a_j	F_j	D_j	$\#\cap$
0	s_0	$\{0\}$	$\varnothing$	0
1	b_{10}	$\{10\}$	$\{24\}$	0
2	s_{29}	$\varnothing$	$\{29\}$	0

The isolated state IDs are $\{10, 24\}$. The refined graph has zero paths projecting to this candidate.

Round 12. The partition has 27 nonempty cells, and the may relation has 28 edges.

The selected path is s_0, b_{11}, s_{29}. The exact number of concrete lifts is 0.

j	a_j	F_j	D_j	$\#\cap$
0	s_0	$\{0\}$	$\varnothing$	0
1	b_{11}	$\{11\}$	$\{25\}$	0
2	s_{29}	$\varnothing$	$\{29\}$	0

The isolated state IDs are $\{11, 25\}$. The refined graph has zero paths projecting to this candidate.

Round 13. The partition has 28 nonempty cells, and the may relation has 28 edges.

The selected path is s_0, b_{12}, s_{29}. The exact number of concrete lifts is 0.

j	a_j	F_j	D_j	$\#\cap$
0	s_0	$\{0\}$	$\varnothing$	0
1	b_{12}	$\{12\}$	$\{26\}$	0
2	s_{29}	$\varnothing$	$\{29\}$	0

The isolated state IDs are $\{12, 26\}$. The refined graph has zero paths projecting to this candidate.

Round 14. The partition has 29 nonempty cells, and the may relation has 28 edges.

The selected path is s_0, b_{13}, s_{29}. The exact number of concrete lifts is 0.

j	a_j	F_j	D_j	$\#\cap$
0	s_0	$\{0\}$	$\varnothing$	0
1	b_{13}	$\{13\}$	$\{27\}$	0
2	s_{29}	$\varnothing$	$\{29\}$	0

The isolated state IDs are $\{13, 27\}$. The refined graph has zero paths projecting to this candidate.

Round 15. The partition has 30 nonempty cells, and the may relation has 28 edges.

The abstract checker found the inductive invariant $\{s_0, s_1, s_2, s_3, s_4, s_5, s_6, s_7, s_8, s_9, s_{10}, s_{11}, s_{12}, s_{13}, s_{14}\}$. It contains $\{s_0\}$ and is disjoint from $\{s_{29}\}$.

Cell	All may successors
s_0	$\{s_1, s_2, s_3, s_4, s_5, s_6, s_7, s_8, s_9, s_{10}, s_{11}, s_{12}, s_{13}, s_{14}\}$
s_1	$\varnothing$
s_2	$\varnothing$
s_3	$\varnothing$
s_4	$\varnothing$
s_5	$\varnothing$
s_6	$\varnothing$
s_7	$\varnothing$
s_8	$\varnothing$
s_9	$\varnothing$
s_{10}	$\varnothing$
s_{11}	$\varnothing$
s_{12}	$\varnothing$

Cell	All may successors
s_{13}	$\varnothing$
s_{14}	$\varnothing$

24.9 F9.9 16 middle cells, concrete branch None

Every intermediate cell has an entry point reachable from the root and a separate exit point. Each failed round isolates precisely one as-yet coarse intermediate cell. The $2 \cdot 16 + 2$ concrete states give $16 + 2$ initial cells, so this case attains the bound of $16 + 1$ abstract rounds.

The concrete carrier has 34 states and 32 edges. Its reachable set has 17 states. The reachable set is disjoint from the bad predicate.

ID	State	Seed	Bd.	d	p	Successors
0	start	0	I	0	–	$\{1,2,3,4,5,6,7,8,9,10,11,12,13,14,15,16\}$
1	(0, 'entry')	1	–	1	0	$\varnothing$
2	(1, 'entry')	2	–	1	0	$\varnothing$
3	(2, 'entry')	3	–	1	0	$\varnothing$
4	(3, 'entry')	4	–	1	0	$\varnothing$
5	(4, 'entry')	5	–	1	0	$\varnothing$
6	(5, 'entry')	6	–	1	0	$\varnothing$
7	(6, 'entry')	7	–	1	0	$\varnothing$
8	(7, 'entry')	8	–	1	0	$\varnothing$
9	(8, 'entry')	9	–	1	0	$\varnothing$
10	(9, 'entry')	10	–	1	0	$\varnothing$
11	(10, 'entry')	11	–	1	0	$\varnothing$
12	(11, 'entry')	12	–	1	0	$\varnothing$
13	(12, 'entry')	13	–	1	0	$\varnothing$
14	(13, 'entry')	14	–	1	0	$\varnothing$
15	(14, 'entry')	15	–	1	0	$\varnothing$
16	(15, 'entry')	16	–	1	0	$\varnothing$
17	(0, 'exit')	1	–	∞	–	$\{33\}$
18	(1, 'exit')	2	–	∞	–	$\{33\}$
19	(2, 'exit')	3	–	∞	–	$\{33\}$
20	(3, 'exit')	4	–	∞	–	$\{33\}$
21	(4, 'exit')	5	–	∞	–	$\{33\}$
22	(5, 'exit')	6	–	∞	–	$\{33\}$
23	(6, 'exit')	7	–	∞	–	$\{33\}$
24	(7, 'exit')	8	–	∞	–	$\{33\}$
25	(8, 'exit')	9	–	∞	–	$\{33\}$
26	(9, 'exit')	10	–	∞	–	$\{33\}$
27	(10, 'exit')	11	–	∞	–	$\{33\}$
28	(11, 'exit')	12	–	∞	–	$\{33\}$
29	(12, 'exit')	13	–	∞	–	$\{33\}$
30	(13, 'exit')	14	–	∞	–	$\{33\}$
31	(14, 'exit')	15	–	∞	–	$\{33\}$
32	(15, 'exit')	16	–	∞	–	$\{33\}$
33	bad	17	B	∞	–	$\varnothing$

Round 1. The partition has 18 nonempty cells, and the may relation has 32 edges.

The selected path is b_0, b_{16}, b_{17}. The exact number of concrete lifts is 0.

j	a_j	F_j	D_j	$\#\cap$
0	b_0	$\{0\}$	$\varnothing$	0
1	b_{16}	$\{16\}$	$\{32\}$	0
2	b_{17}	$\varnothing$	$\{33\}$	0

The isolated state IDs are $\{0, 16, 32, 33\}$. The refined graph has zero paths projecting to this candidate.

Round 2. The partition has 19 nonempty cells, and the may relation has 32 edges.

The selected path is s_0, b_1, s_{33}. The exact number of concrete lifts is 0.

j	a_j	F_j	D_j	$\#\cap$
0	s_0	$\{0\}$	$\varnothing$	0
1	b_1	$\{1\}$	$\{17\}$	0
2	s_{33}	$\varnothing$	$\{33\}$	0

The isolated state IDs are $\{1, 17\}$. The refined graph has zero paths projecting to this candidate.

Round 3. The partition has 20 nonempty cells, and the may relation has 32 edges.

The selected path is s_0, b_2, s_{33}. The exact number of concrete lifts is 0.

j	a_j	F_j	D_j	$\#\cap$
0	s_0	$\{0\}$	$\varnothing$	0
1	b_2	$\{2\}$	$\{18\}$	0
2	s_{33}	$\varnothing$	$\{33\}$	0

The isolated state IDs are $\{2, 18\}$. The refined graph has zero paths projecting to this candidate.

Round 4. The partition has 21 nonempty cells, and the may relation has 32 edges.

The selected path is s_0, b_3, s_{33}. The exact number of concrete lifts is 0.

j	a_j	F_j	D_j	$\#\cap$
0	s_0	$\{0\}$	$\varnothing$	0
1	b_3	$\{3\}$	$\{19\}$	0
2	s_{33}	$\varnothing$	$\{33\}$	0

The isolated state IDs are $\{3, 19\}$. The refined graph has zero paths projecting to this candidate.

Round 5. The partition has 22 nonempty cells, and the may relation has 32 edges.

The selected path is s_0, b_4, s_{33}. The exact number of concrete lifts is 0.

j	a_j	F_j	D_j	$\#\cap$
0	s_0	$\{0\}$	$\varnothing$	0
1	b_4	$\{4\}$	$\{20\}$	0
2	s_{33}	$\varnothing$	$\{33\}$	0

The isolated state IDs are $\{4, 20\}$. The refined graph has zero paths projecting to this candidate.

Round 6. The partition has 23 nonempty cells, and the may relation has 32 edges.

The selected path is s_0, b_5, s_{33}. The exact number of concrete lifts is 0.

j	a_j	F_j	D_j	$\#\cap$
0	s_0	$\{0\}$	$\varnothing$	0
1	b_5	$\{5\}$	$\{21\}$	0
2	s_{33}	$\varnothing$	$\{33\}$	0

The isolated state IDs are $\{5, 21\}$. The refined graph has zero paths projecting to this candidate.

Round 7. The partition has 24 nonempty cells, and the may relation has 32 edges.

The selected path is s_0, b_6, s_{33}. The exact number of concrete lifts is 0.

j	a_j	F_j	D_j	$\#\cap$
0	s_0	$\{0\}$	$\varnothing$	0
1	b_6	$\{6\}$	$\{22\}$	0
2	s_{33}	$\varnothing$	$\{33\}$	0

The isolated state IDs are $\{6, 22\}$. The refined graph has zero paths projecting to this candidate.

Round 8. The partition has 25 nonempty cells, and the may relation has 32 edges.

The selected path is s_0, b_7, s_{33}. The exact number of concrete lifts is 0.

j	a_j	F_j	D_j	$\#\cap$
0	s_0	$\{0\}$	$\varnothing$	0
1	b_7	$\{7\}$	$\{23\}$	0
2	s_{33}	$\varnothing$	$\{33\}$	0

The isolated state IDs are $\{7, 23\}$. The refined graph has zero paths projecting to this candidate.

Round 9. The partition has 26 nonempty cells, and the may relation has 32 edges.

The selected path is s_0, b_8, s_{33}. The exact number of concrete lifts is 0.

j	a_j	F_j	D_j	$\#\cap$
0	s_0	$\{0\}$	$\varnothing$	0
1	b_8	$\{8\}$	$\{24\}$	0
2	s_{33}	$\varnothing$	$\{33\}$	0

The isolated state IDs are $\{8, 24\}$. The refined graph has zero paths projecting to this candidate.

Round 10. The partition has 27 nonempty cells, and the may relation has 32 edges.

The selected path is s_0, b_9, s_{33}. The exact number of concrete lifts is 0.

j	a_j	F_j	D_j	$\#\cap$
0	s_0	$\{0\}$	$\varnothing$	0
1	b_9	$\{9\}$	$\{25\}$	0
2	s_{33}	$\varnothing$	$\{33\}$	0

The isolated state IDs are $\{9, 25\}$. The refined graph has zero paths projecting to this candidate.

Round 11. The partition has 28 nonempty cells, and the may relation has 32 edges.

The selected path is s_0, b_{10}, s_{33}. The exact number of concrete lifts is 0.

j	a_j	F_j	D_j	$\#\cap$
0	s_0	$\{0\}$	$\varnothing$	0
1	b_{10}	$\{10\}$	$\{26\}$	0
2	s_{33}	$\varnothing$	$\{33\}$	0

The isolated state IDs are $\{10, 26\}$. The refined graph has zero paths projecting to this candidate.

Round 12. The partition has 29 nonempty cells, and the may relation has 32 edges.

The selected path is s_0, b_{11}, s_{33}. The exact number of concrete lifts is 0.

j	a_j	F_j	D_j	$\#\cap$
0	s_0	$\{0\}$	$\varnothing$	0
1	b_{11}	$\{11\}$	$\{27\}$	0
2	s_{33}	$\varnothing$	$\{33\}$	0

The isolated state IDs are $\{11, 27\}$. The refined graph has zero paths projecting to this candidate.

Round 13. The partition has 30 nonempty cells, and the may relation has 32 edges.

The selected path is s_0, b_{12}, s_{33}. The exact number of concrete lifts is 0.

j	a_j	F_j	D_j	$\#\cap$
0	s_0	$\{0\}$	$\varnothing$	0
1	b_{12}	$\{12\}$	$\{28\}$	0
2	s_{33}	$\varnothing$	$\{33\}$	0

The isolated state IDs are $\{12, 28\}$. The refined graph has zero paths projecting to this candidate.

Round 14. The partition has 31 nonempty cells, and the may relation has 32 edges.

The selected path is s_0, b_{13}, s_{33}. The exact number of concrete lifts is 0.

j	a_j	F_j	D_j	$\#\cap$
0	s_0	$\{0\}$	$\varnothing$	0
1	b_{13}	$\{13\}$	$\{29\}$	0
2	s_{33}	$\varnothing$	$\{33\}$	0

The isolated state IDs are $\{13, 29\}$. The refined graph has zero paths projecting to this candidate.

Round 15. The partition has 32 nonempty cells, and the may relation has 32 edges.

The selected path is s_0, b_{14}, s_{33}. The exact number of concrete lifts is 0.

j	a_j	F_j	D_j	$\#\cap$
0	s_0	$\{0\}$	$\varnothing$	0
1	b_{14}	$\{14\}$	$\{30\}$	0
2	s_{33}	$\varnothing$	$\{33\}$	0

The isolated state IDs are $\{14, 30\}$. The refined graph has zero paths projecting to this candidate.

Round 16. The partition has 33 nonempty cells, and the may relation has 32 edges.

The selected path is s_0, b_{15}, s_{33}. The exact number of concrete lifts is 0.

j	a_j	F_j	D_j	$\#\cap$
0	s_0	$\{0\}$	$\varnothing$	0
1	b_{15}	$\{15\}$	$\{31\}$	0
2	s_{33}	$\varnothing$	$\{33\}$	0

The isolated state IDs are $\{15, 31\}$. The refined graph has zero paths projecting to this candidate.

Round 17. The partition has 34 nonempty cells, and the may relation has 32 edges.

The abstract checker found the inductive invariant $\{s_0, s_1, s_2, s_3, s_4, s_5, s_6, s_7, s_8, s_9, s_{10}, s_{11}, s_{12}, s_{13}, s_{14}, s_{15}, s_{16}\}$. It contains $\{s_0\}$ and is disjoint from $\{s_{33}\}$.

Cell	All may successors
s_0	$\{s_1, s_2, s_3, s_4, s_5, s_6, s_7, s_8, s_9, s_{10}, s_{11}, s_{12}, s_{13}, s_{14}, s_{15}, s_{16}\}$
s_1	$\varnothing$
s_2	$\varnothing$
s_3	$\varnothing$
s_4	$\varnothing$
s_5	$\varnothing$
s_6	$\varnothing$
s_7	$\varnothing$
s_8	$\varnothing$
s_9	$\varnothing$
s_{10}	$\varnothing$
s_{11}	$\varnothing$
s_{12}	$\varnothing$

Cell	All may successors
s_{13}	$\varnothing$
s_{14}	$\varnothing$
s_{15}	$\varnothing$
s_{16}	$\varnothing$

24.10 F9.10 18 middle cells, concrete branch None

Every intermediate cell has an entry point reachable from the root and a separate exit point. Each failed round isolates precisely one as-yet coarse intermediate cell. The $2 \cdot 18 + 2$ concrete states give $18 + 2$ initial cells, so this case attains the bound of $18 + 1$ abstract rounds.

The concrete carrier has 38 states and 36 edges. Its reachable set has 19 states. The reachable set is disjoint from the bad predicate.

ID	State	Seed	Bd.	d	p	Successors
0	start	0	I	0	–	{1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12, 13, 14, 15, 16, 17, 18}
1	(0, 'entry')	1	–	1	0	$\varnothing$
2	(1, 'entry')	2	–	1	0	$\varnothing$
3	(2, 'entry')	3	–	1	0	$\varnothing$
4	(3, 'entry')	4	–	1	0	$\varnothing$
5	(4, 'entry')	5	–	1	0	$\varnothing$
6	(5, 'entry')	6	–	1	0	$\varnothing$
7	(6, 'entry')	7	–	1	0	$\varnothing$
8	(7, 'entry')	8	–	1	0	$\varnothing$
9	(8, 'entry')	9	–	1	0	$\varnothing$
10	(9, 'entry')	10	–	1	0	$\varnothing$
11	(10, 'entry')	11	–	1	0	$\varnothing$
12	(11, 'entry')	12	–	1	0	$\varnothing$

ID	State	Seed	Bd.	d	p	Successors
13	(12, 'entry')	13	–	1	0	$\varnothing$
14	(13, 'entry')	14	–	1	0	$\varnothing$
15	(14, 'entry')	15	–	1	0	$\varnothing$
16	(15, 'entry')	16	–	1	0	$\varnothing$
17	(16, 'entry')	17	–	1	0	$\varnothing$
18	(17, 'entry')	18	–	1	0	$\varnothing$
19	(0, 'exit')	1	–	∞	–	{37}
20	(1, 'exit')	2	–	∞	–	{37}
21	(2, 'exit')	3	–	∞	–	{37}
22	(3, 'exit')	4	–	∞	–	{37}
23	(4, 'exit')	5	–	∞	–	{37}
24	(5, 'exit')	6	–	∞	–	{37}
25	(6, 'exit')	7	–	∞	–	{37}

ID	State	Seed	Bd.	d	p	Successors
26	(7, 'exit')	8	–	∞	–	{37}
27	(8, 'exit')	9	–	∞	–	{37}
28	(9, 'exit')	10	–	∞	–	{37}
29	(10, 'exit')	11	–	∞	–	{37}
30	(11, 'exit')	12	–	∞	–	{37}
31	(12, 'exit')	13	–	∞	–	{37}
32	(13, 'exit')	14	–	∞	–	{37}
33	(14, 'exit')	15	–	∞	–	{37}
34	(15, 'exit')	16	–	∞	–	{37}
35	(16, 'exit')	17	–	∞	–	{37}
36	(17, 'exit')	18	–	∞	–	{37}
37	bad	19	B	∞	–	$\varnothing$

Round 1. The partition has 20 nonempty cells, and the may relation has 36 edges.

The selected path is b_0, b_{18}, b_{19}. The exact number of concrete lifts is 0.

j	a_j	F_j	D_j	$\#\cap$
0	b_0	$\{0\}$	$\varnothing$	0
1	b_{18}	$\{18\}$	$\{36\}$	0
2	b_{19}	$\varnothing$	$\{37\}$	0

The isolated state IDs are $\{0, 18, 36, 37\}$. The refined graph has zero paths projecting to this candidate.

Round 2. The partition has 21 nonempty cells, and the may relation has 36 edges.

The selected path is s_0, b_1, s_{37}. The exact number of concrete lifts is 0.

j	a_j	F_j	D_j	$\#\cap$
0	s_0	$\{0\}$	$\varnothing$	0
1	b_1	$\{1\}$	$\{19\}$	0
2	s_{37}	$\varnothing$	$\{37\}$	0

The isolated state IDs are $\{1, 19\}$. The refined graph has zero paths projecting to this candidate.

Round 3. The partition has 22 nonempty cells, and the may relation has 36 edges.

The selected path is s_0, b_2, s_{37}. The exact number of concrete lifts is 0.

j	a_j	F_j	D_j	$\#\cap$
0	s_0	$\{0\}$	$\varnothing$	0
1	b_2	$\{2\}$	$\{20\}$	0
2	s_{37}	$\varnothing$	$\{37\}$	0

The isolated state IDs are $\{2, 20\}$. The refined graph has zero paths projecting to this candidate.

Round 4. The partition has 23 nonempty cells, and the may relation has 36 edges.

The selected path is s_0, b_3, s_{37}. The exact number of concrete lifts is 0.

j	a_j	F_j	D_j	$\#\cap$
0	s_0	$\{0\}$	$\varnothing$	0
1	b_3	$\{3\}$	$\{21\}$	0
2	s_{37}	$\varnothing$	$\{37\}$	0

The isolated state IDs are $\{3, 21\}$. The refined graph has zero paths projecting to this candidate.

Round 5. The partition has 24 nonempty cells, and the may relation has 36 edges.

The selected path is s_0, b_4, s_{37}. The exact number of concrete lifts is 0.

j	a_j	F_j	D_j	$\#\cap$
0	s_0	$\{0\}$	$\varnothing$	0
1	b_4	$\{4\}$	$\{22\}$	0
2	s_{37}	$\varnothing$	$\{37\}$	0

The isolated state IDs are $\{4, 22\}$. The refined graph has zero paths projecting to this candidate.

Round 6. The partition has 25 nonempty cells, and the may relation has 36 edges.

The selected path is s_0, b_5, s_{37}. The exact number of concrete lifts is 0.

j	a_j	F_j	D_j	$\#\cap$
0	s_0	$\{0\}$	$\varnothing$	0
1	b_5	$\{5\}$	$\{23\}$	0
2	s_{37}	$\varnothing$	$\{37\}$	0

The isolated state IDs are $\{5, 23\}$. The refined graph has zero paths projecting to this candidate.

Round 7. The partition has 26 nonempty cells, and the may relation has 36 edges.

The selected path is s_0, b_6, s_{37}. The exact number of concrete lifts is 0.

j	a_j	F_j	D_j	$\#\cap$
0	s_0	$\{0\}$	$\varnothing$	0
1	b_6	$\{6\}$	$\{24\}$	0
2	s_{37}	$\varnothing$	$\{37\}$	0

The isolated state IDs are $\{6, 24\}$. The refined graph has zero paths projecting to this candidate.

Round 8. The partition has 27 nonempty cells, and the may relation has 36 edges.

The selected path is s_0, b_7, s_{37}. The exact number of concrete lifts is 0.

j	a_j	F_j	D_j	$\#\cap$
0	s_0	$\{0\}$	$\varnothing$	0
1	b_7	$\{7\}$	$\{25\}$	0
2	s_{37}	$\varnothing$	$\{37\}$	0

The isolated state IDs are $\{7, 25\}$. The refined graph has zero paths projecting to this candidate.

Round 9. The partition has 28 nonempty cells, and the may relation has 36 edges.

The selected path is s_0, b_8, s_{37}. The exact number of concrete lifts is 0.

j	a_j	F_j	D_j	$\#\cap$
0	s_0	$\{0\}$	$\varnothing$	0
1	b_8	$\{8\}$	$\{26\}$	0
2	s_{37}	$\varnothing$	$\{37\}$	0

The isolated state IDs are $\{8, 26\}$. The refined graph has zero paths projecting to this candidate.

Round 10. The partition has 29 nonempty cells, and the may relation has 36 edges.

The selected path is s_0, b_9, s_{37}. The exact number of concrete lifts is 0.

j	a_j	F_j	D_j	$\#\cap$
0	s_0	$\{0\}$	$\varnothing$	0
1	b_9	$\{9\}$	$\{27\}$	0
2	s_{37}	$\varnothing$	$\{37\}$	0

The isolated state IDs are $\{9, 27\}$. The refined graph has zero paths projecting to this candidate.

Round 11. The partition has 30 nonempty cells, and the may relation has 36 edges.

The selected path is s_0, b_{10}, s_{37}. The exact number of concrete lifts is 0.

j	a_j	F_j	D_j	$\#\cap$
0	s_0	$\{0\}$	$\varnothing$	0
1	b_{10}	$\{10\}$	$\{28\}$	0
2	s_{37}	$\varnothing$	$\{37\}$	0

The isolated state IDs are $\{10, 28\}$. The refined graph has zero paths projecting to this candidate.

Round 12. The partition has 31 nonempty cells, and the may relation has 36 edges.

The selected path is s_0, b_{11}, s_{37}. The exact number of concrete lifts is 0.

j	a_j	F_j	D_j	$\#\cap$
0	s_0	$\{0\}$	$\varnothing$	0
1	b_{11}	$\{11\}$	$\{29\}$	0
2	s_{37}	$\varnothing$	$\{37\}$	0

The isolated state IDs are $\{11, 29\}$. The refined graph has zero paths projecting to this candidate.

Round 13. The partition has 32 nonempty cells, and the may relation has 36 edges.

The selected path is s_0, b_{12}, s_{37}. The exact number of concrete lifts is 0.

j	a_j	F_j	D_j	$\#\cap$
0	s_0	$\{0\}$	$\varnothing$	0
1	b_{12}	$\{12\}$	$\{30\}$	0
2	s_{37}	$\varnothing$	$\{37\}$	0

The isolated state IDs are $\{12, 30\}$. The refined graph has zero paths projecting to this candidate.

Round 14. The partition has 33 nonempty cells, and the may relation has 36 edges.

The selected path is s_0, b_{13}, s_{37}. The exact number of concrete lifts is 0.

j	a_j	F_j	D_j	$\#\cap$
0	s_0	$\{0\}$	$\varnothing$	0
1	b_{13}	$\{13\}$	$\{31\}$	0
2	s_{37}	$\varnothing$	$\{37\}$	0

The isolated state IDs are $\{13, 31\}$. The refined graph has zero paths projecting to this candidate.

Round 15. The partition has 34 nonempty cells, and the may relation has 36 edges.

The selected path is s_0, b_{14}, s_{37}. The exact number of concrete lifts is 0.

j	a_j	F_j	D_j	$\#\cap$
0	s_0	$\{0\}$	$\varnothing$	0
1	b_{14}	$\{14\}$	$\{32\}$	0
2	s_{37}	$\varnothing$	$\{37\}$	0

The isolated state IDs are $\{14, 32\}$. The refined graph has zero paths projecting to this candidate.

Round 16. The partition has 35 nonempty cells, and the may relation has 36 edges.

The selected path is s_0, b_{15}, s_{37}. The exact number of concrete lifts is 0.

j	a_j	F_j	D_j	$\#\cap$
0	s_0	$\{0\}$	$\varnothing$	0
1	b_{15}	$\{15\}$	$\{33\}$	0
2	s_{37}	$\varnothing$	$\{37\}$	0

The isolated state IDs are $\{15, 33\}$. The refined graph has zero paths projecting to this candidate.

Round 17. The partition has 36 nonempty cells, and the may relation has 36 edges.

The selected path is s_0, b_{16}, s_{37}. The exact number of concrete lifts is 0.

j	a_j	F_j	D_j	$\#\cap$
0	s_0	$\{0\}$	$\varnothing$	0
1	b_{16}	$\{16\}$	$\{34\}$	0
2	s_{37}	$\varnothing$	$\{37\}$	0

The isolated state IDs are $\{16, 34\}$. The refined graph has zero paths projecting to this candidate.

Round 18. The partition has 37 nonempty cells, and the may relation has 36 edges.

The selected path is s_0, b_{17}, s_{37}. The exact number of concrete lifts is 0.

j	a_j	F_j	D_j	$\#\cap$
0	s_0	$\{0\}$	$\varnothing$	0
1	b_{17}	$\{17\}$	$\{35\}$	0
2	s_{37}	$\varnothing$	$\{37\}$	0

The isolated state IDs are $\{17, 35\}$. The refined graph has zero paths projecting to this candidate.

Round 19. The partition has 38 nonempty cells, and the may relation has 36 edges.

The abstract checker found the inductive invariant $\{s_0, s_1, s_2, s_3, s_4, s_5, s_6, s_7, s_8, s_9, s_{10}, s_{11}, s_{12}, s_{13}, s_{14}, s_{15}, s_{16}, s_{17}, s_{18}\}$. It contains $\{s_0\}$ and is disjoint from $\{s_{37}\}$.

Cell	All may successors
s_0	$\{s_1, s_2, s_3, s_4, s_5, s_6, s_7, s_8, s_9, s_{10}, s_{11}, s_{12}, s_{13}, s_{14}, s_{15}, s_{16}, s_{17}, s_{18}\}$
s_1	$\varnothing$
s_2	$\varnothing$
s_3	$\varnothing$
s_4	$\varnothing$
s_5	$\varnothing$
s_6	$\varnothing$
s_7	$\varnothing$
s_8	$\varnothing$
s_9	$\varnothing$
s_{10}	$\varnothing$
s_{11}	$\varnothing$
s_{12}	$\varnothing$

Cell	All may successors
s_{13}	$\varnothing$
s_{14}	$\varnothing$
s_{15}	$\varnothing$
s_{16}	$\varnothing$
s_{17}	$\varnothing$
s_{18}	$\varnothing$

24.11 F9.11 10 middle cells, concrete branch 1

Only entry 1 has an edge to the bad state. Every abstract choice of a different coarse middle cell is spurious.

The concrete carrier has 22 states and 21 edges. Its reachable set has 12 states. The bad predicate is reached.

ID	State	Seed	Bd.	d	p	Successors
0	start	0	I	0	–	{1, 2, 3, 4, 5, 6, 7, 8, 9, 10}
1	(0, 'entry')	1	–	1	0	$\varnothing$
2	(1, 'entry')	2	–	1	0	{21}
3	(2, 'entry')	3	–	1	0	$\varnothing$
4	(3, 'entry')	4	–	1	0	$\varnothing$
5	(4, 'entry')	5	–	1	0	$\varnothing$
6	(5, 'entry')	6	–	1	0	$\varnothing$
7	(6, 'entry')	7	–	1	0	$\varnothing$
8	(7, 'entry')	8	–	1	0	$\varnothing$
9	(8, 'entry')	9	–	1	0	$\varnothing$
10	(9, 'entry')	10	–	1	0	$\varnothing$
11	(0, 'exit')	1	–	∞	–	{21}
12	(1, 'exit')	2	–	∞	–	{21}

ID	State	Seed	Bd.	d	p	Successors
13	(2, 'exit')	3	–	∞	–	{21}
14	(3, 'exit')	4	–	∞	–	{21}
15	(4, 'exit')	5	–	∞	–	{21}
16	(5, 'exit')	6	–	∞	–	{21}
17	(6, 'exit')	7	–	∞	–	{21}
18	(7, 'exit')	8	–	∞	–	{21}
19	(8, 'exit')	9	–	∞	–	{21}
20	(9, 'exit')	10	–	∞	–	{21}
21	bad	11	B	2	2	$\varnothing$

Round 1. The partition has 12 nonempty cells, and the may relation has 20 edges.

The selected path is b_0, b_{10}, b_{11}. The exact number of concrete lifts is 0.

j	a_j	F_j	D_j	$\#\cap$
0	b_0	{0}	$\varnothing$	0
1	b_{10}	{10}	{20}	0
2	b_{11}	$\varnothing$	{21}	0

The isolated state IDs are $\{0, 10, 20, 21\}$. The refined graph has zero paths projecting to this candidate.

Round 2. The partition has 13 nonempty cells, and the may relation has 20 edges.

The selected path is s_0, b_1, s_{21}. The exact number of concrete lifts is 0.

j	a_j	F_j	D_j	$\#\cap$
0	s_0	{0}	$\varnothing$	0
1	b_1	{1}	{11}	0
2	s_{21}	$\varnothing$	{21}	0

The isolated state IDs are $\{1, 11\}$. The refined graph has zero paths projecting to this candidate.

Round 3. The partition has 14 nonempty cells, and the may relation has 20 edges.

The selected path is s_0, b_2, s_{21}. The exact number of concrete lifts is 1.

j	a_j	F_j	D_j	$\#\cap$
0	s_0	{0}	{0}	1
1	b_2	{2}	{2, 12}	1
2	s_{21}	{21}	{21}	1

A concrete witness is $0, 2, 21$. Each adjacent pair occurs in the concrete successor table.

24.12 F9.12 12 middle cells, concrete branch 8

Only entry 8 has an edge to the bad state. Every abstract choice of a different coarse middle cell is spurious.

The concrete carrier has 26 states and 25 edges. Its reachable set has 14 states. The bad predicate is reached.

ID	State	Seed	Bd.	d	p	Successors
0	start	0	I	0	–	$\{1,2,3,4,5,6,7,8,9,10,11,12\}$
1	(0, 'entry')	1	–	1	0	$\varnothing$
2	(1, 'entry')	2	–	1	0	$\varnothing$
3	(2, 'entry')	3	–	1	0	$\varnothing$
4	(3, 'entry')	4	–	1	0	$\varnothing$
5	(4, 'entry')	5	–	1	0	$\varnothing$
6	(5, 'entry')	6	–	1	0	$\varnothing$
7	(6, 'entry')	7	–	1	0	$\varnothing$
8	(7, 'entry')	8	–	1	0	$\varnothing$
9	(8, 'entry')	9	–	1	0	$\{25\}$
10	(9, 'entry')	10	–	1	0	$\varnothing$
11	(10, 'entry')	11	–	1	0	$\varnothing$
12	(11, 'entry')	12	–	1	0	$\varnothing$

ID	State	Seed	Bd.	d	p	Successors
13	(0, 'exit')	1	–	∞	–	$\{25\}$
14	(1, 'exit')	2	–	∞	–	$\{25\}$
15	(2, 'exit')	3	–	∞	–	$\{25\}$
16	(3, 'exit')	4	–	∞	–	$\{25\}$
17	(4, 'exit')	5	–	∞	–	$\{25\}$
18	(5, 'exit')	6	–	∞	–	$\{25\}$
19	(6, 'exit')	7	–	∞	–	$\{25\}$
20	(7, 'exit')	8	–	∞	–	$\{25\}$
21	(8, 'exit')	9	–	∞	–	$\{25\}$
22	(9, 'exit')	10	–	∞	–	$\{25\}$
23	(10, 'exit')	11	–	∞	–	$\{25\}$
24	(11, 'exit')	12	–	∞	–	$\{25\}$
25	bad	13	B	2	9	$\varnothing$

Round 1. The partition has 14 nonempty cells, and the may relation has 24 edges.

The selected path is b_0, b_{12}, b_{13}. The exact number of concrete lifts is 0.

j	a_j	F_j	D_j	$\#\cap$
0	b_0	$\{0\}$	$\varnothing$	0
1	b_{12}	$\{12\}$	$\{24\}$	0
2	b_{13}	$\varnothing$	$\{25\}$	0

The isolated state IDs are $\{0, 12, 24, 25\}$. The refined graph has zero paths projecting to this candidate.

Round 2. The partition has 15 nonempty cells, and the may relation has 24 edges.

The selected path is s_0, b_1, s_{25}. The exact number of concrete lifts is 0.

j	a_j	F_j	D_j	$\#\cap$
0	s_0	$\{0\}$	$\varnothing$	0
1	b_1	$\{1\}$	$\{13\}$	0
2	s_{25}	$\varnothing$	$\{25\}$	0

The isolated state IDs are $\{1, 13\}$. The refined graph has zero paths projecting to this candidate.

Round 3. The partition has 16 nonempty cells, and the may relation has 24 edges.

The selected path is s_0, b_2, s_{25}. The exact number of concrete lifts is 0.

j	a_j	F_j	D_j	$\#\cap$
0	s_0	$\{0\}$	$\varnothing$	0
1	b_2	$\{2\}$	$\{14\}$	0
2	s_{25}	$\varnothing$	$\{25\}$	0

The isolated state IDs are $\{2, 14\}$. The refined graph has zero paths projecting to this candidate.

Round 4. The partition has 17 nonempty cells, and the may relation has 24 edges.

The selected path is s_0, b_3, s_{25}. The exact number of concrete lifts is 0.

j	a_j	F_j	D_j	$\#\cap$
0	s_0	$\{0\}$	$\varnothing$	0
1	b_3	$\{3\}$	$\{15\}$	0
2	s_{25}	$\varnothing$	$\{25\}$	0

The isolated state IDs are $\{3, 15\}$. The refined graph has zero paths projecting to this candidate.

Round 5. The partition has 18 nonempty cells, and the may relation has 24 edges.

The selected path is s_0, b_4, s_{25}. The exact number of concrete lifts is 0.

j	a_j	F_j	D_j	$\#\cap$
0	s_0	$\{0\}$	$\varnothing$	0
1	b_4	$\{4\}$	$\{16\}$	0
2	s_{25}	$\varnothing$	$\{25\}$	0

The isolated state IDs are $\{4, 16\}$. The refined graph has zero paths projecting to this candidate.

Round 6. The partition has 19 nonempty cells, and the may relation has 24 edges.

The selected path is s_0, b_5, s_{25}. The exact number of concrete lifts is 0.

j	a_j	F_j	D_j	$\#\cap$
0	s_0	$\{0\}$	$\varnothing$	0
1	b_5	$\{5\}$	$\{17\}$	0
2	s_{25}	$\varnothing$	$\{25\}$	0

The isolated state IDs are $\{5, 17\}$. The refined graph has zero paths projecting to this candidate.

Round 7. The partition has 20 nonempty cells, and the may relation has 24 edges.

The selected path is s_0, b_6, s_{25}. The exact number of concrete lifts is 0.

j	a_j	F_j	D_j	$\#\cap$
0	s_0	$\{0\}$	$\varnothing$	0
1	b_6	$\{6\}$	$\{18\}$	0
2	s_{25}	$\varnothing$	$\{25\}$	0

The isolated state IDs are $\{6, 18\}$. The refined graph has zero paths projecting to this candidate.

Round 8. The partition has 21 nonempty cells, and the may relation has 24 edges.

The selected path is s_0, b_7, s_{25}. The exact number of concrete lifts is 0.

j	a_j	F_j	D_j	$\#\cap$
0	s_0	$\{0\}$	$\varnothing$	0
1	b_7	$\{7\}$	$\{19\}$	0
2	s_{25}	$\varnothing$	$\{25\}$	0

The isolated state IDs are $\{7, 19\}$. The refined graph has zero paths projecting to this candidate.

Round 9. The partition has 22 nonempty cells, and the may relation has 24 edges.

The selected path is s_0, b_8, s_{25}. The exact number of concrete lifts is 0.

j	a_j	F_j	D_j	$\#\cap$
0	s_0	$\{0\}$	$\varnothing$	0
1	b_8	$\{8\}$	$\{20\}$	0
2	s_{25}	$\varnothing$	$\{25\}$	0

The isolated state IDs are $\{8, 20\}$. The refined graph has zero paths projecting to this candidate.

Round 10. The partition has 23 nonempty cells, and the may relation has 24 edges.

The selected path is s_0, b_9, s_{25}. The exact number of concrete lifts is 1.

j	a_j	F_j	D_j	$\#\cap$
0	s_0	$\{0\}$	$\{0\}$	1
1	b_9	$\{9\}$	$\{9, 21\}$	1
2	s_{25}	$\{25\}$	$\{25\}$	1

A concrete witness is $0, 9, 25$. Each adjacent pair occurs in the concrete successor table.

24.13 F9.13 16 middle cells, concrete branch 5

Only entry 5 has an edge to the bad state. Every abstract choice of a different coarse middle cell is spurious.

The concrete carrier has 34 states and 33 edges. Its reachable set has 18 states. The bad predicate is reached.

ID	State	Seed	Bd.	d	p	Successors
0	start	0	I	0	–	$\{1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12, 13, 14, 15, 16\}$
1	(0, 'entry')	1	–	1	0	$\varnothing$
2	(1, 'entry')	2	–	1	0	$\varnothing$
3	(2, 'entry')	3	–	1	0	$\varnothing$
4	(3, 'entry')	4	–	1	0	$\varnothing$
5	(4, 'entry')	5	–	1	0	$\varnothing$
6	(5, 'entry')	6	–	1	0	$\{33\}$
7	(6, 'entry')	7	–	1	0	$\varnothing$
8	(7, 'entry')	8	–	1	0	$\varnothing$
9	(8, 'entry')	9	–	1	0	$\varnothing$
10	(9, 'entry')	10	–	1	0	$\varnothing$
11	(10, 'entry')	11	–	1	0	$\varnothing$
12	(11, 'entry')	12	–	1	0	$\varnothing$
13	(12, 'entry')	13	–	1	0	$\varnothing$
14	(13, 'entry')	14	–	1	0	$\varnothing$
15	(14, 'entry')	15	–	1	0	$\varnothing$
16	(15, 'entry')	16	–	1	0	$\varnothing$
17	(0, 'exit')	1	–	∞	–	$\{33\}$
18	(1, 'exit')	2	–	∞	–	$\{33\}$
19	(2, 'exit')	3	–	∞	–	$\{33\}$
20	(3, 'exit')	4	–	∞	–	$\{33\}$
21	(4, 'exit')	5	–	∞	–	$\{33\}$
22	(5, 'exit')	6	–	∞	–	$\{33\}$
23	(6, 'exit')	7	–	∞	–	$\{33\}$
24	(7, 'exit')	8	–	∞	–	$\{33\}$
25	(8, 'exit')	9	–	∞	–	$\{33\}$
26	(9, 'exit')	10	–	∞	–	$\{33\}$
27	(10, 'exit')	11	–	∞	–	$\{33\}$
28	(11, 'exit')	12	–	∞	–	$\{33\}$
29	(12, 'exit')	13	–	∞	–	$\{33\}$
30	(13, 'exit')	14	–	∞	–	$\{33\}$
31	(14, 'exit')	15	–	∞	–	$\{33\}$
32	(15, 'exit')	16	–	∞	–	$\{33\}$
33	bad	17	B	2	6	$\varnothing$

Round 1. The partition has 18 nonempty cells, and the may relation has 32 edges.

The selected path is b_0, b_{16}, b_{17}. The exact number of concrete lifts is 0.

j	a_j	F_j	D_j	$\#\cap$
0	b_0	$\{0\}$	$\varnothing$	0
1	b_{16}	$\{16\}$	$\{32\}$	0
2	b_{17}	$\varnothing$	$\{33\}$	0

The isolated state IDs are $\{0, 16, 32, 33\}$. The refined graph has zero paths projecting to this candidate.

Round 2. The partition has 19 nonempty cells, and the may relation has 32 edges.

The selected path is s_0, b_1, s_{33}. The exact number of concrete lifts is 0.

j	a_j	F_j	D_j	$\#\cap$
0	s_0	$\{0\}$	$\varnothing$	0
1	b_1	$\{1\}$	$\{17\}$	0
2	s_{33}	$\varnothing$	$\{33\}$	0

The isolated state IDs are $\{1, 17\}$. The refined graph has zero paths projecting to this candidate.

Round 3. The partition has 20 nonempty cells, and the may relation has 32 edges.

The selected path is s_0, b_2, s_{33}. The exact number of concrete lifts is 0.

j	a_j	F_j	D_j	$\#\cap$
0	s_0	$\{0\}$	$\varnothing$	0
1	b_2	$\{2\}$	$\{18\}$	0
2	s_{33}	$\varnothing$	$\{33\}$	0

The isolated state IDs are $\{2, 18\}$. The refined graph has zero paths projecting to this candidate.

Round 4. The partition has 21 nonempty cells, and the may relation has 32 edges.

The selected path is s_0, b_3, s_{33}. The exact number of concrete lifts is 0.

j	a_j	F_j	D_j	$\#\cap$
0	s_0	$\{0\}$	$\varnothing$	0
1	b_3	$\{3\}$	$\{19\}$	0
2	s_{33}	$\varnothing$	$\{33\}$	0

The isolated state IDs are $\{3, 19\}$. The refined graph has zero paths projecting to this candidate.

Round 5. The partition has 22 nonempty cells, and the may relation has 32 edges.

The selected path is s_0, b_4, s_{33}. The exact number of concrete lifts is 0.

j	a_j	F_j	D_j	$\#\cap$
0	s_0	$\{0\}$	$\varnothing$	0
1	b_4	$\{4\}$	$\{20\}$	0
2	s_{33}	$\varnothing$	$\{33\}$	0

The isolated state IDs are $\{4, 20\}$. The refined graph has zero paths projecting to this candidate.

Round 6. The partition has 23 nonempty cells, and the may relation has 32 edges.

The selected path is s_0, b_5, s_{33}. The exact number of concrete lifts is 0.

j	a_j	F_j	D_j	$\#\cap$
0	s_0	$\{0\}$	$\varnothing$	0
1	b_5	$\{5\}$	$\{21\}$	0
2	s_{33}	$\varnothing$	$\{33\}$	0

The isolated state IDs are $\{5, 21\}$. The refined graph has zero paths projecting to this candidate.

Round 7. The partition has 24 nonempty cells, and the may relation has 32 edges.

The selected path is s_0, b_6, s_{33}. The exact number of concrete lifts is 1.

j	a_j	F_j	D_j	$\#\cap$
0	s_0	$\{0\}$	$\{0\}$	1
1	b_6	$\{6\}$	$\{6, 22\}$	1
2	s_{33}	$\{33\}$	$\{33\}$	1

A concrete witness is $0, 6, 33$. Each adjacent pair occurs in the concrete successor table.

25 Faces of a Boolean hypercube

The concrete graph is a collection of faces. A seed based on population count or the low two bits can identify vertices in distinct connected components.

Tables enumerate concrete states by integer ID. The d column gives distance from an initial state, and p gives a predecessor attaining that distance. Columns I and B are encoded by the boundary field. Abstract keys b_i are seed blocks and s_x are isolated states.

25.1 F10.1 dimension 4, toggles (0, 1), target 12, mode 0

Coordinates outside the toggle mask 3 remain zero. Every assignment of the permitted coordinates is reachable by toggling its nonzero bits once.

The concrete carrier has 16 states and 32 edges. Its reachable set has 4 states. The reachable set is disjoint from the bad predicate.

ID	State	Seed	Bd.	d	p	Successors
0	0	0	I	0	–	$\{1, 2\}$
1	1	1	–	1	0	$\{0, 3\}$
2	2	1	–	1	0	$\{0, 3\}$
3	3	2	–	2	1	$\{1, 2\}$
4	4	1	–	∞	–	$\{5, 6\}$
5	5	2	–	∞	–	$\{4, 7\}$
6	6	2	–	∞	–	$\{4, 7\}$
7	7	3	–	∞	–	$\{5, 6\}$
8	8	1	–	∞	–	$\{9, 10\}$
9	9	2	–	∞	–	$\{8, 11\}$
10	10	2	–	∞	–	$\{8, 11\}$
11	11	3	–	∞	–	$\{9, 10\}$
12	12	2	B	∞	–	$\{13, 14\}$

ID	State	Seed	Bd.	d	p	Successors
13	13	3	–	∞	–	$\{12, 15\}$
14	14	3	–	∞	–	$\{12, 15\}$
15	15	4	–	∞	–	$\{13, 14\}$

Round 1. The partition has 5 nonempty cells, and the may relation has 8 edges.

The selected path is b_0, b_1, b_2. The exact number of concrete lifts is 0.

j	a_j	F_j	D_j	$\#\cap$
0	b_0	$\{0\}$	$\varnothing$	0
1	b_1	$\{1, 2\}$	$\varnothing$	0
2	b_2	$\{3\}$	$\{12\}$	0

The isolated state IDs are $\{0, 1, 2, 3, 4, 5, 6, 8, 9, 10, 12\}$. The refined graph has zero paths projecting to this candidate.

Round 2. The partition has 13 nonempty cells, and the may relation has 28 edges.

The abstract checker found the inductive invariant $\{s_0, s_1, s_2, s_3\}$. It contains $\{s_0\}$ and is disjoint from $\{s_{12}\}$.

Cell	All may successors
s_0	$\{s_1, s_2\}$
s_1	$\{s_0, s_3\}$
s_2	$\{s_0, s_3\}$
s_3	$\{s_1, s_2\}$

25.2 F10.2 dimension 4, toggles (1, 3), target 10, mode 0

Coordinates outside the toggle mask 10 remain zero. Every assignment of the permitted coordinates is reachable by toggling its nonzero bits once.

The concrete carrier has 16 states and 32 edges. Its reachable set has 4 states. The bad predicate is reached.

ID	State	Seed	Bd.	d	p	Successors
0	0	0	I	0	–	$\{2, 8\}$
1	1	1	–	∞	–	$\{3, 9\}$
2	2	1	–	1	0	$\{0, 10\}$
3	3	2	–	∞	–	$\{1, 11\}$
4	4	1	–	∞	–	$\{6, 12\}$
5	5	2	–	∞	–	$\{7, 13\}$
6	6	2	–	∞	–	$\{4, 14\}$
7	7	3	–	∞	–	$\{5, 15\}$
8	8	1	–	1	0	$\{0, 10\}$
9	9	2	–	∞	–	$\{1, 11\}$
10	10	2	B	2	2	$\{2, 8\}$
11	11	3	–	∞	–	$\{3, 9\}$
12	12	2	–	∞	–	$\{4, 14\}$

ID	State	Seed	Bd.	d	p	Successors
13	13	3	–	∞	–	$\{5, 15\}$
14	14	3	–	∞	–	$\{6, 12\}$
15	15	4	–	∞	–	$\{7, 13\}$

Round 1. The partition has 5 nonempty cells, and the may relation has 8 edges.

The selected path is b_0, b_1, b_2. The exact number of concrete lifts is 2.

j	a_j	F_j	D_j	$\#\cap$
0	b_0	$\{0\}$	$\{0\}$	1
1	b_1	$\{2, 8\}$	$\{2, 8\}$	2
2	b_2	$\{10\}$	$\{10\}$	1

A concrete witness is $0, 2, 10$. Each adjacent pair occurs in the concrete successor table.

25.3 F10.3 dimension 5, toggles (0, 2, 4), target 11, mode 1

Coordinates outside the toggle mask 21 remain zero. Every assignment of the permitted coordinates is reachable by toggling its nonzero bits once.

The concrete carrier has 32 states and 96 edges. Its reachable set has 8 states. The reachable set is disjoint from the bad predicate.

ID	State	Seed	Bd.	d	p	Successors
0	0	0	I	0	–	$\{1, 4, 16\}$
1	1	1	–	1	0	$\{0, 5, 17\}$
2	2	2	–	∞	–	$\{3, 6, 18\}$
3	3	3	–	∞	–	$\{2, 7, 19\}$
4	4	0	–	1	0	$\{0, 5, 20\}$
5	5	1	–	2	1	$\{1, 4, 21\}$
6	6	2	–	∞	–	$\{2, 7, 22\}$
7	7	3	–	∞	–	$\{3, 6, 23\}$
8	8	0	–	∞	–	$\{9, 12, 24\}$
9	9	1	–	∞	–	$\{8, 13, 25\}$
10	10	2	–	∞	–	$\{11, 14, 26\}$
11	11	3	B	∞	–	$\{10, 15, 27\}$
12	12	0	–	∞	–	$\{8, 13, 28\}$

ID	State	Seed	Bd.	d	p	Successors
13	13	1	–	∞	–	$\{9, 12, 29\}$
14	14	2	–	∞	–	$\{10, 15, 30\}$
15	15	3	–	∞	–	$\{11, 14, 31\}$
16	16	0	–	1	0	$\{0, 17, 20\}$
17	17	1	–	2	1	$\{1, 16, 21\}$
18	18	2	–	∞	–	$\{2, 19, 22\}$
19	19	3	–	∞	–	$\{3, 18, 23\}$
20	20	0	–	2	4	$\{4, 16, 21\}$
21	21	1	–	3	5	$\{5, 17, 20\}$
22	22	2	–	∞	–	$\{6, 18, 23\}$
23	23	3	–	∞	–	$\{7, 19, 22\}$
24	24	0	–	∞	–	$\{8, 25, 28\}$
25	25	1	–	∞	–	$\{9, 24, 29\}$

ID	State	Seed	Bd.	d	p	Successors
26	26	2	–	∞	–	$\{10, 27, 30\}$
27	27	3	–	∞	–	$\{11, 26, 31\}$
28	28	0	–	∞	–	$\{12, 24, 29\}$
29	29	1	–	∞	–	$\{13, 25, 28\}$
30	30	2	–	∞	–	$\{14, 26, 31\}$
31	31	3	–	∞	–	$\{15, 27, 30\}$

Round 1. The partition has 4 nonempty cells, and the may relation has 8 edges.

The abstract checker found the inductive invariant $\{b_0, b_1\}$. It contains $\{b_0\}$ and is disjoint from $\{b_3\}$.

Cell	All may successors
b_0	$\{b_0, b_1\}$
b_1	$\{b_0, b_1\}$

25.4 F10.4 dimension 5, toggles (1, 2, 4), target 23, mode 0

Coordinates outside the toggle mask 22 remain zero. Every assignment of the permitted coordinates is reachable by toggling its nonzero bits once.

The concrete carrier has 32 states and 96 edges. Its reachable set has 8 states. The reachable set is disjoint from the bad predicate.

ID	State	Seed	Bd.	d	p	Successors
0	0	0	I	0	–	$\{2, 4, 16\}$
1	1	1	–	∞	–	$\{3, 5, 17\}$
2	2	1	–	1	0	$\{0, 6, 18\}$
3	3	2	–	∞	–	$\{1, 7, 19\}$
4	4	1	–	1	0	$\{0, 6, 20\}$
5	5	2	–	∞	–	$\{1, 7, 21\}$
6	6	2	–	2	2	$\{2, 4, 22\}$
7	7	3	–	∞	–	$\{3, 5, 23\}$
8	8	1	–	∞	–	$\{10, 12, 24\}$
9	9	2	–	∞	–	$\{11, 13, 25\}$
10	10	2	–	∞	–	$\{8, 14, 26\}$
11	11	3	–	∞	–	$\{9, 15, 27\}$
12	12	2	–	∞	–	$\{8, 14, 28\}$

ID	State	Seed	Bd.	d	p	Successors
13	13	3	–	∞	–	$\{9, 15, 29\}$
14	14	3	–	∞	–	$\{10, 12, 30\}$
15	15	4	–	∞	–	$\{11, 13, 31\}$
16	16	1	–	1	0	$\{0, 18, 20\}$
17	17	2	–	∞	–	$\{1, 19, 21\}$
18	18	2	–	2	2	$\{2, 16, 22\}$
19	19	3	–	∞	–	$\{3, 17, 23\}$
20	20	2	–	2	4	$\{4, 16, 22\}$
21	21	3	–	∞	–	$\{5, 17, 23\}$
22	22	3	–	3	6	$\{6, 18, 20\}$
23	23	4	B	∞	–	$\{7, 19, 21\}$
24	24	2	–	∞	–	$\{8, 26, 28\}$
25	25	3	–	∞	–	$\{9, 27, 29\}$

ID	State	Seed	Bd.	d	p	Successors
26	26	3	–	∞	–	$\{10, 24, 30\}$
27	27	4	–	∞	–	$\{11, 25, 31\}$
28	28	3	–	∞	–	$\{12, 24, 30\}$
29	29	4	–	∞	–	$\{13, 25, 31\}$
30	30	4	–	∞	–	$\{14, 26, 28\}$
31	31	5	–	∞	–	$\{15, 27, 29\}$

Round 1. The partition has 6 nonempty cells, and the may relation has 10 edges.

The selected path is b_0, b_1, b_2, b_3, b_4. The exact number of concrete lifts is 0.

j	a_j	F_j	D_j	$\#\cap$
0	b_0	$\{0\}$	$\varnothing$	0
1	b_1	$\{2, 4, 16\}$	$\{1\}$	0
2	b_2	$\{6, 18, 20\}$	$\{3, 5, 17\}$	0
3	b_3	$\{22\}$	$\{7, 19, 21\}$	0
4	b_4	$\varnothing$	$\{23\}$	0

The isolated state IDs are $\{0, 1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12, 13, 14, 15, 16, 17, 18, 19, 20, 21, 22, 23, 24, 25, 26, 27, 28, 29, 30\}$. The refined graph has zero paths projecting to this candidate.

Round 2. The partition has 32 nonempty cells, and the may relation has 96 edges.

The abstract checker found the inductive invariant $\{s_0, s_2, s_4, s_6, s_{16}, s_{18}, s_{20}, s_{22}\}$. It contains $\{s_0\}$ and is disjoint from $\{s_{23}\}$.

Cell	All may successors
s_0	$\{s_2, s_4, s_{16}\}$
s_2	$\{s_0, s_6, s_{18}\}$
s_4	$\{s_0, s_6, s_{20}\}$
s_6	$\{s_2, s_4, s_{22}\}$
s_{16}	$\{s_0, s_{18}, s_{20}\}$
s_{18}	$\{s_2, s_{16}, s_{22}\}$
s_{20}	$\{s_4, s_{16}, s_{22}\}$
s_{22}	$\{s_6, s_{18}, s_{20}\}$

25.5 F10.5 dimension 5, toggles (0, 3, 4), target 25, mode 0

Coordinates outside the toggle mask 25 remain zero. Every assignment of the permitted coordinates is reachable by toggling its nonzero bits once.

The concrete carrier has 32 states and 96 edges. Its reachable set has 8 states. The bad predicate is reached.

ID	State	Seed	Bd.	d	p	Successors
0	0	0	I	0	–	$\{1, 8, 16\}$
1	1	1	–	1	0	$\{0, 9, 17\}$
2	2	1	–	∞	–	$\{3, 10, 18\}$
3	3	2	–	∞	–	$\{2, 11, 19\}$
4	4	1	–	∞	–	$\{5, 12, 20\}$
5	5	2	–	∞	–	$\{4, 13, 21\}$
6	6	2	–	∞	–	$\{7, 14, 22\}$
7	7	3	–	∞	–	$\{6, 15, 23\}$
8	8	1	–	1	0	$\{0, 9, 24\}$
9	9	2	–	2	1	$\{1, 8, 25\}$
10	10	2	–	∞	–	$\{2, 11, 26\}$
11	11	3	–	∞	–	$\{3, 10, 27\}$
12	12	2	–	∞	–	$\{4, 13, 28\}$

ID	State	Seed	Bd.	d	p	Successors
13	13	3	–	∞	–	$\{5, 12, 29\}$
14	14	3	–	∞	–	$\{6, 15, 30\}$
15	15	4	–	∞	–	$\{7, 14, 31\}$
16	16	1	–	1	0	$\{0, 17, 24\}$
17	17	2	–	2	1	$\{1, 16, 25\}$
18	18	2	–	∞	–	$\{2, 19, 26\}$
19	19	3	–	∞	–	$\{3, 18, 27\}$
20	20	2	–	∞	–	$\{4, 21, 28\}$
21	21	3	–	∞	–	$\{5, 20, 29\}$
22	22	3	–	∞	–	$\{6, 23, 30\}$
23	23	4	–	∞	–	$\{7, 22, 31\}$
24	24	2	–	2	8	$\{8, 16, 25\}$
25	25	3	B	3	9	$\{9, 17, 24\}$

ID	State	Seed	Bd.	d	p	Successors
26	26	3	–	∞	–	$\{10, 18, 27\}$
27	27	4	–	∞	–	$\{11, 19, 26\}$
28	28	3	–	∞	–	$\{12, 20, 29\}$
29	29	4	–	∞	–	$\{13, 21, 28\}$
30	30	4	–	∞	–	$\{14, 22, 31\}$
31	31	5	–	∞	–	$\{15, 23, 30\}$

Round 1. The partition has 6 nonempty cells, and the may relation has 10 edges.

The selected path is b_0, b_1, b_2, b_3. The exact number of concrete lifts is 6.

j	a_j	F_j	D_j	$\#\cap$
0	b_0	$\{0\}$	$\{0\}$	1
1	b_1	$\{1, 8, 16\}$	$\{1, 8, 16\}$	3
2	b_2	$\{9, 17, 24\}$	$\{9, 17, 24\}$	3
3	b_3	$\{25\}$	$\{25\}$	1

A concrete witness is $0, 1, 9, 25$. Each adjacent pair occurs in the concrete successor table.

25.6 F10.6 dimension 6, toggles (0, 2, 5), target 37, mode 1

Coordinates outside the toggle mask 37 remain zero. Every assignment of the permitted coordinates is reachable by toggling its nonzero bits once.

The concrete carrier has 64 states and 192 edges. Its reachable set has 8 states. The bad predicate is reached.

ID	State	Seed	Bd.	d	p	Successors
0	0	0	I	0	–	$\{1, 4, 32\}$
1	1	1	–	1	0	$\{0, 5, 33\}$
2	2	2	–	∞	–	$\{3, 6, 34\}$
3	3	3	–	∞	–	$\{2, 7, 35\}$
4	4	0	–	1	0	$\{0, 5, 36\}$
5	5	1	–	2	1	$\{1, 4, 37\}$
6	6	2	–	∞	–	$\{2, 7, 38\}$
7	7	3	–	∞	–	$\{3, 6, 39\}$
8	8	0	–	∞	–	$\{9, 12, 40\}$
9	9	1	–	∞	–	$\{8, 13, 41\}$
10	10	2	–	∞	–	$\{11, 14, 42\}$
11	11	3	–	∞	–	$\{10, 15, 43\}$
12	12	0	–	∞	–	$\{8, 13, 44\}$

ID	State	Seed	Bd.	d	p	Successors
13	13	1	–	∞	–	$\{9, 12, 45\}$
14	14	2	–	∞	–	$\{10, 15, 46\}$
15	15	3	–	∞	–	$\{11, 14, 47\}$
16	16	0	–	∞	–	$\{17, 20, 48\}$
17	17	1	–	∞	–	$\{16, 21, 49\}$
18	18	2	–	∞	–	$\{19, 22, 50\}$
19	19	3	–	∞	–	$\{18, 23, 51\}$
20	20	0	–	∞	–	$\{16, 21, 52\}$
21	21	1	–	∞	–	$\{17, 20, 53\}$
22	22	2	–	∞	–	$\{18, 23, 54\}$
23	23	3	–	∞	–	$\{19, 22, 55\}$
24	24	0	–	∞	–	$\{25, 28, 56\}$
25	25	1	–	∞	–	$\{24, 29, 57\}$

ID	State	Seed	Bd.	d	p	Successors
26	26	2	–	∞	–	$\{27, 30, 58\}$
27	27	3	–	∞	–	$\{26, 31, 59\}$
28	28	0	–	∞	–	$\{24, 29, 60\}$
29	29	1	–	∞	–	$\{25, 28, 61\}$
30	30	2	–	∞	–	$\{26, 31, 62\}$
31	31	3	–	∞	–	$\{27, 30, 63\}$
32	32	0	–	1	0	$\{0, 33, 36\}$
33	33	1	–	2	1	$\{1, 32, 37\}$
34	34	2	–	∞	–	$\{2, 35, 38\}$
35	35	3	–	∞	–	$\{3, 34, 39\}$
36	36	0	–	2	4	$\{4, 32, 37\}$
37	37	1	B	3	5	$\{5, 33, 36\}$
38	38	2	–	∞	–	$\{6, 34, 39\}$

ID	State	Seed	Bd.	d	p	Successors
39	39	3	–	∞	–	$\{7, 35, 38\}$
40	40	0	–	∞	–	$\{8, 41, 44\}$
41	41	1	–	∞	–	$\{9, 40, 45\}$
42	42	2	–	∞	–	$\{10, 43, 46\}$
43	43	3	–	∞	–	$\{11, 42, 47\}$
44	44	0	–	∞	–	$\{12, 40, 45\}$
45	45	1	–	∞	–	$\{13, 41, 44\}$
46	46	2	–	∞	–	$\{14, 42, 47\}$
47	47	3	–	∞	–	$\{15, 43, 46\}$
48	48	0	–	∞	–	$\{16, 49, 52\}$
49	49	1	–	∞	–	$\{17, 48, 53\}$
50	50	2	–	∞	–	$\{18, 51, 54\}$
51	51	3	–	∞	–	$\{19, 50, 55\}$

ID	State	Seed	Bd.	d	p	Successors
52	52	0	–	∞	–	$\{20, 48, 53\}$
53	53	1	–	∞	–	$\{21, 49, 52\}$
54	54	2	–	∞	–	$\{22, 50, 55\}$
55	55	3	–	∞	–	$\{23, 51, 54\}$
56	56	0	–	∞	–	$\{24, 57, 60\}$
57	57	1	–	∞	–	$\{25, 56, 61\}$
58	58	2	–	∞	–	$\{26, 59, 62\}$
59	59	3	–	∞	–	$\{27, 58, 63\}$
60	60	0	–	∞	–	$\{28, 56, 61\}$
61	61	1	–	∞	–	$\{29, 57, 60\}$
62	62	2	–	∞	–	$\{30, 58, 63\}$
63	63	3	–	∞	–	$\{31, 59, 62\}$

Round 1. The partition has 4 nonempty cells, and the may relation has 8 edges.

The selected path is b_0, b_1. The exact number of concrete lifts is 0.

j	a_j	F_j	D_j	$\#\cap$
0	b_0	$\{0\}$	$\{36\}$	0
1	b_1	$\{1\}$	$\{37\}$	0

The isolated state IDs are $\{0, 1, 4, 5, 8, 9, 12, 13, 16, 17, 20, 21, 24, 25, 28, 29, 32, 33, 36, 37, 40, 41, 44, 45, 48, 49, 52, 53, 56, 57, 60, 61\}$. The refined graph has zero paths projecting to this candidate.

Round 2. The partition has 34 nonempty cells, and the may relation has 100 edges.

The selected path is $s_0, s_{32}, s_{36}, s_{37}$. The exact number of concrete lifts is 1.

j	a_j	F_j	D_j	$\#\cap$
0	s_0	$\{0\}$	$\{0\}$	1
1	s_{32}	$\{32\}$	$\{32\}$	1
2	s_{36}	$\{36\}$	$\{36\}$	1
3	s_{37}	$\{37\}$	$\{37\}$	1

A concrete witness is $0, 32, 36, 37$. Each adjacent pair occurs in the concrete successor table.

25.7 F10.7 dimension 6, toggles (1, 2, 3), target 17, mode 0

Coordinates outside the toggle mask 14 remain zero. Every assignment of the permitted coordinates is reachable by toggling its nonzero bits once.

The concrete carrier has 64 states and 192 edges. Its reachable set has 8 states. The reachable set is disjoint from the bad predicate.

ID	State	Seed	Bd.	d	p	Successors
0	0	0	I	0	–	$\{2, 4, 8\}$
1	1	1	–	∞	–	$\{3, 5, 9\}$
2	2	1	–	1	0	$\{0, 6, 10\}$
3	3	2	–	∞	–	$\{1, 7, 11\}$
4	4	1	–	1	0	$\{0, 6, 12\}$
5	5	2	–	∞	–	$\{1, 7, 13\}$
6	6	2	–	2	2	$\{2, 4, 14\}$
7	7	3	–	∞	–	$\{3, 5, 15\}$
8	8	1	–	1	0	$\{0, 10, 12\}$
9	9	2	–	∞	–	$\{1, 11, 13\}$
10	10	2	–	2	2	$\{2, 8, 14\}$
11	11	3	–	∞	–	$\{3, 9, 15\}$
12	12	2	–	2	4	$\{4, 8, 14\}$

ID	State	Seed	Bd.	d	p	Successors
13	13	3	–	∞	–	$\{5, 9, 15\}$
14	14	3	–	3	6	$\{6, 10, 12\}$
15	15	4	–	∞	–	$\{7, 11, 13\}$
16	16	1	–	∞	–	$\{18, 20, 24\}$
17	17	2	B	∞	–	$\{19, 21, 25\}$
18	18	2	–	∞	–	$\{16, 22, 26\}$
19	19	3	–	∞	–	$\{17, 23, 27\}$
20	20	2	–	∞	–	$\{16, 22, 28\}$
21	21	3	–	∞	–	$\{17, 23, 29\}$
22	22	3	–	∞	–	$\{18, 20, 30\}$
23	23	4	–	∞	–	$\{19, 21, 31\}$
24	24	2	–	∞	–	$\{16, 26, 28\}$
25	25	3	–	∞	–	$\{17, 27, 29\}$

ID	State	Seed	Bd.	d	p	Successors
26	26	3	–	∞	–	$\{18, 24, 30\}$
27	27	4	–	∞	–	$\{19, 25, 31\}$
28	28	3	–	∞	–	$\{20, 24, 30\}$
29	29	4	–	∞	–	$\{21, 25, 31\}$
30	30	4	–	∞	–	$\{22, 26, 28\}$
31	31	5	–	∞	–	$\{23, 27, 29\}$
32	32	1	–	∞	–	$\{34, 36, 40\}$
33	33	2	–	∞	–	$\{35, 37, 41\}$
34	34	2	–	∞	–	$\{32, 38, 42\}$
35	35	3	–	∞	–	$\{33, 39, 43\}$
36	36	2	–	∞	–	$\{32, 38, 44\}$
37	37	3	–	∞	–	$\{33, 39, 45\}$
38	38	3	–	∞	–	$\{34, 36, 46\}$

ID	State	Seed	Bd.	d	p	Successors
39	39	4	–	∞	–	$\{35, 37, 47\}$
40	40	2	–	∞	–	$\{32, 42, 44\}$
41	41	3	–	∞	–	$\{33, 43, 45\}$
42	42	3	–	∞	–	$\{34, 40, 46\}$
43	43	4	–	∞	–	$\{35, 41, 47\}$
44	44	3	–	∞	–	$\{36, 40, 46\}$
45	45	4	–	∞	–	$\{37, 41, 47\}$
46	46	4	–	∞	–	$\{38, 42, 44\}$
47	47	5	–	∞	–	$\{39, 43, 45\}$
48	48	2	–	∞	–	$\{50, 52, 56\}$
49	49	3	–	∞	–	$\{51, 53, 57\}$
50	50	3	–	∞	–	$\{48, 54, 58\}$
51	51	4	–	∞	–	$\{49, 55, 59\}$

ID	State	Seed	Bd.	d	p	Successors
52	52	3	–	∞	–	$\{48, 54, 60\}$
53	53	4	–	∞	–	$\{49, 55, 61\}$
54	54	4	–	∞	–	$\{50, 52, 62\}$
55	55	5	–	∞	–	$\{51, 53, 63\}$
56	56	3	–	∞	–	$\{48, 58, 60\}$
57	57	4	–	∞	–	$\{49, 59, 61\}$
58	58	4	–	∞	–	$\{50, 56, 62\}$
59	59	5	–	∞	–	$\{51, 57, 63\}$
60	60	4	–	∞	–	$\{52, 56, 62\}$
61	61	5	–	∞	–	$\{53, 57, 63\}$
62	62	5	–	∞	–	$\{54, 58, 60\}$
63	63	6	–	∞	–	$\{55, 59, 61\}$

Round 1. The partition has 7 nonempty cells, and the may relation has 12 edges.

The selected path is b_0, b_1, b_2. The exact number of concrete lifts is 0.

j	a_j	F_j	D_j	$\#\cap$
0	b_0	$\{0\}$	$\varnothing$	0
1	b_1	$\{2, 4, 8\}$	$\varnothing$	0
2	b_2	$\{6, 10, 12\}$	$\{17\}$	0

The isolated state IDs are $\{0, 1, 2, 3, 4, 5, 6, 8, 9, 10, 12, 16, 17, 18, 20, 24, 32, 33, 34, 36, 40, 48\}$. The refined graph has zero paths projecting to this candidate.

Round 2. The partition has 26 nonempty cells, and the may relation has 72 edges.

The selected path is $s_0, s_8, s_{12}, b_3, s_{17}$. The exact number of concrete lifts is 0.

j	a_j	F_j	D_j	$\#\cap$
0	s_0	$\{0\}$	$\varnothing$	0
1	s_8	$\{8\}$	$\varnothing$	0
2	s_{12}	$\{12\}$	$\varnothing$	0
3	b_3	$\{14\}$	$\{19, 21, 25\}$	0
4	s_{17}	$\varnothing$	$\{17\}$	0

The isolated state IDs are $\{7, 11, 13, 14, 19, 21, 22, 25, 26, 28, 35, 37, 38, 41, 42, 44, 49, 50, 52, 56\}$. The refined graph has zero paths projecting to this candidate.

Round 3. The partition has 45 nonempty cells, and the may relation has 138 edges.

The abstract checker found the inductive invariant $\{s_0, s_2, s_4, s_6, s_8, s_{10}, s_{12}, s_{14}\}$. It contains $\{s_0\}$ and is disjoint from $\{s_{17}\}$.

Cell	All may successors
s_0	$\{s_2, s_4, s_8\}$
s_2	$\{s_0, s_6, s_{10}\}$
s_4	$\{s_0, s_6, s_{12}\}$
s_6	$\{s_2, s_4, s_{14}\}$
s_8	$\{s_0, s_{10}, s_{12}\}$
s_{10}	$\{s_2, s_8, s_{14}\}$
s_{12}	$\{s_4, s_8, s_{14}\}$
s_{14}	$\{s_6, s_{10}, s_{12}\}$

25.8 F10.8 dimension 6, toggles (0, 1, 4), target 28, mode 1

Coordinates outside the toggle mask 19 remain zero. Every assignment of the permitted coordinates is reachable by toggling its nonzero bits once.

The concrete carrier has 64 states and 192 edges. Its reachable set has 8 states. The reachable set is disjoint from the bad predicate.

ID	State	Seed	Bd.	d	p	Successors
0	0	0	I	0	–	$\{1, 2, 16\}$
1	1	1	–	1	0	$\{0, 3, 17\}$
2	2	2	–	1	0	$\{0, 3, 18\}$
3	3	3	–	2	1	$\{1, 2, 19\}$
4	4	0	–	∞	–	$\{5, 6, 20\}$
5	5	1	–	∞	–	$\{4, 7, 21\}$
6	6	2	–	∞	–	$\{4, 7, 22\}$
7	7	3	–	∞	–	$\{5, 6, 23\}$
8	8	0	–	∞	–	$\{9, 10, 24\}$
9	9	1	–	∞	–	$\{8, 11, 25\}$
10	10	2	–	∞	–	$\{8, 11, 26\}$
11	11	3	–	∞	–	$\{9, 10, 27\}$
12	12	0	–	∞	–	$\{13, 14, 28\}$

ID	State	Seed	Bd.	d	p	Successors
13	13	1	–	∞	–	$\{12, 15, 29\}$
14	14	2	–	∞	–	$\{12, 15, 30\}$
15	15	3	–	∞	–	$\{13, 14, 31\}$
16	16	0	–	1	0	$\{0, 17, 18\}$
17	17	1	–	2	1	$\{1, 16, 19\}$
18	18	2	–	2	2	$\{2, 16, 19\}$
19	19	3	–	3	3	$\{3, 17, 18\}$
20	20	0	–	∞	–	$\{4, 21, 22\}$
21	21	1	–	∞	–	$\{5, 20, 23\}$
22	22	2	–	∞	–	$\{6, 20, 23\}$
23	23	3	–	∞	–	$\{7, 21, 22\}$
24	24	0	–	∞	–	$\{8, 25, 26\}$
25	25	1	–	∞	–	$\{9, 24, 27\}$

ID	State	Seed	Bd.	d	p	Successors
26	26	2	–	∞	–	$\{10, 24, 27\}$
27	27	3	–	∞	–	$\{11, 25, 26\}$
28	28	0	B	∞	–	$\{12, 29, 30\}$
29	29	1	–	∞	–	$\{13, 28, 31\}$
30	30	2	–	∞	–	$\{14, 28, 31\}$
31	31	3	–	∞	–	$\{15, 29, 30\}$
32	32	0	–	∞	–	$\{33, 34, 48\}$
33	33	1	–	∞	–	$\{32, 35, 49\}$
34	34	2	–	∞	–	$\{32, 35, 50\}$
35	35	3	–	∞	–	$\{33, 34, 51\}$
36	36	0	–	∞	–	$\{37, 38, 52\}$
37	37	1	–	∞	–	$\{36, 39, 53\}$
38	38	2	–	∞	–	$\{36, 39, 54\}$

ID	State	Seed	Bd.	d	p	Successors
39	39	3	–	∞	–	$\{37, 38, 55\}$
40	40	0	–	∞	–	$\{41, 42, 56\}$
41	41	1	–	∞	–	$\{40, 43, 57\}$
42	42	2	–	∞	–	$\{40, 43, 58\}$
43	43	3	–	∞	–	$\{41, 42, 59\}$
44	44	0	–	∞	–	$\{45, 46, 60\}$
45	45	1	–	∞	–	$\{44, 47, 61\}$
46	46	2	–	∞	–	$\{44, 47, 62\}$
47	47	3	–	∞	–	$\{45, 46, 63\}$
48	48	0	–	∞	–	$\{32, 49, 50\}$
49	49	1	–	∞	–	$\{33, 48, 51\}$
50	50	2	–	∞	–	$\{34, 48, 51\}$
51	51	3	–	∞	–	$\{35, 49, 50\}$

ID	State	Seed	Bd.	d	p	Successors
52	52	0	–	∞	–	$\{36, 53, 54\}$
53	53	1	–	∞	–	$\{37, 52, 55\}$
54	54	2	–	∞	–	$\{38, 52, 55\}$
55	55	3	–	∞	–	$\{39, 53, 54\}$
56	56	0	–	∞	–	$\{40, 57, 58\}$
57	57	1	–	∞	–	$\{41, 56, 59\}$
58	58	2	–	∞	–	$\{42, 56, 59\}$
59	59	3	–	∞	–	$\{43, 57, 58\}$
60	60	0	–	∞	–	$\{44, 61, 62\}$
61	61	1	–	∞	–	$\{45, 60, 63\}$
62	62	2	–	∞	–	$\{46, 60, 63\}$
63	63	3	–	∞	–	$\{47, 61, 62\}$

Round 1. The partition has 4 nonempty cells, and the may relation has 12 edges.

The selected path is b_0. The exact number of concrete lifts is 0.

j	a_j	F_j	D_j	$\#\cap$
0	b_0	$\{0\}$	$\{28\}$	0

The isolated state IDs are $\{0, 4, 8, 12, 16, 20, 24, 28, 32, 36, 40, 44, 48, 52, 56, 60\}$. The refined graph has zero paths projecting to this candidate.

Round 2. The partition has 19 nonempty cells, and the may relation has 87 edges.

The selected path is $s_0, s_{16}, b_1, s_{12}, s_{28}$. The exact number of concrete lifts is 0.

j	a_j	F_j	D_j	$\#\cap$
0	s_0	$\{0\}$	$\varnothing$	0
1	s_{16}	$\{16\}$	$\varnothing$	0
2	b_1	$\{17\}$	$\{13\}$	0
3	s_{12}	$\varnothing$	$\{12\}$	0
4	s_{28}	$\varnothing$	$\{28\}$	0

The isolated state IDs are $\{1, 5, 9, 13, 17, 21, 25, 29, 33, 37, 41, 45, 49, 53, 57, 61\}$. The refined graph has zero paths projecting to this candidate.

Round 3. The partition has 34 nonempty cells, and the may relation has 132 edges.

The selected path is $s_0, s_{16}, b_2, s_{12}, s_{28}$. The exact number of concrete lifts is 0.

j	a_j	F_j	D_j	$\#\cap$
0	s_0	$\{0\}$	$\varnothing$	0
1	s_{16}	$\{16\}$	$\varnothing$	0
2	b_2	$\{18\}$	$\{14\}$	0
3	s_{12}	$\varnothing$	$\{12\}$	0
4	s_{28}	$\varnothing$	$\{28\}$	0

The isolated state IDs are $\{2, 6, 10, 14, 18, 22, 26, 30, 34, 38, 42, 46, 50, 54, 58, 62\}$. The refined graph has zero paths projecting to this candidate.

Round 4. The partition has 49 nonempty cells, and the may relation has 177 edges.

The selected path is $s_0, s_{16}, s_{18}, b_3, s_{13}, s_{12}, s_{28}$. The exact number of concrete lifts is 0.

j	a_j	F_j	D_j	$\#\cap$
0	s_0	$\{0\}$	$\varnothing$	0
1	s_{16}	$\{16\}$	$\varnothing$	0
2	s_{18}	$\{18\}$	$\varnothing$	0
3	b_3	$\{19\}$	$\{15\}$	0
4	s_{13}	$\varnothing$	$\{13\}$	0
5	s_{12}	$\varnothing$	$\{12\}$	0
6	s_{28}	$\varnothing$	$\{28\}$	0

The isolated state IDs are $\{3, 7, 11, 15, 19, 23, 27, 31, 35, 39, 43, 47, 51, 55, 59, 63\}$. The refined graph has zero paths projecting to this candidate.

Round 5. The partition has 64 nonempty cells, and the may relation has 192 edges.

The abstract checker found the inductive invariant $\{s_0, s_1, s_2, s_3, s_{16}, s_{17}, s_{18}, s_{19}\}$. It contains $\{s_0\}$ and is disjoint from $\{s_{28}\}$.

Cell	All may successors
s_0	$\{s_1, s_2, s_{16}\}$
s_1	$\{s_0, s_3, s_{17}\}$
s_2	$\{s_0, s_3, s_{18}\}$
s_3	$\{s_1, s_2, s_{19}\}$
s_{16}	$\{s_0, s_{17}, s_{18}\}$
s_{17}	$\{s_1, s_{16}, s_{19}\}$
s_{18}	$\{s_2, s_{16}, s_{19}\}$
s_{19}	$\{s_3, s_{17}, s_{18}\}$

26 Cyclic rendezvous

One global tick advances both clocks, whose periods can differ. The Chinese remainder condition predicts the exact concrete reachability result.

Tables enumerate concrete states by integer ID. The d column gives distance from an initial state, and p gives a predecessor attaining that distance. Columns I and B are encoded by the boundary field. Abstract keys b_i are seed blocks and s_x are isolated states.

26.1 F11.1 periods 3 and 4, target (2, 1)

A time t has residues (t mod 3, t mod 4). The two target congruences are compatible exactly when their residues agree modulo 1. The sum-mod-five observation can combine incompatible positions.

The concrete carrier has 12 states and 12 edges. Its reachable set has 12 states. The bad predicate is reached.

ID	State	Seed	Bd.	d	p	Successors
0	(0, 0)	0	I	0	–	{5}
1	(0, 1)	1	–	9	8	{6}
2	(0, 2)	2	–	6	9	{7}
3	(0, 3)	3	–	3	10	{4}
4	(1, 0)	1	–	4	3	{9}
5	(1, 1)	2	–	1	0	{10}
6	(1, 2)	3	–	10	1	{11}
7	(1, 3)	4	–	7	2	{8}
8	(2, 0)	2	–	8	7	{1}
9	(2, 1)	3	B	5	4	{2}
10	(2, 2)	4	–	2	5	{3}
11	(2, 3)	0	–	11	6	{0}

Round 1. The partition has 5 nonempty cells, and the may relation has 10 edges.

The selected path is b_0, b_2, b_1, b_3. The exact number of concrete lifts is 0.

j	a_j	F_j	D_j	$\#\cap$
0	b_0	{0}	$\varnothing$	0
1	b_2	{5}	$\varnothing$	0
2	b_1	$\varnothing$	{4}	0
3	b_3	$\varnothing$	{9}	0

The isolated state IDs are $\{0, 1, 2, 3, 4, 5, 6, 8, 9, 11\}$. The refined graph has zero paths projecting to this candidate.

Round 2. The partition has 11 nonempty cells, and the may relation has 12 edges.

The selected path is $s_0, s_5, b_4, s_3, s_4, s_9$. The exact number of concrete lifts is 1.

j	a_j	F_j	D_j	$\#\cap$
0	s_0	{0}	{0}	1
1	s_5	{5}	{5}	1
2	b_4	{10}	{10}	1
3	s_3	{3}	{3}	1
4	s_4	{4}	{4}	1
5	s_9	{9}	{9}	1

A concrete witness is $0, 5, 10, 3, 4, 9$. Each adjacent pair occurs in the concrete successor table.

26.2 F11.2 periods 4 and 6, target (1, 2)

A time t has residues (t mod 4, t mod 6). The two target congruences are compatible exactly when their residues agree modulo 2. The sum-mod-five observation can combine incompatible positions.

The concrete carrier has 24 states and 24 edges. Its reachable set has 12 states. The reachable set is disjoint from the bad predicate.

ID	State	Seed	Bd.	d	p	Successors
0	(0, 0)	0	I	0	–	{7}
1	(0, 1)	1	–	∞	–	{8}
2	(0, 2)	2	–	8	19	{9}
3	(0, 3)	3	–	∞	–	{10}
4	(0, 4)	4	–	4	21	{11}
5	(0, 5)	0	–	∞	–	{6}
6	(1, 0)	1	–	∞	–	{13}
7	(1, 1)	2	–	1	0	{14}
8	(1, 2)	3	B	∞	–	{15}
9	(1, 3)	4	–	9	2	{16}
10	(1, 4)	0	–	∞	–	{17}
11	(1, 5)	1	–	5	4	{12}
12	(2, 0)	2	–	6	11	{19}

ID	State	Seed	Bd.	d	p	Successors
13	(2, 1)	3	–	∞	–	{20}
14	(2, 2)	4	–	2	7	{21}
15	(2, 3)	0	–	∞	–	{22}
16	(2, 4)	1	–	10	9	{23}
17	(2, 5)	2	–	∞	–	{18}
18	(3, 0)	3	–	∞	–	{1}
19	(3, 1)	4	–	7	12	{2}
20	(3, 2)	0	–	∞	–	{3}
21	(3, 3)	1	–	3	14	{4}
22	(3, 4)	2	–	∞	–	{5}
23	(3, 5)	3	–	11	16	{0}

Round 1. The partition has 5 nonempty cells, and the may relation has 13 edges.

The selected path is b_0, b_1, b_2, b_3. The exact number of concrete lifts is 0.

j	a_j	F_j	D_j	$\#\cap$
0	b_0	$\{0\}$	$\varnothing$	0
1	b_1	$\varnothing$	$\varnothing$	0
2	b_2	$\varnothing$	$\varnothing$	0
3	b_3	$\varnothing$	$\{8\}$	0

The isolated state IDs are $\{0, 1, 2, 3, 5, 6, 7, 8, 10, 11, 12, 13, 15, 16, 17, 18, 20, 21, 22, 23\}$. The refined graph has zero paths projecting to this candidate.

Round 2. The partition has 21 nonempty cells, and the may relation has 24 edges.

The abstract checker found the inductive invariant $\{s_0, s_2, s_7, s_{11}, s_{12}, s_{16}, s_{21}, s_{23}, b_4\}$. It contains $\{s_0\}$ and is disjoint from $\{s_8\}$.

Cell	All may successors
s_0	$\{s_7\}$
s_2	$\{b_4\}$
s_7	$\{b_4\}$
s_{11}	$\{s_{12}\}$
s_{12}	$\{b_4\}$
s_{16}	$\{s_{23}\}$
s_{21}	$\{b_4\}$
s_{23}	$\{s_0\}$
b_4	$\{s_2, s_{11}, s_{16}, s_{21}\}$

26.3 F11.3 periods 4 and 8, target (3, 7)

A time t has residues (t mod 4, t mod 8). The two target congruences are compatible exactly when their residues agree modulo 4. The sum-mod-five observation can combine incompatible positions.

The concrete carrier has 32 states and 32 edges. Its reachable set has 8 states. The bad predicate is reached.

ID	State	Seed	Bd.	d	p	Successors
0	(0, 0)	0	I	0	–	{9}
1	(0, 1)	1	–	∞	–	{10}
2	(0, 2)	2	–	∞	–	{11}
3	(0, 3)	3	–	∞	–	{12}
4	(0, 4)	4	–	4	27	{13}
5	(0, 5)	0	–	∞	–	{14}
6	(0, 6)	1	–	∞	–	{15}
7	(0, 7)	2	–	∞	–	{8}
8	(1, 0)	1	–	∞	–	{17}
9	(1, 1)	2	–	1	0	{18}
10	(1, 2)	3	–	∞	–	{19}
11	(1, 3)	4	–	∞	–	{20}
12	(1, 4)	0	–	∞	–	{21}
13	(1, 5)	1	–	5	4	{22}
14	(1, 6)	2	–	∞	–	{23}
15	(1, 7)	3	–	∞	–	{16}
16	(2, 0)	2	–	∞	–	{25}
17	(2, 1)	3	–	∞	–	{26}
18	(2, 2)	4	–	2	9	{27}
19	(2, 3)	0	–	∞	–	{28}
20	(2, 4)	1	–	∞	–	{29}
21	(2, 5)	2	–	∞	–	{30}
22	(2, 6)	3	–	6	13	{31}
23	(2, 7)	4	–	∞	–	{24}
24	(3, 0)	3	–	∞	–	{1}
25	(3, 1)	4	–	∞	–	{2}
26	(3, 2)	0	–	∞	–	{3}
27	(3, 3)	1	–	3	18	{4}
28	(3, 4)	2	–	∞	–	{5}
29	(3, 5)	3	–	∞	–	{6}
30	(3, 6)	4	–	∞	–	{7}
31	(3, 7)	0	B	7	22	{0}

Round 1. The partition has 5 nonempty cells, and the may relation has 14 edges.

The selected path is b_0. The exact number of concrete lifts is 0.

j	a_j	F_j	D_j	$\#\cap$
0	b_0	$\{0\}$	$\{31\}$	0

The isolated state IDs are $\{0, 5, 12, 19, 26, 31\}$. The refined graph has zero paths projecting to this candidate.

Round 2. The partition has 10 nonempty cells, and the may relation has 20 edges.

The selected path is $s_0, b_2, b_1, b_3, s_{31}$. The exact number of concrete lifts is 0.

j	a_j	F_j	D_j	$\#\cap$
0	s_0	$\{0\}$	$\varnothing$	0
1	b_2	$\{9\}$	$\varnothing$	0
2	b_1	$\varnothing$	$\{13\}$	0
3	b_3	$\varnothing$	$\{22\}$	0
4	s_{31}	$\varnothing$	$\{31\}$	0

The isolated state IDs are $\{1, 2, 3, 6, 7, 8, 9, 10, 13, 14, 15, 16, 17, 20, 21, 22, 24, 27, 28, 29\}$. The refined graph has zero paths projecting to this candidate.

Round 3. The partition has 27 nonempty cells, and the may relation has 32 edges.

The selected path is $s_0, s_9, b_4, s_{13}, s_{22}, s_{31}$. The exact number of concrete lifts is 0.

j	a_j	F_j	D_j	$\#\cap$
0	s_0	$\{0\}$	$\varnothing$	0
1	s_9	$\{9\}$	$\varnothing$	0
2	b_4	$\{18\}$	$\{4\}$	0
3	s_{13}	$\varnothing$	$\{13\}$	0
4	s_{22}	$\varnothing$	$\{22\}$	0
5	s_{31}	$\varnothing$	$\{31\}$	0

The isolated state IDs are $\{4, 11, 18, 23, 25, 30\}$. The refined graph has zero paths projecting to this candidate.

Round 4. The partition has 32 nonempty cells, and the may relation has 32 edges.

The selected path is $s_0, s_9, s_{18}, s_{27}, s_4, s_{13}, s_{22}, s_{31}$. The exact number of concrete lifts is 1.

j	a_j	F_j	D_j	$\#\cap$
0	s_0	$\{0\}$	$\{0\}$	1
1	s_9	$\{9\}$	$\{9\}$	1
2	s_{18}	$\{18\}$	$\{18\}$	1
3	s_{27}	$\{27\}$	$\{27\}$	1
4	s_4	$\{4\}$	$\{4\}$	1
5	s_{13}	$\{13\}$	$\{13\}$	1
6	s_{22}	$\{22\}$	$\{22\}$	1
7	s_{31}	$\{31\}$	$\{31\}$	1

A concrete witness is $0, 9, 18, 27, 4, 13, 22, 31$. Each adjacent pair occurs in the concrete successor table.

26.4 F11.4 periods 5 and 7, target (4, 6)

A time t has residues (t mod 5, t mod 7). The two target congruences are compatible exactly when their residues agree modulo 1. The sum-mod-five observation can combine incompatible positions.

The concrete carrier has 35 states and 35 edges. Its reachable set has 35 states. The bad predicate is reached.

ID	State	Seed	Bd.	d	p	Successors
0	(0, 0)	0	I	0	–	{8}
1	(0, 1)	1	–	15	28	{9}
2	(0, 2)	2	–	30	29	{10}
3	(0, 3)	3	–	10	30	{11}
4	(0, 4)	4	–	25	31	{12}
5	(0, 5)	0	–	5	32	{13}
6	(0, 6)	1	–	20	33	{7}
7	(1, 0)	1	–	21	6	{15}
8	(1, 1)	2	–	1	0	{16}
9	(1, 2)	3	–	16	1	{17}
10	(1, 3)	4	–	31	2	{18}
11	(1, 4)	0	–	11	3	{19}
12	(1, 5)	1	–	26	4	{20}
13	(1, 6)	2	–	6	5	{14}
14	(2, 0)	2	–	7	13	{22}
15	(2, 1)	3	–	22	7	{23}
16	(2, 2)	4	–	2	8	{24}
17	(2, 3)	0	–	17	9	{25}
18	(2, 4)	1	–	32	10	{26}
19	(2, 5)	2	–	12	11	{27}
20	(2, 6)	3	–	27	12	{21}
21	(3, 0)	3	–	28	20	{29}
22	(3, 1)	4	–	8	14	{30}
23	(3, 2)	0	–	23	15	{31}
24	(3, 3)	1	–	3	16	{32}
25	(3, 4)	2	–	18	17	{33}
26	(3, 5)	3	–	33	18	{34}
27	(3, 6)	4	–	13	19	{28}
28	(4, 0)	4	–	14	27	{1}
29	(4, 1)	0	–	29	21	{2}
30	(4, 2)	1	–	9	22	{3}
31	(4, 3)	2	–	24	23	{4}
32	(4, 4)	3	–	4	24	{5}
33	(4, 5)	4	–	19	25	{6}
34	(4, 6)	0	B	34	26	{0}

Round 1. The partition has 5 nonempty cells, and the may relation has 10 edges.

The selected path is b_0. The exact number of concrete lifts is 0.

j	a_j	F_j	D_j	$\#\cap$
0	b_0	$\{0\}$	$\{34\}$	0

The isolated state IDs are $\{0, 5, 11, 17, 23, 29, 34\}$. The refined graph has zero paths projecting to this candidate.

Round 2. The partition has 11 nonempty cells, and the may relation has 20 edges.

The selected path is $s_0, b_2, b_4, b_1, b_3, s_{34}$. The exact number of concrete lifts is 0.

j	a_j	F_j	D_j	$\#\cap$
0	s_0	$\{0\}$	$\varnothing$	0
1	b_2	$\{8\}$	$\{2\}$	0
2	b_4	$\{16\}$	$\{10\}$	0
3	b_1	$\{24\}$	$\{18\}$	0
4	b_3	$\{32\}$	$\{26\}$	0
5	s_{34}	$\varnothing$	$\{34\}$	0

The isolated state IDs are $\{1, 2, 3, 4, 6, 7, 8, 9, 10, 12, 13, 14, 15, 16, 18, 19, 20, 21, 22, 24, 25, 26, 27, 28, 30, 31, 32, 33\}$. The refined graph has zero paths projecting to this candidate.

Round 3. The partition has 35 nonempty cells, and the may relation has 35 edges.

The selected path is $s_0, s_8, s_{16}, s_{24}, s_{32}, s_5, s_{13}, s_{14}, s_{22}, s_{30}, s_3, s_{11}, s_{19}, s_{27}, s_{28}, s_1, s_9, s_{17}, s_{25}, s_{33}, s_6, s_7, s_{15}, s_{23}, s_{31}, s_4, s_{12}, s_{20}, s_{21}, s_{29}, s_2, s_{10}, s_{18}, s_{26}, s_{34}$. The exact number of concrete lifts is 1.

j	a_j	F_j	D_j	$\#\cap$
0	s_0	{0}	{0}	1
1	s_8	{8}	{8}	1
2	s_{16}	{16}	{16}	1
3	s_{24}	{24}	{24}	1
4	s_{32}	{32}	{32}	1
5	s_5	{5}	{5}	1
6	s_{13}	{13}	{13}	1
7	s_{14}	{14}	{14}	1
8	s_{22}	{22}	{22}	1
9	s_{30}	{30}	{30}	1
10	s_3	{3}	{3}	1
11	s_{11}	{11}	{11}	1
12	s_{19}	{19}	{19}	1

j	a_j	F_j	D_j	$\#\cap$
13	s_{27}	{27}	{27}	1
14	s_{28}	{28}	{28}	1
15	s_1	{1}	{1}	1
16	s_9	{9}	{9}	1
17	s_{17}	{17}	{17}	1
18	s_{25}	{25}	{25}	1
19	s_{33}	{33}	{33}	1
20	s_6	{6}	{6}	1
21	s_7	{7}	{7}	1
22	s_{15}	{15}	{15}	1
23	s_{23}	{23}	{23}	1
24	s_{31}	{31}	{31}	1
25	s_4	{4}	{4}	1

j	a_j	F_j	D_j	$\#\cap$
26	s_{12}	{12}	{12}	1
27	s_{20}	{20}	{20}	1
28	s_{21}	{21}	{21}	1
29	s_{29}	{29}	{29}	1
30	s_2	{2}	{2}	1
31	s_{10}	{10}	{10}	1
32	s_{18}	{18}	{18}	1
33	s_{26}	{26}	{26}	1
34	s_{34}	{34}	{34}	1

A concrete witness is $0, 8, 16, 24, 32, 5, 13, 14, 22, 30, 3, 11, 19, 27, 28, 1, 9, 17, 25, 33, 6, 7, 15, 23, 31, 4, 12, 20, 21, 29, 2, 10, 18, 26, 34$. Each adjacent pair occurs in the concrete successor table.

26.5 F11.5 periods 5 and 10, target (3, 6)

A time t has residues ($t \bmod 5, t \bmod 10$). The two target congruences are compatible exactly when their residues agree modulo 5. The sum-mod-five observation can combine incompatible positions.

The concrete carrier has 50 states and 50 edges. Its reachable set has 10 states. The reachable set is disjoint from the bad predicate.

ID	State	Seed	Bd.	d	p	Successors
0	(0, 0)	0	I	0	–	{11}
1	(0, 1)	1	–	∞	–	{12}
2	(0, 2)	2	–	∞	–	{13}
3	(0, 3)	3	–	∞	–	{14}
4	(0, 4)	4	–	∞	–	{15}
5	(0, 5)	0	–	5	44	{16}
6	(0, 6)	1	–	∞	–	{17}
7	(0, 7)	2	–	∞	–	{18}
8	(0, 8)	3	–	∞	–	{19}
9	(0, 9)	4	–	∞	–	{10}
10	(1, 0)	1	–	∞	–	{21}
11	(1, 1)	2	–	1	0	{22}
12	(1, 2)	3	–	∞	–	{23}

ID	State	Seed	Bd.	d	p	Successors
13	(1, 3)	4	–	∞	–	{24}
14	(1, 4)	0	–	∞	–	{25}
15	(1, 5)	1	–	∞	–	{26}
16	(1, 6)	2	–	6	5	{27}
17	(1, 7)	3	–	∞	–	{28}
18	(1, 8)	4	–	∞	–	{29}
19	(1, 9)	0	–	∞	–	{20}
20	(2, 0)	2	–	∞	–	{31}
21	(2, 1)	3	–	∞	–	{32}
22	(2, 2)	4	–	2	11	{33}
23	(2, 3)	0	–	∞	–	{34}
24	(2, 4)	1	–	∞	–	{35}
25	(2, 5)	2	–	∞	–	{36}

ID	State	Seed	Bd.	d	p	Successors
26	(2, 6)	3	–	∞	–	{37}
27	(2, 7)	4	–	7	16	{38}
28	(2, 8)	0	–	∞	–	{39}
29	(2, 9)	1	–	∞	–	{30}
30	(3, 0)	3	–	∞	–	{41}
31	(3, 1)	4	–	∞	–	{42}
32	(3, 2)	0	–	∞	–	{43}
33	(3, 3)	1	–	3	22	{44}
34	(3, 4)	2	–	∞	–	{45}
35	(3, 5)	3	–	∞	–	{46}
36	(3, 6)	4	B	∞	–	{47}
37	(3, 7)	0	–	∞	–	{48}
38	(3, 8)	1	–	8	27	{49}

ID	State	Seed	Bd.	d	p	Successors
39	(3, 9)	2	–	∞	–	{40}
40	(4, 0)	4	–	∞	–	{1}
41	(4, 1)	0	–	∞	–	{2}
42	(4, 2)	1	–	∞	–	{3}
43	(4, 3)	2	–	∞	–	{4}
44	(4, 4)	3	–	4	33	{5}
45	(4, 5)	4	–	∞	–	{6}
46	(4, 6)	0	–	∞	–	{7}
47	(4, 7)	1	–	∞	–	{8}
48	(4, 8)	2	–	∞	–	{9}
49	(4, 9)	3	–	9	38	{0}

Round 1. The partition has 5 nonempty cells, and the may relation has 5 edges.

The selected path is b_0, b_2, b_4. The exact number of concrete lifts is 0.

j	a_j	F_j	D_j	$\#\cap$
0	b_0	{0}	{14}	0
1	b_2	{11}	{25}	0
2	b_4	{22}	{36}	0

The isolated state IDs are $\{0, 2, 4, 5, 7, 9, 11, 13, 14, 16, 18, 19, 20, 22, 23, 25, 27, 28, 31, 32, 34, 36, 37, 39, 40, 41, 43, 45, 46, 48\}$. The refined graph has zero paths projecting to this candidate.

Round 2. The partition has 32 nonempty cells, and the may relation has 41 edges.

The selected path is $s_0, s_{11}, s_{22}, b_1, b_3, s_{14}, s_{25}, s_{36}$. The exact number of concrete lifts is 0.

j	a_j	F_j	D_j	$\#\cap$
0	s_0	{0}	$\varnothing$	0
1	s_{11}	{11}	$\varnothing$	0
2	s_{22}	{22}	$\varnothing$	0
3	b_1	{33}	{42}	0
4	b_3	{44}	{3}	0
5	s_{14}	$\varnothing$	{14}	0
6	s_{25}	$\varnothing$	{25}	0
7	s_{36}	$\varnothing$	{36}	0

The isolated state IDs are $\{1, 3, 6, 8, 10, 12, 15, 17, 21, 24, 26, 29, 30, 33, 35, 38, 42, 44, 47, 49\}$. The refined graph has zero paths projecting to this candidate.

Round 3. The partition has 50 nonempty cells, and the may relation has 50 edges.

The abstract checker found the inductive invariant $\{s_0, s_5, s_{11}, s_{16}, s_{22}, s_{27}, s_{33}, s_{38}, s_{44}, s_{49}\}$. It contains $\{s_0\}$ and is disjoint from $\{s_{36}\}$.

Cell	All may successors
s_0	$\{s_{11}\}$
s_5	$\{s_{16}\}$
s_{11}	$\{s_{22}\}$
s_{16}	$\{s_{27}\}$
s_{22}	$\{s_{33}\}$
s_{27}	$\{s_{38}\}$
s_{33}	$\{s_{44}\}$
s_{38}	$\{s_{49}\}$
s_{44}	$\{s_5\}$
s_{49}	$\{s_0\}$

26.6 F11.6 periods 6 and 8, target (5, 6)

A time t has residues $(t \bmod 6, t \bmod 8)$. The two target congruences are compatible exactly when their residues agree modulo 2. The sum-mod-five observation can combine incompatible positions.

The concrete carrier has 48 states and 48 edges. Its reachable set has 24 states. The reachable set is disjoint from the bad predicate.

ID	State	Seed	Bd.	d	p	Successors
0	(0, 0)	0	I	0	–	{9}
1	(0, 1)	1	–	∞	–	{10}
2	(0, 2)	2	–	18	41	{11}
3	(0, 3)	3	–	∞	–	{12}
4	(0, 4)	4	–	12	43	{13}
5	(0, 5)	0	–	∞	–	{14}
6	(0, 6)	1	–	6	45	{15}
7	(0, 7)	2	–	∞	–	{8}
8	(1, 0)	1	–	∞	–	{17}
9	(1, 1)	2	–	1	0	{18}
10	(1, 2)	3	–	∞	–	{19}
11	(1, 3)	4	–	19	2	{20}
12	(1, 4)	0	–	∞	–	{21}

ID	State	Seed	Bd.	d	p	Successors
13	(1, 5)	1	–	13	4	{22}
14	(1, 6)	2	–	∞	–	{23}
15	(1, 7)	3	–	7	6	{16}
16	(2, 0)	2	–	8	15	{25}
17	(2, 1)	3	–	∞	–	{26}
18	(2, 2)	4	–	2	9	{27}
19	(2, 3)	0	–	∞	–	{28}
20	(2, 4)	1	–	20	11	{29}
21	(2, 5)	2	–	∞	–	{30}
22	(2, 6)	3	–	14	13	{31}
23	(2, 7)	4	–	∞	–	{24}
24	(3, 0)	3	–	∞	–	{33}
25	(3, 1)	4	–	9	16	{34}

ID	State	Seed	Bd.	d	p	Successors
26	(3, 2)	0	–	∞	–	{35}
27	(3, 3)	1	–	3	18	{36}
28	(3, 4)	2	–	∞	–	{37}
29	(3, 5)	3	–	21	20	{38}
30	(3, 6)	4	–	∞	–	{39}
31	(3, 7)	0	–	15	22	{32}
32	(4, 0)	4	–	16	31	{41}
33	(4, 1)	0	–	∞	–	{42}
34	(4, 2)	1	–	10	25	{43}
35	(4, 3)	2	–	∞	–	{44}
36	(4, 4)	3	–	4	27	{45}
37	(4, 5)	4	–	∞	–	{46}
38	(4, 6)	0	–	22	29	{47}

ID	State	Seed	Bd.	d	p	Successors
39	(4, 7)	1	–	∞	–	{40}
40	(5, 0)	0	–	∞	–	{1}
41	(5, 1)	1	–	17	32	{2}
42	(5, 2)	2	–	∞	–	{3}
43	(5, 3)	3	–	11	34	{4}
44	(5, 4)	4	–	∞	–	{5}
45	(5, 5)	0	–	5	36	{6}
46	(5, 6)	1	B	∞	–	{7}
47	(5, 7)	2	–	23	38	{0}

Round 1. The partition has 5 nonempty cells, and the may relation has 16 edges.

The selected path is b_0, b_1. The exact number of concrete lifts is 0.

j	a_j	F_j	D_j	$\#\cap$
0	b_0	{0}	$\varnothing$	0
1	b_1	$\varnothing$	{46}	0

The isolated state IDs are $\{0, 1, 5, 6, 8, 12, 13, 19, 20, 26, 27, 31, 33, 34, 38, 39, 40, 41, 45, 46\}$. The refined graph has zero paths projecting to this candidate.

Round 2. The partition has 23 nonempty cells, and the may relation has 42 edges.

The selected path is $s_0, b_2, s_8, b_3, s_{31}, b_4, s_{46}$. The exact number of concrete lifts is 0.

j	a_j	F_j	D_j	$\#\cap$
0	s_0	{0}	$\varnothing$	0
1	b_2	{9}	$\varnothing$	0
2	s_8	$\varnothing$	$\varnothing$	0
3	b_3	$\varnothing$	$\varnothing$	0
4	s_{31}	$\varnothing$	$\varnothing$	0
5	b_4	$\varnothing$	{37}	0
6	s_{46}	$\varnothing$	{46}	0

The isolated state IDs are $\{2, 3, 4, 7, 9, 10, 11, 14, 15, 16, 17, 18, 21, 22, 23, 24, 25, 28, 29, 30, 32, 35, 36, 37, 42, 43, 44, 47\}$. The refined graph has zero paths projecting to this candidate.

Round 3. The partition has 48 nonempty cells, and the may relation has 48 edges.

The abstract checker found the inductive invariant $\{s_0, s_2, s_4, s_6, s_9, s_{11}, s_{13}, s_{15}, s_{16}, s_{18}, s_{20}, s_{22}, s_{25}, s_{27}, s_{29}, s_{31}, s_{32}, s_{34}, s_{36}, s_{38}, s_{41}, s_{43}, s_{45}, s_{47}\}$. It contains $\{s_0\}$ and is disjoint from $\{s_{46}\}$.

Cell	All may successors
s_0	$\{s_9\}$
s_2	$\{s_{11}\}$
s_4	$\{s_{13}\}$
s_6	$\{s_{15}\}$
s_9	$\{s_{18}\}$
s_{11}	$\{s_{20}\}$
s_{13}	$\{s_{22}\}$
s_{15}	$\{s_{16}\}$
s_{16}	$\{s_{25}\}$
s_{18}	$\{s_{27}\}$
s_{20}	$\{s_{29}\}$
s_{22}	$\{s_{31}\}$
s_{25}	$\{s_{34}\}$

Cell	All may successors
s_{27}	$\{s_{36}\}$
s_{29}	$\{s_{38}\}$
s_{31}	$\{s_{32}\}$
s_{32}	$\{s_{41}\}$
s_{34}	$\{s_{43}\}$
s_{36}	$\{s_{45}\}$
s_{38}	$\{s_{47}\}$
s_{41}	$\{s_2\}$
s_{43}	$\{s_4\}$
s_{45}	$\{s_6\}$
s_{47}	$\{s_0\}$

26.7 F11.7 periods 6 and 9, target (4, 7)

A time t has residues $(t \bmod 6, t \bmod 9)$. The two target congruences are compatible exactly when their residues agree modulo 3. The sum-mod-five observation can combine incompatible positions.

The concrete carrier has 54 states and 54 edges. Its reachable set has 18 states. The bad predicate is reached.

ID	State	Seed	Bd.	d	p	Successors
0	(0, 0)	0	I	0	–	{10}
1	(0, 1)	1	–	∞	–	{11}
2	(0, 2)	2	–	∞	–	{12}
3	(0, 3)	3	–	12	47	{13}
4	(0, 4)	4	–	∞	–	{14}
5	(0, 5)	0	–	∞	–	{15}
6	(0, 6)	1	–	6	50	{16}
7	(0, 7)	2	–	∞	–	{17}
8	(0, 8)	3	–	∞	–	{9}
9	(1, 0)	1	–	∞	–	{19}
10	(1, 1)	2	–	1	0	{20}
11	(1, 2)	3	–	∞	–	{21}
12	(1, 3)	4	–	∞	–	{22}

ID	State	Seed	Bd.	d	p	Successors
13	(1, 4)	0	–	13	3	{23}
14	(1, 5)	1	–	∞	–	{24}
15	(1, 6)	2	–	∞	–	{25}
16	(1, 7)	3	–	7	6	{26}
17	(1, 8)	4	–	∞	–	{18}
18	(2, 0)	2	–	∞	–	{28}
19	(2, 1)	3	–	∞	–	{29}
20	(2, 2)	4	–	2	10	{30}
21	(2, 3)	0	–	∞	–	{31}
22	(2, 4)	1	–	∞	–	{32}
23	(2, 5)	2	–	14	13	{33}
24	(2, 6)	3	–	∞	–	{34}
25	(2, 7)	4	–	∞	–	{35}

ID	State	Seed	Bd.	d	p	Successors
26	(2, 8)	0	–	8	16	{27}
27	(3, 0)	3	–	9	26	{37}
28	(3, 1)	4	–	∞	–	{38}
29	(3, 2)	0	–	∞	–	{39}
30	(3, 3)	1	–	3	20	{40}
31	(3, 4)	2	–	∞	–	{41}
32	(3, 5)	3	–	∞	–	{42}
33	(3, 6)	4	–	15	23	{43}
34	(3, 7)	0	–	∞	–	{44}
35	(3, 8)	1	–	∞	–	{36}
36	(4, 0)	4	–	∞	–	{46}
37	(4, 1)	0	–	10	27	{47}
38	(4, 2)	1	–	∞	–	{48}

ID	State	Seed	Bd.	d	p	Successors
39	(4, 3)	2	–	∞	–	{49}
40	(4, 4)	3	–	4	30	{50}
41	(4, 5)	4	–	∞	–	{51}
42	(4, 6)	0	–	∞	–	{52}
43	(4, 7)	1	B	16	33	{53}
44	(4, 8)	2	–	∞	–	{45}
45	(5, 0)	0	–	∞	–	{1}
46	(5, 1)	1	–	∞	–	{2}
47	(5, 2)	2	–	11	37	{3}
48	(5, 3)	3	–	∞	–	{4}
49	(5, 4)	4	–	∞	–	{5}
50	(5, 5)	0	–	5	40	{6}
51	(5, 6)	1	–	∞	–	{7}

ID	State	Seed	Bd.	d	p	Successors
52	(5, 7)	2	–	∞	–	{8}
53	(5, 8)	3	–	17	43	{0}

Round 1. The partition has 5 nonempty cells, and the may relation has 15 edges.

The selected path is b_0, b_1. The exact number of concrete lifts is 0.

j	a_j	F_j	D_j	$\#\cap$
0	b_0	{0}	∅	0
1	b_1	∅	{43}	0

The isolated state IDs are $\{0, 1, 5, 6, 9, 13, 14, 21, 22, 26, 29, 30, 34, 35, 37, 38, 42, 43, 45, 46, 50, 51\}$. The refined graph has zero paths projecting to this candidate.

Round 2. The partition has 25 nonempty cells, and the may relation has 46 edges.

The selected path is $s_0, b_2, s_{45}, s_1, b_3, b_4, s_{43}$. The exact number of concrete lifts is 0.

j	a_j	F_j	D_j	$\#\cap$
0	s_0	{0}	∅	0
1	b_2	{10}	∅	0
2	s_{45}	∅	∅	0
3	s_1	∅	∅	0
4	b_3	∅	∅	0
5	b_4	∅	{33}	0
6	s_{43}	∅	{43}	0

The isolated state IDs are $\{2, 3, 4, 7, 8, 10, 11, 12, 15, 16, 17, 18, 19, 20, 23, 24, 25, 27, 28, 31, 32, 33, 36, 39, 40, 41, 44, 47, 48, 49, 52, 53\}$. The refined graph has zero paths projecting to this candidate.

Round 3. The partition has 54 nonempty cells, and the may relation has 54 edges.

The selected path is $s_0, s_{10}, s_{20}, s_{30}, s_{40}, s_{50}, s_6, s_{16}, s_{26}, s_{27}, s_{37}, s_{47}, s_3, s_{13}, s_{23}, s_{33}, s_{43}$. The exact number of concrete lifts is 1.

j	a_j	F_j	D_j	$\#\cap$
0	s_0	{0}	{0}	1
1	s_{10}	{10}	{10}	1
2	s_{20}	{20}	{20}	1
3	s_{30}	{30}	{30}	1
4	s_{40}	{40}	{40}	1
5	s_{50}	{50}	{50}	1
6	s_6	{6}	{6}	1
7	s_{16}	{16}	{16}	1
8	s_{26}	{26}	{26}	1
9	s_{27}	{27}	{27}	1
10	s_{37}	{37}	{37}	1
11	s_{47}	{47}	{47}	1
12	s_3	{3}	{3}	1

j	a_j	F_j	D_j	$\#\cap$
13	s_{13}	{13}	{13}	1
14	s_{23}	{23}	{23}	1
15	s_{33}	{33}	{33}	1
16	s_{43}	{43}	{43}	1

A concrete witness is $0, 10, 20, 30, 40, 50, 6, 16, 26, 27, 37, 47, 3, 13, 23, 33, 43$. Each adjacent pair occurs in the concrete successor table.

26.8 F11.8 periods 7 and 9, target (6, 8)

A time t has residues $(t \bmod 7, t \bmod 9)$. The two target congruences are compatible exactly when their residues

agree modulo 1. The sum-mod-five observation can combine incompatible positions.

The concrete carrier has 63 states and 63 edges. Its reachable set has 63 states. The bad predicate is reached.

ID	State	Seed	Bd.	d	p	Successors
0	(0, 0)	0	I	0	–	{10}
1	(0, 1)	1	–	28	54	{11}
2	(0, 2)	2	–	56	55	{12}
3	(0, 3)	3	–	21	56	{13}
4	(0, 4)	4	–	49	57	{14}
5	(0, 5)	0	–	14	58	{15}
6	(0, 6)	1	–	42	59	{16}
7	(0, 7)	2	–	7	60	{17}
8	(0, 8)	3	–	35	61	{9}
9	(1, 0)	1	–	36	8	{19}
10	(1, 1)	2	–	1	0	{20}
11	(1, 2)	3	–	29	1	{21}
12	(1, 3)	4	–	57	2	{22}

ID	State	Seed	Bd.	d	p	Successors
13	(1, 4)	0	–	22	3	{23}
14	(1, 5)	1	–	50	4	{24}
15	(1, 6)	2	–	15	5	{25}
16	(1, 7)	3	–	43	6	{26}
17	(1, 8)	4	–	8	7	{18}
18	(2, 0)	2	–	9	17	{28}
19	(2, 1)	3	–	37	9	{29}
20	(2, 2)	4	–	2	10	{30}
21	(2, 3)	0	–	30	11	{31}
22	(2, 4)	1	–	58	12	{32}
23	(2, 5)	2	–	23	13	{33}
24	(2, 6)	3	–	51	14	{34}
25	(2, 7)	4	–	16	15	{35}

ID	State	Seed	Bd.	d	p	Successors
26	(2, 8)	0	–	44	16	{27}
27	(3, 0)	3	–	45	26	{37}
28	(3, 1)	4	–	10	18	{38}
29	(3, 2)	0	–	38	19	{39}
30	(3, 3)	1	–	3	20	{40}
31	(3, 4)	2	–	31	21	{41}
32	(3, 5)	3	–	59	22	{42}
33	(3, 6)	4	–	24	23	{43}
34	(3, 7)	0	–	52	24	{44}
35	(3, 8)	1	–	17	25	{36}
36	(4, 0)	4	–	18	35	{46}
37	(4, 1)	0	–	46	27	{47}
38	(4, 2)	1	–	11	28	{48}

ID	State	Seed	Bd.	d	p	Successors
39	(4, 3)	2	–	39	29	{49}
40	(4, 4)	3	–	4	30	{50}
41	(4, 5)	4	–	32	31	{51}
42	(4, 6)	0	–	60	32	{52}
43	(4, 7)	1	–	25	33	{53}
44	(4, 8)	2	–	53	34	{45}
45	(5, 0)	0	–	54	44	{55}
46	(5, 1)	1	–	19	36	{56}
47	(5, 2)	2	–	47	37	{57}
48	(5, 3)	3	–	12	38	{58}
49	(5, 4)	4	–	40	39	{59}
50	(5, 5)	0	–	5	40	{60}
51	(5, 6)	1	–	33	41	{61}

ID	State	Seed	Bd.	d	p	Successors
52	(5, 7)	2	–	61	42	{62}
53	(5, 8)	3	–	26	43	{54}
54	(6, 0)	1	–	27	53	{1}
55	(6, 1)	2	–	55	45	{2}
56	(6, 2)	3	–	20	46	{3}
57	(6, 3)	4	–	48	47	{4}
58	(6, 4)	0	–	13	48	{5}
59	(6, 5)	1	–	41	49	{6}
60	(6, 6)	2	–	6	50	{7}
61	(6, 7)	3	–	34	51	{8}
62	(6, 8)	4	B	62	52	{0}

Round 1. The partition has 5 nonempty cells, and the may relation has 16 edges.

The selected path is b_0, b_3, b_1, b_4. The exact number of concrete lifts is 0.

j	a_j	F_j	D_j	$\#\cap$
0	b_0	{0}	$\varnothing$	0
1	b_3	$\varnothing$	$\varnothing$	0
2	b_1	$\varnothing$	$\varnothing$	0
3	b_4	$\varnothing$	{62}	0

The isolated state IDs are $\{0, 1, 3, 4, 5, 6, 8, 9, 11, 12, 13, 14, 16, 17, 19, 20, 21, 22, 24, 25, 26, 27, 28, 29, 30, 32, 33, 34, 35, 36, 37, 38, 40, 41, 42, 43, 45, 46, 48, 49, 50, 51, 53, 54, 56, 57, 58, 59, 61, 62\}$. The refined graph has zero paths projecting to this candidate.

Round 2. The partition has 51 nonempty cells, and the may relation has 62 edges.

The selected path is s_0, b_2, s_{62}. The exact number of concrete lifts is 0.

j	a_j	F_j	D_j	$\#\cap$
0	s_0	{0}	$\varnothing$	0
1	b_2	{10}	{52}	0
2	s_{62}	$\varnothing$	{62}	0

The isolated state IDs are $\{2, 7, 10, 15, 18, 23, 31, 39, 44, 47, 52, 55, 60\}$. The refined graph has zero paths projecting to this candidate.

Round 3. The partition has 63 nonempty cells, and the may relation has 63 edges.

The selected path is $s_0, s_{10}, s_{20}, s_{30}, s_{40}, s_{50}, s_{60}, s_7, s_{17}, s_{18}, s_{28}, s_{38}, s_{48}, s_{58}, s_5, s_{15}, s_{25}, s_{35}, s_{36}, s_{46}, s_{56}, s_3, s_{13}, s_{23}, s_{33}, s_{43}, s_{53}, s_{54}, s_1, s_{11}, s_{21}, s_{31}, s_{41}, s_{51}, s_{61}, s_8, s_9, s_{19}, s_{29}, s_{39}, s_{49}, s_{59}, s_6, s_{16}, s_{26}, s_{27}, s_{37}, s_{47}, s_{57}, s_4, s_{14}, s_{24}, s_{34}, s_{44}, s_{45}, s_{55}, s_2, s_{12}, s_{22}, s_{32}, s_{42}, s_{52}, s_{62}$. The exact number of concrete lifts is 1.

j	a_j	F_j	D_j	$\#\cap$
0	s_0	{0}	{0}	1
1	s_{10}	{10}	{10}	1
2	s_{20}	{20}	{20}	1
3	s_{30}	{30}	{30}	1
4	s_{40}	{40}	{40}	1
5	s_{50}	{50}	{50}	1
6	s_{60}	{60}	{60}	1
7	s_7	{7}	{7}	1
8	s_{17}	{17}	{17}	1
9	s_{18}	{18}	{18}	1
10	s_{28}	{28}	{28}	1
11	s_{38}	{38}	{38}	1
12	s_{48}	{48}	{48}	1

j	a_j	F_j	D_j	$\#\cap$
13	s_{58}	{58}	{58}	1
14	s_5	{5}	{5}	1
15	s_{15}	{15}	{15}	1
16	s_{25}	{25}	{25}	1
17	s_{35}	{35}	{35}	1
18	s_{36}	{36}	{36}	1
19	s_{46}	{46}	{46}	1
20	s_{56}	{56}	{56}	1
21	s_3	{3}	{3}	1
22	s_{13}	{13}	{13}	1
23	s_{23}	{23}	{23}	1
24	s_{33}	{33}	{33}	1
25	s_{43}	{43}	{43}	1

j	a_j	F_j	D_j	$\#\cap$
26	s_{53}	{53}	{53}	1
27	s_{54}	{54}	{54}	1
28	s_1	{1}	{1}	1
29	s_{11}	{11}	{11}	1
30	s_{21}	{21}	{21}	1
31	s_{31}	{31}	{31}	1
32	s_{41}	{41}	{41}	1
33	s_{51}	{51}	{51}	1
34	s_{61}	{61}	{61}	1
35	s_8	{8}	{8}	1
36	s_9	{9}	{9}	1
37	s_{19}	{19}	{19}	1
38	s_{29}	{29}	{29}	1

j	a_j	F_j	D_j	$\#\cap$
39	s_{39}	{39}	{39}	1
40	s_{49}	{49}	{49}	1
41	s_{59}	{59}	{59}	1
42	s_6	{6}	{6}	1
43	s_{16}	{16}	{16}	1
44	s_{26}	{26}	{26}	1
45	s_{27}	{27}	{27}	1
46	s_{37}	{37}	{37}	1
47	s_{47}	{47}	{47}	1
48	s_{57}	{57}	{57}	1
49	s_4	{4}	{4}	1
50	s_{14}	{14}	{14}	1
51	s_{24}	{24}	{24}	1

j	a_j	F_j	D_j	$\#\cap$
52	s_{34}	{34}	{34}	1
53	s_{44}	{44}	{44}	1
54	s_{45}	{45}	{45}	1
55	s_{55}	{55}	{55}	1
56	s_2	{2}	{2}	1
57	s_{12}	{12}	{12}	1
58	s_{22}	{22}	{22}	1
59	s_{32}	{32}	{32}	1
60	s_{42}	{42}	{42}	1
61	s_{52}	{52}	{52}	1
62	s_{62}	{62}	{62}	1

A concrete witness is $0, 10, 20, 30, 40, 50, 60, 7, 17, 18, 28, 38, 48, 58, 5, 15, 25, 35, 36, 46, 56, 3, 13, 23, 33, 43, 53, 54, 1, 11, 21, 31, 41, 51, 61, 8, 9, 19, 29, 39, 49, 59, 6, 16, 26, 27, 37, 47, 57, 4, 14, 24, 34, 44, 45, 55, 2, 12, 22, 32, 42, 52, 62$. Each adjacent pair occurs in the concrete successor table.

27 Multiple starts and branch merges

Each root starts a set of possible walks. A fork adds two to the ordinary shift. Different choices may merge at one endpoint, so the witness count and the reachable set size can diverge. Two observations compare phase-only and phase-with-parity precision.

Tables enumerate concrete states by integer ID. The d column gives distance from an initial state, and p gives a predecessor attaining that distance. Columns I and B are encoded by the boundary field. Abstract keys b_i are seed blocks and s_x are isolated states.

27.1 F12.1 width 5, depth 5, starts (0, 2), target 4, seed mode 0

The ordinary shift is 15. At 2 independent fork layers one may add two more. The exact bad-state test is membership in the union of the translated initial sets.

The concrete carrier has 30 states and 35 edges. Its reachable set has 19 states. The bad predicate is reached.

ID	State	Seed	Bd.	d	p	Successors
0	(0, 0)	0	I	0	–	$\{6, 8\}$
1	(0, 1)	0	–	∞	–	$\{7, 9\}$
2	(0, 2)	0	I	0	–	$\{5, 8\}$
3	(0, 3)	0	–	∞	–	$\{6, 9\}$
4	(0, 4)	0	–	∞	–	$\{5, 7\}$
5	(1, 0)	1	–	1	2	$\{12\}$
6	(1, 1)	1	–	1	0	$\{13\}$
7	(1, 2)	1	–	∞	–	$\{14\}$
8	(1, 3)	1	–	1	0	$\{10\}$
9	(1, 4)	1	–	∞	–	$\{11\}$
10	(2, 0)	2	–	2	8	$\{18\}$
11	(2, 1)	2	–	∞	–	$\{19\}$
12	(2, 2)	2	–	2	5	$\{15\}$

ID	State	Seed	Bd.	d	p	Successors
13	(2, 3)	2	–	2	6	$\{16\}$
14	(2, 4)	2	–	∞	–	$\{17\}$
15	(3, 0)	3	–	3	12	$\{21, 24\}$
16	(3, 1)	3	–	3	13	$\{20, 22\}$
17	(3, 2)	3	–	∞	–	$\{21, 23\}$
18	(3, 3)	3	–	3	10	$\{22, 24\}$
19	(3, 4)	3	–	∞	–	$\{20, 23\}$
20	(4, 0)	4	–	4	16	$\{25\}$
21	(4, 1)	4	–	4	15	$\{26\}$
22	(4, 2)	4	–	4	16	$\{27\}$
23	(4, 3)	4	–	∞	–	$\{28\}$
24	(4, 4)	4	–	4	18	$\{29\}$
25	(5, 0)	5	–	5	20	$\varnothing$

ID	State	Seed	Bd.	d	p	Successors
26	(5, 1)	5	–	5	21	$\varnothing$
27	(5, 2)	5	–	5	22	$\varnothing$
28	(5, 3)	5	–	∞	–	$\varnothing$
29	(5, 4)	5	B	5	24	$\varnothing$

Round 1. The partition has 6 nonempty cells, and the may relation has 5 edges.

The selected path is $b_0, b_1, b_2, b_3, b_4, b_5$. The exact number of concrete lifts is 3.

j	a_j	F_j	D_j	$\#\cap$
0	b_0	$\{0, 2\}$	$\{0, 2, 4\}$	2
1	b_1	$\{5, 6, 8\}$	$\{5, 8\}$	2
2	b_2	$\{10, 12, 13\}$	$\{10, 12\}$	2
3	b_3	$\{15, 16, 18\}$	$\{15, 18\}$	2
4	b_4	$\{20, 21, 22, 24\}$	$\{24\}$	1
5	b_5	$\{25, 26, 27, 29\}$	$\{29\}$	1

A concrete witness is $2, 5, 12, 15, 24, 29$. Each adjacent pair occurs in the concrete successor table.

27.2 F12.2 width 6, depth 6, starts (0, 2), target 2, seed mode 1

The ordinary shift is 21. At 2 independent fork layers one may add two more. The exact bad-state test is membership in the union of the translated initial sets.

The concrete carrier has 42 states and 48 edges. Its reachable set has 20 states. The reachable set is disjoint from the bad predicate.

ID	State	Seed	Bd.	d	p	Successors
0	(0, 0)	0	I	0	–	$\{7, 9\}$
1	(0, 1)	1	–	∞	–	$\{8, 10\}$
2	(0, 2)	0	I	0	–	$\{9, 11\}$
3	(0, 3)	1	–	∞	–	$\{6, 10\}$
4	(0, 4)	0	–	∞	–	$\{7, 11\}$
5	(0, 5)	1	–	∞	–	$\{6, 8\}$
6	(1, 0)	2	–	∞	–	$\{14\}$
7	(1, 1)	3	–	1	0	$\{15\}$
8	(1, 2)	2	–	∞	–	$\{16\}$
9	(1, 3)	3	–	1	0	$\{17\}$
10	(1, 4)	2	–	∞	–	$\{12\}$
11	(1, 5)	3	–	1	2	$\{13\}$
12	(2, 0)	4	–	∞	–	$\{21\}$

ID	State	Seed	Bd.	d	p	Successors
13	(2, 1)	5	–	2	11	$\{22\}$
14	(2, 2)	4	–	∞	–	$\{23\}$
15	(2, 3)	5	–	2	7	$\{18\}$
16	(2, 4)	4	–	∞	–	$\{19\}$
17	(2, 5)	5	–	2	9	$\{20\}$
18	(3, 0)	6	–	3	15	$\{24, 28\}$
19	(3, 1)	7	–	∞	–	$\{25, 29\}$
20	(3, 2)	6	–	3	17	$\{24, 26\}$
21	(3, 3)	7	–	∞	–	$\{25, 27\}$
22	(3, 4)	6	–	3	13	$\{26, 28\}$
23	(3, 5)	7	–	∞	–	$\{27, 29\}$
24	(4, 0)	8	–	4	18	$\{35\}$
25	(4, 1)	9	–	∞	–	$\{30\}$

ID	State	Seed	Bd.	d	p	Successors
26	(4, 2)	8	–	4	20	$\{31\}$
27	(4, 3)	9	–	∞	–	$\{32\}$
28	(4, 4)	8	–	4	18	$\{33\}$
29	(4, 5)	9	–	∞	–	$\{34\}$
30	(5, 0)	10	–	∞	–	$\{36\}$
31	(5, 1)	11	–	5	26	$\{37\}$
32	(5, 2)	10	–	∞	–	$\{38\}$
33	(5, 3)	11	–	5	28	$\{39\}$
34	(5, 4)	10	–	∞	–	$\{40\}$
35	(5, 5)	11	–	5	24	$\{41\}$
36	(6, 0)	12	–	∞	–	$\varnothing$
37	(6, 1)	13	–	6	31	$\varnothing$
38	(6, 2)	12	B	∞	–	$\varnothing$

ID	State	Seed	Bd.	d	p	Successors
39	(6, 3)	13	–	6	33	$\varnothing$
40	(6, 4)	12	–	∞	–	$\varnothing$
41	(6, 5)	13	–	6	35	$\varnothing$

Round 1. The partition has 14 nonempty cells, and the may relation has 12 edges.

The abstract checker found the inductive invariant $\{b_0, b_3, b_5, b_6, b_8, b_{11}, b_{13}\}$. It contains $\{b_0\}$ and is disjoint from $\{b_{12}\}$.

Cell	All may successors
b_0	$\{b_3\}$
b_3	$\{b_5\}$
b_5	$\{b_6\}$
b_6	$\{b_8\}$
b_8	$\{b_{11}\}$
b_{11}	$\{b_{13}\}$
b_{13}	$\varnothing$

27.3 F12.3 width 6, depth 7, starts (0, 4), target 1, seed mode 0

The ordinary shift is 28. At 3 independent fork layers one may add two more. The exact bad-state test is membership in the union of the translated initial sets.

The concrete carrier has 48 states and 60 edges. Its reachable set has 23 states. The reachable set is disjoint from the bad predicate.

ID	State	Seed	Bd.	d	p	Successors
0	(0, 0)	0	I	0	–	$\{7, 9\}$
1	(0, 1)	0	–	∞	–	$\{8, 10\}$
2	(0, 2)	0	–	∞	–	$\{9, 11\}$
3	(0, 3)	0	–	∞	–	$\{6, 10\}$
4	(0, 4)	0	I	0	–	$\{7, 11\}$
5	(0, 5)	0	–	∞	–	$\{6, 8\}$
6	(1, 0)	1	–	∞	–	$\{14\}$
7	(1, 1)	1	–	1	0	$\{15\}$
8	(1, 2)	1	–	∞	–	$\{16\}$
9	(1, 3)	1	–	1	0	$\{17\}$
10	(1, 4)	1	–	∞	–	$\{12\}$
11	(1, 5)	1	–	1	4	$\{13\}$
12	(2, 0)	2	–	∞	–	$\{21\}$
13	(2, 1)	2	–	2	11	$\{22\}$
14	(2, 2)	2	–	∞	–	$\{23\}$
15	(2, 3)	2	–	2	7	$\{18\}$
16	(2, 4)	2	–	∞	–	$\{19\}$
17	(2, 5)	2	–	2	9	$\{20\}$
18	(3, 0)	3	–	3	15	$\{24, 28\}$
19	(3, 1)	3	–	∞	–	$\{25, 29\}$
20	(3, 2)	3	–	3	17	$\{24, 26\}$
21	(3, 3)	3	–	∞	–	$\{25, 27\}$
22	(3, 4)	3	–	3	13	$\{26, 28\}$
23	(3, 5)	3	–	∞	–	$\{27, 29\}$
24	(4, 0)	4	–	4	18	$\{35\}$
25	(4, 1)	4	–	∞	–	$\{30\}$
26	(4, 2)	4	–	4	20	$\{31\}$
27	(4, 3)	4	–	∞	–	$\{32\}$
28	(4, 4)	4	–	4	18	$\{33\}$
29	(4, 5)	4	–	∞	–	$\{34\}$
30	(5, 0)	5	–	∞	–	$\{36\}$
31	(5, 1)	5	–	5	26	$\{37\}$
32	(5, 2)	5	–	∞	–	$\{38\}$
33	(5, 3)	5	–	5	28	$\{39\}$
34	(5, 4)	5	–	∞	–	$\{40\}$
35	(5, 5)	5	–	5	24	$\{41\}$
36	(6, 0)	6	–	∞	–	$\{43, 45\}$
37	(6, 1)	6	–	6	31	$\{44, 46\}$
38	(6, 2)	6	–	∞	–	$\{45, 47\}$
39	(6, 3)	6	–	6	33	$\{42, 46\}$
40	(6, 4)	6	–	∞	–	$\{43, 47\}$
41	(6, 5)	6	–	6	35	$\{42, 44\}$
42	(7, 0)	7	–	7	41	$\varnothing$
43	(7, 1)	7	B	∞	–	$\varnothing$
44	(7, 2)	7	–	7	41	$\varnothing$
45	(7, 3)	7	–	∞	–	$\varnothing$
46	(7, 4)	7	–	7	39	$\varnothing$
47	(7, 5)	7	–	∞	–	$\varnothing$

Round 1. The partition has 8 nonempty cells, and the may relation has 7 edges.

The selected path is $b_0, b_1, b_2, b_3, b_4, b_5, b_6, b_7$. The exact number of concrete lifts is 0.

j	a_j	F_j	D_j	$\#\cap$
0	b_0	$\{0, 4\}$	$\{1, 3, 5\}$	0
1	b_1	$\{7, 9, 11\}$	$\{6, 8, 10\}$	0
2	b_2	$\{13, 15, 17\}$	$\{12, 14, 16\}$	0
3	b_3	$\{18, 20, 22\}$	$\{19, 21, 23\}$	0
4	b_4	$\{24, 26, 28\}$	$\{25, 29\}$	0
5	b_5	$\{31, 33, 35\}$	$\{30, 34\}$	0
6	b_6	$\{37, 39, 41\}$	$\{36, 40\}$	0
7	b_7	$\{42, 44, 46\}$	$\{43\}$	0

The isolated state IDs are $\{0, 1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12, 13, 14, 15, 16, 17, 18, 19, 20, 21, 22, 23, 24, 25, 26, 27, 28, 29, 30, 31, 32, 33, 34, 35, 36, 37, 38, 39, 40, 41, 42, 43, 44, 45, 46, 47\}$. The refined graph has zero paths projecting to this candidate.

Round 2. The partition has 48 nonempty cells, and the may relation has 60 edges.

The abstract checker found the inductive invariant $\{s_0, s_4, s_7, s_9, s_{11}, s_{13}, s_{15}, s_{17}, s_{18}, s_{20}, s_{22}, s_{24}, s_{26}, s_{28}, s_{31}, s_{33}, s_{35}, s_{37}, s_{39}, s_{41}, s_{42}, s_{44}, s_{46}\}$. It contains $\{s_0, s_4\}$ and is disjoint from $\{s_{43}\}$.

Cell	All may successors
s_0	$\{s_7, s_9\}$
s_4	$\{s_7, s_{11}\}$
s_7	$\{s_{15}\}$
s_9	$\{s_{17}\}$
s_{11}	$\{s_{13}\}$
s_{13}	$\{s_{22}\}$
s_{15}	$\{s_{18}\}$
s_{17}	$\{s_{20}\}$
s_{18}	$\{s_{24}, s_{28}\}$
s_{20}	$\{s_{24}, s_{26}\}$
s_{22}	$\{s_{26}, s_{28}\}$
s_{24}	$\{s_{35}\}$
s_{26}	$\{s_{31}\}$

Cell	All may successors
s_{28}	$\{s_{33}\}$
s_{31}	$\{s_{37}\}$
s_{33}	$\{s_{39}\}$
s_{35}	$\{s_{41}\}$
s_{37}	$\{s_{44}, s_{46}\}$
s_{39}	$\{s_{42}, s_{46}\}$
s_{41}	$\{s_{42}, s_{44}\}$
s_{42}	$\varnothing$
s_{44}	$\varnothing$
s_{46}	$\varnothing$

27.4 F12.4 width 7, depth 6, starts (1, 3), target 2, seed mode 0

The ordinary shift is 21. At 2 independent fork layers one may add two more. The exact bad-state test is membership in the union of the translated initial sets.

The concrete carrier has 49 states and 56 edges. Its reachable set has 23 states. The reachable set is disjoint from the bad predicate.

ID	State	Seed	Bd.	d	p	Successors
0	(0, 0)	0	–	∞	–	$\{8, 10\}$
1	(0, 1)	0	I	0	–	$\{9, 11\}$
2	(0, 2)	0	–	∞	–	$\{10, 12\}$
3	(0, 3)	0	I	0	–	$\{11, 13\}$
4	(0, 4)	0	–	∞	–	$\{7, 12\}$
5	(0, 5)	0	–	∞	–	$\{8, 13\}$
6	(0, 6)	0	–	∞	–	$\{7, 9\}$
7	(1, 0)	1	–	∞	–	$\{16\}$
8	(1, 1)	1	–	∞	–	$\{17\}$
9	(1, 2)	1	–	1	1	$\{18\}$
10	(1, 3)	1	–	∞	–	$\{19\}$
11	(1, 4)	1	–	1	1	$\{20\}$
12	(1, 5)	1	–	∞	–	$\{14\}$
13	(1, 6)	1	–	1	3	$\{15\}$
14	(2, 0)	2	–	∞	–	$\{24\}$
15	(2, 1)	2	–	2	13	$\{25\}$
16	(2, 2)	2	–	∞	–	$\{26\}$
17	(2, 3)	2	–	∞	–	$\{27\}$
18	(2, 4)	2	–	2	9	$\{21\}$
19	(2, 5)	2	–	∞	–	$\{22\}$
20	(2, 6)	2	–	2	11	$\{23\}$
21	(3, 0)	3	–	3	18	$\{32, 34\}$
22	(3, 1)	3	–	∞	–	$\{28, 33\}$
23	(3, 2)	3	–	3	20	$\{29, 34\}$
24	(3, 3)	3	–	∞	–	$\{28, 30\}$
25	(3, 4)	3	–	3	15	$\{29, 31\}$
26	(3, 5)	3	–	∞	–	$\{30, 32\}$
27	(3, 6)	3	–	∞	–	$\{31, 33\}$
28	(4, 0)	4	–	∞	–	$\{40\}$
29	(4, 1)	4	–	4	23	$\{41\}$
30	(4, 2)	4	–	∞	–	$\{35\}$
31	(4, 3)	4	–	4	25	$\{36\}$
32	(4, 4)	4	–	4	21	$\{37\}$
33	(4, 5)	4	–	∞	–	$\{38\}$
34	(4, 6)	4	–	4	21	$\{39\}$
35	(5, 0)	5	–	∞	–	$\{48\}$
36	(5, 1)	5	–	5	31	$\{42\}$
37	(5, 2)	5	–	5	32	$\{43\}$
38	(5, 3)	5	–	∞	–	$\{44\}$
39	(5, 4)	5	–	5	34	$\{45\}$
40	(5, 5)	5	–	∞	–	$\{46\}$
41	(5, 6)	5	–	5	29	$\{47\}$
42	(6, 0)	6	–	6	36	$\varnothing$
43	(6, 1)	6	–	6	37	$\varnothing$
44	(6, 2)	6	B	∞	–	$\varnothing$
45	(6, 3)	6	–	6	39	$\varnothing$
46	(6, 4)	6	–	∞	–	$\varnothing$
47	(6, 5)	6	–	6	41	$\varnothing$
48	(6, 6)	6	–	∞	–	$\varnothing$

Round 1. The partition has 7 nonempty cells, and the may relation has 6 edges.

The selected path is $b_0, b_1, b_2, b_3, b_4, b_5, b_6$. The exact number of concrete lifts is 0.

j	a_j	F_j	D_j	$\#\cap$
0	b_0	$\{1, 3\}$	$\{0, 2, 5\}$	0
1	b_1	$\{9, 11, 13\}$	$\{8, 10\}$	0
2	b_2	$\{15, 18, 20\}$	$\{17, 19\}$	0
3	b_3	$\{21, 23, 25\}$	$\{22, 27\}$	0
4	b_4	$\{29, 31, 32, 34\}$	$\{33\}$	0
5	b_5	$\{36, 37, 39, 41\}$	$\{38\}$	0
6	b_6	$\{42, 43, 45, 47\}$	$\{44\}$	0

The isolated state IDs are $\{0, 1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12, 13, 14, 15, 16, 17, 18, 19, 20, 21, 22, 23, 24, 25, 26,$

27, 28, 29, 30, 31, 32, 33, 34, 35, 36, 37, 38, 39, 40, 41, 42, 43, 44, 45, 46, 47, 48}. The refined graph has zero paths projecting to this candidate.

Round 2. The partition has 49 nonempty cells, and the may relation has 56 edges.

The abstract checker found the inductive invariant $\{s_1, s_3, s_9, s_{11}, s_{13}, s_{15}, s_{18}, s_{20}, s_{21}, s_{23}, s_{25}, s_{29}, s_{31}, s_{32}, s_{34}, s_{36}, s_{37}, s_{39}, s_{41}, s_{42}, s_{43}, s_{45}, s_{47}\}$. It contains $\{s_1, s_3\}$ and is disjoint from $\{s_{44}\}$.

Cell	All may successors
s_1	$\{s_9, s_{11}\}$
s_3	$\{s_{11}, s_{13}\}$
s_9	$\{s_{18}\}$
s_{11}	$\{s_{20}\}$
s_{13}	$\{s_{15}\}$
s_{15}	$\{s_{25}\}$
s_{18}	$\{s_{21}\}$
s_{20}	$\{s_{23}\}$
s_{21}	$\{s_{32}, s_{34}\}$
s_{23}	$\{s_{29}, s_{34}\}$
s_{25}	$\{s_{29}, s_{31}\}$
s_{29}	$\{s_{41}\}$
s_{31}	$\{s_{36}\}$

Cell	All may successors
s_{32}	$\{s_{37}\}$
s_{34}	$\{s_{39}\}$
s_{36}	$\{s_{42}\}$
s_{37}	$\{s_{43}\}$
s_{39}	$\{s_{45}\}$
s_{41}	$\{s_{47}\}$
s_{42}	$\varnothing$
s_{43}	$\varnothing$
s_{45}	$\varnothing$
s_{47}	$\varnothing$

27.5 F12.5 width 8, depth 7, starts (0, 4), target 1, seed mode 1

The ordinary shift is 28. At 3 independent fork layers one may add two more. The exact bad-state test is membership in the union of the translated initial sets.

The concrete carrier has 64 states and 80 edges. Its reachable set has 30 states. The reachable set is disjoint from the bad predicate.

ID	State	Seed	Bd.	d	p	Successors
0	(0, 0)	0	I	0	–	{9, 11}
1	(0, 1)	1	–	∞	–	{10, 12}
2	(0, 2)	0	–	∞	–	{11, 13}
3	(0, 3)	1	–	∞	–	{12, 14}
4	(0, 4)	0	I	0	–	{13, 15}
5	(0, 5)	1	–	∞	–	{8, 14}
6	(0, 6)	0	–	∞	–	{9, 15}
7	(0, 7)	1	–	∞	–	{8, 10}
8	(1, 0)	2	–	∞	–	{18}
9	(1, 1)	3	–	1	0	{19}
10	(1, 2)	2	–	∞	–	{20}
11	(1, 3)	3	–	1	0	{21}
12	(1, 4)	2	–	∞	–	{22}

ID	State	Seed	Bd.	d	p	Successors
13	(1, 5)	3	–	1	4	{23}
14	(1, 6)	2	–	∞	–	{16}
15	(1, 7)	3	–	1	4	{17}
16	(2, 0)	4	–	∞	–	{27}
17	(2, 1)	5	–	2	15	{28}
18	(2, 2)	4	–	∞	–	{29}
19	(2, 3)	5	–	2	9	{30}
20	(2, 4)	4	–	∞	–	{31}
21	(2, 5)	5	–	2	11	{24}
22	(2, 6)	4	–	∞	–	{25}
23	(2, 7)	5	–	2	13	{26}
24	(3, 0)	6	–	3	21	{36, 38}
25	(3, 1)	7	–	∞	–	{37, 39}

ID	State	Seed	Bd.	d	p	Successors
26	(3, 2)	6	–	3	23	{32, 38}
27	(3, 3)	7	–	∞	–	{33, 39}
28	(3, 4)	6	–	3	17	{32, 34}
29	(3, 5)	7	–	∞	–	{33, 35}
30	(3, 6)	6	–	3	19	{34, 36}
31	(3, 7)	7	–	∞	–	{35, 37}
32	(4, 0)	8	–	4	26	{45}
33	(4, 1)	9	–	∞	–	{46}
34	(4, 2)	8	–	4	30	{47}
35	(4, 3)	9	–	∞	–	{40}
36	(4, 4)	8	–	4	30	{41}
37	(4, 5)	9	–	∞	–	{42}
38	(4, 6)	8	–	4	24	{43}

ID	State	Seed	Bd.	d	p	Successors
39	(4, 7)	9	–	∞	–	{44}
40	(5, 0)	10	–	∞	–	{54}
41	(5, 1)	11	–	5	36	{55}
42	(5, 2)	10	–	∞	–	{48}
43	(5, 3)	11	–	5	38	{49}
44	(5, 4)	10	–	∞	–	{50}
45	(5, 5)	11	–	5	32	{51}
46	(5, 6)	10	–	∞	–	{52}
47	(5, 7)	11	–	5	34	{53}
48	(6, 0)	12	–	∞	–	{57, 63}
49	(6, 1)	13	–	6	43	{56, 58}
50	(6, 2)	12	–	∞	–	{57, 59}
51	(6, 3)	13	–	6	45	{58, 60}

ID	State	Seed	Bd.	d	p	Successors
52	(6, 4)	12	–	∞	–	$\{59, 61\}$
53	(6, 5)	13	–	6	47	$\{60, 62\}$
54	(6, 6)	12	–	∞	–	$\{61, 63\}$
55	(6, 7)	13	–	6	41	$\{56, 62\}$
56	(7, 0)	14	–	7	55	$\varnothing$
57	(7, 1)	15	B	∞	–	$\varnothing$
58	(7, 2)	14	–	7	49	$\varnothing$
59	(7, 3)	15	–	∞	–	$\varnothing$
60	(7, 4)	14	–	7	53	$\varnothing$
61	(7, 5)	15	–	∞	–	$\varnothing$
62	(7, 6)	14	–	7	53	$\varnothing$
63	(7, 7)	15	–	∞	–	$\varnothing$

Round 1. The partition has 16 nonempty cells, and the may relation has 14 edges.

The abstract checker found the inductive invariant $\{b_0, b_3, b_5, b_6, b_8, b_{11}, b_{13}, b_{14}\}$. It contains $\{b_0\}$ and is disjoint from $\{b_{15}\}$.

Cell	All may successors
b_0	$\{b_3\}$
b_3	$\{b_5\}$
b_5	$\{b_6\}$
b_6	$\{b_8\}$
b_8	$\{b_{11}\}$
b_{11}	$\{b_{13}\}$
b_{13}	$\{b_{14}\}$
b_{14}	$\varnothing$

27.6 F12.6 width 8, depth 8, starts (2, 4), target 5, seed mode 0

The ordinary shift is 36. At 3 independent fork layers one may add two more. The exact bad-state test is membership in the union of the translated initial sets.

The concrete carrier has 72 states and 88 edges. Its reachable set has 31 states. The reachable set is disjoint from the bad predicate.

ID	State	Seed	Bd.	d	p	Successors
0	(0, 0)	0	–	∞	–	$\{9, 11\}$
1	(0, 1)	0	–	∞	–	$\{10, 12\}$
2	(0, 2)	0	I	0	–	$\{11, 13\}$
3	(0, 3)	0	–	∞	–	$\{12, 14\}$
4	(0, 4)	0	I	0	–	$\{13, 15\}$
5	(0, 5)	0	–	∞	–	$\{8, 14\}$
6	(0, 6)	0	–	∞	–	$\{9, 15\}$
7	(0, 7)	0	–	∞	–	$\{8, 10\}$
8	(1, 0)	1	–	∞	–	$\{18\}$
9	(1, 1)	1	–	∞	–	$\{19\}$
10	(1, 2)	1	–	∞	–	$\{20\}$
11	(1, 3)	1	–	1	2	$\{21\}$
12	(1, 4)	1	–	∞	–	$\{22\}$
13	(1, 5)	1	–	1	2	$\{23\}$
14	(1, 6)	1	–	∞	–	$\{16\}$
15	(1, 7)	1	–	1	4	$\{17\}$
16	(2, 0)	2	–	∞	–	$\{27\}$
17	(2, 1)	2	–	2	15	$\{28\}$
18	(2, 2)	2	–	∞	–	$\{29\}$
19	(2, 3)	2	–	∞	–	$\{30\}$
20	(2, 4)	2	–	∞	–	$\{31\}$
21	(2, 5)	2	–	2	11	$\{24\}$
22	(2, 6)	2	–	∞	–	$\{25\}$
23	(2, 7)	2	–	2	13	$\{26\}$
24	(3, 0)	3	–	3	21	$\{36, 38\}$
25	(3, 1)	3	–	∞	–	$\{37, 39\}$
26	(3, 2)	3	–	3	23	$\{32, 38\}$
27	(3, 3)	3	–	∞	–	$\{33, 39\}$
28	(3, 4)	3	–	3	17	$\{32, 34\}$
29	(3, 5)	3	–	∞	–	$\{33, 35\}$
30	(3, 6)	3	–	∞	–	$\{34, 36\}$
31	(3, 7)	3	–	∞	–	$\{35, 37\}$
32	(4, 0)	4	–	4	26	$\{45\}$
33	(4, 1)	4	–	∞	–	$\{46\}$
34	(4, 2)	4	–	4	28	$\{47\}$
35	(4, 3)	4	–	∞	–	$\{40\}$
36	(4, 4)	4	–	4	24	$\{41\}$
37	(4, 5)	4	–	∞	–	$\{42\}$
38	(4, 6)	4	–	4	24	$\{43\}$
39	(4, 7)	4	–	∞	–	$\{44\}$
40	(5, 0)	5	–	∞	–	$\{54\}$
41	(5, 1)	5	–	5	36	$\{55\}$
42	(5, 2)	5	–	∞	–	$\{48\}$
43	(5, 3)	5	–	5	38	$\{49\}$
44	(5, 4)	5	–	∞	–	$\{50\}$
45	(5, 5)	5	–	5	32	$\{51\}$
46	(5, 6)	5	–	∞	–	$\{52\}$
47	(5, 7)	5	–	5	34	$\{53\}$
48	(6, 0)	6	–	∞	–	$\{57, 63\}$
49	(6, 1)	6	–	6	43	$\{56, 58\}$
50	(6, 2)	6	–	∞	–	$\{57, 59\}$
51	(6, 3)	6	–	6	45	$\{58, 60\}$
52	(6, 4)	6	–	∞	–	$\{59, 61\}$
53	(6, 5)	6	–	6	47	$\{60, 62\}$
54	(6, 6)	6	–	∞	–	$\{61, 63\}$
55	(6, 7)	6	–	6	41	$\{56, 62\}$
56	(7, 0)	7	–	7	55	$\{64\}$
57	(7, 1)	7	–	∞	–	$\{65\}$
58	(7, 2)	7	–	7	49	$\{66\}$
59	(7, 3)	7	–	∞	–	$\{67\}$
60	(7, 4)	7	–	7	51	$\{68\}$
61	(7, 5)	7	–	∞	–	$\{69\}$
62	(7, 6)	7	–	7	55	$\{70\}$
63	(7, 7)	7	–	∞	–	$\{71\}$
64	(8, 0)	8	–	8	56	$\varnothing$

ID	State	Seed	Bd.	d	p	Successors
65	(8, 1)	8	–	∞	–	$\varnothing$
66	(8, 2)	8	–	8	58	$\varnothing$
67	(8, 3)	8	–	∞	–	$\varnothing$
68	(8, 4)	8	–	8	60	$\varnothing$
69	(8, 5)	8	B	∞	–	$\varnothing$
70	(8, 6)	8	–	8	62	$\varnothing$
71	(8, 7)	8	–	∞	–	$\varnothing$

Round 1. The partition has 9 nonempty cells, and the may relation has 8 edges.

The selected path is $b_0, b_1, b_2, b_3, b_4, b_5, b_6, b_7, b_8$. The exact number of concrete lifts is 0.

j	a_j	F_j	D_j	$\#\cap$
0	b_0	$\{2, 4\}$	$\{1, 3, 5, 7\}$	0
1	b_1	$\{11, 13, 15\}$	$\{8, 10, 14\}$	0
2	b_2	$\{17, 21, 23\}$	$\{16, 18, 20\}$	0
3	b_3	$\{24, 26, 28\}$	$\{27, 29, 31\}$	0
4	b_4	$\{32, 34, 36, 38\}$	$\{33, 35\}$	0
5	b_5	$\{41, 43, 45, 47\}$	$\{40, 46\}$	0
6	b_6	$\{49, 51, 53, 55\}$	$\{52, 54\}$	0
7	b_7	$\{56, 58, 60, 62\}$	$\{61\}$	0
8	b_8	$\{64, 66, 68, 70\}$	$\{69\}$	0

The isolated state IDs are $\{0, 1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12, 13, 14, 15, 16, 17, 18, 19, 20, 21, 22, 23, 24, 25, 26, 27, 28, 29, 30, 31, 32, 33, 34, 35, 36, 37, 38, 39, 40, 41, 42, 43, 44, 45, 46, 47, 48, 49, 50, 51, 52, 53, 54, 55, 56, 57, 58, 59, 60, 61, 62, 63, 64, 65, 66, 67, 68, 69, 70, 71\}$. The refined graph has zero paths projecting to this candidate.

Round 2. The partition has 72 nonempty cells, and the may relation has 88 edges.

The abstract checker found the inductive invariant $\{s_2, s_4, s_{11}, s_{13}, s_{15}, s_{17}, s_{21}, s_{23}, s_{24}, s_{26}, s_{28}, s_{32}, s_{34}, s_{36}, s_{38}, s_{41}, s_{43}, s_{45}, s_{47}, s_{49}, s_{51}, s_{53}, s_{55}, s_{56}, s_{58}, s_{60}, s_{62}, s_{64}, s_{66}, s_{68}, s_{70}\}$. It contains $\{s_2, s_4\}$ and is disjoint from $\{s_{69}\}$.

Cell	All may successors
s_2	$\{s_{11}, s_{13}\}$
s_4	$\{s_{13}, s_{15}\}$
s_{11}	$\{s_{21}\}$
s_{13}	$\{s_{23}\}$
s_{15}	$\{s_{17}\}$
s_{17}	$\{s_{28}\}$
s_{21}	$\{s_{24}\}$
s_{23}	$\{s_{26}\}$
s_{24}	$\{s_{36}, s_{38}\}$
s_{26}	$\{s_{32}, s_{38}\}$
s_{28}	$\{s_{32}, s_{34}\}$
s_{32}	$\{s_{45}\}$
s_{34}	$\{s_{47}\}$
s_{36}	$\{s_{41}\}$
s_{38}	$\{s_{43}\}$
s_{41}	$\{s_{55}\}$
s_{43}	$\{s_{49}\}$
s_{45}	$\{s_{51}\}$
s_{47}	$\{s_{53}\}$
s_{49}	$\{s_{56}, s_{58}\}$
s_{51}	$\{s_{58}, s_{60}\}$
s_{53}	$\{s_{60}, s_{62}\}$
s_{55}	$\{s_{56}, s_{62}\}$
s_{56}	$\{s_{64}\}$
s_{58}	$\{s_{66}\}$
s_{60}	$\{s_{68}\}$
s_{62}	$\{s_{70}\}$
s_{64}	$\varnothing$
s_{66}	$\varnothing$
s_{68}	$\varnothing$
s_{70}	$\varnothing$

27.7 F12.7 width 9, depth 6, starts (0, 3), target 5, seed mode 0

The ordinary shift is 21. At 2 independent fork layers one may add two more. The exact bad-state test is membership in the union of the translated initial sets.

The concrete carrier has 63 states and 72 edges. Its reachable set has 32 states. The bad predicate is reached.

ID	State	Seed	Bd.	d	p	Successors
0	(0, 0)	0	I	0	–	$\{10, 12\}$
1	(0, 1)	0	–	∞	–	$\{11, 13\}$
2	(0, 2)	0	–	∞	–	$\{12, 14\}$
3	(0, 3)	0	I	0	–	$\{13, 15\}$
4	(0, 4)	0	–	∞	–	$\{14, 16\}$
5	(0, 5)	0	–	∞	–	$\{15, 17\}$
6	(0, 6)	0	–	∞	–	$\{9, 16\}$
7	(0, 7)	0	–	∞	–	$\{10, 17\}$
8	(0, 8)	0	–	∞	–	$\{9, 11\}$
9	(1, 0)	1	–	∞	–	$\{20\}$
10	(1, 1)	1	–	1	0	$\{21\}$
11	(1, 2)	1	–	∞	–	$\{22\}$
12	(1, 3)	1	–	1	0	$\{23\}$

ID	State	Seed	Bd.	d	p	Successors
13	(1, 4)	1	–	1	3	{24}
14	(1, 5)	1	–	∞	–	{25}
15	(1, 6)	1	–	1	3	{26}
16	(1, 7)	1	–	∞	–	{18}
17	(1, 8)	1	–	∞	–	{19}
18	(2, 0)	2	–	∞	–	{30}
19	(2, 1)	2	–	∞	–	{31}
20	(2, 2)	2	–	∞	–	{32}
21	(2, 3)	2	–	2	10	{33}
22	(2, 4)	2	–	∞	–	{34}
23	(2, 5)	2	–	2	12	{35}
24	(2, 6)	2	–	2	13	{27}
25	(2, 7)	2	–	∞	–	{28}

ID	State	Seed	Bd.	d	p	Successors
26	(2, 8)	2	–	2	15	{29}
27	(3, 0)	3	–	3	24	{40, 42}
28	(3, 1)	3	–	∞	–	{41, 43}
29	(3, 2)	3	–	3	26	{42, 44}
30	(3, 3)	3	–	∞	–	{36, 43}
31	(3, 4)	3	–	∞	–	{37, 44}
32	(3, 5)	3	–	∞	–	{36, 38}
33	(3, 6)	3	–	3	21	{37, 39}
34	(3, 7)	3	–	∞	–	{38, 40}
35	(3, 8)	3	–	3	23	{39, 41}
36	(4, 0)	4	–	∞	–	{50}
37	(4, 1)	4	–	4	33	{51}
38	(4, 2)	4	–	∞	–	{52}

ID	State	Seed	Bd.	d	p	Successors
39	(4, 3)	4	–	4	33	{53}
40	(4, 4)	4	–	4	27	{45}
41	(4, 5)	4	–	4	35	{46}
42	(4, 6)	4	–	4	27	{47}
43	(4, 7)	4	–	∞	–	{48}
44	(4, 8)	4	–	4	29	{49}
45	(5, 0)	5	–	5	40	{60}
46	(5, 1)	5	–	5	41	{61}
47	(5, 2)	5	–	5	42	{62}
48	(5, 3)	5	–	∞	–	{54}
49	(5, 4)	5	–	5	44	{55}
50	(5, 5)	5	–	∞	–	{56}
51	(5, 6)	5	–	5	37	{57}

ID	State	Seed	Bd.	d	p	Successors
52	(5, 7)	5	–	∞	–	{58}
53	(5, 8)	5	–	5	39	{59}
54	(6, 0)	6	–	∞	–	∅
55	(6, 1)	6	–	6	49	∅
56	(6, 2)	6	–	∞	–	∅
57	(6, 3)	6	–	6	51	∅
58	(6, 4)	6	–	∞	–	∅
59	(6, 5)	6	B	6	53	∅
60	(6, 6)	6	–	6	45	∅
61	(6, 7)	6	–	6	46	∅
62	(6, 8)	6	–	6	47	∅

Round 1. The partition has 7 nonempty cells, and the may relation has 6 edges.

The selected path is $b_0, b_1, b_2, b_3, b_4, b_5, b_6$. The exact number of concrete lifts is 2.

j	a_j	F_j	D_j	$\#\cap$
0	b_0	$\{0, 3\}$	$\{0, 2, 7\}$	1
1	b_1	$\{10, 12, 13, 15\}$	$\{10, 12\}$	2
2	b_2	$\{21, 23, 24, 26\}$	$\{21, 23\}$	2
3	b_3	$\{27, 29, 33, 35\}$	$\{33, 35\}$	2
4	b_4	$\{37, 39, 40, 41, 42, 44\}$	$\{39\}$	1
5	b_5	$\{45, 46, 47, 49, 51, 53\}$	$\{53\}$	1
6	b_6	$\{55, 57, 59, 60, 61, 62\}$	$\{59\}$	1

A concrete witness is $0, 10, 21, 33, 39, 53, 59$. Each adjacent pair occurs in the concrete successor table.

27.8 F12.8 width 10, depth 7, starts (1, 7), target 8, seed mode 1

The ordinary shift is 28. At 3 independent fork layers one may add two more. The exact bad-state test is membership in the union of the translated initial sets.

The concrete carrier has 80 states and 100 edges. Its reachable set has 34 states. The reachable set is disjoint from the bad predicate.

ID	State	Seed	Bd.	d	p	Successors
0	(0, 0)	0	–	∞	–	{11, 13}
1	(0, 1)	1	I	0	–	{12, 14}
2	(0, 2)	0	–	∞	–	{13, 15}
3	(0, 3)	1	–	∞	–	{14, 16}
4	(0, 4)	0	–	∞	–	{15, 17}
5	(0, 5)	1	–	∞	–	{16, 18}
6	(0, 6)	0	–	∞	–	{17, 19}
7	(0, 7)	1	I	0	–	{10, 18}
8	(0, 8)	0	–	∞	–	{11, 19}
9	(0, 9)	1	–	∞	–	{10, 12}
10	(1, 0)	2	–	1	7	{22}
11	(1, 1)	3	–	∞	–	{23}
12	(1, 2)	2	–	1	1	{24}

ID	State	Seed	Bd.	d	p	Successors
13	(1, 3)	3	–	∞	–	{25}
14	(1, 4)	2	–	1	1	{26}
15	(1, 5)	3	–	∞	–	{27}
16	(1, 6)	2	–	∞	–	{28}
17	(1, 7)	3	–	∞	–	{29}
18	(1, 8)	2	–	1	7	{20}
19	(1, 9)	3	–	∞	–	{21}
20	(2, 0)	4	–	2	18	{33}
21	(2, 1)	5	–	∞	–	{34}
22	(2, 2)	4	–	2	10	{35}
23	(2, 3)	5	–	∞	–	{36}
24	(2, 4)	4	–	2	12	{37}
25	(2, 5)	5	–	∞	–	{38}

ID	State	Seed	Bd.	d	p	Successors
26	(2, 6)	4	–	2	14	{39}
27	(2, 7)	5	–	∞	–	{30}
28	(2, 8)	4	–	∞	–	{31}
29	(2, 9)	5	–	∞	–	{32}
30	(3, 0)	6	–	∞	–	{44, 46}
31	(3, 1)	7	–	∞	–	{45, 47}
32	(3, 2)	6	–	∞	–	{46, 48}
33	(3, 3)	7	–	3	20	{47, 49}
34	(3, 4)	6	–	∞	–	{40, 48}
35	(3, 5)	7	–	3	22	{41, 49}
36	(3, 6)	6	–	∞	–	{40, 42}
37	(3, 7)	7	–	3	24	{41, 43}
38	(3, 8)	6	–	∞	–	{42, 44}

ID	State	Seed	Bd.	d	p	Successors
39	(3, 9)	7	–	3	26	{43, 45}
40	(4, 0)	8	–	∞	–	{55}
41	(4, 1)	9	–	4	37	{56}
42	(4, 2)	8	–	∞	–	{57}
43	(4, 3)	9	–	4	37	{58}
44	(4, 4)	8	–	∞	–	{59}
45	(4, 5)	9	–	4	39	{50}
46	(4, 6)	8	–	∞	–	{51}
47	(4, 7)	9	–	4	33	{52}
48	(4, 8)	8	–	∞	–	{53}
49	(4, 9)	9	–	4	35	{54}
50	(5, 0)	10	–	5	45	{66}
51	(5, 1)	11	–	∞	–	{67}

ID	State	Seed	Bd.	d	p	Successors
52	(5, 2)	10	–	5	47	{68}
53	(5, 3)	11	–	∞	–	{69}
54	(5, 4)	10	–	5	49	{60}
55	(5, 5)	11	–	∞	–	{61}
56	(5, 6)	10	–	5	41	{62}
57	(5, 7)	11	–	∞	–	{63}
58	(5, 8)	10	–	5	43	{64}
59	(5, 9)	11	–	∞	–	{65}
60	(6, 0)	12	–	6	54	{77, 79}
61	(6, 1)	13	–	∞	–	{70, 78}
62	(6, 2)	12	–	6	56	{71, 79}
63	(6, 3)	13	–	∞	–	{70, 72}
64	(6, 4)	12	–	6	58	{71, 73}

ID	State	Seed	Bd.	d	p	Successors
65	(6, 5)	13	–	∞	–	{72, 74}
66	(6, 6)	12	–	6	50	{73, 75}
67	(6, 7)	13	–	∞	–	{74, 76}
68	(6, 8)	12	–	6	52	{75, 77}
69	(6, 9)	13	–	∞	–	{76, 78}
70	(7, 0)	14	–	∞	–	∅
71	(7, 1)	15	–	7	62	∅
72	(7, 2)	14	–	∞	–	∅
73	(7, 3)	15	–	7	64	∅
74	(7, 4)	14	–	∞	–	∅
75	(7, 5)	15	–	7	66	∅
76	(7, 6)	14	–	∞	–	∅
77	(7, 7)	15	–	7	60	∅

ID	State	Seed	Bd.	d	p	Successors
78	(7, 8)	14	B	∞	–	∅
79	(7, 9)	15	–	7	62	∅

Round 1. The partition has 16 nonempty cells, and the may relation has 14 edges.

The abstract checker found the inductive invariant $\{b_1, b_2, b_4, b_7, b_9, b_{10}, b_{12}, b_{15}\}$. It contains $\{b_1\}$ and is disjoint from $\{b_{14}\}$.

Cell	All may successors
b_1	$\{b_2\}$
b_2	$\{b_4\}$
b_4	$\{b_7\}$
b_7	$\{b_9\}$
b_9	$\{b_{10}\}$
b_{10}	$\{b_{12}\}$
b_{12}	$\{b_{15}\}$
b_{15}	∅

V. Predicate refinement cases

Exact prefix and suffix calculations, layered predicates, and certificate checks for labelled words.

28 Labelled translations

Each phase adds the displacement of the selected label. The bad condition fixes one final residue. The complete transition rows make the two affine endpoint sequences checkable.

A state printed as $i : v$ has phase i and residue v. The successor table prints residues at the next phase. A blank column for another label means there is no such edge at that phase.

28.1 P1.1 Translation modulo 7, length 7, target 6

The selected word determines a unique endpoint 5. The two direction vectors are translates of the initial and bad residues.

The prescribed label sequence is a, b, c, a, b, c, a and its observed state sequence is $0, 1, 2, 3, 4, 5, 6, 7$.

The initial set is $\{0 : 0\}$ and the bad set is $\{7 : 6\}$.

The exact lift count is 0. Allowing any label at each layer of this same observed state sequence gives 0 lifts.

i	Exact prefix	Exact suffix	Join
0	$\{0 : 0\}$	$\{0 : 1\}$	0
1	$\{1 : 1\}$	$\{1 : 2\}$	0
2	$\{2 : 4\}$	$\{2 : 5\}$	0
3	$\{3 : 2\}$	$\{3 : 3\}$	0
4	$\{4 : 3\}$	$\{4 : 4\}$	0
5	$\{5 : 6\}$	$\{5 : 0\}$	0
6	$\{6 : 4\}$	$\{6 : 5\}$	0
7	$\{7 : 5\}$	$\{7 : 6\}$	0

Certificates. The least, extended, and greatest sequences give 16, 16, and 16 realized keys, respectively. All three refined labelled graphs have zero descendants of the selected path.

i	$\lvert A_i\rvert$	$\lvert J_i^{\min}\rvert$	$J_i^{\mathrm{ext}} \setminus J_i^{\min}$	$\lvert J_i^{\max}\rvert$
0	7	1	$\{0 : 2\}$	6
1	7	1	$\{1 : 3\}$	6
2	7	1	$\{2 : 6\}$	6
3	7	1	$\{3 : 4\}$	6
4	7	1	$\{4 : 5\}$	6
5	7	1	$\{5 : 1\}$	6
6	7	1	$\{6 : 6\}$	6
7	7	1	$\{7 : 0\}$	6

Transition rows. Identical successor rows within a phase are grouped when they have the same selected-label and other-label outputs.

Phase 0, label a.

Source residues	Selected targets	Other targets
{0}	{1}	–
{1}	{2}	–
{2}	{3}	–
{3}	{4}	–
{4}	{5}	–
{5}	{6}	–
{6}	{0}	–

Phase 1, label b.

Source residues	Selected targets	Other targets
{0}	{3}	–
{1}	{4}	–
{2}	{5}	–
{3}	{6}	–
{4}	{0}	–
{5}	{1}	–
{6}	{2}	–

Phase 2, label c.

Source residues	Selected targets	Other targets
{0}	{5}	–
{1}	{6}	–
{2}	{0}	–
{3}	{1}	–
{4}	{2}	–
{5}	{3}	–
{6}	{4}	–

Phase 3, label a.

Source residues	Selected targets	Other targets
{0}	{1}	–
{1}	{2}	–
{2}	{3}	–
{3}	{4}	–
{4}	{5}	–
{5}	{6}	–
{6}	{0}	–

Phase 4, label b.

Source residues	Selected targets	Other targets
{0}	{3}	–
{1}	{4}	–
{2}	{5}	–
{3}	{6}	–
{4}	{0}	–
{5}	{1}	–
{6}	{2}	–

Phase 5, label c.

Source residues	Selected targets	Other targets
{0}	{5}	–
{1}	{6}	–
{2}	{0}	–
{3}	{1}	–
{4}	{2}	–
{5}	{3}	–
{6}	{4}	–

Phase 6, label a.

Source residues	Selected targets	Other targets
{0}	{1}	–
{1}	{2}	–
{2}	{3}	–
{3}	{4}	–
{4}	{5}	–
{5}	{6}	–
{6}	{0}	–

28.2 P1.2 Translation modulo 8, length 8, target 0

The selected word determines a unique endpoint 0. The two direction vectors are translates of the initial and bad residues.

The prescribed label sequence is a, b, c, a, b, c, a, b and its observed state sequence is $0, 1, 2, 3, 4, 5, 6, 7, 8$.

The initial set is $\{0 : 0\}$ and the bad set is $\{8 : 0\}$.

The exact lift count is 1. Allowing any label at each layer of this same observed state sequence gives 1 lifts.

i	Exact prefix	Exact suffix	Join
0	$\{0 : 0\}$	$\{0 : 0\}$	1
1	$\{1 : 1\}$	$\{1 : 1\}$	1
2	$\{2 : 4\}$	$\{2 : 4\}$	1
3	$\{3 : 2\}$	$\{3 : 2\}$	1
4	$\{4 : 3\}$	$\{4 : 3\}$	1
5	$\{5 : 6\}$	$\{5 : 6\}$	1
6	$\{6 : 4\}$	$\{6 : 4\}$	1
7	$\{7 : 5\}$	$\{7 : 5\}$	1
8	$\{8 : 0\}$	$\{8 : 0\}$	1

Witness. $0 : 0, 1 : 1, 2 : 4, 3 : 2, 4 : 3, 5 : 6, 6 : 4, 7 : 5, 8 : 0$ follows the selected label at every step.

Transition rows. Identical successor rows within a phase are grouped when they have the same selected-label and other-label outputs.

Phase 0, label a.

Source residues	Selected targets	Other targets
{0}	{1}	–
{1}	{2}	–
{2}	{3}	–
{3}	{4}	–
{4}	{5}	–
{5}	{6}	–
{6}	{7}	–
{7}	{0}	–

Phase 1, label b.

Source residues	Selected targets	Other targets
{0}	{3}	–
{1}	{4}	–
{2}	{5}	–
{3}	{6}	–
{4}	{7}	–
{5}	{0}	–
{6}	{1}	–
{7}	{2}	–

Phase 2, label c.

Source residues	Selected targets	Other targets
{0}	{6}	–
{1}	{7}	–
{2}	{0}	–
{3}	{1}	–
{4}	{2}	–
{5}	{3}	–
{6}	{4}	–
{7}	{5}	–

Phase 3, label a.

Source residues	Selected targets	Other targets
{0}	{1}	–
{1}	{2}	–
{2}	{3}	–
{3}	{4}	–
{4}	{5}	–
{5}	{6}	–
{6}	{7}	–
{7}	{0}	–

Phase 4, label b.

Source residues	Selected targets	Other targets
{0}	{3}	–
{1}	{4}	–
{2}	{5}	–
{3}	{6}	–
{4}	{7}	–
{5}	{0}	–
{6}	{1}	–
{7}	{2}	–

Phase 5, label c.

Source residues	Selected targets	Other targets
{0}	{6}	–
{1}	{7}	–
{2}	{0}	–
{3}	{1}	–
{4}	{2}	–
{5}	{3}	–
{6}	{4}	–
{7}	{5}	–

Phase 6, label a.

Source residues	Selected targets	Other targets
{0}	{1}	–
{1}	{2}	–
{2}	{3}	–
{3}	{4}	–
{4}	{5}	–
{5}	{6}	–
{6}	{7}	–
{7}	{0}	–

Phase 7, label b.

Source residues	Selected targets	Other targets
{0}	{3}	–
{1}	{4}	–
{2}	{5}	–
{3}	{6}	–
{4}	{7}	–
{5}	{0}	–
{6}	{1}	–
{7}	{2}	–

28.3 P1.3 Translation modulo 9, length 10, target 1

The selected word determines a unique endpoint 7. The two direction vectors are translates of the initial and bad residues.

The prescribed label sequence is $a, b, c, a, b, c, a, b, c, a$ and its observed state sequence is $0, 1, 2, 3, 4, 5, 6, 7, 8, 9, 10$.

The initial set is $\{0 : 0\}$ and the bad set is $\{10 : 1\}$.

The exact lift count is 0. Allowing any label at each layer of this same observed state sequence gives 0 lifts.

i	Exact prefix	Exact suffix	Join
0	$\{0 : 0\}$	$\{0 : 3\}$	0
1	$\{1 : 1\}$	$\{1 : 4\}$	0
2	$\{2 : 4\}$	$\{2 : 7\}$	0
3	$\{3 : 2\}$	$\{3 : 5\}$	0
4	$\{4 : 3\}$	$\{4 : 6\}$	0
5	$\{5 : 6\}$	$\{5 : 0\}$	0
6	$\{6 : 4\}$	$\{6 : 7\}$	0
7	$\{7 : 5\}$	$\{7 : 8\}$	0
8	$\{8 : 8\}$	$\{8 : 2\}$	0
9	$\{9 : 6\}$	$\{9 : 0\}$	0
10	$\{10 : 7\}$	$\{10 : 1\}$	0

Certificates. The least, extended, and greatest sequences give 22, 22, and 22 realized keys, respectively. All three refined labelled graphs have zero descendants of the selected path.

i	$\lvert A_i\rvert$	$\lvert J_i^{\min}\rvert$	$J_i^{\mathrm{ext}} \setminus J_i^{\min}$	$\lvert J_i^{\max}\rvert$
0	9	1	$\{0 : 1\}$	8
1	9	1	$\{1 : 2\}$	8
2	9	1	$\{2 : 5\}$	8
3	9	1	$\{3 : 3\}$	8
4	9	1	$\{4 : 4\}$	8
5	9	1	$\{5 : 7\}$	8
6	9	1	$\{6 : 5\}$	8
7	9	1	$\{7 : 6\}$	8
8	9	1	$\{8 : 0\}$	8
9	9	1	$\{9 : 7\}$	8
10	9	1	$\{10 : 8\}$	8

Transition rows. Identical successor rows within a phase are grouped when they have the same selected-label and other-label outputs.

Phase 0, label a.

Source residues	Selected targets	Other targets
{0}	{1}	–
{1}	{2}	–
{2}	{3}	–
{3}	{4}	–
{4}	{5}	–
{5}	{6}	–
{6}	{7}	–
{7}	{8}	–
{8}	{0}	–

Phase 1, label b.

Source residues	Selected targets	Other targets
{0}	{3}	–
{1}	{4}	–
{2}	{5}	–
{3}	{6}	–
{4}	{7}	–
{5}	{8}	–
{6}	{0}	–
{7}	{1}	–
{8}	{2}	–

Phase 2, label c.

Source residues	Selected targets	Other targets
{0}	{7}	–
{1}	{8}	–
{2}	{0}	–
{3}	{1}	–
{4}	{2}	–
{5}	{3}	–
{6}	{4}	–
{7}	{5}	–
{8}	{6}	–

Phase 3, label a.

Source residues	Selected targets	Other targets
{0}	{1}	–
{1}	{2}	–
{2}	{3}	–
{3}	{4}	–
{4}	{5}	–
{5}	{6}	–
{6}	{7}	–
{7}	{8}	–
{8}	{0}	–

Phase 4, label b.

Source residues	Selected targets	Other targets
{0}	{3}	–
{1}	{4}	–
{2}	{5}	–
{3}	{6}	–
{4}	{7}	–
{5}	{8}	–
{6}	{0}	–
{7}	{1}	–
{8}	{2}	–

Phase 5, label c.

Source residues	Selected targets	Other targets
{0}	{7}	–
{1}	{8}	–
{2}	{0}	–
{3}	{1}	–
{4}	{2}	–
{5}	{3}	–
{6}	{4}	–
{7}	{5}	–
{8}	{6}	–

Phase 6, label a.

Source residues	Selected targets	Other targets
{0}	{1}	–
{1}	{2}	–
{2}	{3}	–
{3}	{4}	–
{4}	{5}	–
{5}	{6}	–
{6}	{7}	–
{7}	{8}	–
{8}	{0}	–

Phase 7, label b.

Source residues	Selected targets	Other targets
{0}	{3}	–
{1}	{4}	–
{2}	{5}	–
{3}	{6}	–
{4}	{7}	–
{5}	{8}	–
{6}	{0}	–
{7}	{1}	–
{8}	{2}	–

Phase 8, label c.

Source residues	Selected targets	Other targets
{0}	{7}	–
{1}	{8}	–
{2}	{0}	–
{3}	{1}	–
{4}	{2}	–
{5}	{3}	–
{6}	{4}	–
{7}	{5}	–
{8}	{6}	–

Phase 9, label a.

Source residues	Selected targets	Other targets
{0}	{1}	–
{1}	{2}	–
{2}	{3}	–
{3}	{4}	–
{4}	{5}	–
{5}	{6}	–
{6}	{7}	–
{7}	{8}	–
{8}	{0}	–

28.4 P1.4 Translation modulo 10, length 11, target 1

The selected word determines a unique endpoint 0. The two direction vectors are translates of the initial and bad residues.

The prescribed label sequence is $a, b, c, a, b, c, a, b, c, a, b$ and its observed state sequence is $0, 1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11$.

The initial set is $\{0 : 0\}$ and the bad set is $\{11 : 1\}$.

The exact lift count is 0. Allowing any label at each layer of this same observed state sequence gives 0 lifts.

i	Exact prefix	Exact suffix	Join
0	{0 : 0}	{0 : 1}	0
1	{1 : 1}	{1 : 2}	0
2	{2 : 4}	{2 : 5}	0
3	{3 : 2}	{3 : 3}	0
4	{4 : 3}	{4 : 4}	0
5	{5 : 6}	{5 : 7}	0
6	{6 : 4}	{6 : 5}	0
7	{7 : 5}	{7 : 6}	0
8	{8 : 8}	{8 : 9}	0
9	{9 : 6}	{9 : 7}	0
10	{10 : 7}	{10 : 8}	0

i	Exact prefix	Exact suffix	Join
11	{11 : 0}	{11 : 1}	0

Certificates. The least, extended, and greatest sequences give 24, 24, and 24 realized keys, respectively. All three refined labelled graphs have zero descendants of the selected path.

i	$\lvert A_i\rvert$	$\lvert J_i^{\min}\rvert$	$J_i^{\mathrm{ext}} \setminus J_i^{\min}$	$\lvert J_i^{\max}\rvert$
0	10	1	{0 : 2}	9
1	10	1	{1 : 3}	9
2	10	1	{2 : 6}	9
3	10	1	{3 : 4}	9
4	10	1	{4 : 5}	9
5	10	1	{5 : 8}	9
6	10	1	{6 : 6}	9
7	10	1	{7 : 7}	9
8	10	1	{8 : 0}	9
9	10	1	{9 : 8}	9
10	10	1	{10 : 9}	9

i	$\lvert A_i\rvert$	$\lvert J_i^{\min}\rvert$	$J_i^{\mathrm{ext}} \setminus J_i^{\min}$	$\lvert J_i^{\max}\rvert$
11	10	1	{11 : 2}	9

Transition rows. Identical successor rows within a phase are grouped when they have the same selected-label and other-label outputs.

Phase 0, label a.

Source residues	Selected targets	Other targets
{0}	{1}	–
{1}	{2}	–
{2}	{3}	–
{3}	{4}	–
{4}	{5}	–
{5}	{6}	–
{6}	{7}	–
{7}	{8}	–
{8}	{9}	–
{9}	{0}	–

Phase 1, label b.

Source residues	Selected targets	Other targets
{0}	{3}	–
{1}	{4}	–
{2}	{5}	–
{3}	{6}	–
{4}	{7}	–
{5}	{8}	–
{6}	{9}	–
{7}	{0}	–
{8}	{1}	–
{9}	{2}	–

Phase 2, label c.

Source residues	Selected targets	Other targets
{0}	{8}	–
{1}	{9}	–
{2}	{0}	–
{3}	{1}	–
{4}	{2}	–
{5}	{3}	–
{6}	{4}	–
{7}	{5}	–
{8}	{6}	–
{9}	{7}	–

Phase 3, label a.

Source residues	Selected targets	Other targets
{0}	{1}	–
{1}	{2}	–
{2}	{3}	–
{3}	{4}	–
{4}	{5}	–
{5}	{6}	–
{6}	{7}	–
{7}	{8}	–
{8}	{9}	–
{9}	{0}	–

Phase 4, label b.

Source residues	Selected targets	Other targets
{0}	{3}	–
{1}	{4}	–
{2}	{5}	–
{3}	{6}	–
{4}	{7}	–
{5}	{8}	–
{6}	{9}	–
{7}	{0}	–
{8}	{1}	–
{9}	{2}	–

Phase 5, label c.

Source residues	Selected targets	Other targets
{0}	{8}	–
{1}	{9}	–
{2}	{0}	–
{3}	{1}	–
{4}	{2}	–
{5}	{3}	–
{6}	{4}	–
{7}	{5}	–
{8}	{6}	–
{9}	{7}	–

Phase 6, label a.

Source residues	Selected targets	Other targets
{0}	{1}	–
{1}	{2}	–
{2}	{3}	–
{3}	{4}	–
{4}	{5}	–
{5}	{6}	–
{6}	{7}	–
{7}	{8}	–
{8}	{9}	–
{9}	{0}	–

Phase 7, label b.

Source residues	Selected targets	Other targets
{0}	{3}	–
{1}	{4}	–
{2}	{5}	–
{3}	{6}	–
{4}	{7}	–
{5}	{8}	–
{6}	{9}	–
{7}	{0}	–
{8}	{1}	–
{9}	{2}	–

Phase 8, label c.

Source residues	Selected targets	Other targets
{0}	{8}	–
{1}	{9}	–
{2}	{0}	–
{3}	{1}	–
{4}	{2}	–
{5}	{3}	–
{6}	{4}	–
{7}	{5}	–
{8}	{6}	–
{9}	{7}	–

Phase 9, label a.

Source residues	Selected targets	Other targets
{0}	{1}	–
{1}	{2}	–
{2}	{3}	–
{3}	{4}	–
{4}	{5}	–
{5}	{6}	–
{6}	{7}	–
{7}	{8}	–
{8}	{9}	–
{9}	{0}	–

Phase 10, label b.

Source residues	Selected targets	Other targets
{0}	{3}	–
{1}	{4}	–
{2}	{5}	–
{3}	{6}	–
{4}	{7}	–
{5}	{8}	–
{6}	{9}	–
{7}	{0}	–
{8}	{1}	–
{9}	{2}	–

28.5 P1.5 Translation modulo 11, length 12, target 8

The selected word determines a unique endpoint 8. The two direction vectors are translates of the initial and bad residues.

The prescribed label sequence is $a, b, c, a, b, c, a, b, c, a, b, c$ and its observed state sequence is $0, 1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12$.

The initial set is $\{0 : 0\}$ and the bad set is $\{12 : 8\}$.

The exact lift count is 1. Allowing any label at each layer of this same observed state sequence gives 1 lifts.

i	Exact prefix	Exact suffix	Join
0	$\{0 : 0\}$	$\{0 : 0\}$	1
1	$\{1 : 1\}$	$\{1 : 1\}$	1
2	$\{2 : 4\}$	$\{2 : 4\}$	1
3	$\{3 : 2\}$	$\{3 : 2\}$	1
4	$\{4 : 3\}$	$\{4 : 3\}$	1
5	$\{5 : 6\}$	$\{5 : 6\}$	1
6	$\{6 : 4\}$	$\{6 : 4\}$	1
7	$\{7 : 5\}$	$\{7 : 5\}$	1
8	$\{8 : 8\}$	$\{8 : 8\}$	1
9	$\{9 : 6\}$	$\{9 : 6\}$	1
10	$\{10 : 7\}$	$\{10 : 7\}$	1
11	$\{11 : 10\}$	$\{11 : 10\}$	1
12	$\{12 : 8\}$	$\{12 : 8\}$	1

Witness. $0 : 0, 1 : 1, 2 : 4, 3 : 2, 4 : 3, 5 : 6, 6 : 4, 7 : 5, 8 : 8, 9 : 6, 10 : 7, 11 : 10, 12 : 8$ follows the selected label at every step.

Transition rows. Identical successor rows within a phase are grouped when they have the same selected-label and other-label outputs.

Phase 0, label a.

Source residues	Selected targets	Other targets
{0}	{1}	–
{1}	{2}	–
{2}	{3}	–
{3}	{4}	–
{4}	{5}	–
{5}	{6}	–
{6}	{7}	–
{7}	{8}	–
{8}	{9}	–
{9}	{10}	–
{10}	{0}	–

Phase 1, label b.

Source residues	Selected targets	Other targets
{0}	{3}	–
{1}	{4}	–
{2}	{5}	–
{3}	{6}	–
{4}	{7}	–
{5}	{8}	–
{6}	{9}	–
{7}	{10}	–
{8}	{0}	–
{9}	{1}	–
{10}	{2}	–

Phase 2, label c.

Source residues	Selected targets	Other targets
{0}	{9}	–
{1}	{10}	–
{2}	{0}	–
{3}	{1}	–
{4}	{2}	–
{5}	{3}	–
{6}	{4}	–
{7}	{5}	–
{8}	{6}	–
{9}	{7}	–
{10}	{8}	–

Phase 3, label a.

Source residues	Selected targets	Other targets
{0}	{1}	–
{1}	{2}	–
{2}	{3}	–
{3}	{4}	–
{4}	{5}	–
{5}	{6}	–
{6}	{7}	–
{7}	{8}	–
{8}	{9}	–
{9}	{10}	–
{10}	{0}	–

Phase 4, label b.

Source residues	Selected targets	Other targets
{0}	{3}	–
{1}	{4}	–
{2}	{5}	–
{3}	{6}	–
{4}	{7}	–
{5}	{8}	–
{6}	{9}	–
{7}	{10}	–
{8}	{0}	–
{9}	{1}	–
{10}	{2}	–

Phase 5, label c.

Source residues	Selected targets	Other targets
{0}	{9}	–
{1}	{10}	–
{2}	{0}	–
{3}	{1}	–
{4}	{2}	–
{5}	{3}	–
{6}	{4}	–
{7}	{5}	–
{8}	{6}	–
{9}	{7}	–
{10}	{8}	–

Phase 6, label a.

Source residues	Selected targets	Other targets
{0}	{1}	–
{1}	{2}	–
{2}	{3}	–
{3}	{4}	–
{4}	{5}	–
{5}	{6}	–
{6}	{7}	–
{7}	{8}	–
{8}	{9}	–
{9}	{10}	–
{10}	{0}	–

Phase 7, label b.

Source residues	Selected targets	Other targets
{0}	{3}	–
{1}	{4}	–
{2}	{5}	–
{3}	{6}	–
{4}	{7}	–
{5}	{8}	–
{6}	{9}	–
{7}	{10}	–
{8}	{0}	–
{9}	{1}	–
{10}	{2}	–

Phase 8, label c.

Source residues	Selected targets	Other targets
{0}	{9}	–
{1}	{10}	–
{2}	{0}	–
{3}	{1}	–
{4}	{2}	–
{5}	{3}	–
{6}	{4}	–
{7}	{5}	–
{8}	{6}	–
{9}	{7}	–
{10}	{8}	–

Phase 9, label a.

Source residues	Selected targets	Other targets
{0}	{1}	–
{1}	{2}	–
{2}	{3}	–
{3}	{4}	–
{4}	{5}	–
{5}	{6}	–
{6}	{7}	–
{7}	{8}	–
{8}	{9}	–
{9}	{10}	–
{10}	{0}	–

Phase 10, label b.

Source residues	Selected targets	Other targets
{0}	{3}	–
{1}	{4}	–
{2}	{5}	–
{3}	{6}	–
{4}	{7}	–
{5}	{8}	–
{6}	{9}	–
{7}	{10}	–
{8}	{0}	–
{9}	{1}	–
{10}	{2}	–

Phase 11, label c.

Source residues	Selected targets	Other targets
{0}	{9}	–
{1}	{10}	–
{2}	{0}	–
{3}	{1}	–
{4}	{2}	–
{5}	{3}	–
{6}	{4}	–
{7}	{5}	–
{8}	{6}	–
{9}	{7}	–
{10}	{8}	–

28.6 P1.6 Translation modulo 12, length 13, target 2

The selected word determines a unique endpoint 9. The two direction vectors are translates of the initial and bad residues.

The prescribed label sequence is $a, b, c, a, b, c, a, b, c, a, b, c, a$ and its observed state sequence is $0, 1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12, 13$.

The initial set is $\{0 : 0\}$ and the bad set is $\{13 : 2\}$.

The exact lift count is 0. Allowing any label at each layer of this same observed state sequence gives 0 lifts.

i	Exact prefix	Exact suffix	Join
0	$\{0 : 0\}$	$\{0 : 5\}$	0
1	$\{1 : 1\}$	$\{1 : 6\}$	0
2	$\{2 : 4\}$	$\{2 : 9\}$	0
3	$\{3 : 2\}$	$\{3 : 7\}$	0
4	$\{4 : 3\}$	$\{4 : 8\}$	0
5	$\{5 : 6\}$	$\{5 : 11\}$	0
6	$\{6 : 4\}$	$\{6 : 9\}$	0
7	$\{7 : 5\}$	$\{7 : 10\}$	0
8	$\{8 : 8\}$	$\{8 : 1\}$	0
9	$\{9 : 6\}$	$\{9 : 11\}$	0
10	$\{10 : 7\}$	$\{10 : 0\}$	0

i	Exact prefix	Exact suffix	Join
11	$\{11 : 10\}$	$\{11 : 3\}$	0
12	$\{12 : 8\}$	$\{12 : 1\}$	0
13	$\{13 : 9\}$	$\{13 : 2\}$	0

Certificates. The least, extended, and greatest sequences give 28, 28, and 28 realized keys, respectively. All three refined labelled graphs have zero descendants of the selected path.

i	$\lvert A_i\rvert$	$\lvert J_i^{\min}\rvert$	$J_i^{\mathrm{ext}} \setminus J_i^{\min}$	$\lvert J_i^{\max}\rvert$
0	12	1	$\{0 : 1\}$	11
1	12	1	$\{1 : 2\}$	11
2	12	1	$\{2 : 5\}$	11
3	12	1	$\{3 : 3\}$	11
4	12	1	$\{4 : 4\}$	11
5	12	1	$\{5 : 7\}$	11
6	12	1	$\{6 : 5\}$	11
7	12	1	$\{7 : 6\}$	11
8	12	1	$\{8 : 9\}$	11
9	12	1	$\{9 : 7\}$	11
10	12	1	$\{10 : 8\}$	11

i	$\lvert A_i\rvert$	$\lvert J_i^{\min}\rvert$	$J_i^{\mathrm{ext}} \setminus J_i^{\min}$	$\lvert J_i^{\max}\rvert$
11	12	1	$\{11 : 11\}$	11
12	12	1	$\{12 : 9\}$	11
13	12	1	$\{13 : 10\}$	11

Transition rows. Identical successor rows within a phase are grouped when they have the same selected-label and other-label outputs.

Phase 0, label a.

Source residues	Selected targets	Other targets
{0}	{1}	–
{1}	{2}	–
{2}	{3}	–
{3}	{4}	–
{4}	{5}	–
{5}	{6}	–
{6}	{7}	–
{7}	{8}	–
{8}	{9}	–
{9}	{10}	–
{10}	{11}	–
{11}	{0}	–

Phase 1, label b.

Source residues	Selected targets	Other targets
{0}	{3}	–
{1}	{4}	–
{2}	{5}	–
{3}	{6}	–
{4}	{7}	–
{5}	{8}	–
{6}	{9}	–
{7}	{10}	–
{8}	{11}	–
{9}	{0}	–
{10}	{1}	–
{11}	{2}	–

Phase 2, label c.

Source residues	Selected targets	Other targets
{0}	{10}	–
{1}	{11}	–
{2}	{0}	–
{3}	{1}	–
{4}	{2}	–
{5}	{3}	–
{6}	{4}	–
{7}	{5}	–
{8}	{6}	–
{9}	{7}	–
{10}	{8}	–
{11}	{9}	–

Phase 3, label a.

Source residues	Selected targets	Other targets
{0}	{1}	–
{1}	{2}	–
{2}	{3}	–
{3}	{4}	–
{4}	{5}	–
{5}	{6}	–
{6}	{7}	–
{7}	{8}	–
{8}	{9}	–
{9}	{10}	–
{10}	{11}	–
{11}	{0}	–

Phase 4, label b.

Source residues	Selected targets	Other targets
{0}	{3}	–
{1}	{4}	–
{2}	{5}	–
{3}	{6}	–
{4}	{7}	–
{5}	{8}	–
{6}	{9}	–
{7}	{10}	–
{8}	{11}	–
{9}	{0}	–
{10}	{1}	–
{11}	{2}	–

Phase 5, label c.

Source residues	Selected targets	Other targets
{0}	{10}	–
{1}	{11}	–
{2}	{0}	–
{3}	{1}	–
{4}	{2}	–
{5}	{3}	–
{6}	{4}	–
{7}	{5}	–
{8}	{6}	–
{9}	{7}	–
{10}	{8}	–
{11}	{9}	–

Phase 6, label a.

Source residues	Selected targets	Other targets
{0}	{1}	–
{1}	{2}	–
{2}	{3}	–
{3}	{4}	–
{4}	{5}	–
{5}	{6}	–
{6}	{7}	–
{7}	{8}	–
{8}	{9}	–
{9}	{10}	–
{10}	{11}	–
{11}	{0}	–

Phase 7, label b.

Source residues	Selected targets	Other targets
{0}	{3}	–
{1}	{4}	–
{2}	{5}	–
{3}	{6}	–
{4}	{7}	–
{5}	{8}	–
{6}	{9}	–
{7}	{10}	–
{8}	{11}	–
{9}	{0}	–
{10}	{1}	–
{11}	{2}	–

Phase 8, label c.

Source residues	Selected targets	Other targets
{0}	{10}	–
{1}	{11}	–
{2}	{0}	–
{3}	{1}	–
{4}	{2}	–
{5}	{3}	–
{6}	{4}	–
{7}	{5}	–
{8}	{6}	–
{9}	{7}	–
{10}	{8}	–
{11}	{9}	–

Phase 9, label a.

Source residues	Selected targets	Other targets
{0}	{1}	–
{1}	{2}	–
{2}	{3}	–
{3}	{4}	–
{4}	{5}	–
{5}	{6}	–
{6}	{7}	–
{7}	{8}	–
{8}	{9}	–
{9}	{10}	–
{10}	{11}	–
{11}	{0}	–

Phase 10, label b.

Source residues	Selected targets	Other targets
{0}	{3}	–
{1}	{4}	–
{2}	{5}	–
{3}	{6}	–
{4}	{7}	–
{5}	{8}	–
{6}	{9}	–
{7}	{10}	–
{8}	{11}	–
{9}	{0}	–
{10}	{1}	–
{11}	{2}	–

Phase 11, label c.

Source residues	Selected targets	Other targets
{0}	{10}	–
{1}	{11}	–
{2}	{0}	–
{3}	{1}	–
{4}	{2}	–
{5}	{3}	–
{6}	{4}	–
{7}	{5}	–
{8}	{6}	–
{9}	{7}	–
{10}	{8}	–
{11}	{9}	–

Phase 12, label a.

Source residues	Selected targets	Other targets
{0}	{1}	–
{1}	{2}	–
{2}	{3}	–
{3}	{4}	–
{4}	{5}	–
{5}	{6}	–
{6}	{7}	–
{7}	{8}	–
{8}	{9}	–
{9}	{10}	–
{10}	{11}	–
{11}	{0}	–

28.7 P1.7 Translation modulo 13, length 14, target 3

The selected word determines a unique endpoint 12. The two direction vectors are translates of the initial and bad residues.

The prescribed label sequence is $a, b, c, a, b, c, a, b, c, a, b, c, a, b$ and its observed state sequence is $0, 1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12, 13, 14$.

The initial set is $\{0 : 0\}$ and the bad set is $\{14 : 3\}$.

The exact lift count is 0. Allowing any label at each layer of this same observed state sequence gives 0 lifts.

i	Exact prefix	Exact suffix	Join
0	$\{0 : 0\}$	$\{0 : 4\}$	0
1	$\{1 : 1\}$	$\{1 : 5\}$	0
2	$\{2 : 4\}$	$\{2 : 8\}$	0
3	$\{3 : 2\}$	$\{3 : 6\}$	0
4	$\{4 : 3\}$	$\{4 : 7\}$	0
5	$\{5 : 6\}$	$\{5 : 10\}$	0
6	$\{6 : 4\}$	$\{6 : 8\}$	0
7	$\{7 : 5\}$	$\{7 : 9\}$	0
8	$\{8 : 8\}$	$\{8 : 12\}$	0
9	$\{9 : 6\}$	$\{9 : 10\}$	0
10	$\{10 : 7\}$	$\{10 : 11\}$	0

i	Exact prefix	Exact suffix	Join
11	$\{11 : 10\}$	$\{11 : 1\}$	0
12	$\{12 : 8\}$	$\{12 : 12\}$	0
13	$\{13 : 9\}$	$\{13 : 0\}$	0
14	$\{14 : 12\}$	$\{14 : 3\}$	0

Certificates. The least, extended, and greatest sequences give 30, 30, and 30 realized keys, respectively. All three refined labelled graphs have zero descendants of the selected path.

i	$\lvert A_i\rvert$	$\lvert J_i^{\min}\rvert$	$J_i^{\mathrm{ext}} \setminus J_i^{\min}$	$\lvert J_i^{\max}\rvert$
0	13	1	$\{0 : 1\}$	12
1	13	1	$\{1 : 2\}$	12
2	13	1	$\{2 : 5\}$	12
3	13	1	$\{3 : 3\}$	12
4	13	1	$\{4 : 4\}$	12
5	13	1	$\{5 : 7\}$	12
6	13	1	$\{6 : 5\}$	12
7	13	1	$\{7 : 6\}$	12
8	13	1	$\{8 : 9\}$	12
9	13	1	$\{9 : 7\}$	12
10	13	1	$\{10 : 8\}$	12

i	$\lvert A_i\rvert$	$\lvert J_i^{\min}\rvert$	$J_i^{\mathrm{ext}} \setminus J_i^{\min}$	$\lvert J_i^{\max}\rvert$
11	13	1	$\{11 : 11\}$	12
12	13	1	$\{12 : 9\}$	12
13	13	1	$\{13 : 10\}$	12
14	13	1	$\{14 : 0\}$	12

Transition rows. Identical successor rows within a phase are grouped when they have the same selected-label and other-label outputs.

Phase 0, label a.

Source residues	Selected targets	Other targets
{0}	{1}	–
{1}	{2}	–
{2}	{3}	–
{3}	{4}	–
{4}	{5}	–
{5}	{6}	–
{6}	{7}	–
{7}	{8}	–
{8}	{9}	–
{9}	{10}	–
{10}	{11}	–
{11}	{12}	–

Phase 0, label a (continued).

Source residues	Selected targets	Other targets
{12}	{0}	–

Phase 1, label b.

Source residues	Selected targets	Other targets
{0}	{3}	–
{1}	{4}	–
{2}	{5}	–
{3}	{6}	–
{4}	{7}	–
{5}	{8}	–
{6}	{9}	–
{7}	{10}	–
{8}	{11}	–
{9}	{12}	–
{10}	{0}	–
{11}	{1}	–

Phase 1, label b (continued).

Source residues	Selected targets	Other targets
{12}	{2}	–

Phase 2, label c.

Source residues	Selected targets	Other targets
{0}	{11}	–
{1}	{12}	–
{2}	{0}	–
{3}	{1}	–
{4}	{2}	–
{5}	{3}	–
{6}	{4}	–
{7}	{5}	–
{8}	{6}	–
{9}	{7}	–
{10}	{8}	–
{11}	{9}	–

Phase 2, label c (continued).

Source residues	Selected targets	Other targets
{12}	{10}	–

Phase 3, label a.

Source residues	Selected targets	Other targets
{0}	{1}	–
{1}	{2}	–
{2}	{3}	–
{3}	{4}	–
{4}	{5}	–
{5}	{6}	–
{6}	{7}	–
{7}	{8}	–
{8}	{9}	–
{9}	{10}	–
{10}	{11}	–
{11}	{12}	–

Phase 3, label a (continued).

Source residues	Selected targets	Other targets
{12}	{0}	–

Phase 4, label b.

Source residues	Selected targets	Other targets
{0}	{3}	–
{1}	{4}	–
{2}	{5}	–
{3}	{6}	–
{4}	{7}	–
{5}	{8}	–
{6}	{9}	–
{7}	{10}	–
{8}	{11}	–
{9}	{12}	–
{10}	{0}	–
{11}	{1}	–

Phase 4, label b (continued).

Source residues	Selected targets	Other targets
{12}	{2}	–

Phase 5, label c.

Source residues	Selected targets	Other targets
{0}	{11}	–
{1}	{12}	–
{2}	{0}	–
{3}	{1}	–
{4}	{2}	–
{5}	{3}	–
{6}	{4}	–
{7}	{5}	–
{8}	{6}	–
{9}	{7}	–
{10}	{8}	–
{11}	{9}	–

Phase 5, label c (continued).

Source residues	Selected targets	Other targets
{12}	{10}	–

Phase 6, label a.

Source residues	Selected targets	Other targets
{0}	{1}	–
{1}	{2}	–
{2}	{3}	–
{3}	{4}	–
{4}	{5}	–
{5}	{6}	–
{6}	{7}	–
{7}	{8}	–
{8}	{9}	–
{9}	{10}	–
{10}	{11}	–
{11}	{12}	–

Phase 6, label a (continued).

Source residues	Selected targets	Other targets
{12}	{0}	–

Phase 7, label b.

Source residues	Selected targets	Other targets
{0}	{3}	–
{1}	{4}	–
{2}	{5}	–
{3}	{6}	–
{4}	{7}	–
{5}	{8}	–
{6}	{9}	–
{7}	{10}	–
{8}	{11}	–
{9}	{12}	–
{10}	{0}	–
{11}	{1}	–

Phase 7, label b (continued).

Source residues	Selected targets	Other targets
{12}	{2}	–

Phase 8, label c.

Source residues	Selected targets	Other targets
{0}	{11}	–
{1}	{12}	–
{2}	{0}	–
{3}	{1}	–
{4}	{2}	–
{5}	{3}	–
{6}	{4}	–
{7}	{5}	–
{8}	{6}	–
{9}	{7}	–
{10}	{8}	–
{11}	{9}	–

Phase 8, label c (continued).

Source residues	Selected targets	Other targets
{12}	{10}	–

Phase 9, label a.

Source residues	Selected targets	Other targets
{0}	{1}	–
{1}	{2}	–
{2}	{3}	–
{3}	{4}	–
{4}	{5}	–
{5}	{6}	–
{6}	{7}	–
{7}	{8}	–
{8}	{9}	–
{9}	{10}	–
{10}	{11}	–
{11}	{12}	–

Phase 9, label a (continued).

Source residues	Selected targets	Other targets
{12}	{0}	–

Phase 10, label b.

Source residues	Selected targets	Other targets
{0}	{3}	–
{1}	{4}	–
{2}	{5}	–
{3}	{6}	–
{4}	{7}	–
{5}	{8}	–
{6}	{9}	–
{7}	{10}	–
{8}	{11}	–
{9}	{12}	–
{10}	{0}	–
{11}	{1}	–

Phase 10, label b (continued).

Source residues	Selected targets	Other targets
{12}	{2}	–

Phase 11, label c.

Source residues	Selected targets	Other targets
{0}	{11}	–
{1}	{12}	–
{2}	{0}	–
{3}	{1}	–
{4}	{2}	–
{5}	{3}	–
{6}	{4}	–
{7}	{5}	–
{8}	{6}	–
{9}	{7}	–
{10}	{8}	–
{11}	{9}	–

Phase 11, label c (continued).

Source residues	Selected targets	Other targets
{12}	{10}	–

Phase 12, label a.

Source residues	Selected targets	Other targets
{0}	{1}	–
{1}	{2}	–
{2}	{3}	–
{3}	{4}	–
{4}	{5}	–
{5}	{6}	–
{6}	{7}	–
{7}	{8}	–
{8}	{9}	–
{9}	{10}	–
{10}	{11}	–
{11}	{12}	–

Phase 12, label a (continued).

Source residues	Selected targets	Other targets
{12}	{0}	–

Phase 13, label b.

Source residues	Selected targets	Other targets
{0}	{3}	–
{1}	{4}	–
{2}	{5}	–
{3}	{6}	–
{4}	{7}	–
{5}	{8}	–
{6}	{9}	–
{7}	{10}	–
{8}	{11}	–
{9}	{12}	–
{10}	{0}	–
{11}	{1}	–

Phase 13, label b (continued).

Source residues	Selected targets	Other targets
{12}	{2}	–

28.8 P1.8 Translation modulo 14, length 15, target 12

The selected word determines a unique endpoint 10. The two direction vectors are translates of the initial and bad residues.

The prescribed label sequence is $a, b, c, a, b, c, a, b, c, a, b, c, a, b, c$ and its observed state sequence is $0, 1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12, 13, 14, 15$.

The initial set is $\{0 : 0\}$ and the bad set is $\{15 : 12\}$.

The exact lift count is 0. Allowing any label at each layer of this same observed state sequence gives 0 lifts.

i	Exact prefix	Exact suffix	Join
0	$\{0 : 0\}$	$\{0 : 2\}$	0
1	$\{1 : 1\}$	$\{1 : 3\}$	0
2	$\{2 : 4\}$	$\{2 : 6\}$	0
3	$\{3 : 2\}$	$\{3 : 4\}$	0
4	$\{4 : 3\}$	$\{4 : 5\}$	0
5	$\{5 : 6\}$	$\{5 : 8\}$	0
6	$\{6 : 4\}$	$\{6 : 6\}$	0
7	$\{7 : 5\}$	$\{7 : 7\}$	0
8	$\{8 : 8\}$	$\{8 : 10\}$	0
9	$\{9 : 6\}$	$\{9 : 8\}$	0
10	$\{10 : 7\}$	$\{10 : 9\}$	0

i	Exact prefix	Exact suffix	Join
11	$\{11 : 10\}$	$\{11 : 12\}$	0
12	$\{12 : 8\}$	$\{12 : 10\}$	0
13	$\{13 : 9\}$	$\{13 : 11\}$	0
14	$\{14 : 12\}$	$\{14 : 0\}$	0
15	$\{15 : 10\}$	$\{15 : 12\}$	0

Certificates. The least, extended, and greatest sequences give 32, 32, and 32 realized keys, respectively. All three refined labelled graphs have zero descendants of the selected path.

i	$\lvert A_i\rvert$	$\lvert J_i^{\min}\rvert$	$J_i^{\mathrm{ext}} \setminus J_i^{\min}$	$\lvert J_i^{\max}\rvert$
0	14	1	$\{0 : 1\}$	13
1	14	1	$\{1 : 2\}$	13
2	14	1	$\{2 : 5\}$	13
3	14	1	$\{3 : 3\}$	13
4	14	1	$\{4 : 4\}$	13
5	14	1	$\{5 : 7\}$	13
6	14	1	$\{6 : 5\}$	13
7	14	1	$\{7 : 6\}$	13
8	14	1	$\{8 : 9\}$	13
9	14	1	$\{9 : 7\}$	13
10	14	1	$\{10 : 8\}$	13

i	$\lvert A_i\rvert$	$\lvert J_i^{\min}\rvert$	$J_i^{\mathrm{ext}} \setminus J_i^{\min}$	$\lvert J_i^{\max}\rvert$
11	14	1	$\{11 : 11\}$	13
12	14	1	$\{12 : 9\}$	13
13	14	1	$\{13 : 10\}$	13
14	14	1	$\{14 : 13\}$	13
15	14	1	$\{15 : 11\}$	13

Transition rows. Identical successor rows within a phase are grouped when they have the same selected-label and other-label outputs.

Phase 0, label a.

Source residues	Selected targets	Other targets
{0}	{1}	–
{1}	{2}	–
{2}	{3}	–
{3}	{4}	–
{4}	{5}	–
{5}	{6}	–
{6}	{7}	–
{7}	{8}	–
{8}	{9}	–
{9}	{10}	–
{10}	{11}	–
{11}	{12}	–

Phase 0, label a (continued).

Source residues	Selected targets	Other targets
{12}	{13}	–
{13}	{0}	–

Phase 1, label b.

Source residues	Selected targets	Other targets
{0}	{3}	–
{1}	{4}	–
{2}	{5}	–
{3}	{6}	–
{4}	{7}	–
{5}	{8}	–
{6}	{9}	–
{7}	{10}	–
{8}	{11}	–
{9}	{12}	–
{10}	{13}	–
{11}	{0}	–

Phase 1, label b (continued).

Source residues	Selected targets	Other targets
{12}	{1}	–
{13}	{2}	–

Phase 2, label c.

Source residues	Selected targets	Other targets
{0}	{12}	–
{1}	{13}	–
{2}	{0}	–
{3}	{1}	–
{4}	{2}	–
{5}	{3}	–
{6}	{4}	–
{7}	{5}	–
{8}	{6}	–
{9}	{7}	–
{10}	{8}	–
{11}	{9}	–

Phase 2, label c (continued).

Source residues	Selected targets	Other targets
{12}	{10}	–
{13}	{11}	–

Phase 3, label a.

Source residues	Selected targets	Other targets
{0}	{1}	–
{1}	{2}	–
{2}	{3}	–
{3}	{4}	–
{4}	{5}	–
{5}	{6}	–
{6}	{7}	–
{7}	{8}	–
{8}	{9}	–
{9}	{10}	–
{10}	{11}	–
{11}	{12}	–

Phase 3, label a (continued).

Source residues	Selected targets	Other targets
{12}	{13}	–
{13}	{0}	–

Phase 4, label b.

Source residues	Selected targets	Other targets
{0}	{3}	–
{1}	{4}	–
{2}	{5}	–
{3}	{6}	–
{4}	{7}	–
{5}	{8}	–
{6}	{9}	–
{7}	{10}	–
{8}	{11}	–
{9}	{12}	–
{10}	{13}	–
{11}	{0}	–

Phase 4, label b (continued).

Source residues	Selected targets	Other targets
{12}	{1}	–
{13}	{2}	–

Phase 5, label c.

Source residues	Selected targets	Other targets
{0}	{12}	–
{1}	{13}	–
{2}	{0}	–
{3}	{1}	–
{4}	{2}	–
{5}	{3}	–
{6}	{4}	–
{7}	{5}	–
{8}	{6}	–
{9}	{7}	–
{10}	{8}	–
{11}	{9}	–

Phase 5, label c (continued).

Source residues	Selected targets	Other targets
{12}	{10}	–
{13}	{11}	–

Phase 6, label a.

Source residues	Selected targets	Other targets
{0}	{1}	–
{1}	{2}	–
{2}	{3}	–
{3}	{4}	–
{4}	{5}	–
{5}	{6}	–
{6}	{7}	–
{7}	{8}	–
{8}	{9}	–
{9}	{10}	–
{10}	{11}	–
{11}	{12}	–

Phase 6, label a (continued).

Source residues	Selected targets	Other targets
{12}	{13}	–
{13}	{0}	–

Phase 7, label b.

Source residues	Selected targets	Other targets
{0}	{3}	–
{1}	{4}	–
{2}	{5}	–
{3}	{6}	–
{4}	{7}	–
{5}	{8}	–
{6}	{9}	–
{7}	{10}	–
{8}	{11}	–
{9}	{12}	–
{10}	{13}	–
{11}	{0}	–

Phase 7, label b (continued).

Source residues	Selected targets	Other targets
{12}	{1}	–
{13}	{2}	–

Phase 8, label c.

Source residues	Selected targets	Other targets
{0}	{12}	–
{1}	{13}	–
{2}	{0}	–
{3}	{1}	–
{4}	{2}	–
{5}	{3}	–
{6}	{4}	–
{7}	{5}	–
{8}	{6}	–
{9}	{7}	–
{10}	{8}	–
{11}	{9}	–

Phase 8, label c (continued).

Source residues	Selected targets	Other targets
{12}	{10}	–
{13}	{11}	–

Phase 9, label a.

Source residues	Selected targets	Other targets
{0}	{1}	–
{1}	{2}	–
{2}	{3}	–
{3}	{4}	–
{4}	{5}	–
{5}	{6}	–
{6}	{7}	–
{7}	{8}	–
{8}	{9}	–
{9}	{10}	–
{10}	{11}	–
{11}	{12}	–

Phase 9, label a (continued).

Source residues	Selected targets	Other targets
{12}	{13}	–
{13}	{0}	–

Phase 10, label b.

Source residues	Selected targets	Other targets
{0}	{3}	–
{1}	{4}	–
{2}	{5}	–
{3}	{6}	–
{4}	{7}	–
{5}	{8}	–
{6}	{9}	–
{7}	{10}	–
{8}	{11}	–
{9}	{12}	–
{10}	{13}	–
{11}	{0}	–

Phase 10, label b (continued).

Source residues	Selected targets	Other targets
{12}	{1}	–
{13}	{2}	–

Phase 11, label c.

Source residues	Selected targets	Other targets
{0}	{12}	–
{1}	{13}	–
{2}	{0}	–
{3}	{1}	–
{4}	{2}	–
{5}	{3}	–
{6}	{4}	–
{7}	{5}	–
{8}	{6}	–
{9}	{7}	–
{10}	{8}	–
{11}	{9}	–

Phase 11, label c (continued).

Source residues	Selected targets	Other targets
{12}	{10}	–
{13}	{11}	–

Phase 12, label a.

Source residues	Selected targets	Other targets
{0}	{1}	–
{1}	{2}	–
{2}	{3}	–
{3}	{4}	–
{4}	{5}	–
{5}	{6}	–
{6}	{7}	–
{7}	{8}	–
{8}	{9}	–
{9}	{10}	–
{10}	{11}	–
{11}	{12}	–

Phase 12, label a (continued).

Source residues	Selected targets	Other targets
{12}	{13}	–
{13}	{0}	–

Phase 13, label b.

Source residues	Selected targets	Other targets
{0}	{3}	–
{1}	{4}	–
{2}	{5}	–
{3}	{6}	–
{4}	{7}	–
{5}	{8}	–
{6}	{9}	–
{7}	{10}	–
{8}	{11}	–
{9}	{12}	–
{10}	{13}	–
{11}	{0}	–

Phase 13, label b (continued).

Source residues	Selected targets	Other targets
{12}	{1}	–
{13}	{2}	–

Phase 14, label c.

Source residues	Selected targets	Other targets
{0}	{12}	–
{1}	{13}	–
{2}	{0}	–
{3}	{1}	–
{4}	{2}	–
{5}	{3}	–
{6}	{4}	–
{7}	{5}	–
{8}	{6}	–
{9}	{7}	–
{10}	{8}	–
{11}	{9}	–

Phase 14, label c (continued).

Source residues	Selected targets	Other targets
{12}	{10}	–
{13}	{11}	–

29 Branching and parity

The carrier is a layered residue space. Several edges can share a target, so lift counts can be much larger than the number of feasible endpoints.

A state printed as $i : v$ has phase i and residue v. The successor table prints residues at the next phase. A blank column for another label means there is no such edge at that phase.

29.1 P2.1 Nondeterministic parity, width 6, length 6, target parity 1

All labelled edges preserve parity. Branching changes the set and number of feasible prefixes while preserving the parity obstruction when the target is odd.

The prescribed label sequence is g, s, s, g, s, s and its observed state sequence is $0, 1, 0, 1, 0, 1, 0$.

The initial set is $\{0 : 0, 0 : 2\}$ and the bad set is $\{6 : 1, 6 : 3\}$.

The exact lift count is 0. Allowing any label at each layer of this same observed state sequence gives 0 lifts.

i	Exact prefix	Exact suffix	Join
0	$\{0:0, 0:2\}$	$\{0:1, 0:3, 0:5\}$	0
1	$\{1:0, 1:2, 1:4\}$	$\{1:1, 1:3, 1:5\}$	0
2	$\{2:0, 2:2, 2:4\}$	$\{2:1, 2:3, 2:5\}$	0
3	$\{3:0, 3:2, 3:4\}$	$\{3:1, 3:3, 3:5\}$	0
4	$\{4:0, 4:2, 4:4\}$	$\{4:1, 4:3, 4:5\}$	0
5	$\{5:0, 5:2, 5:4\}$	$\{5:1, 5:3, 5:5\}$	0
6	$\{6:0, 6:2, 6:4\}$	$\{6:1, 6:3\}$	0

Certificates. The least, extended, and greatest sequences give 9, 9, and 9 realized keys, respectively. All three refined labelled graphs have zero descendants of the selected path.

i	$\lvert A_i\rvert$	$\lvert J_i^{\min}\rvert$	$J_i^{\mathrm{ext}} \setminus J_i^{\min}$	$\lvert J_i^{\max}\rvert$
0	24	2	$\{0:4\}$	21
1	18	3	$\varnothing$	15
2	24	3	$\varnothing$	21
3	18	3	$\varnothing$	15
4	24	3	$\varnothing$	21
5	18	3	$\varnothing$	15
6	24	3	$\varnothing$	22

Transition rows. Identical successor rows within a phase are grouped when they have the same selected-label and other-label outputs.

Phase 0, label g.

Source residues	Selected targets	Other targets
{0}	{0, 2}	–
{1}	{1, 3}	–
{2}	{2, 4}	–
{3}	{3, 5}	–
{4}	{0, 4}	–
{5}	{1, 5}	–

Phase 1, label s.

Source residues	Selected targets	Other targets
$\{0\}$	$\{2,4\}$	–
$\{1\}$	$\{3,5\}$	–
$\{2\}$	$\{0,4\}$	–
$\{3\}$	$\{1,5\}$	–
$\{4\}$	$\{0,2\}$	–
$\{5\}$	$\{1,3\}$	–

Phase 2, label s.

Source residues	Selected targets	Other targets
$\{0\}$	$\{2,4\}$	–
$\{1\}$	$\{3,5\}$	–
$\{2\}$	$\{0,4\}$	–
$\{3\}$	$\{1,5\}$	–
$\{4\}$	$\{0,2\}$	–
$\{5\}$	$\{1,3\}$	–

Phase 3, label g.

Source residues	Selected targets	Other targets
$\{0\}$	$\{0,2\}$	–
$\{1\}$	$\{1,3\}$	–
$\{2\}$	$\{2,4\}$	–
$\{3\}$	$\{3,5\}$	–
$\{4\}$	$\{0,4\}$	–
$\{5\}$	$\{1,5\}$	–

Phase 4, label s.

Source residues	Selected targets	Other targets
$\{0\}$	$\{2,4\}$	–
$\{1\}$	$\{3,5\}$	–
$\{2\}$	$\{0,4\}$	–
$\{3\}$	$\{1,5\}$	–
$\{4\}$	$\{0,2\}$	–
$\{5\}$	$\{1,3\}$	–

Phase 5, label s.

Source residues	Selected targets	Other targets
$\{0\}$	$\{2,4\}$	–
$\{1\}$	$\{3,5\}$	–
$\{2\}$	$\{0,4\}$	–
$\{3\}$	$\{1,5\}$	–
$\{4\}$	$\{0,2\}$	–
$\{5\}$	$\{1,3\}$	–

29.2 P2.2 Nondeterministic parity, width 8, length 7, target parity 0

All labelled edges preserve parity. Branching changes the set and number of feasible prefixes while preserving the parity obstruction when the target is odd.

The prescribed label sequence is g, s, s, g, s, s, g and its observed state sequence is $0, 1, 0, 1, 0, 1, 0, 1$.

The initial set is $\{0:0, 0:2\}$ and the bad set is $\{7:0, 7:2, 7:4\}$.

The exact lift count is 972. Allowing any label at each layer of this same observed state sequence gives 972 lifts.

i	Exact prefix	Exact suffix	Join
0	$\{0:0, 0:2\}$	$\{0:0, 0:2, 0:4, 0:6\}$	2
1	$\{1:0, 1:2, 1:4\}$	$\{1:0, 1:2, 1:4, 1:6\}$	3
2	$\{2:0, 2:2, 2:4, 2:6\}$	$\{2:0, 2:2, 2:4, 2:6\}$	4
3	$\{3:0, 3:2, 3:4, 3:6\}$	$\{3:0, 3:2, 3:4, 3:6\}$	4
4	$\{4:0, 4:2, 4:4, 4:6\}$	$\{4:0, 4:2, 4:4, 4:6\}$	4
5	$\{5:0, 5:2, 5:4, 5:6\}$	$\{5:0, 5:2, 5:4, 5:6\}$	4
6	$\{6:0, 6:2, 6:4, 6:6\}$	$\{6:0, 6:2, 6:4, 6:6\}$	4
7	$\{7:0, 7:2, 7:4, 7:6\}$	$\{7:0, 7:2, 7:4\}$	3

Witness. $0:0, 1:0, 2:2, 3:0, 4:0, 5:2, 6:0, 7:0$ follows the selected label at every step.

Transition rows. Identical successor rows within a phase are grouped when they have the same selected-label and other-label outputs.

Phase 0, label g.

Source residues	Selected targets	Other targets
$\{0\}$	$\{0,2\}$	–
$\{1\}$	$\{1,3\}$	–
$\{2\}$	$\{2,4\}$	–
$\{3\}$	$\{3,5\}$	–
$\{4\}$	$\{4,6\}$	–
$\{5\}$	$\{5,7\}$	–
$\{6\}$	$\{0,6\}$	–
$\{7\}$	$\{1,7\}$	–

Phase 1, label s.

Source residues	Selected targets	Other targets
$\{0\}$	$\{2,4,6\}$	–
$\{1\}$	$\{3,5,7\}$	–
$\{2\}$	$\{0,4,6\}$	–
$\{3\}$	$\{1,5,7\}$	–
$\{4\}$	$\{0,2,6\}$	–
$\{5\}$	$\{1,3,7\}$	–
$\{6\}$	$\{0,2,4\}$	–
$\{7\}$	$\{1,3,5\}$	–

Phase 2, label s.

Source residues	Selected targets	Other targets
$\{0\}$	$\{2,4,6\}$	–
$\{1\}$	$\{3,5,7\}$	–
$\{2\}$	$\{0,4,6\}$	–
$\{3\}$	$\{1,5,7\}$	–
$\{4\}$	$\{0,2,6\}$	–
$\{5\}$	$\{1,3,7\}$	–
$\{6\}$	$\{0,2,4\}$	–
$\{7\}$	$\{1,3,5\}$	–

Phase 3, label g.

Source residues	Selected targets	Other targets
$\{0\}$	$\{0,2\}$	–
$\{1\}$	$\{1,3\}$	–
$\{2\}$	$\{2,4\}$	–
$\{3\}$	$\{3,5\}$	–
$\{4\}$	$\{4,6\}$	–
$\{5\}$	$\{5,7\}$	–
$\{6\}$	$\{0,6\}$	–
$\{7\}$	$\{1,7\}$	–

Phase 4, label s.

Source residues	Selected targets	Other targets
$\{0\}$	$\{2,4,6\}$	–
$\{1\}$	$\{3,5,7\}$	–
$\{2\}$	$\{0,4,6\}$	–
$\{3\}$	$\{1,5,7\}$	–
$\{4\}$	$\{0,2,6\}$	–
$\{5\}$	$\{1,3,7\}$	–
$\{6\}$	$\{0,2,4\}$	–
$\{7\}$	$\{1,3,5\}$	–

Phase 5, label s.

Source residues	Selected targets	Other targets
$\{0\}$	$\{2,4,6\}$	–
$\{1\}$	$\{3,5,7\}$	–
$\{2\}$	$\{0,4,6\}$	–
$\{3\}$	$\{1,5,7\}$	–
$\{4\}$	$\{0,2,6\}$	–
$\{5\}$	$\{1,3,7\}$	–
$\{6\}$	$\{0,2,4\}$	–
$\{7\}$	$\{1,3,5\}$	–

Phase 6, label g.

Source residues	Selected targets	Other targets
$\{0\}$	$\{0,2\}$	–
$\{1\}$	$\{1,3\}$	–
$\{2\}$	$\{2,4\}$	–
$\{3\}$	$\{3,5\}$	–
$\{4\}$	$\{4,6\}$	–
$\{5\}$	$\{5,7\}$	–
$\{6\}$	$\{0,6\}$	–
$\{7\}$	$\{1,7\}$	–

29.3 P2.3 Nondeterministic parity, width 8, length 9, target parity 1

All labelled edges preserve parity. Branching changes the set and number of feasible prefixes while preserving the parity obstruction when the target is odd.

The prescribed label sequence is $g, s, s, g, s, s, g, s, s$ and its observed state sequence is $0, 1, 2, 0, 1, 2, 0, 1, 2, 0$.

The initial set is $\{0:0, 0:2\}$ and the bad set is $\{9:1, 9:3\}$.

The exact lift count is 0. Allowing any label at each layer of this same observed state sequence gives 0 lifts.

i	Exact prefix	Exact suffix	Join
0	$\{0:0, 0:2\}$	$\{0:1, 0:3, 0:5, 0:7\}$	0
1	$\{1:0, 1:2, 1:4\}$	$\{1:1, 1:3, 1:5, 1:7\}$	0
2	$\{2:0, 2:2, 2:4, 2:6\}$	$\{2:1, 2:3, 2:5, 2:7\}$	0
3	$\{3:0, 3:2, 3:4, 3:6\}$	$\{3:1, 3:3, 3:5, 3:7\}$	0
4	$\{4:0, 4:2, 4:4, 4:6\}$	$\{4:1, 4:3, 4:5, 4:7\}$	0
5	$\{5:0, 5:2, 5:4, 5:6\}$	$\{5:1, 5:3, 5:5, 5:7\}$	0
6	$\{6:0, 6:2, 6:4, 6:6\}$	$\{6:1, 6:3, 6:5, 6:7\}$	0
7	$\{7:0, 7:2, 7:4, 7:6\}$	$\{7:1, 7:3, 7:5, 7:7\}$	0
8	$\{8:0, 8:2, 8:4, 8:6\}$	$\{8:1, 8:3, 8:5, 8:7\}$	0
9	$\{9:0, 9:2, 9:4, 9:6\}$	$\{9:1, 9:3\}$	0

Certificates. The least, extended, and greatest sequences give 13, 13, and 13 realized keys, respectively. All three refined labelled graphs have zero descendants of the selected path.

i	$\lvert A_i\rvert$	$\lvert J_i^{\min}\rvert$	$J_i^{\text{ext}} \setminus J_i^{\min}$	$\lvert J_i^{\max}\rvert$
0	32	2	$\{0:4\}$	28
1	24	3	$\{1:6\}$	20
2	24	4	$\varnothing$	20
3	32	4	$\varnothing$	28
4	24	4	$\varnothing$	20
5	24	4	$\varnothing$	20
6	32	4	$\varnothing$	28
7	24	4	$\varnothing$	20
8	24	4	$\varnothing$	20
9	32	4	$\varnothing$	30

Transition rows. Identical successor rows within a phase are grouped when they have the same selected-label and other-label outputs.

Phase 0, label g.

Source residues	Selected targets	Other targets
$\{0\}$	$\{0,2\}$	–
$\{1\}$	$\{1,3\}$	–
$\{2\}$	$\{2,4\}$	–
$\{3\}$	$\{3,5\}$	–
$\{4\}$	$\{4,6\}$	–
$\{5\}$	$\{5,7\}$	–
$\{6\}$	$\{0,6\}$	–
$\{7\}$	$\{1,7\}$	–

Phase 1, label s.

Source residues	Selected targets	Other targets
$\{0\}$	$\{2,4,6\}$	–
$\{1\}$	$\{3,5,7\}$	–
$\{2\}$	$\{0,4,6\}$	–
$\{3\}$	$\{1,5,7\}$	–
$\{4\}$	$\{0,2,6\}$	–
$\{5\}$	$\{1,3,7\}$	–
$\{6\}$	$\{0,2,4\}$	–
$\{7\}$	$\{1,3,5\}$	–

Phase 2, label s.

Source residues	Selected targets	Other targets
$\{0\}$	$\{2,4,6\}$	–
$\{1\}$	$\{3,5,7\}$	–
$\{2\}$	$\{0,4,6\}$	–
$\{3\}$	$\{1,5,7\}$	–
$\{4\}$	$\{0,2,6\}$	–
$\{5\}$	$\{1,3,7\}$	–
$\{6\}$	$\{0,2,4\}$	–
$\{7\}$	$\{1,3,5\}$	–

Phase 3, label g.

Source residues	Selected targets	Other targets
$\{0\}$	$\{0,2\}$	–
$\{1\}$	$\{1,3\}$	–
$\{2\}$	$\{2,4\}$	–
$\{3\}$	$\{3,5\}$	–
$\{4\}$	$\{4,6\}$	–
$\{5\}$	$\{5,7\}$	–
$\{6\}$	$\{0,6\}$	–
$\{7\}$	$\{1,7\}$	–

Phase 4, label s.

Source residues	Selected targets	Other targets
$\{0\}$	$\{2,4,6\}$	–
$\{1\}$	$\{3,5,7\}$	–
$\{2\}$	$\{0,4,6\}$	–
$\{3\}$	$\{1,5,7\}$	–
$\{4\}$	$\{0,2,6\}$	–
$\{5\}$	$\{1,3,7\}$	–
$\{6\}$	$\{0,2,4\}$	–
$\{7\}$	$\{1,3,5\}$	–

Phase 5, label s.

Source residues	Selected targets	Other targets
$\{0\}$	$\{2,4,6\}$	–
$\{1\}$	$\{3,5,7\}$	–
$\{2\}$	$\{0,4,6\}$	–
$\{3\}$	$\{1,5,7\}$	–
$\{4\}$	$\{0,2,6\}$	–
$\{5\}$	$\{1,3,7\}$	–
$\{6\}$	$\{0,2,4\}$	–
$\{7\}$	$\{1,3,5\}$	–

Phase 6, label g.

Source residues	Selected targets	Other targets
$\{0\}$	$\{0,2\}$	–
$\{1\}$	$\{1,3\}$	–
$\{2\}$	$\{2,4\}$	–
$\{3\}$	$\{3,5\}$	–
$\{4\}$	$\{4,6\}$	–
$\{5\}$	$\{5,7\}$	–
$\{6\}$	$\{0,6\}$	–
$\{7\}$	$\{1,7\}$	–

Phase 7, label s.

Source residues	Selected targets	Other targets
$\{0\}$	$\{2,4,6\}$	–
$\{1\}$	$\{3,5,7\}$	–
$\{2\}$	$\{0,4,6\}$	–
$\{3\}$	$\{1,5,7\}$	–
$\{4\}$	$\{0,2,6\}$	–
$\{5\}$	$\{1,3,7\}$	–
$\{6\}$	$\{0,2,4\}$	–
$\{7\}$	$\{1,3,5\}$	–

Phase 8, label s.

Source residues	Selected targets	Other targets
$\{0\}$	$\{2,4,6\}$	–
$\{1\}$	$\{3,5,7\}$	–
$\{2\}$	$\{0,4,6\}$	–
$\{3\}$	$\{1,5,7\}$	–
$\{4\}$	$\{0,2,6\}$	–
$\{5\}$	$\{1,3,7\}$	–
$\{6\}$	$\{0,2,4\}$	–
$\{7\}$	$\{1,3,5\}$	–

29.4 P2.4 Nondeterministic parity, width 10, length 9, target parity 1

All labelled edges preserve parity. Branching changes the set and number of feasible prefixes while preserving the parity obstruction when the target is odd.

The prescribed label sequence is $g, s, s, g, s, s, g, s, s$ and its observed state sequence is $0, 1, 0, 1, 0, 1, 0, 1, 0, 1$.

The initial set is $\{0:0, 0:2\}$ and the bad set is $\{9:1, 9:3\}$.

The exact lift count is 0. Allowing any label at each layer of this same observed state sequence gives 0 lifts.

i	Exact prefix	Exact suffix	Join
0	$\{0:0, 0:2\}$	$\{0:1, 0:3, 0:5, 0:7, 0:9\}$	0
1	$\{1:0, 1:2, 1:4\}$	$\{1:1, 1:3, 1:5, 1:7, 1:9\}$	0
2	$\{2:0, 2:2, 2:4, 2:6, 2:8\}$	$\{2:1, 2:3, 2:5, 2:7, 2:9\}$	0
3	$\{3:0, 3:2, 3:4, 3:6, 3:8\}$	$\{3:1, 3:3, 3:5, 3:7, 3:9\}$	0
4	$\{4:0, 4:2, 4:4, 4:6, 4:8\}$	$\{4:1, 4:3, 4:5, 4:7, 4:9\}$	0
5	$\{5:0, 5:2, 5:4, 5:6, 5:8\}$	$\{5:1, 5:3, 5:5, 5:7, 5:9\}$	0
6	$\{6:0, 6:2, 6:4, 6:6, 6:8\}$	$\{6:1, 6:3, 6:5, 6:7, 6:9\}$	0
7	$\{7:0, 7:2, 7:4, 7:6, 7:8\}$	$\{7:1, 7:3, 7:5, 7:7, 7:9\}$	0
8	$\{8:0, 8:2, 8:4, 8:6, 8:8\}$	$\{8:1, 8:3, 8:5, 8:7, 8:9\}$	0
9	$\{9:0, 9:2, 9:4, 9:6, 9:8\}$	$\{9:1, 9:3\}$	0

Certificates. The least, extended, and greatest sequences give 12, 12, and 12 realized keys, respectively. All three refined labelled graphs have zero descendants of the selected path.

i	$\lvert A_i\rvert$	$\lvert J_i^{\min}\rvert$	$J_i^{\mathrm{ext}} \setminus J_i^{\min}$	$\lvert J_i^{\max}\rvert$
0	50	2	$\{0:4\}$	45
1	50	3	$\{1:6\}$	45
2	50	5	$\varnothing$	45
3	50	5	$\varnothing$	45
4	50	5	$\varnothing$	45
5	50	5	$\varnothing$	45
6	50	5	$\varnothing$	45
7	50	5	$\varnothing$	45
8	50	5	$\varnothing$	45
9	50	5	$\varnothing$	48

Transition rows. Identical successor rows within a phase are grouped when they have the same selected-label and other-label outputs.

Phase 0, label g.

Source residues	Selected targets	Other targets
$\{0\}$	$\{0,2\}$	–
$\{1\}$	$\{1,3\}$	–
$\{2\}$	$\{2,4\}$	–
$\{3\}$	$\{3,5\}$	–
$\{4\}$	$\{4,6\}$	–
$\{5\}$	$\{5,7\}$	–
$\{6\}$	$\{6,8\}$	–
$\{7\}$	$\{7,9\}$	–
$\{8\}$	$\{0,8\}$	–
$\{9\}$	$\{1,9\}$	–

Phase 1, label s.

Source residues	Selected targets	Other targets
{0}	{2, 4, 8}	–
{1}	{3, 5, 9}	–
{2}	{0, 4, 6}	–
{3}	{1, 5, 7}	–
{4}	{2, 6, 8}	–
{5}	{3, 7, 9}	–
{6}	{0, 4, 8}	–
{7}	{1, 5, 9}	–
{8}	{0, 2, 6}	–
{9}	{1, 3, 7}	–

Phase 2, label s.

Source residues	Selected targets	Other targets
{0}	{2, 4, 8}	–
{1}	{3, 5, 9}	–
{2}	{0, 4, 6}	–
{3}	{1, 5, 7}	–
{4}	{2, 6, 8}	–
{5}	{3, 7, 9}	–
{6}	{0, 4, 8}	–
{7}	{1, 5, 9}	–
{8}	{0, 2, 6}	–
{9}	{1, 3, 7}	–

Phase 3, label g.

Source residues	Selected targets	Other targets
{0}	{0, 2}	–
{1}	{1, 3}	–
{2}	{2, 4}	–
{3}	{3, 5}	–
{4}	{4, 6}	–
{5}	{5, 7}	–
{6}	{6, 8}	–
{7}	{7, 9}	–
{8}	{0, 8}	–
{9}	{1, 9}	–

Phase 4, label s.

Source residues	Selected targets	Other targets
{0}	{2, 4, 8}	–
{1}	{3, 5, 9}	–
{2}	{0, 4, 6}	–
{3}	{1, 5, 7}	–
{4}	{2, 6, 8}	–
{5}	{3, 7, 9}	–
{6}	{0, 4, 8}	–
{7}	{1, 5, 9}	–
{8}	{0, 2, 6}	–
{9}	{1, 3, 7}	–

Phase 5, label s.

Source residues	Selected targets	Other targets
{0}	{2, 4, 8}	–
{1}	{3, 5, 9}	–
{2}	{0, 4, 6}	–
{3}	{1, 5, 7}	–
{4}	{2, 6, 8}	–
{5}	{3, 7, 9}	–
{6}	{0, 4, 8}	–
{7}	{1, 5, 9}	–
{8}	{0, 2, 6}	–
{9}	{1, 3, 7}	–

Phase 6, label g.

Source residues	Selected targets	Other targets
{0}	{0, 2}	–
{1}	{1, 3}	–
{2}	{2, 4}	–
{3}	{3, 5}	–
{4}	{4, 6}	–
{5}	{5, 7}	–
{6}	{6, 8}	–
{7}	{7, 9}	–
{8}	{0, 8}	–
{9}	{1, 9}	–

Phase 7, label s.

Source residues	Selected targets	Other targets
{0}	{2, 4, 8}	–
{1}	{3, 5, 9}	–
{2}	{0, 4, 6}	–
{3}	{1, 5, 7}	–
{4}	{2, 6, 8}	–
{5}	{3, 7, 9}	–
{6}	{0, 4, 8}	–
{7}	{1, 5, 9}	–
{8}	{0, 2, 6}	–
{9}	{1, 3, 7}	–

Phase 8, label s.

Source residues	Selected targets	Other targets
{0}	{2, 4, 8}	–
{1}	{3, 5, 9}	–
{2}	{0, 4, 6}	–
{3}	{1, 5, 7}	–
{4}	{2, 6, 8}	–
{5}	{3, 7, 9}	–
{6}	{0, 4, 8}	–
{7}	{1, 5, 9}	–
{8}	{0, 2, 6}	–
{9}	{1, 3, 7}	–

29.5 P2.5 Nondeterministic parity, width 10, length 10, target parity 0

All labelled edges preserve parity. Branching changes the set and number of feasible prefixes while preserving the parity obstruction when the target is odd.

The prescribed label sequence is $g, s, s, g, s, s, g, s, s, g$ and its observed state sequence is $0, 1, 2, 0, 1, 2, 0, 1, 2, 0, 1$.

The initial set is $\{0 : 0, 0 : 2\}$ and the bad set is $\{10 : 0, 10 : 2, 10 : 4\}$.

The exact lift count is 13997. Allowing any label at each layer of this same observed state sequence gives 13997 lifts.

i	Exact prefix	Exact suffix	Join
0	$\{0:0,0:2\}$	$\{0:0,0:2,0:4,0:6,0:8\}$	2
1	$\{1:0,1:2,1:4\}$	$\{1:0,1:2,1:4,1:6,1:8\}$	3
2	$\{2:0,2:2,2:4,2:6,2:8\}$	$\{2:0,2:2,2:4,2:6,2:8\}$	5
3	$\{3:0,3:2,3:4,3:6,3:8\}$	$\{3:0,3:2,3:4,3:6,3:8\}$	5
4	$\{4:0,4:2,4:4,4:6,4:8\}$	$\{4:0,4:2,4:4,4:6,4:8\}$	5
5	$\{5:0,5:2,5:4,5:6,5:8\}$	$\{5:0,5:2,5:4,5:6,5:8\}$	5
6	$\{6:0,6:2,6:4,6:6,6:8\}$	$\{6:0,6:2,6:4,6:6,6:8\}$	5
7	$\{7:0,7:2,7:4,7:6,7:8\}$	$\{7:0,7:2,7:4,7:6,7:8\}$	5
8	$\{8:0,8:2,8:4,8:6,8:8\}$	$\{8:0,8:2,8:4,8:6,8:8\}$	5
9	$\{9:0,9:2,9:4,9:6,9:8\}$	$\{9:0,9:2,9:4,9:8\}$	4
10	$\{10:0,10:2,10:4,10:6,10:8\}$	$\{10:0,10:2,10:4\}$	3

Witness. $0:0, 1:0, 2:2, 3:0, 4:0, 5:2, 6:0, 7:0, 8:2, 9:0, 10:0$ follows the selected label at every step.

Transition rows. Identical successor rows within a phase are grouped when they have the same selected-label and other-label outputs.

Phase 0, label g.

Source residues	Selected targets	Other targets
$\{0\}$	$\{0,2\}$	–
$\{1\}$	$\{1,3\}$	–
$\{2\}$	$\{2,4\}$	–
$\{3\}$	$\{3,5\}$	–
$\{4\}$	$\{4,6\}$	–
$\{5\}$	$\{5,7\}$	–
$\{6\}$	$\{6,8\}$	–
$\{7\}$	$\{7,9\}$	–
$\{8\}$	$\{0,8\}$	–
$\{9\}$	$\{1,9\}$	–

Phase 1, label s.

Source residues	Selected targets	Other targets
$\{0\}$	$\{2,4,8\}$	–
$\{1\}$	$\{3,5,9\}$	–
$\{2\}$	$\{0,4,6\}$	–
$\{3\}$	$\{1,5,7\}$	–
$\{4\}$	$\{2,6,8\}$	–
$\{5\}$	$\{3,7,9\}$	–
$\{6\}$	$\{0,4,8\}$	–
$\{7\}$	$\{1,5,9\}$	–
$\{8\}$	$\{0,2,6\}$	–
$\{9\}$	$\{1,3,7\}$	–

Phase 2, label s.

Source residues	Selected targets	Other targets
$\{0\}$	$\{2,4,8\}$	–
$\{1\}$	$\{3,5,9\}$	–
$\{2\}$	$\{0,4,6\}$	–
$\{3\}$	$\{1,5,7\}$	–
$\{4\}$	$\{2,6,8\}$	–
$\{5\}$	$\{3,7,9\}$	–
$\{6\}$	$\{0,4,8\}$	–
$\{7\}$	$\{1,5,9\}$	–
$\{8\}$	$\{0,2,6\}$	–
$\{9\}$	$\{1,3,7\}$	–

Phase 3, label g.

Source residues	Selected targets	Other targets
$\{0\}$	$\{0,2\}$	–
$\{1\}$	$\{1,3\}$	–
$\{2\}$	$\{2,4\}$	–
$\{3\}$	$\{3,5\}$	–
$\{4\}$	$\{4,6\}$	–
$\{5\}$	$\{5,7\}$	–
$\{6\}$	$\{6,8\}$	–
$\{7\}$	$\{7,9\}$	–
$\{8\}$	$\{0,8\}$	–
$\{9\}$	$\{1,9\}$	–

Phase 4, label s.

Source residues	Selected targets	Other targets
$\{0\}$	$\{2,4,8\}$	–
$\{1\}$	$\{3,5,9\}$	–
$\{2\}$	$\{0,4,6\}$	–
$\{3\}$	$\{1,5,7\}$	–
$\{4\}$	$\{2,6,8\}$	–
$\{5\}$	$\{3,7,9\}$	–
$\{6\}$	$\{0,4,8\}$	–
$\{7\}$	$\{1,5,9\}$	–
$\{8\}$	$\{0,2,6\}$	–
$\{9\}$	$\{1,3,7\}$	–

Phase 5, label s.

Source residues	Selected targets	Other targets
$\{0\}$	$\{2,4,8\}$	–
$\{1\}$	$\{3,5,9\}$	–
$\{2\}$	$\{0,4,6\}$	–
$\{3\}$	$\{1,5,7\}$	–
$\{4\}$	$\{2,6,8\}$	–
$\{5\}$	$\{3,7,9\}$	–
$\{6\}$	$\{0,4,8\}$	–
$\{7\}$	$\{1,5,9\}$	–
$\{8\}$	$\{0,2,6\}$	–
$\{9\}$	$\{1,3,7\}$	–

Phase 6, label g.

Source residues	Selected targets	Other targets
$\{0\}$	$\{0,2\}$	–
$\{1\}$	$\{1,3\}$	–
$\{2\}$	$\{2,4\}$	–
$\{3\}$	$\{3,5\}$	–
$\{4\}$	$\{4,6\}$	–
$\{5\}$	$\{5,7\}$	–
$\{6\}$	$\{6,8\}$	–
$\{7\}$	$\{7,9\}$	–
$\{8\}$	$\{0,8\}$	–
$\{9\}$	$\{1,9\}$	–

Phase 7, label s.

Source residues	Selected targets	Other targets
$\{0\}$	$\{2,4,8\}$	–
$\{1\}$	$\{3,5,9\}$	–
$\{2\}$	$\{0,4,6\}$	–
$\{3\}$	$\{1,5,7\}$	–
$\{4\}$	$\{2,6,8\}$	–
$\{5\}$	$\{3,7,9\}$	–
$\{6\}$	$\{0,4,8\}$	–
$\{7\}$	$\{1,5,9\}$	–
$\{8\}$	$\{0,2,6\}$	–
$\{9\}$	$\{1,3,7\}$	–

Phase 8, label s.

Source residues	Selected targets	Other targets
$\{0\}$	$\{2,4,8\}$	–
$\{1\}$	$\{3,5,9\}$	–
$\{2\}$	$\{0,4,6\}$	–
$\{3\}$	$\{1,5,7\}$	–
$\{4\}$	$\{2,6,8\}$	–
$\{5\}$	$\{3,7,9\}$	–
$\{6\}$	$\{0,4,8\}$	–
$\{7\}$	$\{1,5,9\}$	–
$\{8\}$	$\{0,2,6\}$	–
$\{9\}$	$\{1,3,7\}$	–

Phase 9, label g.

Source residues	Selected targets	Other targets
$\{0\}$	$\{0,2\}$	–
$\{1\}$	$\{1,3\}$	–
$\{2\}$	$\{2,4\}$	–
$\{3\}$	$\{3,5\}$	–
$\{4\}$	$\{4,6\}$	–
$\{5\}$	$\{5,7\}$	–
$\{6\}$	$\{6,8\}$	–
$\{7\}$	$\{7,9\}$	–
$\{8\}$	$\{0,8\}$	–
$\{9\}$	$\{1,9\}$	–

29.6 P2.6 Nondeterministic parity, width 12, length 11, target parity 1

All labelled edges preserve parity. Branching changes the set and number of feasible prefixes while preserving the parity obstruction when the target is odd.

The prescribed label sequence is $g, s, s, g, s, s, g, s, s, g, s$ and its observed state sequence is $0, 1, 0, 1, 0, 1, 0, 1, 0, 1, 0, 1$.

The initial set is $\{0:0, 0:2\}$ and the bad set is $\{11:1, 11:3\}$.

The exact lift count is 0. Allowing any label at each layer of this same observed state sequence gives 0 lifts.

i	Exact prefix	Exact suffix	Join
0	$\{0:0, 0:2\}$	$\{0:1, 0:3, 0:5, 0:7, 0:9, 0:11\}$	0
1	$\{1:0, 1:2, 1:4\}$	$\{1:1, 1:3, 1:5, 1:7, 1:9, 1:11\}$	0
2	$\{2:0, 2:2, 2:4, 2:6, 2:8, 2:10\}$	$\{2:1, 2:3, 2:5, 2:7, 2:9, 2:11\}$	0
3	$\{3:0, 3:2, 3:4, 3:6, 3:8, 3:10\}$	$\{3:1, 3:3, 3:5, 3:7, 3:9, 3:11\}$	0
4	$\{4:0, 4:2, 4:4, 4:6, 4:8, 4:10\}$	$\{4:1, 4:3, 4:5, 4:7, 4:9, 4:11\}$	0
5	$\{5:0, 5:2, 5:4, 5:6, 5:8, 5:10\}$	$\{5:1, 5:3, 5:5, 5:7, 5:9, 5:11\}$	0
6	$\{6:0, 6:2, 6:4, 6:6, 6:8, 6:10\}$	$\{6:1, 6:3, 6:5, 6:7, 6:9, 6:11\}$	0
7	$\{7:0, 7:2, 7:4, 7:6, 7:8, 7:10\}$	$\{7:1, 7:3, 7:5, 7:7, 7:9, 7:11\}$	0
8	$\{8:0, 8:2, 8:4, 8:6, 8:8, 8:10\}$	$\{8:1, 8:3, 8:5, 8:7, 8:9, 8:11\}$	0
9	$\{9:0, 9:2, 9:4, 9:6, 9:8, 9:10\}$	$\{9:1, 9:3, 9:5, 9:7, 9:9, 9:11\}$	0
10	$\{10:0, 10:2, 10:4, 10:6, 10:8, 10:10\}$	$\{10:1, 10:3, 10:5, 10:9, 10:11\}$	0
11	$\{11:0, 11:2, 11:4, 11:6, 11:8, 11:10\}$	$\{11:1, 11:3\}$	0

Certificates. The least, extended, and greatest sequences give 14, 14, and 14 realized keys, respectively. All three refined labelled graphs have zero descendants of the selected path.

i	$\lvert A_i\rvert$	$\lvert J_i^{\min}\rvert$	$J_i^{\text{ext}} \setminus J_i^{\min}$	$\lvert J_i^{\max}\rvert$
0	72	2	$\{0:4\}$	66
1	72	3	$\{1:6\}$	66
2	72	6	$\varnothing$	66
3	72	6	$\varnothing$	66
4	72	6	$\varnothing$	66
5	72	6	$\varnothing$	66
6	72	6	$\varnothing$	66
7	72	6	$\varnothing$	66
8	72	6	$\varnothing$	66
9	72	6	$\varnothing$	66
10	72	6	$\varnothing$	67

i	$\lvert A_i\rvert$	$\lvert J_i^{\min}\rvert$	$J_i^{\text{ext}} \setminus J_i^{\min}$	$\lvert J_i^{\max}\rvert$
11	72	6	$\varnothing$	70

Transition rows. Identical successor rows within a phase are grouped when they have the same selected-label and other-label outputs.

Phase 0, label g.

Source residues	Selected targets	Other targets
$\{0\}$	$\{0,2\}$	–
$\{1\}$	$\{1,3\}$	–
$\{2\}$	$\{2,4\}$	–
$\{3\}$	$\{3,5\}$	–
$\{4\}$	$\{4,6\}$	–
$\{5\}$	$\{5,7\}$	–
$\{6\}$	$\{6,8\}$	–
$\{7\}$	$\{7,9\}$	–
$\{8\}$	$\{8,10\}$	–
$\{9\}$	$\{9,11\}$	–
$\{10\}$	$\{0,10\}$	–
$\{11\}$	$\{1,11\}$	–

Phase 1, label s.

Source residues	Selected targets	Other targets
$\{0\}$	$\{2,4,10\}$	–
$\{1\}$	$\{3,5,11\}$	–
$\{2\}$	$\{0,4,6\}$	–
$\{3\}$	$\{1,5,7\}$	–
$\{4\}$	$\{2,6,8\}$	–
$\{5\}$	$\{3,7,9\}$	–
$\{6\}$	$\{4,8,10\}$	–
$\{7\}$	$\{5,9,11\}$	–
$\{8\}$	$\{0,6,10\}$	–
$\{9\}$	$\{1,7,11\}$	–
$\{10\}$	$\{0,2,8\}$	–
$\{11\}$	$\{1,3,9\}$	–

Phase 2, label s.

Source residues	Selected targets	Other targets
$\{0\}$	$\{2,4,10\}$	–
$\{1\}$	$\{3,5,11\}$	–
$\{2\}$	$\{0,4,6\}$	–
$\{3\}$	$\{1,5,7\}$	–
$\{4\}$	$\{2,6,8\}$	–
$\{5\}$	$\{3,7,9\}$	–
$\{6\}$	$\{4,8,10\}$	–
$\{7\}$	$\{5,9,11\}$	–
$\{8\}$	$\{0,6,10\}$	–
$\{9\}$	$\{1,7,11\}$	–
$\{10\}$	$\{0,2,8\}$	–
$\{11\}$	$\{1,3,9\}$	–

Phase 3, label g.

Source residues	Selected targets	Other targets
$\{0\}$	$\{0,2\}$	–
$\{1\}$	$\{1,3\}$	–
$\{2\}$	$\{2,4\}$	–
$\{3\}$	$\{3,5\}$	–
$\{4\}$	$\{4,6\}$	–
$\{5\}$	$\{5,7\}$	–
$\{6\}$	$\{6,8\}$	–
$\{7\}$	$\{7,9\}$	–
$\{8\}$	$\{8,10\}$	–
$\{9\}$	$\{9,11\}$	–
$\{10\}$	$\{0,10\}$	–
$\{11\}$	$\{1,11\}$	–

Phase 4, label s.

Source residues	Selected targets	Other targets
$\{0\}$	$\{2,4,10\}$	–
$\{1\}$	$\{3,5,11\}$	–
$\{2\}$	$\{0,4,6\}$	–
$\{3\}$	$\{1,5,7\}$	–
$\{4\}$	$\{2,6,8\}$	–
$\{5\}$	$\{3,7,9\}$	–
$\{6\}$	$\{4,8,10\}$	–
$\{7\}$	$\{5,9,11\}$	–
$\{8\}$	$\{0,6,10\}$	–
$\{9\}$	$\{1,7,11\}$	–
$\{10\}$	$\{0,2,8\}$	–
$\{11\}$	$\{1,3,9\}$	–

Phase 5, label s.

Source residues	Selected targets	Other targets
$\{0\}$	$\{2, 4, 10\}$	–
$\{1\}$	$\{3, 5, 11\}$	–
$\{2\}$	$\{0, 4, 6\}$	–
$\{3\}$	$\{1, 5, 7\}$	–
$\{4\}$	$\{2, 6, 8\}$	–
$\{5\}$	$\{3, 7, 9\}$	–
$\{6\}$	$\{4, 8, 10\}$	–
$\{7\}$	$\{5, 9, 11\}$	–
$\{8\}$	$\{0, 6, 10\}$	–
$\{9\}$	$\{1, 7, 11\}$	–
$\{10\}$	$\{0, 2, 8\}$	–
$\{11\}$	$\{1, 3, 9\}$	–

Phase 6, label g.

Source residues	Selected targets	Other targets
$\{0\}$	$\{0, 2\}$	–
$\{1\}$	$\{1, 3\}$	–
$\{2\}$	$\{2, 4\}$	–
$\{3\}$	$\{3, 5\}$	–
$\{4\}$	$\{4, 6\}$	–
$\{5\}$	$\{5, 7\}$	–
$\{6\}$	$\{6, 8\}$	–
$\{7\}$	$\{7, 9\}$	–
$\{8\}$	$\{8, 10\}$	–
$\{9\}$	$\{9, 11\}$	–
$\{10\}$	$\{0, 10\}$	–
$\{11\}$	$\{1, 11\}$	–

Phase 7, label s.

Source residues	Selected targets	Other targets
$\{0\}$	$\{2, 4, 10\}$	–
$\{1\}$	$\{3, 5, 11\}$	–
$\{2\}$	$\{0, 4, 6\}$	–
$\{3\}$	$\{1, 5, 7\}$	–
$\{4\}$	$\{2, 6, 8\}$	–
$\{5\}$	$\{3, 7, 9\}$	–
$\{6\}$	$\{4, 8, 10\}$	–
$\{7\}$	$\{5, 9, 11\}$	–
$\{8\}$	$\{0, 6, 10\}$	–
$\{9\}$	$\{1, 7, 11\}$	–
$\{10\}$	$\{0, 2, 8\}$	–
$\{11\}$	$\{1, 3, 9\}$	–

Phase 8, label s.

Source residues	Selected targets	Other targets
$\{0\}$	$\{2, 4, 10\}$	–
$\{1\}$	$\{3, 5, 11\}$	–
$\{2\}$	$\{0, 4, 6\}$	–
$\{3\}$	$\{1, 5, 7\}$	–
$\{4\}$	$\{2, 6, 8\}$	–
$\{5\}$	$\{3, 7, 9\}$	–
$\{6\}$	$\{4, 8, 10\}$	–
$\{7\}$	$\{5, 9, 11\}$	–
$\{8\}$	$\{0, 6, 10\}$	–
$\{9\}$	$\{1, 7, 11\}$	–
$\{10\}$	$\{0, 2, 8\}$	–
$\{11\}$	$\{1, 3, 9\}$	–

Phase 9, label g.

Source residues	Selected targets	Other targets
$\{0\}$	$\{0, 2\}$	–
$\{1\}$	$\{1, 3\}$	–
$\{2\}$	$\{2, 4\}$	–
$\{3\}$	$\{3, 5\}$	–
$\{4\}$	$\{4, 6\}$	–
$\{5\}$	$\{5, 7\}$	–
$\{6\}$	$\{6, 8\}$	–
$\{7\}$	$\{7, 9\}$	–
$\{8\}$	$\{8, 10\}$	–
$\{9\}$	$\{9, 11\}$	–
$\{10\}$	$\{0, 10\}$	–
$\{11\}$	$\{1, 11\}$	–

Phase 10, label s.

Source residues	Selected targets	Other targets
$\{0\}$	$\{2, 4, 10\}$	–
$\{1\}$	$\{3, 5, 11\}$	–
$\{2\}$	$\{0, 4, 6\}$	–
$\{3\}$	$\{1, 5, 7\}$	–
$\{4\}$	$\{2, 6, 8\}$	–
$\{5\}$	$\{3, 7, 9\}$	–
$\{6\}$	$\{4, 8, 10\}$	–
$\{7\}$	$\{5, 9, 11\}$	–
$\{8\}$	$\{0, 6, 10\}$	–
$\{9\}$	$\{1, 7, 11\}$	–
$\{10\}$	$\{0, 2, 8\}$	–
$\{11\}$	$\{1, 3, 9\}$	–

29.7 P2.7 Nondeterministic parity, width 12, length 12, target parity 0

All labelled edges preserve parity. Branching changes the set and number of feasible prefixes while preserving the parity obstruction when the target is odd.

The prescribed label sequence is $g, s, s, g, s, s, g, s, s, g, s, s$ and its observed state sequence is $0, 1, 2, 3, 0, 1, 2, 3, 0, 1, 2, 3, 0$.

The initial set is $\{0:0, 0:2\}$ and the bad set is $\{12:0, 12:2, 12:4\}$.

The exact lift count is 104967. Allowing any label at each layer of this same observed state sequence gives 104967 lifts.

i	Exact prefix	Exact suffix	Join
0	$\{0:0, 0:2\}$	$\{0:0, 0:2, 0:4, 0:6, 0:8, 0:10\}$	2
1	$\{1:0, 1:2, 1:4\}$	$\{1:0, 1:2, 1:4, 1:6, 1:8, 1:10\}$	3
2	$\{2:0, 2:2, 2:4, 2:6, 2:8, 2:10\}$	$\{2:0, 2:2, 2:4, 2:6, 2:8, 2:10\}$	6
3	$\{3:0, 3:2, 3:4, 3:6, 3:8, 3:10\}$	$\{3:0, 3:2, 3:4, 3:6, 3:8, 3:10\}$	6
4	$\{4:0, 4:2, 4:4, 4:6, 4:8, 4:10\}$	$\{4:0, 4:2, 4:4, 4:6, 4:8, 4:10\}$	6
5	$\{5:0, 5:2, 5:4, 5:6, 5:8, 5:10\}$	$\{5:0, 5:2, 5:4, 5:6, 5:8, 5:10\}$	6
6	$\{6:0, 6:2, 6:4, 6:6, 6:8, 6:10\}$	$\{6:0, 6:2, 6:4, 6:6, 6:8, 6:10\}$	6
7	$\{7:0, 7:2, 7:4, 7:6, 7:8, 7:10\}$	$\{7:0, 7:2, 7:4, 7:6, 7:8, 7:10\}$	6
8	$\{8:0, 8:2, 8:4, 8:6, 8:8, 8:10\}$	$\{8:0, 8:2, 8:4, 8:6, 8:8, 8:10\}$	6
9	$\{9:0, 9:2, 9:4, 9:6, 9:8, 9:10\}$	$\{9:0, 9:2, 9:4, 9:6, 9:8, 9:10\}$	6
10	$\{10:0, 10:2, 10:4, 10:6, 10:8, 10:10\}$	$\{10:0, 10:2, 10:4, 10:6, 10:8, 10:10\}$	6
11	$\{11:0, 11:2, 11:4, 11:6, 11:8, 11:10\}$	$\{11:0, 11:2, 11:4, 11:6, 11:8, 11:10\}$	6
12	$\{12:0, 12:2, 12:4, 12:6, 12:8, 12:10\}$	$\{12:0, 12:2, 12:4\}$	3

Witness. $0:0, 1:0, 2:2, 3:0, 4:0, 5:2, 6:0, 7:0, 8:2, 9:0, 10:0, 11:2, 12:0$ follows the selected label at every step.

Transition rows. Identical successor rows within a phase are grouped when they have the same selected-label and other-label outputs.

Phase 0, label g.

Source residues	Selected targets	Other targets
$\{0\}$	$\{0, 2\}$	–
$\{1\}$	$\{1, 3\}$	–
$\{2\}$	$\{2, 4\}$	–
$\{3\}$	$\{3, 5\}$	–
$\{4\}$	$\{4, 6\}$	–
$\{5\}$	$\{5, 7\}$	–
$\{6\}$	$\{6, 8\}$	–
$\{7\}$	$\{7, 9\}$	–
$\{8\}$	$\{8, 10\}$	–
$\{9\}$	$\{9, 11\}$	–
$\{10\}$	$\{0, 10\}$	–
$\{11\}$	$\{1, 11\}$	–

Phase 1, label s.

Source residues	Selected targets	Other targets
$\{0\}$	$\{2, 4, 10\}$	–
$\{1\}$	$\{3, 5, 11\}$	–
$\{2\}$	$\{0, 4, 6\}$	–
$\{3\}$	$\{1, 5, 7\}$	–
$\{4\}$	$\{2, 6, 8\}$	–
$\{5\}$	$\{3, 7, 9\}$	–
$\{6\}$	$\{4, 8, 10\}$	–
$\{7\}$	$\{5, 9, 11\}$	–
$\{8\}$	$\{0, 6, 10\}$	–
$\{9\}$	$\{1, 7, 11\}$	–
$\{10\}$	$\{0, 2, 8\}$	–
$\{11\}$	$\{1, 3, 9\}$	–

Phase 2, label s.

Source residues	Selected targets	Other targets
$\{0\}$	$\{2, 4, 10\}$	–
$\{1\}$	$\{3, 5, 11\}$	–
$\{2\}$	$\{0, 4, 6\}$	–
$\{3\}$	$\{1, 5, 7\}$	–
$\{4\}$	$\{2, 6, 8\}$	–
$\{5\}$	$\{3, 7, 9\}$	–
$\{6\}$	$\{4, 8, 10\}$	–
$\{7\}$	$\{5, 9, 11\}$	–
$\{8\}$	$\{0, 6, 10\}$	–
$\{9\}$	$\{1, 7, 11\}$	–
$\{10\}$	$\{0, 2, 8\}$	–
$\{11\}$	$\{1, 3, 9\}$	–

Phase 3, label g.

Source residues	Selected targets	Other targets
{0}	{0, 2}	–
{1}	{1, 3}	–
{2}	{2, 4}	–
{3}	{3, 5}	–
{4}	{4, 6}	–
{5}	{5, 7}	–
{6}	{6, 8}	–
{7}	{7, 9}	–
{8}	{8, 10}	–
{9}	{9, 11}	–
{10}	{0, 10}	–
{11}	{1, 11}	–

Phase 4, label s.

Source residues	Selected targets	Other targets
{0}	{2, 4, 10}	–
{1}	{3, 5, 11}	–
{2}	{0, 4, 6}	–
{3}	{1, 5, 7}	–
{4}	{2, 6, 8}	–
{5}	{3, 7, 9}	–
{6}	{4, 8, 10}	–
{7}	{5, 9, 11}	–
{8}	{0, 6, 10}	–
{9}	{1, 7, 11}	–
{10}	{0, 2, 8}	–
{11}	{1, 3, 9}	–

Phase 5, label s.

Source residues	Selected targets	Other targets
{0}	{2, 4, 10}	–
{1}	{3, 5, 11}	–
{2}	{0, 4, 6}	–
{3}	{1, 5, 7}	–
{4}	{2, 6, 8}	–
{5}	{3, 7, 9}	–
{6}	{4, 8, 10}	–
{7}	{5, 9, 11}	–
{8}	{0, 6, 10}	–
{9}	{1, 7, 11}	–
{10}	{0, 2, 8}	–
{11}	{1, 3, 9}	–

Phase 6, label g.

Source residues	Selected targets	Other targets
{0}	{0, 2}	–
{1}	{1, 3}	–
{2}	{2, 4}	–
{3}	{3, 5}	–
{4}	{4, 6}	–
{5}	{5, 7}	–
{6}	{6, 8}	–
{7}	{7, 9}	–
{8}	{8, 10}	–
{9}	{9, 11}	–
{10}	{0, 10}	–
{11}	{1, 11}	–

Phase 7, label s.

Source residues	Selected targets	Other targets
{0}	{2, 4, 10}	–
{1}	{3, 5, 11}	–
{2}	{0, 4, 6}	–
{3}	{1, 5, 7}	–
{4}	{2, 6, 8}	–
{5}	{3, 7, 9}	–
{6}	{4, 8, 10}	–
{7}	{5, 9, 11}	–
{8}	{0, 6, 10}	–
{9}	{1, 7, 11}	–
{10}	{0, 2, 8}	–
{11}	{1, 3, 9}	–

Phase 8, label s.

Source residues	Selected targets	Other targets
{0}	{2, 4, 10}	–
{1}	{3, 5, 11}	–
{2}	{0, 4, 6}	–
{3}	{1, 5, 7}	–
{4}	{2, 6, 8}	–
{5}	{3, 7, 9}	–
{6}	{4, 8, 10}	–
{7}	{5, 9, 11}	–
{8}	{0, 6, 10}	–
{9}	{1, 7, 11}	–
{10}	{0, 2, 8}	–
{11}	{1, 3, 9}	–

Phase 9, label g.

Source residues	Selected targets	Other targets
$\{0\}$	$\{0, 2\}$	–
$\{1\}$	$\{1, 3\}$	–
$\{2\}$	$\{2, 4\}$	–
$\{3\}$	$\{3, 5\}$	–
$\{4\}$	$\{4, 6\}$	–
$\{5\}$	$\{5, 7\}$	–
$\{6\}$	$\{6, 8\}$	–
$\{7\}$	$\{7, 9\}$	–
$\{8\}$	$\{8, 10\}$	–
$\{9\}$	$\{9, 11\}$	–
$\{10\}$	$\{0, 10\}$	–
$\{11\}$	$\{1, 11\}$	–

Phase 10, label s.

Source residues	Selected targets	Other targets
$\{0\}$	$\{2, 4, 10\}$	–
$\{1\}$	$\{3, 5, 11\}$	–
$\{2\}$	$\{0, 4, 6\}$	–
$\{3\}$	$\{1, 5, 7\}$	–
$\{4\}$	$\{2, 6, 8\}$	–
$\{5\}$	$\{3, 7, 9\}$	–
$\{6\}$	$\{4, 8, 10\}$	–
$\{7\}$	$\{5, 9, 11\}$	–
$\{8\}$	$\{0, 6, 10\}$	–
$\{9\}$	$\{1, 7, 11\}$	–
$\{10\}$	$\{0, 2, 8\}$	–
$\{11\}$	$\{1, 3, 9\}$	–

Phase 11, label s.

Source residues	Selected targets	Other targets
$\{0\}$	$\{2, 4, 10\}$	–
$\{1\}$	$\{3, 5, 11\}$	–
$\{2\}$	$\{0, 4, 6\}$	–
$\{3\}$	$\{1, 5, 7\}$	–
$\{4\}$	$\{2, 6, 8\}$	–
$\{5\}$	$\{3, 7, 9\}$	–
$\{6\}$	$\{4, 8, 10\}$	–
$\{7\}$	$\{5, 9, 11\}$	–
$\{8\}$	$\{0, 6, 10\}$	–
$\{9\}$	$\{1, 7, 11\}$	–
$\{10\}$	$\{0, 2, 8\}$	–
$\{11\}$	$\{1, 3, 9\}$	–

29.8 P2.8 Nondeterministic parity, width 14, length 13, target parity 1

All labelled edges preserve parity. Branching changes the set and number of feasible prefixes while preserving the parity obstruction when the target is odd.

The prescribed label sequence is $g, s, s, g, s, s, g, s, s, g, s, s, g$ and its observed state sequence is $0, 1, 2, 0, 1, 2, 0, 1, 2, 0, 1, 2, 0, 1$.

The initial set is $\{0 : 0, 0 : 2\}$ and the bad set is $\{13 : 1, 13 : 3\}$.

The exact lift count is 0. Allowing any label at each layer of this same observed state sequence gives 0 lifts.

i	Exact prefix	Exact suffix	Join
0	$\{0:0, 0:2\}$	$\{0:1, 0:3, 0:5, 0:7, 0:9, 0:11, 0:13\}$	0
1	$\{1:0, 1:2, 1:4\}$	$\{1:1, 1:3, 1:5, 1:7, 1:9, 1:11, 1:13\}$	0
2	$\{2:0, 2:2, 2:4, 2:6, 2:8, 2:12\}$	$\{2:1, 2:3, 2:5, 2:7, 2:9, 2:11, 2:13\}$	0
3	$\{3:0, 3:2, 3:4, 3:6, 3:8, 3:10, 3:12\}$	$\{3:1, 3:3, 3:5, 3:7, 3:9, 3:11, 3:13\}$	0
4	$\{4:0, 4:2, 4:4, 4:6, 4:8, 4:10, 4:12\}$	$\{4:1, 4:3, 4:5, 4:7, 4:9, 4:11, 4:13\}$	0
5	$\{5:0, 5:2, 5:4, 5:6, 5:8, 5:10, 5:12\}$	$\{5:1, 5:3, 5:5, 5:7, 5:9, 5:11, 5:13\}$	0
6	$\{6:0, 6:2, 6:4, 6:6, 6:8, 6:10, 6:12\}$	$\{6:1, 6:3, 6:5, 6:7, 6:9, 6:11, 6:13\}$	0
7	$\{7:0, 7:2, 7:4, 7:6, 7:8, 7:10, 7:12\}$	$\{7:1, 7:3, 7:5, 7:7, 7:9, 7:11, 7:13\}$	0
8	$\{8:0, 8:2, 8:4, 8:6, 8:8, 8:10, 8:12\}$	$\{8:1, 8:3, 8:5, 8:7, 8:9, 8:11, 8:13\}$	0
9	$\{9:0, 9:2, 9:4, 9:6, 9:8, 9:10, 9:12\}$	$\{9:1, 9:3, 9:5, 9:7, 9:9, 9:11, 9:13\}$	0
10	$\{10:0, 10:2, 10:4, 10:6, 10:8, 10:10, 10:12\}$	$\{10:1, 10:3, 10:5, 10:7, 10:9, 10:11, 10:13\}$	0
11	$\{11:0, 11:2, 11:4, 11:6, 11:8, 11:10, 11:12\}$	$\{11:1, 11:3, 11:5, 11:9, 11:11, 11:13\}$	0
12	$\{12:0, 12:2, 12:4, 12:6, 12:8, 12:10, 12:12\}$	$\{12:1, 12:3, 12:13\}$	0
13	$\{13:0, 13:2, 13:4, 13:6, 13:8, 13:10, 13:12\}$	$\{13:1, 13:3\}$	0

Certificates. The least, extended, and greatest sequences give 17, 17, and 17 realized keys, respectively. All three refined labelled graphs have zero descendants of the selected path.

i	$\lvert A_i\rvert$	$\lvert J_i^{\min}\rvert$	$J_i^{\text{ext}} \setminus J_i^{\min}$	$\lvert J_i^{\max}\rvert$
0	70	2	$\{0:4\}$	63
1	70	3	$\{1:6\}$	63
2	56	6	$\{2:10\}$	49
3	70	7	$\varnothing$	63
4	70	7	$\varnothing$	63
5	56	7	$\varnothing$	49
6	70	7	$\varnothing$	63
7	70	7	$\varnothing$	63
8	56	7	$\varnothing$	49
9	70	7	$\varnothing$	63
10	70	7	$\varnothing$	63

i	$\lvert A_i\rvert$	$\lvert J_i^{\min}\rvert$	$J_i^{\text{ext}} \setminus J_i^{\min}$	$\lvert J_i^{\max}\rvert$
11	56	7	$\varnothing$	50
12	70	7	$\varnothing$	67
13	70	7	$\varnothing$	68

Transition rows. Identical successor rows within a phase are grouped when they have the same selected-label and other-label outputs.

Phase 0, label g.

Source residues	Selected targets	Other targets
$\{0\}$	$\{0,2\}$	–
$\{1\}$	$\{1,3\}$	–
$\{2\}$	$\{2,4\}$	–
$\{3\}$	$\{3,5\}$	–
$\{4\}$	$\{4,6\}$	–
$\{5\}$	$\{5,7\}$	–
$\{6\}$	$\{6,8\}$	–
$\{7\}$	$\{7,9\}$	–
$\{8\}$	$\{8,10\}$	–
$\{9\}$	$\{9,11\}$	–
$\{10\}$	$\{10,12\}$	–
$\{11\}$	$\{11,13\}$	–

Phase 0, label g (continued).

Source residues	Selected targets	Other targets
$\{12\}$	$\{0,12\}$	–
$\{13\}$	$\{1,13\}$	–

Phase 1, label s.

Source residues	Selected targets	Other targets
$\{0\}$	$\{2,4,12\}$	–
$\{1\}$	$\{3,5,13\}$	–
$\{2\}$	$\{0,4,6\}$	–
$\{3\}$	$\{1,5,7\}$	–
$\{4\}$	$\{2,6,8\}$	–
$\{5\}$	$\{3,7,9\}$	–
$\{6\}$	$\{4,8,10\}$	–
$\{7\}$	$\{5,9,11\}$	–
$\{8\}$	$\{6,10,12\}$	–
$\{9\}$	$\{7,11,13\}$	–
$\{10\}$	$\{0,8,12\}$	–
$\{11\}$	$\{1,9,13\}$	–

Phase 1, label s (continued).

Source residues	Selected targets	Other targets
$\{12\}$	$\{0,2,10\}$	–
$\{13\}$	$\{1,3,11\}$	–

Phase 2, label s.

Source residues	Selected targets	Other targets
$\{0\}$	$\{2,4,12\}$	–
$\{1\}$	$\{3,5,13\}$	–
$\{2\}$	$\{0,4,6\}$	–
$\{3\}$	$\{1,5,7\}$	–
$\{4\}$	$\{2,6,8\}$	–
$\{5\}$	$\{3,7,9\}$	–
$\{6\}$	$\{4,8,10\}$	–
$\{7\}$	$\{5,9,11\}$	–
$\{8\}$	$\{6,10,12\}$	–
$\{9\}$	$\{7,11,13\}$	–
$\{10\}$	$\{0,8,12\}$	–
$\{11\}$	$\{1,9,13\}$	–

Phase 2, label s (continued).

Source residues	Selected targets	Other targets
$\{12\}$	$\{0,2,10\}$	–
$\{13\}$	$\{1,3,11\}$	–

Phase 3, label g.

Source residues	Selected targets	Other targets
{0}	{0, 2}	–
{1}	{1, 3}	–
{2}	{2, 4}	–
{3}	{3, 5}	–
{4}	{4, 6}	–
{5}	{5, 7}	–
{6}	{6, 8}	–
{7}	{7, 9}	–
{8}	{8, 10}	–
{9}	{9, 11}	–
{10}	{10, 12}	–
{11}	{11, 13}	–

Phase 3, label g (continued).

Source residues	Selected targets	Other targets
{12}	{0, 12}	–
{13}	{1, 13}	–

Phase 4, label s.

Source residues	Selected targets	Other targets
{0}	{2, 4, 12}	–
{1}	{3, 5, 13}	–
{2}	{0, 4, 6}	–
{3}	{1, 5, 7}	–
{4}	{2, 6, 8}	–
{5}	{3, 7, 9}	–
{6}	{4, 8, 10}	–
{7}	{5, 9, 11}	–
{8}	{6, 10, 12}	–
{9}	{7, 11, 13}	–
{10}	{0, 8, 12}	–
{11}	{1, 9, 13}	–

Phase 4, label s (continued).

Source residues	Selected targets	Other targets
{12}	{0, 2, 10}	–
{13}	{1, 3, 11}	–

Phase 5, label s.

Source residues	Selected targets	Other targets
{0}	{2, 4, 12}	–
{1}	{3, 5, 13}	–
{2}	{0, 4, 6}	–
{3}	{1, 5, 7}	–
{4}	{2, 6, 8}	–
{5}	{3, 7, 9}	–
{6}	{4, 8, 10}	–
{7}	{5, 9, 11}	–
{8}	{6, 10, 12}	–
{9}	{7, 11, 13}	–
{10}	{0, 8, 12}	–
{11}	{1, 9, 13}	–

Phase 5, label s (continued).

Source residues	Selected targets	Other targets
{12}	{0, 2, 10}	–
{13}	{1, 3, 11}	–

Phase 6, label g.

Source residues	Selected targets	Other targets
{0}	{0, 2}	–
{1}	{1, 3}	–
{2}	{2, 4}	–
{3}	{3, 5}	–
{4}	{4, 6}	–
{5}	{5, 7}	–
{6}	{6, 8}	–
{7}	{7, 9}	–
{8}	{8, 10}	–
{9}	{9, 11}	–
{10}	{10, 12}	–
{11}	{11, 13}	–

Phase 6, label g (continued).

Source residues	Selected targets	Other targets
{12}	{0, 12}	–
{13}	{1, 13}	–

Phase 7, label s.

Source residues	Selected targets	Other targets
{0}	{2, 4, 12}	–
{1}	{3, 5, 13}	–
{2}	{0, 4, 6}	–
{3}	{1, 5, 7}	–
{4}	{2, 6, 8}	–
{5}	{3, 7, 9}	–
{6}	{4, 8, 10}	–
{7}	{5, 9, 11}	–
{8}	{6, 10, 12}	–
{9}	{7, 11, 13}	–
{10}	{0, 8, 12}	–
{11}	{1, 9, 13}	–

Phase 7, label s (continued).

Source residues	Selected targets	Other targets
{12}	{0, 2, 10}	–
{13}	{1, 3, 11}	–

Phase 8, label s.

Source residues	Selected targets	Other targets
{0}	{2, 4, 12}	–
{1}	{3, 5, 13}	–
{2}	{0, 4, 6}	–
{3}	{1, 5, 7}	–
{4}	{2, 6, 8}	–
{5}	{3, 7, 9}	–
{6}	{4, 8, 10}	–
{7}	{5, 9, 11}	–
{8}	{6, 10, 12}	–
{9}	{7, 11, 13}	–
{10}	{0, 8, 12}	–
{11}	{1, 9, 13}	–

Phase 8, label s (continued).

Source residues	Selected targets	Other targets
{12}	{0, 2, 10}	–
{13}	{1, 3, 11}	–

Phase 9, label g.

Source residues	Selected targets	Other targets
{0}	{0, 2}	–
{1}	{1, 3}	–
{2}	{2, 4}	–
{3}	{3, 5}	–
{4}	{4, 6}	–
{5}	{5, 7}	–
{6}	{6, 8}	–
{7}	{7, 9}	–
{8}	{8, 10}	–
{9}	{9, 11}	–
{10}	{10, 12}	–
{11}	{11, 13}	–

Phase 9, label g (continued).

Source residues	Selected targets	Other targets
{12}	{0, 12}	–
{13}	{1, 13}	–

Phase 10, label s.

Source residues	Selected targets	Other targets
{0}	{2, 4, 12}	–
{1}	{3, 5, 13}	–
{2}	{0, 4, 6}	–
{3}	{1, 5, 7}	–
{4}	{2, 6, 8}	–
{5}	{3, 7, 9}	–
{6}	{4, 8, 10}	–
{7}	{5, 9, 11}	–
{8}	{6, 10, 12}	–
{9}	{7, 11, 13}	–
{10}	{0, 8, 12}	–
{11}	{1, 9, 13}	–

Phase 10, label s (continued).

Source residues	Selected targets	Other targets
{12}	{0, 2, 10}	–
{13}	{1, 3, 11}	–

Phase 11, label s.

Source residues	Selected targets	Other targets
{0}	{2, 4, 12}	–
{1}	{3, 5, 13}	–
{2}	{0, 4, 6}	–
{3}	{1, 5, 7}	–
{4}	{2, 6, 8}	–
{5}	{3, 7, 9}	–
{6}	{4, 8, 10}	–
{7}	{5, 9, 11}	–
{8}	{6, 10, 12}	–
{9}	{7, 11, 13}	–
{10}	{0, 8, 12}	–
{11}	{1, 9, 13}	–

Phase 11, label s (continued).

Source residues	Selected targets	Other targets
{12}	{0, 2, 10}	–
{13}	{1, 3, 11}	–

Phase 12, label g.

Source residues	Selected targets	Other targets
{0}	{0, 2}	–
{1}	{1, 3}	–
{2}	{2, 4}	–
{3}	{3, 5}	–
{4}	{4, 6}	–
{5}	{5, 7}	–
{6}	{6, 8}	–
{7}	{7, 9}	–
{8}	{8, 10}	–
{9}	{9, 11}	–
{10}	{10, 12}	–
{11}	{11, 13}	–

Phase 12, label g (continued).

Source residues	Selected targets	Other targets
{12}	{0, 12}	–
{13}	{1, 13}	–

30 Repeated labels and guards

The guard uses one label and the ordinary step another. Some instances contain a third label for a bypass. The prescribed word cannot silently use it.

A state printed as $i : v$ has phase i and residue v. The successor table prints residues at the next phase. A blank column for another label means there is no such edge at that phase.

30.1 P3.1 Guard at layer 3, width 6, target 1, bypass 0

Before the guard, every reachable residue is zero modulo 2. The guard accepts residue one. A bypass with a different label can change the verdict in the union graph while leaving the prescribed word infeasible.

The prescribed label sequence is s, s, g, s, s, s and its observed state sequence is $0, 1, 0, 1, 0, 1, 0$.

The initial set is $\{0 : 0\}$ and the bad set is $\{6 : 1\}$.

The exact lift count is 0. Allowing any label at each layer of this same observed state sequence gives 0 lifts.

i	Exact prefix	Exact suffix	Join
0	$\{0 : 0\}$	$\varnothing$	0
1	$\{1 : 0, 1 : 2\}$	$\varnothing$	0
2	$\{2 : 0, 2 : 2, 2 : 4\}$	$\varnothing$	0
3	$\varnothing$	$\{3 : 1, 3 : 3, 3 : 5\}$	0
4	$\varnothing$	$\{4 : 1, 4 : 3, 4 : 5\}$	0
5	$\varnothing$	$\{5 : 1, 5 : 5\}$	0
6	$\varnothing$	$\{6 : 1\}$	0

Certificates. The least, extended, and greatest sequences give 5, 9, and 6 realized keys, respectively. All three refined labelled graphs have zero descendants of the selected path.

i	$\lvert A_i \rvert$	$\lvert J_i^{\min} \rvert$	$J_i^{\mathrm{ext}} \setminus J_i^{\min}$	$\lvert J_i^{\max} \rvert$
0	24	1	$\{0 : 1\}$	24
1	18	2	$\{1 : 1, 1 : 3\}$	18
2	24	3	$\{2 : 1, 2 : 3, 2 : 5\}$	24
3	18	0	$\{3 : 0, 3 : 2, 3 : 4\}$	15
4	24	0	$\{4 : 0, 4 : 2, 4 : 4\}$	21
5	18	0	$\{5 : 0, 5 : 2, 5 : 4\}$	16
6	24	0	$\{6 : 0, 6 : 2, 6 : 4\}$	23

Transition rows. Identical successor rows within a phase are grouped when they have the same selected-label and other-label outputs.

Phase 0, label s.

Source residues	Selected targets	Other targets
{0}	{0, 2}	–
{1}	{1, 3}	–
{2}	{2, 4}	–
{3}	{3, 5}	–
{4}	{0, 4}	–
{5}	{1, 5}	–

Phase 1, label s.

Source residues	Selected targets	Other targets
$\{0\}$	$\{0, 2\}$	–
$\{1\}$	$\{1, 3\}$	–
$\{2\}$	$\{2, 4\}$	–
$\{3\}$	$\{3, 5\}$	–
$\{4\}$	$\{0, 4\}$	–
$\{5\}$	$\{1, 5\}$	–

Phase 2, label g.

Source residues	Selected targets	Other targets
$\{0, 2, 4\}$	$\varnothing$	–
$\{1\}$	$\{2\}$	–
$\{3\}$	$\{4\}$	–
$\{5\}$	$\{0\}$	–

Phase 3, label s.

Source residues	Selected targets	Other targets
$\{0\}$	$\{0, 2\}$	–
$\{1\}$	$\{1, 3\}$	–
$\{2\}$	$\{2, 4\}$	–
$\{3\}$	$\{3, 5\}$	–
$\{4\}$	$\{0, 4\}$	–
$\{5\}$	$\{1, 5\}$	–

Phase 4, label s.

Source residues	Selected targets	Other targets
$\{0\}$	$\{0, 2\}$	–
$\{1\}$	$\{1, 3\}$	–
$\{2\}$	$\{2, 4\}$	–
$\{3\}$	$\{3, 5\}$	–
$\{4\}$	$\{0, 4\}$	–
$\{5\}$	$\{1, 5\}$	–

Phase 5, label s.

Source residues	Selected targets	Other targets
$\{0\}$	$\{0, 2\}$	–
$\{1\}$	$\{1, 3\}$	–
$\{2\}$	$\{2, 4\}$	–
$\{3\}$	$\{3, 5\}$	–
$\{4\}$	$\{0, 4\}$	–
$\{5\}$	$\{1, 5\}$	–

30.2 P3.2 Guard at layer 4, width 6, target 1, bypass 1

Before the guard, every reachable residue is zero modulo 2. The guard accepts residue one. A bypass with a different label can change the verdict in the union graph while leaving the prescribed word infeasible.

The prescribed label sequence is s, s, s, g, s, s, s and its observed state sequence is $0, 1, 0, 1, 0, 1, 0, 1$.

The initial set is $\{0 : 0\}$ and the bad set is $\{7 : 1\}$.

The exact lift count is 0. Allowing any label at each layer of this same observed state sequence gives 22 lifts.

i	Exact prefix	Exact suffix	Join
0	$\{0 : 0\}$	$\varnothing$	0
1	$\{1 : 0, 1 : 2\}$	$\varnothing$	0
2	$\{2 : 0, 2 : 2, 2 : 4\}$	$\varnothing$	0
3	$\{3 : 0, 3 : 2, 3 : 4\}$	$\varnothing$	0
4	$\varnothing$	$\{4 : 1, 4 : 3, 4 : 5\}$	0
5	$\varnothing$	$\{5 : 1, 5 : 3, 5 : 5\}$	0
6	$\varnothing$	$\{6 : 1, 6 : 5\}$	0
7	$\varnothing$	$\{7 : 1\}$	0

Certificates. The least, extended, and greatest sequences give 6, 10, and 6 realized keys, respectively. All three refined labelled graphs have zero descendants of the selected path.

i	$\lvert A_i \rvert$	$\lvert J_i^{\min} \rvert$	$J_i^{\text{ext}} \setminus J_i^{\min}$	$\lvert J_i^{\max} \rvert$
0	24	1	$\{0 : 1\}$	24
1	24	2	$\{1 : 1, 1 : 3\}$	24
2	24	3	$\{2 : 1, 2 : 3, 2 : 5\}$	24
3	24	3	$\{3 : 1, 3 : 3, 3 : 5\}$	24
4	24	0	$\{4 : 0, 4 : 2, 4 : 4\}$	21
5	24	0	$\{5 : 0, 5 : 2, 5 : 4\}$	21
6	24	0	$\{6 : 0, 6 : 2, 6 : 4\}$	22
7	24	0	$\{7 : 0, 7 : 2, 7 : 4\}$	23

Transition rows. Identical successor rows within a phase are grouped when they have the same selected-label and other-label outputs.

Phase 0, label s.

Source residues	Selected targets	Other targets
$\{0\}$	$\{0, 2\}$	–
$\{1\}$	$\{1, 3\}$	–
$\{2\}$	$\{2, 4\}$	–
$\{3\}$	$\{3, 5\}$	–
$\{4\}$	$\{0, 4\}$	–
$\{5\}$	$\{1, 5\}$	–

Phase 1, label s.

Source residues	Selected targets	Other targets
$\{0\}$	$\{0, 2\}$	–
$\{1\}$	$\{1, 3\}$	–
$\{2\}$	$\{2, 4\}$	–
$\{3\}$	$\{3, 5\}$	–
$\{4\}$	$\{0, 4\}$	–
$\{5\}$	$\{1, 5\}$	–

Phase 2, label s.

Source residues	Selected targets	Other targets
$\{0\}$	$\{0, 2\}$	–
$\{1\}$	$\{1, 3\}$	–
$\{2\}$	$\{2, 4\}$	–
$\{3\}$	$\{3, 5\}$	–
$\{4\}$	$\{0, 4\}$	–
$\{5\}$	$\{1, 5\}$	–

Phase 3, label g.

Source residues	Selected targets	Other targets
$\{0\}$	$\varnothing$	b $\{1\}$
$\{1\}$	$\{2\}$	b $\{2\}$
$\{2\}$	$\varnothing$	b $\{3\}$
$\{3\}$	$\{4\}$	b $\{4\}$
$\{4\}$	$\varnothing$	b $\{5\}$
$\{5\}$	$\{0\}$	b $\{0\}$

Phase 4, label s.

Source residues	Selected targets	Other targets
$\{0\}$	$\{0, 2\}$	–
$\{1\}$	$\{1, 3\}$	–
$\{2\}$	$\{2, 4\}$	–
$\{3\}$	$\{3, 5\}$	–
$\{4\}$	$\{0, 4\}$	–
$\{5\}$	$\{1, 5\}$	–

Phase 5, label s.

Source residues	Selected targets	Other targets
$\{0\}$	$\{0, 2\}$	–
$\{1\}$	$\{1, 3\}$	–
$\{2\}$	$\{2, 4\}$	–
$\{3\}$	$\{3, 5\}$	–
$\{4\}$	$\{0, 4\}$	–
$\{5\}$	$\{1, 5\}$	–

Phase 6, label s.

Source residues	Selected targets	Other targets
$\{0\}$	$\{0, 2\}$	–
$\{1\}$	$\{1, 3\}$	–
$\{2\}$	$\{2, 4\}$	–
$\{3\}$	$\{3, 5\}$	–
$\{4\}$	$\{0, 4\}$	–
$\{5\}$	$\{1, 5\}$	–

30.3 P3.3 Guard at layer 2, width 8, target 5, bypass 1

Before the guard, every reachable residue is zero modulo 4. The guard accepts residue one. A bypass with a different label can change the verdict in the union graph while leaving the prescribed word infeasible.

The prescribed label sequence is s, g, s, s, s, s, s, s and its observed state sequence is $0, 1, 0, 1, 0, 1, 0, 1, 0$.

The initial set is $\{0 : 0\}$ and the bad set is $\{8 : 5\}$.

The exact lift count is 0. Allowing any label at each layer of this same observed state sequence gives 64 lifts.

i	Exact prefix	Exact suffix	Join
0	$\{0 : 0\}$	$\varnothing$	0
1	$\{1 : 0, 1 : 4\}$	$\varnothing$	0
2	$\varnothing$	$\{2 : 1, 2 : 5\}$	0
3	$\varnothing$	$\{3 : 1, 3 : 5\}$	0
4	$\varnothing$	$\{4 : 1, 4 : 5\}$	0
5	$\varnothing$	$\{5 : 1, 5 : 5\}$	0
6	$\varnothing$	$\{6 : 1, 6 : 5\}$	0
7	$\varnothing$	$\{7 : 1, 7 : 5\}$	0
8	$\varnothing$	$\{8 : 5\}$	0

Certificates. The least, extended, and greatest sequences give 4, 11, and 9 realized keys, respectively. All three refined labelled graphs have zero descendants of the selected path.

i	$\lvert A_i \rvert$	$\lvert J_i^{\min} \rvert$	$J_i^{\mathrm{ext}} \setminus J_i^{\min}$	$\lvert J_i^{\max} \rvert$
0	40	1	$\{0 : 1\}$	40
1	32	2	$\{1 : 1, 1 : 5\}$	32
2	40	0	$\{2 : 2, 2 : 6\}$	38
3	32	0	$\{3 : 2, 3 : 6\}$	30
4	40	0	$\{4 : 2, 4 : 6\}$	38
5	32	0	$\{5 : 2, 5 : 6\}$	30
6	40	0	$\{6 : 2, 6 : 6\}$	38
7	32	0	$\{7 : 2, 7 : 6\}$	30
8	40	0	$\{8 : 2, 8 : 6\}$	39

Transition rows. Identical successor rows within a phase are grouped when they have the same selected-label and other-label outputs.

Phase 0, label s.

Source residues	Selected targets	Other targets
$\{0, 4\}$	$\{0, 4\}$	–
$\{1, 5\}$	$\{1, 5\}$	–
$\{2, 6\}$	$\{2, 6\}$	–
$\{3, 7\}$	$\{3, 7\}$	–

Phase 1, label g.

Source residues	Selected targets	Other targets
{0}	∅	b {1}
{1}	{2}	b {2}
{2}	∅	b {3}
{3}	∅	b {4}
{4}	∅	b {5}
{5}	{6}	b {6}
{6}	∅	b {7}
{7}	∅	b {0}

Phase 2, label s.

Source residues	Selected targets	Other targets
{0, 4}	{0, 4}	–
{1, 5}	{1, 5}	–
{2, 6}	{2, 6}	–
{3, 7}	{3, 7}	–

Phase 3, label s.

Source residues	Selected targets	Other targets
{0, 4}	{0, 4}	–
{1, 5}	{1, 5}	–
{2, 6}	{2, 6}	–
{3, 7}	{3, 7}	–

Phase 4, label s.

Source residues	Selected targets	Other targets
{0, 4}	{0, 4}	–
{1, 5}	{1, 5}	–
{2, 6}	{2, 6}	–
{3, 7}	{3, 7}	–

Phase 5, label s.

Source residues	Selected targets	Other targets
{0, 4}	{0, 4}	–
{1, 5}	{1, 5}	–
{2, 6}	{2, 6}	–
{3, 7}	{3, 7}	–

Phase 6, label s.

Source residues	Selected targets	Other targets
{0, 4}	{0, 4}	–
{1, 5}	{1, 5}	–
{2, 6}	{2, 6}	–
{3, 7}	{3, 7}	–

Phase 7, label s.

Source residues	Selected targets	Other targets
{0, 4}	{0, 4}	–
{1, 5}	{1, 5}	–
{2, 6}	{2, 6}	–
{3, 7}	{3, 7}	–

30.4 P3.4 Guard at layer 5, width 9, target 4, bypass 1

Before the guard, every reachable residue is zero modulo 3. The guard accepts residue one. A bypass with a different label can change the verdict in the union graph while leaving the prescribed word infeasible.

The prescribed label sequence is $s, s, s, s, g, s, s, s, s$ and its observed state sequence is $0, 1, 0, 1, 0, 1, 0, 1, 0, 1$.

The initial set is $\{0 : 0\}$ and the bad set is $\{9 : 4\}$.

The exact lift count is 0. Allowing any label at each layer of this same observed state sequence gives 86 lifts.

i	Exact prefix	Exact suffix	Join
0	$\{0:0\}$	∅	0
1	$\{1:0, 1:3\}$	∅	0
2	$\{2:0, 2:3, 2:6\}$	∅	0
3	$\{3:0, 3:3, 3:6\}$	∅	0
4	$\{4:0, 4:3, 4:6\}$	∅	0
5	∅	$\{5:1, 5:4, 5:7\}$	0
6	∅	$\{6:1, 6:4, 6:7\}$	0
7	∅	$\{7:1, 7:4, 7:7\}$	0
8	∅	$\{8:1, 8:4\}$	0
9	∅	$\{9:4\}$	0

Certificates. The least, extended, and greatest sequences give 7, 12, and 7 realized keys, respectively. All three refined labelled graphs have zero descendants of the selected path.

i	$\lvert A_i\rvert$	$\lvert J_i^{\min}\rvert$	$J_i^{\mathrm{ext}} \setminus J_i^{\min}$	$\lvert J_i^{\max}\rvert$
0	45	1	$\{0:1\}$	45
1	45	2	$\{1:1, 1:4\}$	45
2	45	3	$\{2:1, 2:4, 2:7\}$	45
3	45	3	$\{3:1, 3:4, 3:7\}$	45
4	45	3	$\{4:1, 4:4, 4:7\}$	45
5	45	0	$\{5:2, 5:5, 5:8\}$	42
6	45	0	$\{6:2, 6:5, 6:8\}$	42
7	45	0	$\{7:2, 7:5, 7:8\}$	42
8	45	0	$\{8:2, 8:5, 8:8\}$	43
9	45	0	$\{9:2, 9:5, 9:8\}$	44

Transition rows. Identical successor rows within a phase are grouped when they have the same selected-label and other-label outputs.

Phase 0, label s.

Source residues	Selected targets	Other targets
{0}	{0, 3}	–
{1}	{1, 4}	–
{2}	{2, 5}	–
{3}	{3, 6}	–
{4}	{4, 7}	–
{5}	{5, 8}	–
{6}	{0, 6}	–
{7}	{1, 7}	–
{8}	{2, 8}	–

Phase 1, label s.

Source residues	Selected targets	Other targets
{0}	{0, 3}	–
{1}	{1, 4}	–
{2}	{2, 5}	–
{3}	{3, 6}	–
{4}	{4, 7}	–
{5}	{5, 8}	–
{6}	{0, 6}	–
{7}	{1, 7}	–
{8}	{2, 8}	–

Phase 2, label s.

Source residues	Selected targets	Other targets
{0}	{0, 3}	–
{1}	{1, 4}	–
{2}	{2, 5}	–
{3}	{3, 6}	–
{4}	{4, 7}	–
{5}	{5, 8}	–
{6}	{0, 6}	–
{7}	{1, 7}	–
{8}	{2, 8}	–

Phase 3, label s.

Source residues	Selected targets	Other targets
{0}	{0, 3}	–
{1}	{1, 4}	–
{2}	{2, 5}	–
{3}	{3, 6}	–
{4}	{4, 7}	–
{5}	{5, 8}	–
{6}	{0, 6}	–
{7}	{1, 7}	–
{8}	{2, 8}	–

Phase 4, label g.

Source residues	Selected targets	Other targets
{0}	∅	b {1}
{1}	{2}	b {2}
{2}	∅	b {3}
{3}	∅	b {4}
{4}	{5}	b {5}
{5}	∅	b {6}
{6}	∅	b {7}
{7}	{8}	b {8}
{8}	∅	b {0}

Phase 5, label s.

Source residues	Selected targets	Other targets
{0}	{0, 3}	–
{1}	{1, 4}	–
{2}	{2, 5}	–
{3}	{3, 6}	–
{4}	{4, 7}	–
{5}	{5, 8}	–
{6}	{0, 6}	–
{7}	{1, 7}	–
{8}	{2, 8}	–

Phase 6, label s.

Source residues	Selected targets	Other targets
{0}	{0, 3}	–
{1}	{1, 4}	–
{2}	{2, 5}	–
{3}	{3, 6}	–
{4}	{4, 7}	–
{5}	{5, 8}	–
{6}	{0, 6}	–
{7}	{1, 7}	–
{8}	{2, 8}	–

Phase 7, label s.

Source residues	Selected targets	Other targets
{0}	{0, 3}	–
{1}	{1, 4}	–
{2}	{2, 5}	–
{3}	{3, 6}	–
{4}	{4, 7}	–
{5}	{5, 8}	–
{6}	{0, 6}	–
{7}	{1, 7}	–
{8}	{2, 8}	–

Phase 8, label s.

Source residues	Selected targets	Other targets
$\{0\}$	$\{0, 3\}$	–
$\{1\}$	$\{1, 4\}$	–
$\{2\}$	$\{2, 5\}$	–
$\{3\}$	$\{3, 6\}$	–
$\{4\}$	$\{4, 7\}$	–
$\{5\}$	$\{5, 8\}$	–
$\{6\}$	$\{0, 6\}$	–
$\{7\}$	$\{1, 7\}$	–
$\{8\}$	$\{2, 8\}$	–

30.5 P3.5 Guard at layer 6, width 10, target 1, bypass 0

Before the guard, every reachable residue is zero modulo 2. The guard accepts residue one. A bypass with a different label can change the verdict in the union graph while leaving the prescribed word infeasible.

The prescribed label sequence is $s, s, s, s, s, g, s, s, s, s$ and its observed state sequence is $0, 1, 0, 1, 0, 1, 0, 1, 0, 1, 0$.

The initial set is $\{0 : 0\}$ and the bad set is $\{10 : 1\}$.

The exact lift count is 0. Allowing any label at each layer of this same observed state sequence gives 0 lifts.

i	Exact prefix	Exact suffix	Join
0	$\{0 : 0\}$	$\varnothing$	0
1	$\{1 : 0, 1 : 2\}$	$\varnothing$	0
2	$\{2 : 0, 2 : 2, 2 : 4\}$	$\varnothing$	0
3	$\{3 : 0, 3 : 2, 3 : 4, 3 : 6\}$	$\varnothing$	0
4	$\{4 : 0, 4 : 2, 4 : 4, 4 : 6, 4 : 8\}$	$\varnothing$	0
5	$\{5 : 0, 5 : 2, 5 : 4, 5 : 6, 5 : 8\}$	$\varnothing$	0
6	$\varnothing$	$\{6 : 1, 6 : 3, 6 : 5, 6 : 7, 6 : 9\}$	0
7	$\varnothing$	$\{7 : 1, 7 : 5, 7 : 7, 7 : 9\}$	0
8	$\varnothing$	$\{8 : 1, 8 : 7, 8 : 9\}$	0
9	$\varnothing$	$\{9 : 1, 9 : 9\}$	0
10	$\varnothing$	$\{10 : 1\}$	0

Certificates. The least, extended, and greatest sequences give 8, 13, and 7 realized keys, respectively. All three refined labelled graphs have zero descendants of the selected path.

i	$\lvert A_i\rvert$	$\lvert J_i^{\min}\rvert$	$J_i^{\text{ext}} \setminus J_i^{\min}$	$\lvert J_i^{\max}\rvert$
0	60	1	$\{0 : 1\}$	60
1	50	2	$\{1 : 1, 1 : 3\}$	50
2	60	3	$\{2 : 1, 2 : 3, 2 : 5\}$	60
3	50	4	$\{3 : 1, 3 : 3, 3 : 5, 3 : 7\}$	50
4	60	5	$\{4 : 1, 4 : 3, 4 : 5, 4 : 7, 4 : 9\}$	60
5	50	5	$\{5 : 1, 5 : 3, 5 : 5, 5 : 7, 5 : 9\}$	50
6	60	0	$\{6 : 0, 6 : 2, 6 : 4, 6 : 6, 6 : 8\}$	55
7	50	0	$\{7 : 0, 7 : 2, 7 : 4, 7 : 6, 7 : 8\}$	46
8	60	0	$\{8 : 0, 8 : 2, 8 : 4, 8 : 6, 8 : 8\}$	57
9	50	0	$\{9 : 0, 9 : 2, 9 : 4, 9 : 6, 9 : 8\}$	48
10	60	0	$\{10 : 0, 10 : 2, 10 : 4, 10 : 6, 10 : 8\}$	59

Transition rows. Identical successor rows within a phase are grouped when they have the same selected-label and other-label outputs.

Phase 0, label s.

Source residues	Selected targets	Other targets
$\{0\}$	$\{0, 2\}$	–
$\{1\}$	$\{1, 3\}$	–
$\{2\}$	$\{2, 4\}$	–
$\{3\}$	$\{3, 5\}$	–
$\{4\}$	$\{4, 6\}$	–
$\{5\}$	$\{5, 7\}$	–
$\{6\}$	$\{6, 8\}$	–
$\{7\}$	$\{7, 9\}$	–
$\{8\}$	$\{0, 8\}$	–
$\{9\}$	$\{1, 9\}$	–

Phase 1, label s.

Source residues	Selected targets	Other targets
$\{0\}$	$\{0, 2\}$	–
$\{1\}$	$\{1, 3\}$	–
$\{2\}$	$\{2, 4\}$	–
$\{3\}$	$\{3, 5\}$	–
$\{4\}$	$\{4, 6\}$	–
$\{5\}$	$\{5, 7\}$	–
$\{6\}$	$\{6, 8\}$	–
$\{7\}$	$\{7, 9\}$	–
$\{8\}$	$\{0, 8\}$	–
$\{9\}$	$\{1, 9\}$	–

Phase 2, label s.

Source residues	Selected targets	Other targets
$\{0\}$	$\{0, 2\}$	–
$\{1\}$	$\{1, 3\}$	–
$\{2\}$	$\{2, 4\}$	–
$\{3\}$	$\{3, 5\}$	–
$\{4\}$	$\{4, 6\}$	–
$\{5\}$	$\{5, 7\}$	–
$\{6\}$	$\{6, 8\}$	–
$\{7\}$	$\{7, 9\}$	–
$\{8\}$	$\{0, 8\}$	–
$\{9\}$	$\{1, 9\}$	–

Phase 3, label s.

Source residues	Selected targets	Other targets
$\{0\}$	$\{0, 2\}$	–
$\{1\}$	$\{1, 3\}$	–
$\{2\}$	$\{2, 4\}$	–
$\{3\}$	$\{3, 5\}$	–
$\{4\}$	$\{4, 6\}$	–
$\{5\}$	$\{5, 7\}$	–
$\{6\}$	$\{6, 8\}$	–
$\{7\}$	$\{7, 9\}$	–
$\{8\}$	$\{0, 8\}$	–
$\{9\}$	$\{1, 9\}$	–

Phase 4, label s.

Source residues	Selected targets	Other targets
$\{0\}$	$\{0, 2\}$	–
$\{1\}$	$\{1, 3\}$	–
$\{2\}$	$\{2, 4\}$	–
$\{3\}$	$\{3, 5\}$	–
$\{4\}$	$\{4, 6\}$	–
$\{5\}$	$\{5, 7\}$	–
$\{6\}$	$\{6, 8\}$	–
$\{7\}$	$\{7, 9\}$	–
$\{8\}$	$\{0, 8\}$	–
$\{9\}$	$\{1, 9\}$	–

Phase 5, label g.

Source residues	Selected targets	Other targets
$\{0, 2, 4, 6, 8\}$	$\varnothing$	–
$\{1\}$	$\{2\}$	–
$\{3\}$	$\{4\}$	–
$\{5\}$	$\{6\}$	–
$\{7\}$	$\{8\}$	–
$\{9\}$	$\{0\}$	–

Phase 6, label s.

Source residues	Selected targets	Other targets
$\{0\}$	$\{0, 2\}$	–
$\{1\}$	$\{1, 3\}$	–
$\{2\}$	$\{2, 4\}$	–
$\{3\}$	$\{3, 5\}$	–
$\{4\}$	$\{4, 6\}$	–
$\{5\}$	$\{5, 7\}$	–
$\{6\}$	$\{6, 8\}$	–
$\{7\}$	$\{7, 9\}$	–
$\{8\}$	$\{0, 8\}$	–
$\{9\}$	$\{1, 9\}$	–

Phase 7, label s.

Source residues	Selected targets	Other targets
$\{0\}$	$\{0, 2\}$	–
$\{1\}$	$\{1, 3\}$	–
$\{2\}$	$\{2, 4\}$	–
$\{3\}$	$\{3, 5\}$	–
$\{4\}$	$\{4, 6\}$	–
$\{5\}$	$\{5, 7\}$	–
$\{6\}$	$\{6, 8\}$	–
$\{7\}$	$\{7, 9\}$	–
$\{8\}$	$\{0, 8\}$	–
$\{9\}$	$\{1, 9\}$	–

Phase 8, label s.

Source residues	Selected targets	Other targets
$\{0\}$	$\{0, 2\}$	–
$\{1\}$	$\{1, 3\}$	–
$\{2\}$	$\{2, 4\}$	–
$\{3\}$	$\{3, 5\}$	–
$\{4\}$	$\{4, 6\}$	–
$\{5\}$	$\{5, 7\}$	–
$\{6\}$	$\{6, 8\}$	–
$\{7\}$	$\{7, 9\}$	–
$\{8\}$	$\{0, 8\}$	–
$\{9\}$	$\{1, 9\}$	–

Phase 9, label s.

Source residues	Selected targets	Other targets
$\{0\}$	$\{0, 2\}$	–
$\{1\}$	$\{1, 3\}$	–
$\{2\}$	$\{2, 4\}$	–
$\{3\}$	$\{3, 5\}$	–
$\{4\}$	$\{4, 6\}$	–
$\{5\}$	$\{5, 7\}$	–
$\{6\}$	$\{6, 8\}$	–
$\{7\}$	$\{7, 9\}$	–
$\{8\}$	$\{0, 8\}$	–
$\{9\}$	$\{1, 9\}$	–

30.6 P3.6 Guard at layer 7, width 12, target 4, bypass 1

Before the guard, every reachable residue is zero modulo 3. The guard accepts residue one. A bypass with a different label can change the verdict in the union graph while leaving the prescribed word infeasible.

The prescribed label sequence is $s, s, s, s, s, s, g, s, s, s, s$ and its observed state sequence is $0, 1, 0, 1, 0, 1, 0, 1, 0, 1, 0, 1$.

The initial set is $\{0 : 0\}$ and the bad set is $\{11 : 4\}$.

The exact lift count is 0. Allowing any label at each layer of this same observed state sequence gives 272 lifts.

i	Exact prefix	Exact suffix	Join
0	$\{0:0\}$	$\varnothing$	0
1	$\{1:0, 1:3\}$	$\varnothing$	0
2	$\{2:0, 2:3, 2:6\}$	$\varnothing$	0
3	$\{3:0, 3:3, 3:6, 3:9\}$	$\varnothing$	0
4	$\{4:0, 4:3, 4:6, 4:9\}$	$\varnothing$	0
5	$\{5:0, 5:3, 5:6, 5:9\}$	$\varnothing$	0
6	$\{6:0, 6:3, 6:6, 6:9\}$	$\varnothing$	0
7	$\varnothing$	$\{7:1, 7:4, 7:7, 7:10\}$	0
8	$\varnothing$	$\{8:1, 8:4, 8:7, 8:10\}$	0
9	$\varnothing$	$\{9:1, 9:4, 9:10\}$	0
10	$\varnothing$	$\{10:1, 10:4\}$	0

i	Exact prefix	Exact suffix	Join
11	$\varnothing$	$\{11:4\}$	0

Certificates. The least, extended, and greatest sequences give 9, 14, and 7 realized keys, respectively. All three refined labelled graphs have zero descendants of the selected path.

i	$\lvert A_i\rvert$	$\lvert J_i^{\min}\rvert$	$J_i^{\text{ext}} \setminus J_i^{\min}$	$\lvert J_i^{\max}\rvert$
0	72	1	$\{0:1\}$	72
1	72	2	$\{1:1, 1:4\}$	72
2	72	3	$\{2:1, 2:4, 2:7\}$	72
3	72	4	$\{3:1, 3:4, 3:7, 3:10\}$	72
4	72	4	$\{4:1, 4:4, 4:7, 4:10\}$	72
5	72	4	$\{5:1, 5:4, 5:7, 5:10\}$	72
6	72	4	$\{6:1, 6:4, 6:7, 6:10\}$	72
7	72	0	$\{7:2, 7:5, 7:8, 7:11\}$	68
8	72	0	$\{8:2, 8:5, 8:8, 8:11\}$	68
9	72	0	$\{9:2, 9:5, 9:8, 9:11\}$	69
10	72	0	$\{10:2, 10:5, 10:8, 10:11\}$	70

i	$\lvert A_i\rvert$	$\lvert J_i^{\min}\rvert$	$J_i^{\text{ext}} \setminus J_i^{\min}$	$\lvert J_i^{\max}\rvert$
11	72	0	$\{11:2, 11:5, 11:8, 11:11\}$	71

Transition rows. Identical successor rows within a phase are grouped when they have the same selected-label and other-label outputs.

Phase 0, label s.

Source residues	Selected targets	Other targets
$\{0\}$	$\{0, 3\}$	–
$\{1\}$	$\{1, 4\}$	–
$\{2\}$	$\{2, 5\}$	–
$\{3\}$	$\{3, 6\}$	–
$\{4\}$	$\{4, 7\}$	–
$\{5\}$	$\{5, 8\}$	–
$\{6\}$	$\{6, 9\}$	–
$\{7\}$	$\{7, 10\}$	–
$\{8\}$	$\{8, 11\}$	–
$\{9\}$	$\{0, 9\}$	–
$\{10\}$	$\{1, 10\}$	–
$\{11\}$	$\{2, 11\}$	–

Phase 1, label s.

Source residues	Selected targets	Other targets
{0}	{0, 3}	–
{1}	{1, 4}	–
{2}	{2, 5}	–
{3}	{3, 6}	–
{4}	{4, 7}	–
{5}	{5, 8}	–
{6}	{6, 9}	–
{7}	{7, 10}	–
{8}	{8, 11}	–
{9}	{0, 9}	–
{10}	{1, 10}	–
{11}	{2, 11}	–

Phase 2, label s.

Source residues	Selected targets	Other targets
{0}	{0, 3}	–
{1}	{1, 4}	–
{2}	{2, 5}	–
{3}	{3, 6}	–
{4}	{4, 7}	–
{5}	{5, 8}	–
{6}	{6, 9}	–
{7}	{7, 10}	–
{8}	{8, 11}	–
{9}	{0, 9}	–
{10}	{1, 10}	–
{11}	{2, 11}	–

Phase 3, label s.

Source residues	Selected targets	Other targets
{0}	{0, 3}	–
{1}	{1, 4}	–
{2}	{2, 5}	–
{3}	{3, 6}	–
{4}	{4, 7}	–
{5}	{5, 8}	–
{6}	{6, 9}	–
{7}	{7, 10}	–
{8}	{8, 11}	–
{9}	{0, 9}	–
{10}	{1, 10}	–
{11}	{2, 11}	–

Phase 4, label s.

Source residues	Selected targets	Other targets
{0}	{0, 3}	–
{1}	{1, 4}	–
{2}	{2, 5}	–
{3}	{3, 6}	–
{4}	{4, 7}	–
{5}	{5, 8}	–
{6}	{6, 9}	–
{7}	{7, 10}	–
{8}	{8, 11}	–
{9}	{0, 9}	–
{10}	{1, 10}	–
{11}	{2, 11}	–

Phase 5, label s.

Source residues	Selected targets	Other targets
{0}	{0, 3}	–
{1}	{1, 4}	–
{2}	{2, 5}	–
{3}	{3, 6}	–
{4}	{4, 7}	–
{5}	{5, 8}	–
{6}	{6, 9}	–
{7}	{7, 10}	–
{8}	{8, 11}	–
{9}	{0, 9}	–
{10}	{1, 10}	–
{11}	{2, 11}	–

Phase 6, label g.

Source residues	Selected targets	Other targets
{0}	∅	b {1}
{1}	{2}	b {2}
{2}	∅	b {3}
{3}	∅	b {4}
{4}	{5}	b {5}
{5}	∅	b {6}
{6}	∅	b {7}
{7}	{8}	b {8}
{8}	∅	b {9}
{9}	∅	b {10}
{10}	{11}	b {11}
{11}	∅	b {0}

Phase 7, label s.

Source residues	Selected targets	Other targets
$\{0\}$	$\{0, 3\}$	–
$\{1\}$	$\{1, 4\}$	–
$\{2\}$	$\{2, 5\}$	–
$\{3\}$	$\{3, 6\}$	–
$\{4\}$	$\{4, 7\}$	–
$\{5\}$	$\{5, 8\}$	–
$\{6\}$	$\{6, 9\}$	–
$\{7\}$	$\{7, 10\}$	–
$\{8\}$	$\{8, 11\}$	–
$\{9\}$	$\{0, 9\}$	–
$\{10\}$	$\{1, 10\}$	–
$\{11\}$	$\{2, 11\}$	–

Phase 8, label s.

Source residues	Selected targets	Other targets
$\{0\}$	$\{0, 3\}$	–
$\{1\}$	$\{1, 4\}$	–
$\{2\}$	$\{2, 5\}$	–
$\{3\}$	$\{3, 6\}$	–
$\{4\}$	$\{4, 7\}$	–
$\{5\}$	$\{5, 8\}$	–
$\{6\}$	$\{6, 9\}$	–
$\{7\}$	$\{7, 10\}$	–
$\{8\}$	$\{8, 11\}$	–
$\{9\}$	$\{0, 9\}$	–
$\{10\}$	$\{1, 10\}$	–
$\{11\}$	$\{2, 11\}$	–

Phase 9, label s.

Source residues	Selected targets	Other targets
$\{0\}$	$\{0, 3\}$	–
$\{1\}$	$\{1, 4\}$	–
$\{2\}$	$\{2, 5\}$	–
$\{3\}$	$\{3, 6\}$	–
$\{4\}$	$\{4, 7\}$	–
$\{5\}$	$\{5, 8\}$	–
$\{6\}$	$\{6, 9\}$	–
$\{7\}$	$\{7, 10\}$	–
$\{8\}$	$\{8, 11\}$	–
$\{9\}$	$\{0, 9\}$	–
$\{10\}$	$\{1, 10\}$	–
$\{11\}$	$\{2, 11\}$	–

Phase 10, label s.

Source residues	Selected targets	Other targets
$\{0\}$	$\{0, 3\}$	–
$\{1\}$	$\{1, 4\}$	–
$\{2\}$	$\{2, 5\}$	–
$\{3\}$	$\{3, 6\}$	–
$\{4\}$	$\{4, 7\}$	–
$\{5\}$	$\{5, 8\}$	–
$\{6\}$	$\{6, 9\}$	–
$\{7\}$	$\{7, 10\}$	–
$\{8\}$	$\{8, 11\}$	–
$\{9\}$	$\{0, 9\}$	–
$\{10\}$	$\{1, 10\}$	–
$\{11\}$	$\{2, 11\}$	–

30.7 P3.7 Guard at layer 4, width 12, target 9, bypass 1

Before the guard, every reachable residue is zero modulo 4. The guard accepts residue one. A bypass with a different label can change the verdict in the union graph while leaving the prescribed word infeasible.

The prescribed label sequence is $s, s, s, g, s, s, s, s, s, s, s, s$ and its observed state sequence is $0, 1, 0, 1, 0, 1, 0, 1, 0, 1, 0, 1, 0$.

The initial set is $\{0 : 0\}$ and the bad set is $\{12 : 9\}$.

The exact lift count is 0. Allowing any label at each layer of this same observed state sequence gives 683 lifts.

i	Exact prefix	Exact suffix	Join
0	$\{0 : 0\}$	$\varnothing$	0
1	$\{1 : 0, 1 : 4\}$	$\varnothing$	0
2	$\{2 : 0, 2 : 4, 2 : 8\}$	$\varnothing$	0
3	$\{3 : 0, 3 : 4, 3 : 8\}$	$\varnothing$	0
4	$\varnothing$	$\{4 : 1, 4 : 5, 4 : 9\}$	0
5	$\varnothing$	$\{5 : 1, 5 : 5, 5 : 9\}$	0
6	$\varnothing$	$\{6 : 1, 6 : 5, 6 : 9\}$	0
7	$\varnothing$	$\{7 : 1, 7 : 5, 7 : 9\}$	0
8	$\varnothing$	$\{8 : 1, 8 : 5, 8 : 9\}$	0
9	$\varnothing$	$\{9 : 1, 9 : 5, 9 : 9\}$	0
10	$\varnothing$	$\{10 : 1, 10 : 5, 10 : 9\}$	0
11	$\varnothing$	$\{11 : 5, 11 : 9\}$	0
12	$\varnothing$	$\{12 : 9\}$	0

Certificates. The least, extended, and greatest sequences give 6, 15, and 11 realized keys, respectively. All three refined labelled graphs have zero descendants of the selected path.

i	$\lvert A_i\rvert$	$\lvert J_i^{\min}\rvert$	$J_i^{\text{ext}} \setminus J_i^{\min}$	$\lvert J_i^{\max}\rvert$
0	84	1	$\{0:1\}$	84
1	72	2	$\{1:1, 1:5\}$	72
2	84	3	$\{2:1, 2:5, 2:9\}$	84
3	72	3	$\{3:1, 3:5, 3:9\}$	72
4	84	0	$\{4:2, 4:6, 4:10\}$	81
5	72	0	$\{5:2, 5:6, 5:10\}$	69
6	84	0	$\{6:2, 6:6, 6:10\}$	81
7	72	0	$\{7:2, 7:6, 7:10\}$	69
8	84	0	$\{8:2, 8:6, 8:10\}$	81
9	72	0	$\{9:2, 9:6, 9:10\}$	69
10	84	0	$\{10:2, 10:6, 10:10\}$	81
11	72	0	$\{11:2, 11:6, 11:10\}$	70
12	84	0	$\{12:2, 12:6, 12:10\}$	83

Transition rows. Identical successor rows within a phase are grouped when they have the same selected-label and other-label outputs.

Phase 0, label s.

Source residues	Selected targets	Other targets
$\{0\}$	$\{0, 4\}$	–
$\{1\}$	$\{1, 5\}$	–
$\{2\}$	$\{2, 6\}$	–
$\{3\}$	$\{3, 7\}$	–
$\{4\}$	$\{4, 8\}$	–
$\{5\}$	$\{5, 9\}$	–
$\{6\}$	$\{6, 10\}$	–
$\{7\}$	$\{7, 11\}$	–
$\{8\}$	$\{0, 8\}$	–
$\{9\}$	$\{1, 9\}$	–
$\{10\}$	$\{2, 10\}$	–
$\{11\}$	$\{3, 11\}$	–

Phase 1, label s.

Source residues	Selected targets	Other targets
$\{0\}$	$\{0, 4\}$	–
$\{1\}$	$\{1, 5\}$	–
$\{2\}$	$\{2, 6\}$	–
$\{3\}$	$\{3, 7\}$	–
$\{4\}$	$\{4, 8\}$	–
$\{5\}$	$\{5, 9\}$	–
$\{6\}$	$\{6, 10\}$	–
$\{7\}$	$\{7, 11\}$	–
$\{8\}$	$\{0, 8\}$	–
$\{9\}$	$\{1, 9\}$	–
$\{10\}$	$\{2, 10\}$	–
$\{11\}$	$\{3, 11\}$	–

Phase 2, label s.

Source residues	Selected targets	Other targets
$\{0\}$	$\{0, 4\}$	–
$\{1\}$	$\{1, 5\}$	–
$\{2\}$	$\{2, 6\}$	–
$\{3\}$	$\{3, 7\}$	–
$\{4\}$	$\{4, 8\}$	–
$\{5\}$	$\{5, 9\}$	–
$\{6\}$	$\{6, 10\}$	–
$\{7\}$	$\{7, 11\}$	–
$\{8\}$	$\{0, 8\}$	–
$\{9\}$	$\{1, 9\}$	–
$\{10\}$	$\{2, 10\}$	–
$\{11\}$	$\{3, 11\}$	–

Phase 3, label g.

Source residues	Selected targets	Other targets
$\{0\}$	$\varnothing$	b $\{1\}$
$\{1\}$	$\{2\}$	b $\{2\}$
$\{2\}$	$\varnothing$	b $\{3\}$
$\{3\}$	$\varnothing$	b $\{4\}$
$\{4\}$	$\varnothing$	b $\{5\}$
$\{5\}$	$\{6\}$	b $\{6\}$
$\{6\}$	$\varnothing$	b $\{7\}$
$\{7\}$	$\varnothing$	b $\{8\}$
$\{8\}$	$\varnothing$	b $\{9\}$
$\{9\}$	$\{10\}$	b $\{10\}$
$\{10\}$	$\varnothing$	b $\{11\}$
$\{11\}$	$\varnothing$	b $\{0\}$

Phase 4, label s.

Source residues	Selected targets	Other targets
{0}	{0, 4}	–
{1}	{1, 5}	–
{2}	{2, 6}	–
{3}	{3, 7}	–
{4}	{4, 8}	–
{5}	{5, 9}	–
{6}	{6, 10}	–
{7}	{7, 11}	–
{8}	{0, 8}	–
{9}	{1, 9}	–
{10}	{2, 10}	–
{11}	{3, 11}	–

Phase 5, label s.

Source residues	Selected targets	Other targets
{0}	{0, 4}	–
{1}	{1, 5}	–
{2}	{2, 6}	–
{3}	{3, 7}	–
{4}	{4, 8}	–
{5}	{5, 9}	–
{6}	{6, 10}	–
{7}	{7, 11}	–
{8}	{0, 8}	–
{9}	{1, 9}	–
{10}	{2, 10}	–
{11}	{3, 11}	–

Phase 6, label s.

Source residues	Selected targets	Other targets
{0}	{0, 4}	–
{1}	{1, 5}	–
{2}	{2, 6}	–
{3}	{3, 7}	–
{4}	{4, 8}	–
{5}	{5, 9}	–
{6}	{6, 10}	–
{7}	{7, 11}	–
{8}	{0, 8}	–
{9}	{1, 9}	–
{10}	{2, 10}	–
{11}	{3, 11}	–

Phase 7, label s.

Source residues	Selected targets	Other targets
{0}	{0, 4}	–
{1}	{1, 5}	–
{2}	{2, 6}	–
{3}	{3, 7}	–
{4}	{4, 8}	–
{5}	{5, 9}	–
{6}	{6, 10}	–
{7}	{7, 11}	–
{8}	{0, 8}	–
{9}	{1, 9}	–
{10}	{2, 10}	–
{11}	{3, 11}	–

Phase 8, label s.

Source residues	Selected targets	Other targets
{0}	{0, 4}	–
{1}	{1, 5}	–
{2}	{2, 6}	–
{3}	{3, 7}	–
{4}	{4, 8}	–
{5}	{5, 9}	–
{6}	{6, 10}	–
{7}	{7, 11}	–
{8}	{0, 8}	–
{9}	{1, 9}	–
{10}	{2, 10}	–
{11}	{3, 11}	–

Phase 9, label s.

Source residues	Selected targets	Other targets
{0}	{0, 4}	–
{1}	{1, 5}	–
{2}	{2, 6}	–
{3}	{3, 7}	–
{4}	{4, 8}	–
{5}	{5, 9}	–
{6}	{6, 10}	–
{7}	{7, 11}	–
{8}	{0, 8}	–
{9}	{1, 9}	–
{10}	{2, 10}	–
{11}	{3, 11}	–

Phase 10, label s.

Source residues	Selected targets	Other targets
$\{0\}$	$\{0, 4\}$	–
$\{1\}$	$\{1, 5\}$	–
$\{2\}$	$\{2, 6\}$	–
$\{3\}$	$\{3, 7\}$	–
$\{4\}$	$\{4, 8\}$	–
$\{5\}$	$\{5, 9\}$	–
$\{6\}$	$\{6, 10\}$	–
$\{7\}$	$\{7, 11\}$	–
$\{8\}$	$\{0, 8\}$	–
$\{9\}$	$\{1, 9\}$	–
$\{10\}$	$\{2, 10\}$	–
$\{11\}$	$\{3, 11\}$	–

Phase 11, label s.

Source residues	Selected targets	Other targets
$\{0\}$	$\{0, 4\}$	–
$\{1\}$	$\{1, 5\}$	–
$\{2\}$	$\{2, 6\}$	–
$\{3\}$	$\{3, 7\}$	–
$\{4\}$	$\{4, 8\}$	–
$\{5\}$	$\{5, 9\}$	–
$\{6\}$	$\{6, 10\}$	–
$\{7\}$	$\{7, 11\}$	–
$\{8\}$	$\{0, 8\}$	–
$\{9\}$	$\{1, 9\}$	–
$\{10\}$	$\{2, 10\}$	–
$\{11\}$	$\{3, 11\}$	–

30.8 P3.8 Guard at layer 10, width 15, target 6, bypass 0

Before the guard, every reachable residue is zero modulo 5. The guard accepts residue one. A bypass with a different label can change the verdict in the union graph while leaving the prescribed word infeasible.

The prescribed label sequence is $s, s, s, s, s, s, s, s, s, g, s, s, s$ and its observed state sequence is $0, 1, 0, 1, 0, 1, 0, 1, 0, 1, 0, 1, 0, 1$.

The initial set is $\{0 : 0\}$ and the bad set is $\{13 : 6\}$.

The exact lift count is 0. Allowing any label at each layer of this same observed state sequence gives 0 lifts.

i	Exact prefix	Exact suffix	Join
0	$\{0 : 0\}$	$\varnothing$	0
1	$\{1 : 0, 1 : 5\}$	$\varnothing$	0
2	$\{2 : 0, 2 : 5, 2 : 10\}$	$\varnothing$	0
3	$\{3 : 0, 3 : 5, 3 : 10\}$	$\varnothing$	0
4	$\{4 : 0, 4 : 5, 4 : 10\}$	$\varnothing$	0
5	$\{5 : 0, 5 : 5, 5 : 10\}$	$\varnothing$	0
6	$\{6 : 0, 6 : 5, 6 : 10\}$	$\varnothing$	0
7	$\{7 : 0, 7 : 5, 7 : 10\}$	$\varnothing$	0
8	$\{8 : 0, 8 : 5, 8 : 10\}$	$\varnothing$	0
9	$\{9 : 0, 9 : 5, 9 : 10\}$	$\varnothing$	0
10	$\varnothing$	$\{10 : 1, 10 : 6, 10 : 11\}$	0

i	Exact prefix	Exact suffix	Join
11	$\varnothing$	$\{11 : 1, 11 : 6, 11 : 11\}$	0
12	$\varnothing$	$\{12 : 1, 12 : 6\}$	0
13	$\varnothing$	$\{13 : 6\}$	0

Certificates. The least, extended, and greatest sequences give 12, 16, and 6 realized keys, respectively. All three refined labelled graphs have zero descendants of the selected path.

i	$\lvert A_i\rvert$	$\lvert J_i^{\min}\rvert$	$J_i^{\mathrm{ext}} \setminus J_i^{\min}$	$\lvert J_i^{\max}\rvert$
0	105	1	$\{0 : 1\}$	105
1	105	2	$\{1 : 1, 1 : 6\}$	105
2	105	3	$\{2 : 1, 2 : 6, 2 : 11\}$	105
3	105	3	$\{3 : 1, 3 : 6, 3 : 11\}$	105
4	105	3	$\{4 : 1, 4 : 6, 4 : 11\}$	105
5	105	3	$\{5 : 1, 5 : 6, 5 : 11\}$	105
6	105	3	$\{6 : 1, 6 : 6, 6 : 11\}$	105
7	105	3	$\{7 : 1, 7 : 6, 7 : 11\}$	105
8	105	3	$\{8 : 1, 8 : 6, 8 : 11\}$	105
9	105	3	$\{9 : 1, 9 : 6, 9 : 11\}$	105
10	105	0	$\{10 : 2, 10 : 7, 10 : 12\}$	102

i	$\lvert A_i\rvert$	$\lvert J_i^{\min}\rvert$	$J_i^{\mathrm{ext}} \setminus J_i^{\min}$	$\lvert J_i^{\max}\rvert$
11	105	0	$\{11 : 2, 11 : 7, 11 : 12\}$	102
12	105	0	$\{12 : 2, 12 : 7, 12 : 12\}$	103
13	105	0	$\{13 : 2, 13 : 7, 13 : 12\}$	104

Transition rows. Identical successor rows within a phase are grouped when they have the same selected-label and other-label outputs.

Phase 0, label s.

Source residues	Selected targets	Other targets
{0}	{0, 5}	–
{1}	{1, 6}	–
{2}	{2, 7}	–
{3}	{3, 8}	–
{4}	{4, 9}	–
{5}	{5, 10}	–
{6}	{6, 11}	–
{7}	{7, 12}	–
{8}	{8, 13}	–
{9}	{9, 14}	–
{10}	{0, 10}	–
{11}	{1, 11}	–

Phase 0, label s (continued).

Source residues	Selected targets	Other targets
{12}	{2, 12}	–
{13}	{3, 13}	–
{14}	{4, 14}	–

Phase 1, label s.

Source residues	Selected targets	Other targets
{0}	{0, 5}	–
{1}	{1, 6}	–
{2}	{2, 7}	–
{3}	{3, 8}	–
{4}	{4, 9}	–
{5}	{5, 10}	–
{6}	{6, 11}	–
{7}	{7, 12}	–
{8}	{8, 13}	–
{9}	{9, 14}	–
{10}	{0, 10}	–
{11}	{1, 11}	–

Phase 1, label s (continued).

Source residues	Selected targets	Other targets
{12}	{2, 12}	–
{13}	{3, 13}	–
{14}	{4, 14}	–

Phase 2, label s.

Source residues	Selected targets	Other targets
{0}	{0, 5}	–
{1}	{1, 6}	–
{2}	{2, 7}	–
{3}	{3, 8}	–
{4}	{4, 9}	–
{5}	{5, 10}	–
{6}	{6, 11}	–
{7}	{7, 12}	–
{8}	{8, 13}	–
{9}	{9, 14}	–
{10}	{0, 10}	–
{11}	{1, 11}	–

Phase 2, label s (continued).

Source residues	Selected targets	Other targets
{12}	{2, 12}	–
{13}	{3, 13}	–
{14}	{4, 14}	–

Phase 3, label s.

Source residues	Selected targets	Other targets
{0}	{0, 5}	–
{1}	{1, 6}	–
{2}	{2, 7}	–
{3}	{3, 8}	–
{4}	{4, 9}	–
{5}	{5, 10}	–
{6}	{6, 11}	–
{7}	{7, 12}	–
{8}	{8, 13}	–
{9}	{9, 14}	–
{10}	{0, 10}	–
{11}	{1, 11}	–

Phase 3, label s (continued).

Source residues	Selected targets	Other targets
{12}	{2, 12}	–
{13}	{3, 13}	–
{14}	{4, 14}	–

Phase 4, label s.

Source residues	Selected targets	Other targets
{0}	{0, 5}	–
{1}	{1, 6}	–
{2}	{2, 7}	–
{3}	{3, 8}	–
{4}	{4, 9}	–
{5}	{5, 10}	–
{6}	{6, 11}	–
{7}	{7, 12}	–
{8}	{8, 13}	–
{9}	{9, 14}	–
{10}	{0, 10}	–
{11}	{1, 11}	–

Phase 4, label s (continued).

Source residues	Selected targets	Other targets
{12}	{2, 12}	–
{13}	{3, 13}	–
{14}	{4, 14}	–

Phase 5, label s.

Source residues	Selected targets	Other targets
{0}	{0, 5}	–
{1}	{1, 6}	–
{2}	{2, 7}	–
{3}	{3, 8}	–
{4}	{4, 9}	–
{5}	{5, 10}	–
{6}	{6, 11}	–
{7}	{7, 12}	–
{8}	{8, 13}	–
{9}	{9, 14}	–
{10}	{0, 10}	–
{11}	{1, 11}	–

Phase 5, label s (continued).

Source residues	Selected targets	Other targets
{12}	{2, 12}	–
{13}	{3, 13}	–
{14}	{4, 14}	–

Phase 6, label s.

Source residues	Selected targets	Other targets
{0}	{0, 5}	–
{1}	{1, 6}	–
{2}	{2, 7}	–
{3}	{3, 8}	–
{4}	{4, 9}	–
{5}	{5, 10}	–
{6}	{6, 11}	–
{7}	{7, 12}	–
{8}	{8, 13}	–
{9}	{9, 14}	–
{10}	{0, 10}	–
{11}	{1, 11}	–

Phase 6, label s (continued).

Source residues	Selected targets	Other targets
{12}	{2, 12}	–
{13}	{3, 13}	–
{14}	{4, 14}	–

Phase 7, label s.

Source residues	Selected targets	Other targets
{0}	{0, 5}	–
{1}	{1, 6}	–
{2}	{2, 7}	–
{3}	{3, 8}	–
{4}	{4, 9}	–
{5}	{5, 10}	–
{6}	{6, 11}	–
{7}	{7, 12}	–
{8}	{8, 13}	–
{9}	{9, 14}	–
{10}	{0, 10}	–
{11}	{1, 11}	–

Phase 7, label s (continued).

Source residues	Selected targets	Other targets
{12}	{2, 12}	–
{13}	{3, 13}	–
{14}	{4, 14}	–

Phase 8, label s.

Source residues	Selected targets	Other targets
$\{0\}$	$\{0, 5\}$	–
$\{1\}$	$\{1, 6\}$	–
$\{2\}$	$\{2, 7\}$	–
$\{3\}$	$\{3, 8\}$	–
$\{4\}$	$\{4, 9\}$	–
$\{5\}$	$\{5, 10\}$	–
$\{6\}$	$\{6, 11\}$	–
$\{7\}$	$\{7, 12\}$	–
$\{8\}$	$\{8, 13\}$	–
$\{9\}$	$\{9, 14\}$	–
$\{10\}$	$\{0, 10\}$	–
$\{11\}$	$\{1, 11\}$	–

Phase 8, label s (continued).

Source residues	Selected targets	Other targets
$\{12\}$	$\{2, 12\}$	–
$\{13\}$	$\{3, 13\}$	–
$\{14\}$	$\{4, 14\}$	–

Phase 9, label g.

Source residues	Selected targets	Other targets
$\{0, 2, 3, 4, 5, 7, 8, 9, 10, 12, 13, 14\}$	$\varnothing$	–
$\{1\}$	$\{2\}$	–
$\{6\}$	$\{7\}$	–
$\{11\}$	$\{12\}$	–

Phase 10, label s.

Source residues	Selected targets	Other targets
$\{0\}$	$\{0, 5\}$	–
$\{1\}$	$\{1, 6\}$	–
$\{2\}$	$\{2, 7\}$	–
$\{3\}$	$\{3, 8\}$	–
$\{4\}$	$\{4, 9\}$	–
$\{5\}$	$\{5, 10\}$	–
$\{6\}$	$\{6, 11\}$	–
$\{7\}$	$\{7, 12\}$	–
$\{8\}$	$\{8, 13\}$	–
$\{9\}$	$\{9, 14\}$	–
$\{10\}$	$\{0, 10\}$	–
$\{11\}$	$\{1, 11\}$	–

Phase 10, label s (continued).

Source residues	Selected targets	Other targets
$\{12\}$	$\{2, 12\}$	–
$\{13\}$	$\{3, 13\}$	–
$\{14\}$	$\{4, 14\}$	–

Phase 11, label s.

Source residues	Selected targets	Other targets
$\{0\}$	$\{0, 5\}$	–
$\{1\}$	$\{1, 6\}$	–
$\{2\}$	$\{2, 7\}$	–
$\{3\}$	$\{3, 8\}$	–
$\{4\}$	$\{4, 9\}$	–
$\{5\}$	$\{5, 10\}$	–
$\{6\}$	$\{6, 11\}$	–
$\{7\}$	$\{7, 12\}$	–
$\{8\}$	$\{8, 13\}$	–
$\{9\}$	$\{9, 14\}$	–
$\{10\}$	$\{0, 10\}$	–
$\{11\}$	$\{1, 11\}$	–

Phase 11, label s (continued).

Source residues	Selected targets	Other targets
$\{12\}$	$\{2, 12\}$	–
$\{13\}$	$\{3, 13\}$	–
$\{14\}$	$\{4, 14\}$	–

Phase 12, label s.

Source residues	Selected targets	Other targets
$\{0\}$	$\{0, 5\}$	–
$\{1\}$	$\{1, 6\}$	–
$\{2\}$	$\{2, 7\}$	–
$\{3\}$	$\{3, 8\}$	–
$\{4\}$	$\{4, 9\}$	–
$\{5\}$	$\{5, 10\}$	–
$\{6\}$	$\{6, 11\}$	–
$\{7\}$	$\{7, 12\}$	–
$\{8\}$	$\{8, 13\}$	–
$\{9\}$	$\{9, 14\}$	–
$\{10\}$	$\{0, 10\}$	–
$\{11\}$	$\{1, 11\}$	–

Phase 12, label s (continued).

Source residues	Selected targets	Other targets
$\{12\}$	$\{2, 12\}$	–
$\{13\}$	$\{3, 13\}$	–
$\{14\}$	$\{4, 14\}$	–

31 Forks and merging resets

The forward computation counts each path entering a reset separately. Repeated observations identify phases that remain distinct in the selected word.

A state printed as $i : v$ has phase i and residue v. The successor table prints residues at the next phase. A blank column for another label means there is no such edge at that phase.

31.1 P4.1 Fork and reset, width 6, length 7, target 1

The fork can increase the count of distinct prefixes even when their endpoints merge at the reset. At that layer the endpoint retains parity alone.

The prescribed label sequence is s, f, s, s, j, s, s and its observed state sequence is $0, 1, 2, 0, 1, 2, 0, 1$.

The initial set is $\{0 : 0\}$ and the bad set is $\{7 : 1\}$.

The exact lift count is 0. Allowing any label at each layer of this same observed state sequence gives 0 lifts.

i	Exact prefix	Exact suffix	Join
0	$\{0 : 0\}$	$\varnothing$	0
1	$\{1 : 2\}$	$\varnothing$	0
2	$\{2 : 2, 2 : 4\}$	$\varnothing$	0
3	$\{3 : 0, 3 : 4\}$	$\varnothing$	0
4	$\{4 : 0, 4 : 2\}$	$\varnothing$	0
5	$\{5 : 0\}$	$\{5 : 3\}$	0
6	$\{6 : 2\}$	$\{6 : 5\}$	0
7	$\{7 : 4\}$	$\{7 : 1\}$	0

Certificates. The least, extended, and greatest sequences give 11, 11, and 6 realized keys, respectively. All three refined labelled graphs have zero descendants of the selected path.

i	$\lvert A_i\rvert$	$\lvert J_i^{\min}\rvert$	$J_i^{\text{ext}} \setminus J_i^{\min}$	$\lvert J_i^{\max}\rvert$
0	18	1	$\{0 : 1\}$	18
1	18	1	$\{1 : 3\}$	18
2	12	2	$\{2 : 3, 2 : 5\}$	12
3	18	2	$\{3 : 1, 3 : 5\}$	18
4	18	2	$\{4 : 1, 4 : 3\}$	18
5	12	1	$\{5 : 1\}$	11
6	18	1	$\{6 : 3\}$	17
7	18	1	$\{7 : 5\}$	17

Transition rows. Identical successor rows within a phase are grouped when they have the same selected-label and other-label outputs.

Phase 0, label s.

Source residues	Selected targets	Other targets
$\{0\}$	$\{2\}$	–
$\{1\}$	$\{3\}$	–
$\{2\}$	$\{4\}$	–
$\{3\}$	$\{5\}$	–
$\{4\}$	$\{0\}$	–
$\{5\}$	$\{1\}$	–

Phase 1, label f.

Source residues	Selected targets	Other targets
$\{0\}$	$\{0, 2\}$	–
$\{1\}$	$\{1, 3\}$	–
$\{2\}$	$\{2, 4\}$	–
$\{3\}$	$\{3, 5\}$	–
$\{4\}$	$\{0, 4\}$	–
$\{5\}$	$\{1, 5\}$	–

Phase 2, label s.

Source residues	Selected targets	Other targets
$\{0\}$	$\{2\}$	–
$\{1\}$	$\{3\}$	–
$\{2\}$	$\{4\}$	–
$\{3\}$	$\{5\}$	–
$\{4\}$	$\{0\}$	–
$\{5\}$	$\{1\}$	–

Phase 3, label s.

Source residues	Selected targets	Other targets
$\{0\}$	$\{2\}$	–
$\{1\}$	$\{3\}$	–
$\{2\}$	$\{4\}$	–
$\{3\}$	$\{5\}$	–
$\{4\}$	$\{0\}$	–
$\{5\}$	$\{1\}$	–

Phase 4, label j.

Source residues	Selected targets	Other targets
$\{0, 2, 4\}$	$\{0\}$	–
$\{1, 3, 5\}$	$\{1\}$	–

Phase 5, label s.

Source residues	Selected targets	Other targets
$\{0\}$	$\{2\}$	–
$\{1\}$	$\{3\}$	–
$\{2\}$	$\{4\}$	–
$\{3\}$	$\{5\}$	–
$\{4\}$	$\{0\}$	–
$\{5\}$	$\{1\}$	–

Phase 6, label s.

Source residues	Selected targets	Other targets
$\{0\}$	$\{2\}$	–
$\{1\}$	$\{3\}$	–
$\{2\}$	$\{4\}$	–
$\{3\}$	$\{5\}$	–
$\{4\}$	$\{0\}$	–
$\{5\}$	$\{1\}$	–

31.2 P4.2 Fork and reset, width 8, length 8, target 2

The fork can increase the count of distinct prefixes even when their endpoints merge at the reset. At that layer the endpoint retains parity alone.

The prescribed label sequence is f, s, s, s, s, s, j, s and its observed state sequence is $0, 1, 2, 0, 1, 2, 0, 1, 2$.

The initial set is $\{0 : 0\}$ and the bad set is $\{8 : 2\}$.

The exact lift count is 2. Allowing any label at each layer of this same observed state sequence gives 2 lifts.

i	Exact prefix	Exact suffix	Join
0	$\{0:0\}$	$\{0:0, 0:2, 0:4, 0:6\}$	1
1	$\{1:0, 1:2\}$	$\{1:0, 1:2, 1:4, 1:6\}$	2
2	$\{2:2, 2:4\}$	$\{2:0, 2:2, 2:4, 2:6\}$	2
3	$\{3:4, 3:6\}$	$\{3:0, 3:2, 3:4, 3:6\}$	2
4	$\{4:0, 4:6\}$	$\{4:0, 4:2, 4:4, 4:6\}$	2
5	$\{5:0, 5:2\}$	$\{5:0, 5:2, 5:4, 5:6\}$	2
6	$\{6:2, 6:4\}$	$\{6:0, 6:2, 6:4, 6:6\}$	2
7	$\{7:0\}$	$\{7:0\}$	1
8	$\{8:2\}$	$\{8:2\}$	1

Witness. $0:0, 1:0, 2:2, 3:4, 4:6, 5:0, 6:2, 7:0, 8:2$ follows the selected label at every step.

Transition rows. Identical successor rows within a phase are grouped when they have the same selected-label and other-label outputs.

Phase 0, label f.

Source residues	Selected targets	Other targets
$\{0\}$	$\{0, 2\}$	–
$\{1\}$	$\{1, 3\}$	–
$\{2\}$	$\{2, 4\}$	–
$\{3\}$	$\{3, 5\}$	–
$\{4\}$	$\{4, 6\}$	–
$\{5\}$	$\{5, 7\}$	–
$\{6\}$	$\{0, 6\}$	–
$\{7\}$	$\{1, 7\}$	–

Phase 1, label s.

Source residues	Selected targets	Other targets
$\{0\}$	$\{2\}$	–
$\{1\}$	$\{3\}$	–
$\{2\}$	$\{4\}$	–
$\{3\}$	$\{5\}$	–
$\{4\}$	$\{6\}$	–
$\{5\}$	$\{7\}$	–
$\{6\}$	$\{0\}$	–
$\{7\}$	$\{1\}$	–

Phase 2, label s.

Source residues	Selected targets	Other targets
$\{0\}$	$\{2\}$	–
$\{1\}$	$\{3\}$	–
$\{2\}$	$\{4\}$	–
$\{3\}$	$\{5\}$	–
$\{4\}$	$\{6\}$	–
$\{5\}$	$\{7\}$	–
$\{6\}$	$\{0\}$	–
$\{7\}$	$\{1\}$	–

Phase 3, label s.

Source residues	Selected targets	Other targets
$\{0\}$	$\{2\}$	–
$\{1\}$	$\{3\}$	–
$\{2\}$	$\{4\}$	–
$\{3\}$	$\{5\}$	–
$\{4\}$	$\{6\}$	–
$\{5\}$	$\{7\}$	–
$\{6\}$	$\{0\}$	–
$\{7\}$	$\{1\}$	–

Phase 4, label s.

Source residues	Selected targets	Other targets
$\{0\}$	$\{2\}$	–
$\{1\}$	$\{3\}$	–
$\{2\}$	$\{4\}$	–
$\{3\}$	$\{5\}$	–
$\{4\}$	$\{6\}$	–
$\{5\}$	$\{7\}$	–
$\{6\}$	$\{0\}$	–
$\{7\}$	$\{1\}$	–

Phase 5, label s.

Source residues	Selected targets	Other targets
{0}	{2}	–
{1}	{3}	–
{2}	{4}	–
{3}	{5}	–
{4}	{6}	–
{5}	{7}	–
{6}	{0}	–
{7}	{1}	–

Phase 6, label j.

Source residues	Selected targets	Other targets
{0, 2, 4, 6}	{0}	–
{1, 3, 5, 7}	{1}	–

Phase 7, label s.

Source residues	Selected targets	Other targets
{0}	{2}	–
{1}	{3}	–
{2}	{4}	–
{3}	{5}	–
{4}	{6}	–
{5}	{7}	–
{6}	{0}	–
{7}	{1}	–

31.3 P4.3 Fork and reset, width 8, length 9, target 1

The fork can increase the count of distinct prefixes even when their endpoints merge at the reset. At that layer the endpoint retains parity alone.

The prescribed label sequence is $s, s, s, f, s, s, s, j, s$ and its observed state sequence is $0, 1, 2, 0, 1, 2, 0, 1, 2, 0$.

The initial set is $\{0 : 0\}$ and the bad set is $\{9 : 1\}$.

The exact lift count is 0. Allowing any label at each layer of this same observed state sequence gives 0 lifts.

i	Exact prefix	Exact suffix	Join
0	$\{0 : 0\}$	$\varnothing$	0
1	$\{1 : 2\}$	$\varnothing$	0
2	$\{2 : 4\}$	$\varnothing$	0
3	$\{3 : 6\}$	$\varnothing$	0
4	$\{4 : 0, 4 : 6\}$	$\varnothing$	0
5	$\{5 : 0, 5 : 2\}$	$\varnothing$	0
6	$\{6 : 2, 6 : 4\}$	$\varnothing$	0
7	$\{7 : 4, 7 : 6\}$	$\varnothing$	0
8	$\{8 : 0\}$	$\{8 : 7\}$	0
9	$\{9 : 2\}$	$\{9 : 1\}$	0

Certificates. The least, extended, and greatest sequences give 13, 13, and 5 realized keys, respectively. All three refined labelled graphs have zero descendants of the selected path.

i	$\lvert A_i\rvert$	$\lvert J_i^{\min}\rvert$	$J_i^{\mathrm{ext}} \setminus J_i^{\min}$	$\lvert J_i^{\max}\rvert$
0	32	1	$\{0 : 1\}$	32
1	24	1	$\{1 : 3\}$	24
2	24	1	$\{2 : 5\}$	24
3	32	1	$\{3 : 7\}$	32
4	24	2	$\{4 : 1, 4 : 7\}$	24
5	24	2	$\{5 : 1, 5 : 3\}$	24
6	32	2	$\{6 : 3, 6 : 5\}$	32
7	24	2	$\{7 : 5, 7 : 7\}$	24
8	24	1	$\{8 : 1\}$	23
9	32	1	$\{9 : 3\}$	31

Transition rows. Identical successor rows within a phase are grouped when they have the same selected-label and other-label outputs.

Phase 0, label s.

Source residues	Selected targets	Other targets
{0}	{2}	–
{1}	{3}	–
{2}	{4}	–
{3}	{5}	–
{4}	{6}	–
{5}	{7}	–
{6}	{0}	–
{7}	{1}	–

Phase 1, label s.

Source residues	Selected targets	Other targets
{0}	{2}	–
{1}	{3}	–
{2}	{4}	–
{3}	{5}	–
{4}	{6}	–
{5}	{7}	–
{6}	{0}	–
{7}	{1}	–

Phase 2, label s.

Source residues	Selected targets	Other targets
{0}	{2}	–
{1}	{3}	–
{2}	{4}	–
{3}	{5}	–
{4}	{6}	–
{5}	{7}	–
{6}	{0}	–
{7}	{1}	–

Phase 3, label f.

Source residues	Selected targets	Other targets
$\{0\}$	$\{0, 2\}$	–
$\{1\}$	$\{1, 3\}$	–
$\{2\}$	$\{2, 4\}$	–
$\{3\}$	$\{3, 5\}$	–
$\{4\}$	$\{4, 6\}$	–
$\{5\}$	$\{5, 7\}$	–
$\{6\}$	$\{0, 6\}$	–
$\{7\}$	$\{1, 7\}$	–

Phase 4, label s.

Source residues	Selected targets	Other targets
$\{0\}$	$\{2\}$	–
$\{1\}$	$\{3\}$	–
$\{2\}$	$\{4\}$	–
$\{3\}$	$\{5\}$	–
$\{4\}$	$\{6\}$	–
$\{5\}$	$\{7\}$	–
$\{6\}$	$\{0\}$	–
$\{7\}$	$\{1\}$	–

Phase 5, label s.

Source residues	Selected targets	Other targets
$\{0\}$	$\{2\}$	–
$\{1\}$	$\{3\}$	–
$\{2\}$	$\{4\}$	–
$\{3\}$	$\{5\}$	–
$\{4\}$	$\{6\}$	–
$\{5\}$	$\{7\}$	–
$\{6\}$	$\{0\}$	–
$\{7\}$	$\{1\}$	–

Phase 6, label s.

Source residues	Selected targets	Other targets
$\{0\}$	$\{2\}$	–
$\{1\}$	$\{3\}$	–
$\{2\}$	$\{4\}$	–
$\{3\}$	$\{5\}$	–
$\{4\}$	$\{6\}$	–
$\{5\}$	$\{7\}$	–
$\{6\}$	$\{0\}$	–
$\{7\}$	$\{1\}$	–

Phase 7, label j.

Source residues	Selected targets	Other targets
$\{0, 2, 4, 6\}$	$\{0\}$	–
$\{1, 3, 5, 7\}$	$\{1\}$	–

Phase 8, label s.

Source residues	Selected targets	Other targets
$\{0\}$	$\{2\}$	–
$\{1\}$	$\{3\}$	–
$\{2\}$	$\{4\}$	–
$\{3\}$	$\{5\}$	–
$\{4\}$	$\{6\}$	–
$\{5\}$	$\{7\}$	–
$\{6\}$	$\{0\}$	–
$\{7\}$	$\{1\}$	–

31.4 P4.4 Fork and reset, width 10, length 9, target 4

The fork can increase the count of distinct prefixes even when their endpoints merge at the reset. At that layer the endpoint retains parity alone.

The prescribed label sequence is $s, s, f, s, s, s, j, s, s$ and its observed state sequence is $0, 1, 2, 0, 1, 2, 0, 1, 2, 0$.

The initial set is $\{0 : 0\}$ and the bad set is $\{9 : 4\}$.

The exact lift count is 2. Allowing any label at each layer of this same observed state sequence gives 2 lifts.

i	Exact prefix	Exact suffix	Join
0	$\{0:0\}$	$\{0:0, 0:2, 0:4, 0:6, 0:8\}$	1
1	$\{1:2\}$	$\{1:0, 1:2, 1:4, 1:6, 1:8\}$	1
2	$\{2:4\}$	$\{2:0, 2:2, 2:4, 2:6, 2:8\}$	1
3	$\{3:4, 3:6\}$	$\{3:0, 3:2, 3:4, 3:6, 3:8\}$	2
4	$\{4:6, 4:8\}$	$\{4:0, 4:2, 4:4, 4:6, 4:8\}$	2
5	$\{5:0, 5:8\}$	$\{5:0, 5:2, 5:4, 5:6, 5:8\}$	2
6	$\{6:0, 6:2\}$	$\{6:0, 6:2, 6:4, 6:6, 6:8\}$	2
7	$\{7:0\}$	$\{7:0\}$	1
8	$\{8:2\}$	$\{8:2\}$	1
9	$\{9:4\}$	$\{9:4\}$	1

Witness. $0:0, 1:2, 2:4, 3:4, 4:6, 5:8, 6:0, 7:0, 8:2, 9:4$ follows the selected label at every step.

Transition rows. Identical successor rows within a phase are grouped when they have the same selected-label and other-label outputs.

Phase 0, label s.

Source residues	Selected targets	Other targets
{0}	{2}	–
{1}	{3}	–
{2}	{4}	–
{3}	{5}	–
{4}	{6}	–
{5}	{7}	–
{6}	{8}	–
{7}	{9}	–
{8}	{0}	–
{9}	{1}	–

Phase 1, label s.

Source residues	Selected targets	Other targets
{0}	{2}	–
{1}	{3}	–
{2}	{4}	–
{3}	{5}	–
{4}	{6}	–
{5}	{7}	–
{6}	{8}	–
{7}	{9}	–
{8}	{0}	–
{9}	{1}	–

Phase 2, label f.

Source residues	Selected targets	Other targets
{0}	{0, 2}	–
{1}	{1, 3}	–
{2}	{2, 4}	–
{3}	{3, 5}	–
{4}	{4, 6}	–
{5}	{5, 7}	–
{6}	{6, 8}	–
{7}	{7, 9}	–
{8}	{0, 8}	–
{9}	{1, 9}	–

Phase 3, label s.

Source residues	Selected targets	Other targets
{0}	{2}	–
{1}	{3}	–
{2}	{4}	–
{3}	{5}	–
{4}	{6}	–
{5}	{7}	–
{6}	{8}	–
{7}	{9}	–
{8}	{0}	–
{9}	{1}	–

Phase 4, label s.

Source residues	Selected targets	Other targets
{0}	{2}	–
{1}	{3}	–
{2}	{4}	–
{3}	{5}	–
{4}	{6}	–
{5}	{7}	–
{6}	{8}	–
{7}	{9}	–
{8}	{0}	–
{9}	{1}	–

Phase 5, label s.

Source residues	Selected targets	Other targets
{0}	{2}	–
{1}	{3}	–
{2}	{4}	–
{3}	{5}	–
{4}	{6}	–
{5}	{7}	–
{6}	{8}	–
{7}	{9}	–
{8}	{0}	–
{9}	{1}	–

Phase 6, label j.

Source residues	Selected targets	Other targets
{0, 2, 4, 6, 8}	{0}	–
{1, 3, 5, 7, 9}	{1}	–

Phase 7, label s.

Source residues	Selected targets	Other targets
{0}	{2}	–
{1}	{3}	–
{2}	{4}	–
{3}	{5}	–
{4}	{6}	–
{5}	{7}	–
{6}	{8}	–
{7}	{9}	–
{8}	{0}	–
{9}	{1}	–

Phase 8, label s.

Source residues	Selected targets	Other targets
{0}	{2}	–
{1}	{3}	–
{2}	{4}	–
{3}	{5}	–
{4}	{6}	–
{5}	{7}	–
{6}	{8}	–
{7}	{9}	–
{8}	{0}	–
{9}	{1}	–

31.5 P4.5 Fork and reset, width 10, length 10, target 3

The fork can increase the count of distinct prefixes even when their endpoints merge at the reset. At that layer the endpoint retains parity alone.

The prescribed label sequence is $s, s, f, s, s, j, s, s, s, s$ and its observed state sequence is $0, 1, 2, 0, 1, 2, 0, 1, 2, 0, 1$.

The initial set is $\{0 : 0\}$ and the bad set is $\{10 : 3\}$.

The exact lift count is 0. Allowing any label at each layer of this same observed state sequence gives 0 lifts.

i	Exact prefix	Exact suffix	Join
0	$\{0 : 0\}$	$\varnothing$	0
1	$\{1 : 2\}$	$\varnothing$	0
2	$\{2 : 4\}$	$\varnothing$	0
3	$\{3 : 4, 3 : 6\}$	$\varnothing$	0
4	$\{4 : 6, 4 : 8\}$	$\varnothing$	0
5	$\{5 : 0, 5 : 8\}$	$\varnothing$	0
6	$\{6 : 0\}$	$\{6 : 5\}$	0
7	$\{7 : 2\}$	$\{7 : 7\}$	0
8	$\{8 : 4\}$	$\{8 : 9\}$	0
9	$\{9 : 6\}$	$\{9 : 1\}$	0
10	$\{10 : 8\}$	$\{10 : 3\}$	0

Certificates. The least, extended, and greatest sequences give 14, 14, and 8 realized keys, respectively. All three refined labelled graphs have zero descendants of the selected path.

i	$\lvert A_i\rvert$	$\lvert J_i^{\min}\rvert$	$J_i^{\text{ext}} \setminus J_i^{\min}$	$\lvert J_i^{\max}\rvert$
0	40	1	$\{0 : 1\}$	40
1	40	1	$\{1 : 3\}$	40
2	30	1	$\{2 : 5\}$	30
3	40	2	$\{3 : 5, 3 : 7\}$	40
4	40	2	$\{4 : 7, 4 : 9\}$	40
5	30	2	$\{5 : 1, 5 : 9\}$	30
6	40	1	$\{6 : 1\}$	39
7	40	1	$\{7 : 3\}$	39
8	30	1	$\{8 : 5\}$	29
9	40	1	$\{9 : 7\}$	39
10	40	1	$\{10 : 9\}$	39

Transition rows. Identical successor rows within a phase are grouped when they have the same selected-label and other-label outputs.

Phase 0, label s.

Source residues	Selected targets	Other targets
{0}	{2}	–
{1}	{3}	–
{2}	{4}	–
{3}	{5}	–
{4}	{6}	–
{5}	{7}	–
{6}	{8}	–
{7}	{9}	–
{8}	{0}	–
{9}	{1}	–

Phase 1, label s.

Source residues	Selected targets	Other targets
{0}	{2}	–
{1}	{3}	–
{2}	{4}	–
{3}	{5}	–
{4}	{6}	–
{5}	{7}	–
{6}	{8}	–
{7}	{9}	–
{8}	{0}	–
{9}	{1}	–

Phase 2, label f.

Source residues	Selected targets	Other targets
{0}	{0, 2}	–
{1}	{1, 3}	–
{2}	{2, 4}	–
{3}	{3, 5}	–
{4}	{4, 6}	–
{5}	{5, 7}	–
{6}	{6, 8}	–
{7}	{7, 9}	–
{8}	{0, 8}	–
{9}	{1, 9}	–

Phase 3, label s.

Source residues	Selected targets	Other targets
{0}	{2}	–
{1}	{3}	–
{2}	{4}	–
{3}	{5}	–
{4}	{6}	–
{5}	{7}	–
{6}	{8}	–
{7}	{9}	–
{8}	{0}	–
{9}	{1}	–

Phase 4, label s.

Source residues	Selected targets	Other targets
{0}	{2}	–
{1}	{3}	–
{2}	{4}	–
{3}	{5}	–
{4}	{6}	–
{5}	{7}	–
{6}	{8}	–
{7}	{9}	–
{8}	{0}	–
{9}	{1}	–

Phase 5, label j.

Source residues	Selected targets	Other targets
{0, 2, 4, 6, 8}	{0}	–
{1, 3, 5, 7, 9}	{1}	–

Phase 6, label s.

Source residues	Selected targets	Other targets
{0}	{2}	–
{1}	{3}	–
{2}	{4}	–
{3}	{5}	–
{4}	{6}	–
{5}	{7}	–
{6}	{8}	–
{7}	{9}	–
{8}	{0}	–
{9}	{1}	–

Phase 7, label s.

Source residues	Selected targets	Other targets
{0}	{2}	–
{1}	{3}	–
{2}	{4}	–
{3}	{5}	–
{4}	{6}	–
{5}	{7}	–
{6}	{8}	–
{7}	{9}	–
{8}	{0}	–
{9}	{1}	–

Phase 8, label s.

Source residues	Selected targets	Other targets
{0}	{2}	–
{1}	{3}	–
{2}	{4}	–
{3}	{5}	–
{4}	{6}	–
{5}	{7}	–
{6}	{8}	–
{7}	{9}	–
{8}	{0}	–
{9}	{1}	–

Phase 9, label s.

Source residues	Selected targets	Other targets
{0}	{2}	–
{1}	{3}	–
{2}	{4}	–
{3}	{5}	–
{4}	{6}	–
{5}	{7}	–
{6}	{8}	–
{7}	{9}	–
{8}	{0}	–
{9}	{1}	–

31.6 P4.6 Fork and reset, width 12, length 11, target 2

The fork can increase the count of distinct prefixes even when their endpoints merge at the reset. At that layer the endpoint retains parity alone.

The prescribed label sequence is $f, s, s, s, s, s, s, s, j, s, s$ and its observed state sequence is $0, 1, 2, 0, 1, 2, 0, 1, 2, 0, 1, 2$.

The initial set is $\{0 : 0\}$ and the bad set is $\{11 : 2\}$.

The exact lift count is 0. Allowing any label at each layer of this same observed state sequence gives 0 lifts.

i	Exact prefix	Exact suffix	Join
0	$\{0:0\}$	$\varnothing$	0
1	$\{1:0, 1:2\}$	$\varnothing$	0
2	$\{2:2, 2:4\}$	$\varnothing$	0
3	$\{3:4, 3:6\}$	$\varnothing$	0
4	$\{4:6, 4:8\}$	$\varnothing$	0
5	$\{5:8, 5:10\}$	$\varnothing$	0
6	$\{6:0, 6:10\}$	$\varnothing$	0
7	$\{7:0, 7:2\}$	$\varnothing$	0
8	$\{8:2, 8:4\}$	$\varnothing$	0
9	$\{9:0\}$	$\{9:10\}$	0
10	$\{10:2\}$	$\{10:0\}$	0

i	Exact prefix	Exact suffix	Join
11	$\{11:4\}$	$\{11:2\}$	0

Certificates. The least, extended, and greatest sequences give 15, 15, and 6 realized keys, respectively. All three refined labelled graphs have zero descendants of the selected path.

i	$\lvert A_i\rvert$	$\lvert J_i^{\min}\rvert$	$J_i^{\text{ext}} \setminus J_i^{\min}$	$\lvert J_i^{\max}\rvert$
0	48	1	$\{0:1\}$	48
1	48	2	$\{1:1, 1:3\}$	48
2	48	2	$\{2:3, 2:5\}$	48
3	48	2	$\{3:5, 3:7\}$	48
4	48	2	$\{4:7, 4:9\}$	48
5	48	2	$\{5:9, 5:11\}$	48
6	48	2	$\{6:1, 6:11\}$	48
7	48	2	$\{7:1, 7:3\}$	48
8	48	2	$\{8:3, 8:5\}$	48
9	48	1	$\{9:1\}$	47
10	48	1	$\{10:3\}$	47

i	$\lvert A_i\rvert$	$\lvert J_i^{\min}\rvert$	$J_i^{\text{ext}} \setminus J_i^{\min}$	$\lvert J_i^{\max}\rvert$
11	48	1	$\{11:5\}$	47

Transition rows. Identical successor rows within a phase are grouped when they have the same selected-label and other-label outputs.

Phase 0, label f.

Source residues	Selected targets	Other targets
$\{0\}$	$\{0, 2\}$	–
$\{1\}$	$\{1, 3\}$	–
$\{2\}$	$\{2, 4\}$	–
$\{3\}$	$\{3, 5\}$	–
$\{4\}$	$\{4, 6\}$	–
$\{5\}$	$\{5, 7\}$	–
$\{6\}$	$\{6, 8\}$	–
$\{7\}$	$\{7, 9\}$	–
$\{8\}$	$\{8, 10\}$	–
$\{9\}$	$\{9, 11\}$	–
$\{10\}$	$\{0, 10\}$	–
$\{11\}$	$\{1, 11\}$	–

Phase 1, label s.

Source residues	Selected targets	Other targets
$\{0\}$	$\{2\}$	–
$\{1\}$	$\{3\}$	–
$\{2\}$	$\{4\}$	–
$\{3\}$	$\{5\}$	–
$\{4\}$	$\{6\}$	–
$\{5\}$	$\{7\}$	–
$\{6\}$	$\{8\}$	–
$\{7\}$	$\{9\}$	–
$\{8\}$	$\{10\}$	–
$\{9\}$	$\{11\}$	–
$\{10\}$	$\{0\}$	–
$\{11\}$	$\{1\}$	–

Phase 2, label s.

Source residues	Selected targets	Other targets
$\{0\}$	$\{2\}$	–
$\{1\}$	$\{3\}$	–
$\{2\}$	$\{4\}$	–
$\{3\}$	$\{5\}$	–
$\{4\}$	$\{6\}$	–
$\{5\}$	$\{7\}$	–
$\{6\}$	$\{8\}$	–
$\{7\}$	$\{9\}$	–
$\{8\}$	$\{10\}$	–
$\{9\}$	$\{11\}$	–
$\{10\}$	$\{0\}$	–
$\{11\}$	$\{1\}$	–

Phase 3, label s.

Source residues	Selected targets	Other targets
$\{0\}$	$\{2\}$	–
$\{1\}$	$\{3\}$	–
$\{2\}$	$\{4\}$	–
$\{3\}$	$\{5\}$	–
$\{4\}$	$\{6\}$	–
$\{5\}$	$\{7\}$	–
$\{6\}$	$\{8\}$	–
$\{7\}$	$\{9\}$	–
$\{8\}$	$\{10\}$	–
$\{9\}$	$\{11\}$	–
$\{10\}$	$\{0\}$	–
$\{11\}$	$\{1\}$	–

Phase 4, label s.

Source residues	Selected targets	Other targets
{0}	{2}	–
{1}	{3}	–
{2}	{4}	–
{3}	{5}	–
{4}	{6}	–
{5}	{7}	–
{6}	{8}	–
{7}	{9}	–
{8}	{10}	–
{9}	{11}	–
{10}	{0}	–
{11}	{1}	–

Phase 5, label s.

Source residues	Selected targets	Other targets
{0}	{2}	–
{1}	{3}	–
{2}	{4}	–
{3}	{5}	–
{4}	{6}	–
{5}	{7}	–
{6}	{8}	–
{7}	{9}	–
{8}	{10}	–
{9}	{11}	–
{10}	{0}	–
{11}	{1}	–

Phase 6, label s.

Source residues	Selected targets	Other targets
{0}	{2}	–
{1}	{3}	–
{2}	{4}	–
{3}	{5}	–
{4}	{6}	–
{5}	{7}	–
{6}	{8}	–
{7}	{9}	–
{8}	{10}	–
{9}	{11}	–
{10}	{0}	–
{11}	{1}	–

Phase 7, label s.

Source residues	Selected targets	Other targets
{0}	{2}	–
{1}	{3}	–
{2}	{4}	–
{3}	{5}	–
{4}	{6}	–
{5}	{7}	–
{6}	{8}	–
{7}	{9}	–
{8}	{10}	–
{9}	{11}	–
{10}	{0}	–
{11}	{1}	–

Phase 8, label j.

Source residues	Selected targets	Other targets
{0, 2, 4, 6, 8, 10}	{0}	–
{1, 3, 5, 7, 9, 11}	{1}	–

Phase 9, label s.

Source residues	Selected targets	Other targets
{0}	{2}	–
{1}	{3}	–
{2}	{4}	–
{3}	{5}	–
{4}	{6}	–
{5}	{7}	–
{6}	{8}	–
{7}	{9}	–
{8}	{10}	–
{9}	{11}	–
{10}	{0}	–
{11}	{1}	–

Phase 10, label s.

Source residues	Selected targets	Other targets
{0}	{2}	–
{1}	{3}	–
{2}	{4}	–
{3}	{5}	–
{4}	{6}	–
{5}	{7}	–
{6}	{8}	–
{7}	{9}	–
{8}	{10}	–
{9}	{11}	–
{10}	{0}	–
{11}	{1}	–

31.7 P4.7 Fork and reset, width 12, length 12, target 1

The fork can increase the count of distinct prefixes even when their endpoints merge at the reset. At that layer the endpoint retains parity alone.

The prescribed label sequence is $s, s, s, s, f, s, s, s, s, j, s, s$ and its observed state sequence is $0, 1, 2, 0, 1, 2, 0, 1, 2, 0, 1, 2, 0$.

The initial set is $\{0 : 0\}$ and the bad set is $\{12 : 1\}$.

The exact lift count is 0. Allowing any label at each layer of this same observed state sequence gives 0 lifts.

i	Exact prefix	Exact suffix	Join
0	$\{0 : 0\}$	$\varnothing$	0
1	$\{1 : 2\}$	$\varnothing$	0
2	$\{2 : 4\}$	$\varnothing$	0
3	$\{3 : 6\}$	$\varnothing$	0
4	$\{4 : 8\}$	$\varnothing$	0
5	$\{5 : 8, 5 : 10\}$	$\varnothing$	0
6	$\{6 : 0, 6 : 10\}$	$\varnothing$	0
7	$\{7 : 0, 7 : 2\}$	$\varnothing$	0
8	$\{8 : 2, 8 : 4\}$	$\varnothing$	0
9	$\{9 : 4, 9 : 6\}$	$\varnothing$	0
10	$\{10 : 0\}$	$\{10 : 9\}$	0

i	Exact prefix	Exact suffix	Join
11	$\{11 : 2\}$	$\{11 : 11\}$	0
12	$\{12 : 4\}$	$\{12 : 1\}$	0

Certificates. The least, extended, and greatest sequences give 16, 16, and 6 realized keys, respectively. All three refined labelled graphs have zero descendants of the selected path.

i	$\lvert A_i\rvert$	$\lvert J_i^{\min}\rvert$	$J_i^{\mathrm{ext}} \setminus J_i^{\min}$	$\lvert J_i^{\max}\rvert$
0	60	1	$\{0 : 1\}$	60
1	48	1	$\{1 : 3\}$	48
2	48	1	$\{2 : 5\}$	48
3	60	1	$\{3 : 7\}$	60
4	48	1	$\{4 : 9\}$	48
5	48	2	$\{5 : 9, 5 : 11\}$	48
6	60	2	$\{6 : 1, 6 : 11\}$	60
7	48	2	$\{7 : 1, 7 : 3\}$	48
8	48	2	$\{8 : 3, 8 : 5\}$	48
9	60	2	$\{9 : 5, 9 : 7\}$	60
10	48	1	$\{10 : 1\}$	47

i	$\lvert A_i\rvert$	$\lvert J_i^{\min}\rvert$	$J_i^{\mathrm{ext}} \setminus J_i^{\min}$	$\lvert J_i^{\max}\rvert$
11	48	1	$\{11 : 3\}$	47
12	60	1	$\{12 : 5\}$	59

Transition rows. Identical successor rows within a phase are grouped when they have the same selected-label and other-label outputs.

Phase 0, label s.

Source residues	Selected targets	Other targets
{0}	{2}	–
{1}	{3}	–
{2}	{4}	–
{3}	{5}	–
{4}	{6}	–
{5}	{7}	–
{6}	{8}	–
{7}	{9}	–
{8}	{10}	–
{9}	{11}	–
{10}	{0}	–
{11}	{1}	–

Phase 1, label s.

Source residues	Selected targets	Other targets
{0}	{2}	–
{1}	{3}	–
{2}	{4}	–
{3}	{5}	–
{4}	{6}	–
{5}	{7}	–
{6}	{8}	–
{7}	{9}	–
{8}	{10}	–
{9}	{11}	–
{10}	{0}	–
{11}	{1}	–

Phase 2, label s.

Source residues	Selected targets	Other targets
{0}	{2}	–
{1}	{3}	–
{2}	{4}	–
{3}	{5}	–
{4}	{6}	–
{5}	{7}	–
{6}	{8}	–
{7}	{9}	–
{8}	{10}	–
{9}	{11}	–
{10}	{0}	–
{11}	{1}	–

Phase 3, label s.

Source residues	Selected targets	Other targets
{0}	{2}	–
{1}	{3}	–
{2}	{4}	–
{3}	{5}	–
{4}	{6}	–
{5}	{7}	–
{6}	{8}	–
{7}	{9}	–
{8}	{10}	–
{9}	{11}	–
{10}	{0}	–
{11}	{1}	–

Phase 4, label f.

Source residues	Selected targets	Other targets
{0}	{0, 2}	–
{1}	{1, 3}	–
{2}	{2, 4}	–
{3}	{3, 5}	–
{4}	{4, 6}	–
{5}	{5, 7}	–
{6}	{6, 8}	–
{7}	{7, 9}	–
{8}	{8, 10}	–
{9}	{9, 11}	–
{10}	{0, 10}	–
{11}	{1, 11}	–

Phase 5, label s.

Source residues	Selected targets	Other targets
{0}	{2}	–
{1}	{3}	–
{2}	{4}	–
{3}	{5}	–
{4}	{6}	–
{5}	{7}	–
{6}	{8}	–
{7}	{9}	–
{8}	{10}	–
{9}	{11}	–
{10}	{0}	–
{11}	{1}	–

Phase 6, label s.

Source residues	Selected targets	Other targets
{0}	{2}	–
{1}	{3}	–
{2}	{4}	–
{3}	{5}	–
{4}	{6}	–
{5}	{7}	–
{6}	{8}	–
{7}	{9}	–
{8}	{10}	–
{9}	{11}	–
{10}	{0}	–
{11}	{1}	–

Phase 7, label s.

Source residues	Selected targets	Other targets
{0}	{2}	–
{1}	{3}	–
{2}	{4}	–
{3}	{5}	–
{4}	{6}	–
{5}	{7}	–
{6}	{8}	–
{7}	{9}	–
{8}	{10}	–
{9}	{11}	–
{10}	{0}	–
{11}	{1}	–

Phase 8, label s.

Source residues	Selected targets	Other targets
{0}	{2}	–
{1}	{3}	–
{2}	{4}	–
{3}	{5}	–
{4}	{6}	–
{5}	{7}	–
{6}	{8}	–
{7}	{9}	–
{8}	{10}	–
{9}	{11}	–
{10}	{0}	–
{11}	{1}	–

Phase 9, label j.

Source residues	Selected targets	Other targets
{0, 2, 4, 6, 8, 10}	{0}	–
{1, 3, 5, 7, 9, 11}	{1}	–

Phase 10, label s.

Source residues	Selected targets	Other targets
{0}	{2}	–
{1}	{3}	–
{2}	{4}	–
{3}	{5}	–
{4}	{6}	–
{5}	{7}	–
{6}	{8}	–
{7}	{9}	–
{8}	{10}	–
{9}	{11}	–
{10}	{0}	–
{11}	{1}	–

Phase 11, label s.

Source residues	Selected targets	Other targets
{0}	{2}	–
{1}	{3}	–
{2}	{4}	–
{3}	{5}	–
{4}	{6}	–
{5}	{7}	–
{6}	{8}	–
{7}	{9}	–
{8}	{10}	–
{9}	{11}	–
{10}	{0}	–
{11}	{1}	–

31.8 P4.8 Fork and reset, width 14, length 13, target 2

The fork can increase the count of distinct prefixes even when their endpoints merge at the reset. At that layer the endpoint retains parity alone.

The prescribed label sequence is $s, s, s, s, s, f, s, s, s, s, s, j, s$ and its observed state sequence is $0, 1, 2, 0, 1, 2, 0, 1, 2, 0, 1, 2, 0, 1$.

The initial set is $\{0 : 0\}$ and the bad set is $\{13 : 2\}$.

The exact lift count is 2. Allowing any label at each layer of this same observed state sequence gives 2 lifts.

i	Exact prefix	Exact suffix	Join
0	$\{0:0\}$	$\{0:0, 0:2, 0:4, 0:6, 0:8, 0:10, 0:12\}$	1
1	$\{1:2\}$	$\{1:0, 1:2, 1:4, 1:6, 1:8, 1:10, 1:12\}$	1
2	$\{2:4\}$	$\{2:0, 2:2, 2:4, 2:6, 2:8, 2:10, 2:12\}$	1
3	$\{3:6\}$	$\{3:0, 3:2, 3:4, 3:6, 3:8, 3:10, 3:12\}$	1
4	$\{4:8\}$	$\{4:0, 4:2, 4:4, 4:6, 4:8, 4:10, 4:12\}$	1
5	$\{5:10\}$	$\{5:0, 5:2, 5:4, 5:6, 5:8, 5:10, 5:12\}$	1
6	$\{6:10, 6:12\}$	$\{6:0, 6:2, 6:4, 6:6, 6:8, 6:10, 6:12\}$	2
7	$\{7:0, 7:12\}$	$\{7:0, 7:2, 7:4, 7:6, 7:8, 7:10, 7:12\}$	2
8	$\{8:0, 8:2\}$	$\{8:0, 8:2, 8:4, 8:6, 8:8, 8:10, 8:12\}$	2
9	$\{9:2, 9:4\}$	$\{9:0, 9:2, 9:4, 9:6, 9:8, 9:10, 9:12\}$	2
10	$\{10:4, 10:6\}$	$\{10:0, 10:2, 10:4, 10:6, 10:8, 10:10, 10:12\}$	2
11	$\{11:6, 11:8\}$	$\{11:0, 11:2, 11:4, 11:6, 11:8, 11:10, 11:12\}$	2
12	$\{12:0\}$	$\{12:0\}$	1
13	$\{13:2\}$	$\{13:2\}$	1

Witness. $0:0, 1:2, 2:4, 3:6, 4:8, 5:10, 6:10, 7:12, 8:0, 9:2, 10:4, 11:6, 12:0, 13:2$ follows the selected label at every step.

Transition rows. Identical successor rows within a phase are grouped when they have the same selected-label and other-label outputs.

Phase 0, label s.

Source residues	Selected targets	Other targets
{0}	{2}	–
{1}	{3}	–
{2}	{4}	–
{3}	{5}	–
{4}	{6}	–
{5}	{7}	–
{6}	{8}	–
{7}	{9}	–
{8}	{10}	–
{9}	{11}	–
{10}	{12}	–
{11}	{13}	–

Phase 0, label s (continued).

Source residues	Selected targets	Other targets
{12}	{0}	–
{13}	{1}	–

Phase 1, label s.

Source residues	Selected targets	Other targets
{0}	{2}	–
{1}	{3}	–
{2}	{4}	–
{3}	{5}	–
{4}	{6}	–
{5}	{7}	–
{6}	{8}	–
{7}	{9}	–
{8}	{10}	–
{9}	{11}	–
{10}	{12}	–
{11}	{13}	–

Phase 1, label s (continued).

Source residues	Selected targets	Other targets
{12}	{0}	–
{13}	{1}	–

Phase 2, label s.

Source residues	Selected targets	Other targets
{0}	{2}	–
{1}	{3}	–
{2}	{4}	–
{3}	{5}	–
{4}	{6}	–
{5}	{7}	–
{6}	{8}	–
{7}	{9}	–
{8}	{10}	–
{9}	{11}	–
{10}	{12}	–
{11}	{13}	–

Phase 2, label s (continued).

Source residues	Selected targets	Other targets
{12}	{0}	–
{13}	{1}	–

Phase 3, label s.

Source residues	Selected targets	Other targets
{0}	{2}	–
{1}	{3}	–
{2}	{4}	–
{3}	{5}	–
{4}	{6}	–
{5}	{7}	–
{6}	{8}	–
{7}	{9}	–
{8}	{10}	–
{9}	{11}	–
{10}	{12}	–
{11}	{13}	–

Phase 3, label s (continued).

Source residues	Selected targets	Other targets
{12}	{0}	–
{13}	{1}	–

Phase 4, label s.

Source residues	Selected targets	Other targets
{0}	{2}	–
{1}	{3}	–
{2}	{4}	–
{3}	{5}	–
{4}	{6}	–
{5}	{7}	–
{6}	{8}	–
{7}	{9}	–
{8}	{10}	–
{9}	{11}	–
{10}	{12}	–
{11}	{13}	–

Phase 4, label s (continued).

Source residues	Selected targets	Other targets
{12}	{0}	–
{13}	{1}	–

Phase 5, label f.

Source residues	Selected targets	Other targets
{0}	{0, 2}	–
{1}	{1, 3}	–
{2}	{2, 4}	–
{3}	{3, 5}	–
{4}	{4, 6}	–
{5}	{5, 7}	–
{6}	{6, 8}	–
{7}	{7, 9}	–
{8}	{8, 10}	–
{9}	{9, 11}	–
{10}	{10, 12}	–
{11}	{11, 13}	–

Phase 5, label f (continued).

Source residues	Selected targets	Other targets
{12}	{0, 12}	–
{13}	{1, 13}	–

Phase 6, label s.

Source residues	Selected targets	Other targets
{0}	{2}	–
{1}	{3}	–
{2}	{4}	–
{3}	{5}	–
{4}	{6}	–
{5}	{7}	–
{6}	{8}	–
{7}	{9}	–
{8}	{10}	–
{9}	{11}	–
{10}	{12}	–
{11}	{13}	–

Phase 6, label s (continued).

Source residues	Selected targets	Other targets
{12}	{0}	–
{13}	{1}	–

Phase 7, label s.

Source residues	Selected targets	Other targets
{0}	{2}	–
{1}	{3}	–
{2}	{4}	–
{3}	{5}	–
{4}	{6}	–
{5}	{7}	–
{6}	{8}	–
{7}	{9}	–
{8}	{10}	–
{9}	{11}	–
{10}	{12}	–
{11}	{13}	–

Phase 7, label s (continued).

Source residues	Selected targets	Other targets
{12}	{0}	–
{13}	{1}	–

Phase 8, label s.

Source residues	Selected targets	Other targets
{0}	{2}	–
{1}	{3}	–
{2}	{4}	–
{3}	{5}	–
{4}	{6}	–
{5}	{7}	–
{6}	{8}	–
{7}	{9}	–
{8}	{10}	–
{9}	{11}	–
{10}	{12}	–
{11}	{13}	–

Phase 8, label s (continued).

Source residues	Selected targets	Other targets
{12}	{0}	–
{13}	{1}	–

Phase 9, label s.

Source residues	Selected targets	Other targets
{0}	{2}	–
{1}	{3}	–
{2}	{4}	–
{3}	{5}	–
{4}	{6}	–
{5}	{7}	–
{6}	{8}	–
{7}	{9}	–
{8}	{10}	–
{9}	{11}	–
{10}	{12}	–
{11}	{13}	–

Phase 9, label s (continued).

Source residues	Selected targets	Other targets
{12}	{0}	–
{13}	{1}	–

Phase 10, label s.

Source residues	Selected targets	Other targets
{0}	{2}	–
{1}	{3}	–
{2}	{4}	–
{3}	{5}	–
{4}	{6}	–
{5}	{7}	–
{6}	{8}	–
{7}	{9}	–
{8}	{10}	–
{9}	{11}	–
{10}	{12}	–
{11}	{13}	–

Phase 10, label s (continued).

Source residues	Selected targets	Other targets
{12}	{0}	–
{13}	{1}	–

Phase 11, label j.

Source residues	Selected targets	Other targets
{0, 2, 4, 6, 8, 10, 12}	{0}	–
{1, 3, 5, 7, 9, 11, 13}	{1}	–

Phase 12, label s.

Source residues	Selected targets	Other targets
{0}	{2}	–
{1}	{3}	–
{2}	{4}	–
{3}	{5}	–
{4}	{6}	–
{5}	{7}	–
{6}	{8}	–
{7}	{9}	–
{8}	{10}	–
{9}	{11}	–
{10}	{12}	–
{11}	{13}	–

Phase 12, label s (continued).

Source residues	Selected targets	Other targets
{12}	{0}	–
{13}	{1}	–

VI. Higher fibers and descent cases

Computed actions and homotopy fibers illustrating the difference between support, coherent lifts, and compatible local data.

32 Fibers of classifying-space maps

For a finite group homomorphism $\phi : G \to H$, a model of the homotopy fiber of $B\phi$ over the sole target object is the action groupoid for left multiplication by $\phi(G)$, with G acting through ϕ. Indeed a fiber object is an identification in BH. Write h for the inverse of the path from the image of the source point to the base point. The commuting triangle for a source arrow $g \in G$ then reads $h' = \phi(g)h$.

Its components are the left-action orbits, and the automorphism group at every object is $\ker \phi$. For a chosen orbit representative r, the transporter set $\{g : \phi(g)h = r\}$ is a coset of the kernel. Each component is equivalent to $B \ker \phi$, by inclusion of the full subgroupoid on its representative. The support of the fiber over the point is terminal in every example.

An element of D_n is (r, e) with $r \in \mathbb{Z}/n$ and $e \in \mathbb{Z}/2$. The product is $(r, e)(s, f) = (r + (-1)^e s, e + f)$. Its numeric ID is $2r + e$.

32.1 H1.1 $C_4 \to C_8$

The cyclic homomorphism sends 1 to 2 modulo 8. Its image has index 2.

The source and target orders are 4 and 8. The image has order 4, the kernel has order 1, and the residual has 2 components.

The image IDs are $\{0, 2, 4, 6\}$. The kernel IDs are $\{0\}$.

The table gives the orbit representative r of each target element h and all source arrows carrying h to r. The least arrow gives an explicit equivalence to the representative.

ID	Element	r	Least	Count	Transporters
0	0	0	0	1	$\{0\}$
2	2	0	3	1	$\{3\}$
4	4	0	2	1	$\{2\}$
6	6	0	1	1	$\{1\}$
1	1	1	0	1	$\{0\}$
3	3	1	3	1	$\{3\}$
5	5	1	2	1	$\{2\}$
7	7	1	1	1	$\{1\}$

32.2 H1.2 $C_6 \to C_9$

The cyclic homomorphism sends 1 to 3 modulo 9. Its image has index 3.

The source and target orders are 6 and 9. The image has order 3, the kernel has order 2, and the residual has 3 components.

The image IDs are $\{0, 3, 6\}$. The kernel IDs are $\{0, 3\}$.

The table gives the orbit representative r of each target element h and all source arrows carrying h to r. The least arrow gives an explicit equivalence to the representative.

ID	Element	r	Least	Count	Transporters
0	0	0	0	2	$\{0, 3\}$
3	3	0	2	2	$\{2, 5\}$
6	6	0	1	2	$\{1, 4\}$
1	1	1	0	2	$\{0, 3\}$
4	4	1	2	2	$\{2, 5\}$
7	7	1	1	2	$\{1, 4\}$
2	2	2	0	2	$\{0, 3\}$
5	5	2	2	2	$\{2, 5\}$
8	8	2	1	2	$\{1, 4\}$

32.3 H1.3 $C_8 \to C_{12}$

The cyclic homomorphism sends 1 to 3 modulo 12. Its image has index 3.

The source and target orders are 8 and 12. The image has order 4, the kernel has order 2, and the residual has 3 components.

The image IDs are $\{0, 3, 6, 9\}$. The kernel IDs are $\{0, 4\}$.

The table gives the orbit representative r of each target element h and all source arrows carrying h to r. The least arrow gives an explicit equivalence to the representative.

ID	Element	r	Least	Count	Transporters
0	0	0	0	2	$\{0, 4\}$
3	3	0	3	2	$\{3, 7\}$
6	6	0	2	2	$\{2, 6\}$
9	9	0	1	2	$\{1, 5\}$
1	1	1	0	2	$\{0, 4\}$
4	4	1	3	2	$\{3, 7\}$
7	7	1	2	2	$\{2, 6\}$
10	10	1	1	2	$\{1, 5\}$
2	2	2	0	2	$\{0, 4\}$
5	5	2	3	2	$\{3, 7\}$
8	8	2	2	2	$\{2, 6\}$
11	11	2	1	2	$\{1, 5\}$

32.4 H1.4 $C_9 \to C_{12}$

The cyclic homomorphism sends 1 to 4 modulo 12. Its image has index 4.

The source and target orders are 9 and 12. The image has order 3, the kernel has order 3, and the residual has 4 components.

The image IDs are $\{0, 4, 8\}$. The kernel IDs are $\{0, 3, 6\}$.

The table gives the orbit representative r of each target element h and all source arrows carrying h to r. The least arrow gives an explicit equivalence to the representative.

ID	Element	r	Least	Count	Transporters
0	0	0	0	3	$\{0,3,6\}$
4	4	0	2	3	$\{2,5,8\}$
8	8	0	1	3	$\{1,4,7\}$
1	1	1	0	3	$\{0,3,6\}$
5	5	1	2	3	$\{2,5,8\}$
9	9	1	1	3	$\{1,4,7\}$
2	2	2	0	3	$\{0,3,6\}$
6	6	2	2	3	$\{2,5,8\}$
10	10	2	1	3	$\{1,4,7\}$
3	3	3	0	3	$\{0,3,6\}$
7	7	3	2	3	$\{2,5,8\}$
11	11	3	1	3	$\{1,4,7\}$

32.5 H1.5 $C_{10} \to C_{15}$

The cyclic homomorphism sends 1 to 3 modulo 15. Its image has index 3.

The source and target orders are 10 and 15. The image has order 5, the kernel has order 2, and the residual has 3 components.

The image IDs are $\{0,3,6,9,12\}$. The kernel IDs are $\{0,5\}$.

The table gives the orbit representative r of each target element h and all source arrows carrying h to r. The least arrow gives an explicit equivalence to the representative.

ID	Element	r	Least	Count	Transporters
0	0	0	0	2	$\{0,5\}$
3	3	0	4	2	$\{4,9\}$
6	6	0	3	2	$\{3,8\}$
9	9	0	2	2	$\{2,7\}$
12	12	0	1	2	$\{1,6\}$
1	1	1	0	2	$\{0,5\}$
4	4	1	4	2	$\{4,9\}$
7	7	1	3	2	$\{3,8\}$
10	10	1	2	2	$\{2,7\}$
13	13	1	1	2	$\{1,6\}$
2	2	2	0	2	$\{0,5\}$
5	5	2	4	2	$\{4,9\}$
8	8	2	3	2	$\{3,8\}$
11	11	2	2	2	$\{2,7\}$
14	14	2	1	2	$\{1,6\}$

32.6 H1.6 $C_{12} \to C_{18}$

The cyclic homomorphism sends 1 to 6 modulo 18. Its image has index 6.

The source and target orders are 12 and 18. The image has order 3, the kernel has order 4, and the residual has 6 components.

The image IDs are $\{0,6,12\}$. The kernel IDs are $\{0,3,6,9\}$.

The table gives the orbit representative r of each target element h and all source arrows carrying h to r. The least arrow gives an explicit equivalence to the representative.

ID	Element	r	Least	Count	Transporters
0	0	0	0	4	$\{0,3,6,9\}$
6	6	0	2	4	$\{2,5,8,11\}$
12	12	0	1	4	$\{1,4,7,10\}$
1	1	1	0	4	$\{0,3,6,9\}$
7	7	1	2	4	$\{2,5,8,11\}$
13	13	1	1	4	$\{1,4,7,10\}$
2	2	2	0	4	$\{0,3,6,9\}$
8	8	2	2	4	$\{2,5,8,11\}$
14	14	2	1	4	$\{1,4,7,10\}$
3	3	3	0	4	$\{0,3,6,9\}$
9	9	3	2	4	$\{2,5,8,11\}$
15	15	3	1	4	$\{1,4,7,10\}$
4	4	4	0	4	$\{0,3,6,9\}$
10	10	4	2	4	$\{2,5,8,11\}$
16	16	4	1	4	$\{1,4,7,10\}$

ID	Element	r	Least	Count	Transporters
5	5	5	0	4	$\{0,3,6,9\}$
11	11	5	2	4	$\{2,5,8,11\}$
17	17	5	1	4	$\{1,4,7,10\}$

32.7 H1.7 $C_{14} \to C_{21}$

The cyclic homomorphism sends 1 to 3 modulo 21. Its image has index 3.

The source and target orders are 14 and 21. The image has order 7, the kernel has order 2, and the residual has 3 components.

The image IDs are $\{0,3,6,9,12,15,18\}$. The kernel IDs are $\{0,7\}$.

The table gives the orbit representative r of each target element h and all source arrows carrying h to r. The least arrow gives an explicit equivalence to the representative.

ID	Element	r	Least	Count	Transporters
0	0	0	0	2	$\{0,7\}$
3	3	0	6	2	$\{6,13\}$
6	6	0	5	2	$\{5,12\}$
9	9	0	4	2	$\{4,11\}$
12	12	0	3	2	$\{3,10\}$
15	15	0	2	2	$\{2,9\}$
18	18	0	1	2	$\{1,8\}$
1	1	1	0	2	$\{0,7\}$
4	4	1	6	2	$\{6,13\}$
7	7	1	5	2	$\{5,12\}$
10	10	1	4	2	$\{4,11\}$
13	13	1	3	2	$\{3,10\}$
16	16	1	2	2	$\{2,9\}$
19	19	1	1	2	$\{1,8\}$
2	2	2	0	2	$\{0,7\}$

ID	Element	r	Least	Count	Transporters
5	5	2	6	2	$\{6,13\}$
8	8	2	5	2	$\{5,12\}$
11	11	2	4	2	$\{4,11\}$
14	14	2	3	2	$\{3,10\}$
17	17	2	2	2	$\{2,9\}$
20	20	2	1	2	$\{1,8\}$

32.8 H1.8 $C_{15} \to C_{20}$

The cyclic homomorphism sends 1 to 8 modulo 20. Its image has index 4.

The source and target orders are 15 and 20. The image has order 5, the kernel has order 3, and the residual has 4 components.

The image IDs are $\{0,4,8,12,16\}$. The kernel IDs are $\{0,5,10\}$.

The table gives the orbit representative r of each target element h and all source arrows carrying h to r. The least arrow gives an explicit equivalence to the representative.

ID	Element	r	Least	Count	Transporters
0	0	0	0	3	$\{0,5,10\}$
4	4	0	2	3	$\{2,7,12\}$
8	8	0	4	3	$\{4,9,14\}$
12	12	0	1	3	$\{1,6,11\}$
16	16	0	3	3	$\{3,8,13\}$
1	1	1	0	3	$\{0,5,10\}$
5	5	1	2	3	$\{2,7,12\}$
9	9	1	4	3	$\{4,9,14\}$
13	13	1	1	3	$\{1,6,11\}$
17	17	1	3	3	$\{3,8,13\}$
2	2	2	0	3	$\{0,5,10\}$
6	6	2	2	3	$\{2,7,12\}$
10	10	2	4	3	$\{4,9,14\}$
14	14	2	1	3	$\{1,6,11\}$
18	18	2	3	3	$\{3,8,13\}$

ID	Element	r	Least	Count	Transporters
3	3	3	0	3	$\{0,5,10\}$
7	7	3	2	3	$\{2,7,12\}$
11	11	3	4	3	$\{4,9,14\}$
15	15	3	1	3	$\{1,6,11\}$
19	19	3	3	3	$\{3,8,13\}$

32.9 H1.9 $C_{18} \to C_{24}$

The cyclic homomorphism sends 1 to 4 modulo 24. Its image has index 4.

The source and target orders are 18 and 24. The image has order 6, the kernel has order 3, and the residual has 4 components.

The image IDs are $\{0,4,8,12,16,20\}$. The kernel IDs are $\{0,6,12\}$.

The table gives the orbit representative r of each target element h and all source arrows carrying h to r. The least arrow gives an explicit equivalence to the representative.

ID	Element	r	Least	Count	Transporters
0	0	0	0	3	$\{0,6,12\}$
4	4	0	5	3	$\{5,11,17\}$
8	8	0	4	3	$\{4,10,16\}$
12	12	0	3	3	$\{3,9,15\}$
16	16	0	2	3	$\{2,8,14\}$
20	20	0	1	3	$\{1,7,13\}$
1	1	1	0	3	$\{0,6,12\}$
5	5	1	5	3	$\{5,11,17\}$
9	9	1	4	3	$\{4,10,16\}$
13	13	1	3	3	$\{3,9,15\}$
17	17	1	2	3	$\{2,8,14\}$
21	21	1	1	3	$\{1,7,13\}$
2	2	2	0	3	$\{0,6,12\}$
6	6	2	5	3	$\{5,11,17\}$
10	10	2	4	3	$\{4,10,16\}$

ID	Element	r	Least	Count	Transporters
14	14	2	3	3	$\{3,9,15\}$
18	18	2	2	3	$\{2,8,14\}$
22	22	2	1	3	$\{1,7,13\}$
3	3	3	0	3	$\{0,6,12\}$
7	7	3	5	3	$\{5,11,17\}$
11	11	3	4	3	$\{4,10,16\}$
15	15	3	3	3	$\{3,9,15\}$
19	19	3	2	3	$\{2,8,14\}$
23	23	3	1	3	$\{1,7,13\}$

32.10 H1.10 $C_{21} \to C_{28}$

The cyclic homomorphism sends 1 to 4 modulo 28. Its image has index 4.

The source and target orders are 21 and 28. The image has order 7, the kernel has order 3, and the residual has 4 components.

The image IDs are $\{0,4,8,12,16,20,24\}$. The kernel IDs are $\{0,7,14\}$.

The table gives the orbit representative r of each target element h and all source arrows carrying h to r. The least arrow gives an explicit equivalence to the representative.

ID	Element	r	Least	Count	Transporters
0	0	0	0	3	$\{0,7,14\}$
4	4	0	6	3	$\{6,13,20\}$
8	8	0	5	3	$\{5,12,19\}$
12	12	0	4	3	$\{4,11,18\}$
16	16	0	3	3	$\{3,10,17\}$
20	20	0	2	3	$\{2,9,16\}$
24	24	0	1	3	$\{1,8,15\}$
1	1	1	0	3	$\{0,7,14\}$
5	5	1	6	3	$\{6,13,20\}$
9	9	1	5	3	$\{5,12,19\}$
13	13	1	4	3	$\{4,11,18\}$
17	17	1	3	3	$\{3,10,17\}$
21	21	1	2	3	$\{2,9,16\}$
25	25	1	1	3	$\{1,8,15\}$
2	2	2	0	3	$\{0,7,14\}$

ID	Element	r	Least	Count	Transporters
6	6	2	6	3	$\{6, 13, 20\}$
10	10	2	5	3	$\{5, 12, 19\}$
14	14	2	4	3	$\{4, 11, 18\}$
18	18	2	3	3	$\{3, 10, 17\}$
22	22	2	2	3	$\{2, 9, 16\}$
26	26	2	1	3	$\{1, 8, 15\}$
3	3	3	0	3	$\{0, 7, 14\}$
7	7	3	6	3	$\{6, 13, 20\}$
11	11	3	5	3	$\{5, 12, 19\}$
15	15	3	4	3	$\{4, 11, 18\}$
19	19	3	3	3	$\{3, 10, 17\}$
23	23	3	2	3	$\{2, 9, 16\}$
27	27	3	1	3	$\{1, 8, 15\}$

32.11 H1.11 $C_{24} \to C_{30}$

The cyclic homomorphism sends 1 to 5 modulo 30. Its image has index 5.

The source and target orders are 24 and 30. The image has order 6, the kernel has order 4, and the residual has 5 components.

The image IDs are $\{0, 5, 10, 15, 20, 25\}$. The kernel IDs are $\{0, 6, 12, 18\}$.

The table gives the orbit representative r of each target element h and all source arrows carrying h to r. The least arrow gives an explicit equivalence to the representative.

ID	Element	r	Least	Count	Transporters
0	0	0	0	4	$\{0, 6, 12, 18\}$
5	5	0	5	4	$\{5, 11, 17, 23\}$
10	10	0	4	4	$\{4, 10, 16, 22\}$
15	15	0	3	4	$\{3, 9, 15, 21\}$
20	20	0	2	4	$\{2, 8, 14, 20\}$
25	25	0	1	4	$\{1, 7, 13, 19\}$
1	1	1	0	4	$\{0, 6, 12, 18\}$
6	6	1	5	4	$\{5, 11, 17, 23\}$
11	11	1	4	4	$\{4, 10, 16, 22\}$
16	16	1	3	4	$\{3, 9, 15, 21\}$
21	21	1	2	4	$\{2, 8, 14, 20\}$
26	26	1	1	4	$\{1, 7, 13, 19\}$
2	2	2	0	4	$\{0, 6, 12, 18\}$
7	7	2	5	4	$\{5, 11, 17, 23\}$
12	12	2	4	4	$\{4, 10, 16, 22\}$
17	17	2	3	4	$\{3, 9, 15, 21\}$
22	22	2	2	4	$\{2, 8, 14, 20\}$
27	27	2	1	4	$\{1, 7, 13, 19\}$
3	3	3	0	4	$\{0, 6, 12, 18\}$
8	8	3	5	4	$\{5, 11, 17, 23\}$
13	13	3	4	4	$\{4, 10, 16, 22\}$
18	18	3	3	4	$\{3, 9, 15, 21\}$
23	23	3	2	4	$\{2, 8, 14, 20\}$
28	28	3	1	4	$\{1, 7, 13, 19\}$
4	4	4	0	4	$\{0, 6, 12, 18\}$
9	9	4	5	4	$\{5, 11, 17, 23\}$
14	14	4	4	4	$\{4, 10, 16, 22\}$
19	19	4	3	4	$\{3, 9, 15, 21\}$
24	24	4	2	4	$\{2, 8, 14, 20\}$
29	29	4	1	4	$\{1, 7, 13, 19\}$

32.12 H1.12 $C_{16} \to C_{16}$

The cyclic homomorphism sends 1 to 0 modulo 16. Its image has index 16.

The source and target orders are 16 and 16. The image has order 1, the kernel has order 16, and the residual has 16 components.

The image IDs are $\{0\}$. The kernel IDs are $\{0, 1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12, 13, 14, 15\}$.

The table gives the orbit representative r of each target element h and all source arrows carrying h to r. The least arrow gives an explicit equivalence to the representative.

ID	Element	r	Least	Count	Transporters
0	0	0	0	16	$\{0,1,2,3,4,5,6,7,8,9,10,11,12,13,14,15\}$
1	1	1	0	16	$\{0,1,2,3,4,5,6,7,8,9,10,11,12,13,14,15\}$
2	2	2	0	16	$\{0,1,2,3,4,5,6,7,8,9,10,11,12,13,14,15\}$
3	3	3	0	16	$\{0,1,2,3,4,5,6,7,8,9,10,11,12,13,14,15\}$
4	4	4	0	16	$\{0,1,2,3,4,5,6,7,8,9,10,11,12,13,14,15\}$
5	5	5	0	16	$\{0,1,2,3,4,5,6,7,8,9,10,11,12,13,14,15\}$
6	6	6	0	16	$\{0,1,2,3,4,5,6,7,8,9,10,11,12,13,14,15\}$
7	7	7	0	16	$\{0,1,2,3,4,5,6,7,8,9,10,11,12,13,14,15\}$
8	8	8	0	16	$\{0,1,2,3,4,5,6,7,8,9,10,11,12,13,14,15\}$
9	9	9	0	16	$\{0,1,2,3,4,5,6,7,8,9,10,11,12,13,14,15\}$
10	10	10	0	16	$\{0,1,2,3,4,5,6,7,8,9,10,11,12,13,14,15\}$
11	11	11	0	16	$\{0,1,2,3,4,5,6,7,8,9,10,11,12,13,14,15\}$
12	12	12	0	16	$\{0,1,2,3,4,5,6,7,8,9,10,11,12,13,14,15\}$
13	13	13	0	16	$\{0,1,2,3,4,5,6,7,8,9,10,11,12,13,14,15\}$
14	14	14	0	16	$\{0,1,2,3,4,5,6,7,8,9,10,11,12,13,14,15\}$
15	15	15	0	16	$\{0,1,2,3,4,5,6,7,8,9,10,11,12,13,14,15\}$

32.13 H1.13 $D_8 \to D_4$

The rotation coordinate is reduced modulo 4; the reflection bit is unchanged. The kernel has two elements and the target action is transitive.

The source and target orders are 16 and 8. The image has order 8, the kernel has order 2, and the residual has 1 components.

The image IDs are $\{0,1,2,3,4,5,6,7\}$. The kernel IDs are $\{0,8\}$.

The table gives the orbit representative r of each target element h and all source arrows carrying h to r. The least arrow gives an explicit equivalence to the representative.

ID	Element	r	Least	Count	Transporters
0	$(0,0)$	0	0	2	$\{0,8\}$
1	$(0,1)$	0	1	2	$\{1,9\}$
2	$(1,0)$	0	6	2	$\{6,14\}$
3	$(1,1)$	0	3	2	$\{3,11\}$
4	$(2,0)$	0	4	2	$\{4,12\}$
5	$(2,1)$	0	5	2	$\{5,13\}$
6	$(3,0)$	0	2	2	$\{2,10\}$
7	$(3,1)$	0	7	2	$\{7,15\}$

32.14 H1.14 $D_{10} \to D_5$

The rotation coordinate is reduced modulo 5; the reflection bit is unchanged. The kernel has two elements and the target action is transitive.

The source and target orders are 20 and 10. The image has order 10, the kernel has order 2, and the residual has 1 components.

The image IDs are $\{0,1,2,3,4,5,6,7,8,9\}$. The kernel IDs are $\{0,10\}$.

The table gives the orbit representative r of each target element h and all source arrows carrying h to r. The least arrow gives an explicit equivalence to the representative.

ID	Element	r	Least	Count	Transporters
0	$(0,0)$	0	0	2	$\{0,10\}$
1	$(0,1)$	0	1	2	$\{1,11\}$
2	$(1,0)$	0	8	2	$\{8,18\}$
3	$(1,1)$	0	3	2	$\{3,13\}$
4	$(2,0)$	0	6	2	$\{6,16\}$
5	$(2,1)$	0	5	2	$\{5,15\}$
6	$(3,0)$	0	4	2	$\{4,14\}$
7	$(3,1)$	0	7	2	$\{7,17\}$
8	$(4,0)$	0	2	2	$\{2,12\}$
9	$(4,1)$	0	9	2	$\{9,19\}$

32.15 H1.15 $D_{12} \to D_6$

The rotation coordinate is reduced modulo 6; the reflection bit is unchanged. The kernel has two elements and the target action is transitive.

The source and target orders are 24 and 12. The image has order 12, the kernel has order 2, and the residual has 1 components.

The image IDs are $\{0,1,2,3,4,5,6,7,8,9,10,11\}$. The kernel IDs are $\{0,12\}$.

The table gives the orbit representative r of each target element h and all source arrows carrying h to r. The least arrow gives an explicit equivalence to the representative.

ID	Element	r	Least	Count	Transporters
0	$(0,0)$	0	0	2	$\{0,12\}$
1	$(0,1)$	0	1	2	$\{1,13\}$
2	$(1,0)$	0	10	2	$\{10,22\}$
3	$(1,1)$	0	3	2	$\{3,15\}$
4	$(2,0)$	0	8	2	$\{8,20\}$
5	$(2,1)$	0	5	2	$\{5,17\}$
6	$(3,0)$	0	6	2	$\{6,18\}$
7	$(3,1)$	0	7	2	$\{7,19\}$
8	$(4,0)$	0	4	2	$\{4,16\}$
9	$(4,1)$	0	9	2	$\{9,21\}$
10	$(5,0)$	0	2	2	$\{2,14\}$
11	$(5,1)$	0	11	2	$\{11,23\}$

32.16 H1.16 $D_{14} \to D_7$

The rotation coordinate is reduced modulo 7; the reflection bit is unchanged. The kernel has two elements and the target action is transitive.

The source and target orders are 28 and 14. The image has order 14, the kernel has order 2, and the residual has 1 components.

The image IDs are $\{0,1,2,3,4,5,6,7,8,9,10,11,12,13\}$. The kernel IDs are $\{0,14\}$.

The table gives the orbit representative r of each target element h and all source arrows carrying h to r. The least arrow gives an explicit equivalence to the representative.

ID	Element	r	Least	Count	Transporters
0	$(0,0)$	0	0	2	$\{0,14\}$
1	$(0,1)$	0	1	2	$\{1,15\}$
2	$(1,0)$	0	12	2	$\{12,26\}$
3	$(1,1)$	0	3	2	$\{3,17\}$
4	$(2,0)$	0	10	2	$\{10,24\}$
5	$(2,1)$	0	5	2	$\{5,19\}$
6	$(3,0)$	0	8	2	$\{8,22\}$
7	$(3,1)$	0	7	2	$\{7,21\}$
8	$(4,0)$	0	6	2	$\{6,20\}$
9	$(4,1)$	0	9	2	$\{9,23\}$
10	$(5,0)$	0	4	2	$\{4,18\}$
11	$(5,1)$	0	11	2	$\{11,25\}$
12	$(6,0)$	0	2	2	$\{2,16\}$
13	$(6,1)$	0	13	2	$\{13,27\}$

32.17 H1.17 $D_{16} \to D_8$

The rotation coordinate is reduced modulo 8; the reflection bit is unchanged. The kernel has two elements and the target action is transitive.

The source and target orders are 32 and 16. The image has order 16, the kernel has order 2, and the residual has 1 components.

The image IDs are $\{0,1,2,3,4,5,6,7,8,9,10,11,12,13,14,15\}$. The kernel IDs are $\{0,16\}$.

The table gives the orbit representative r of each target element h and all source arrows carrying h to r. The least arrow gives an explicit equivalence to the representative.

ID	Element	r	Least	Count	Transporters
0	$(0,0)$	0	0	2	$\{0,16\}$
1	$(0,1)$	0	1	2	$\{1,17\}$
2	$(1,0)$	0	14	2	$\{14,30\}$
3	$(1,1)$	0	3	2	$\{3,19\}$
4	$(2,0)$	0	12	2	$\{12,28\}$
5	$(2,1)$	0	5	2	$\{5,21\}$
6	$(3,0)$	0	10	2	$\{10,26\}$
7	$(3,1)$	0	7	2	$\{7,23\}$
8	$(4,0)$	0	8	2	$\{8,24\}$
9	$(4,1)$	0	9	2	$\{9,25\}$
10	$(5,0)$	0	6	2	$\{6,22\}$
11	$(5,1)$	0	11	2	$\{11,27\}$
12	$(6,0)$	0	4	2	$\{4,20\}$
13	$(6,1)$	0	13	2	$\{13,29\}$
14	$(7,0)$	0	2	2	$\{2,18\}$

ID	Element	r	Least	Count	Transporters
15	$(7,1)$	0	15	2	$\{15,31\}$

32.18 H1.18 $D_{18} \to D_9$

The rotation coordinate is reduced modulo 9; the reflection bit is unchanged. The kernel has two elements and the target action is transitive.

The source and target orders are 36 and 18. The image has order 18, the kernel has order 2, and the residual has 1 components.

The image IDs are $\{0,1,2,3,4,5,6,7,8,9,10,11,12,13,14,15,16,17\}$. The kernel IDs are $\{0,18\}$.

The table gives the orbit representative r of each target element h and all source arrows carrying h to r. The least arrow gives an explicit equivalence to the representative.

ID	Element	r	Least	Count	Transporters
0	$(0,0)$	0	0	2	$\{0,18\}$
1	$(0,1)$	0	1	2	$\{1,19\}$
2	$(1,0)$	0	16	2	$\{16,34\}$
3	$(1,1)$	0	3	2	$\{3,21\}$
4	$(2,0)$	0	14	2	$\{14,32\}$
5	$(2,1)$	0	5	2	$\{5,23\}$
6	$(3,0)$	0	12	2	$\{12,30\}$
7	$(3,1)$	0	7	2	$\{7,25\}$
8	$(4,0)$	0	10	2	$\{10,28\}$
9	$(4,1)$	0	9	2	$\{9,27\}$
10	$(5,0)$	0	8	2	$\{8,26\}$
11	$(5,1)$	0	11	2	$\{11,29\}$
12	$(6,0)$	0	6	2	$\{6,24\}$
13	$(6,1)$	0	13	2	$\{13,31\}$
14	$(7,0)$	0	4	2	$\{4,22\}$

ID	Element	r	Least	Count	Transporters
15	$(7,1)$	0	15	2	$\{15,33\}$
16	$(8,0)$	0	2	2	$\{2,20\}$
17	$(8,1)$	0	17	2	$\{17,35\}$

32.19 H1.19 $C_4 \to D_4$

The image is the rotations. Its two orbits consist of rotations and reflections. Every isotropy group is trivial because the map is injective.

The source and target orders are 4 and 8. The image has order 4, the kernel has order 1, and the residual has 2 components.

The image IDs are $\{0, 2, 4, 6\}$. The kernel IDs are $\{0\}$.

The table gives the orbit representative r of each target element h and all source arrows carrying h to r. The least arrow gives an explicit equivalence to the representative.

ID	Element	r	Least	Count	Transporters
0	$(0,0)$	0	0	1	$\{0\}$
2	$(1,0)$	0	3	1	$\{3\}$
4	$(2,0)$	0	2	1	$\{2\}$
6	$(3,0)$	0	1	1	$\{1\}$
1	$(0,1)$	1	0	1	$\{0\}$
3	$(1,1)$	1	3	1	$\{3\}$
5	$(2,1)$	1	2	1	$\{2\}$
7	$(3,1)$	1	1	1	$\{1\}$

32.20 H1.20 $C_5 \to D_5$

The image is the rotations. Its two orbits consist of rotations and reflections. Every isotropy group is trivial because the map is injective.

The source and target orders are 5 and 10. The image has order 5, the kernel has order 1, and the residual has 2 components.

The image IDs are $\{0, 2, 4, 6, 8\}$. The kernel IDs are $\{0\}$.

The table gives the orbit representative r of each target element h and all source arrows carrying h to r. The least arrow gives an explicit equivalence to the representative.

ID	Element	r	Least	Count	Transporters
0	$(0,0)$	0	0	1	$\{0\}$
2	$(1,0)$	0	4	1	$\{4\}$
4	$(2,0)$	0	3	1	$\{3\}$
6	$(3,0)$	0	2	1	$\{2\}$
8	$(4,0)$	0	1	1	$\{1\}$
1	$(0,1)$	1	0	1	$\{0\}$
3	$(1,1)$	1	4	1	$\{4\}$
5	$(2,1)$	1	3	1	$\{3\}$
7	$(3,1)$	1	2	1	$\{2\}$
9	$(4,1)$	1	1	1	$\{1\}$

32.21 H1.21 $C_7 \to D_7$

The image is the rotations. Its two orbits consist of rotations and reflections. Every isotropy group is trivial because the map is injective.

The source and target orders are 7 and 14. The image has order 7, the kernel has order 1, and the residual has 2 components.

The image IDs are $\{0, 2, 4, 6, 8, 10, 12\}$. The kernel IDs are $\{0\}$.

The table gives the orbit representative r of each target element h and all source arrows carrying h to r. The least arrow gives an explicit equivalence to the representative.

ID	Element	r	Least	Count	Transporters
0	$(0,0)$	0	0	1	$\{0\}$
2	$(1,0)$	0	6	1	$\{6\}$
4	$(2,0)$	0	5	1	$\{5\}$
6	$(3,0)$	0	4	1	$\{4\}$
8	$(4,0)$	0	3	1	$\{3\}$
10	$(5,0)$	0	2	1	$\{2\}$
12	$(6,0)$	0	1	1	$\{1\}$
1	$(0,1)$	1	0	1	$\{0\}$
3	$(1,1)$	1	6	1	$\{6\}$
5	$(2,1)$	1	5	1	$\{5\}$
7	$(3,1)$	1	4	1	$\{4\}$
9	$(4,1)$	1	3	1	$\{3\}$
11	$(5,1)$	1	2	1	$\{2\}$
13	$(6,1)$	1	1	1	$\{1\}$

32.22 H1.22 $C_8 \to D_8$

The image is the rotations. Its two orbits consist of rotations and reflections. Every isotropy group is trivial because the map is injective.

The source and target orders are 8 and 16. The image has order 8, the kernel has order 1, and the residual has 2 components.

The image IDs are $\{0, 2, 4, 6, 8, 10, 12, 14\}$. The kernel IDs are $\{0\}$.

The table gives the orbit representative r of each target element h and all source arrows carrying h to r. The least arrow gives an explicit equivalence to the representative.

ID	Element	r	Least	Count	Transporters
0	$(0,0)$	0	0	1	$\{0\}$
2	$(1,0)$	0	7	1	$\{7\}$
4	$(2,0)$	0	6	1	$\{6\}$
6	$(3,0)$	0	5	1	$\{5\}$
8	$(4,0)$	0	4	1	$\{4\}$
10	$(5,0)$	0	3	1	$\{3\}$
12	$(6,0)$	0	2	1	$\{2\}$
14	$(7,0)$	0	1	1	$\{1\}$
1	$(0,1)$	1	0	1	$\{0\}$
3	$(1,1)$	1	7	1	$\{7\}$
5	$(2,1)$	1	6	1	$\{6\}$
7	$(3,1)$	1	5	1	$\{5\}$
9	$(4,1)$	1	4	1	$\{4\}$
11	$(5,1)$	1	3	1	$\{3\}$
13	$(6,1)$	1	2	1	$\{2\}$

ID	Element	r	Least	Count	Transporters
15	$(7,1)$	1	1	1	$\{1\}$

32.23 H1.23 $C_{10} \to D_{10}$

The image is the rotations. Its two orbits consist of rotations and reflections. Every isotropy group is trivial because the map is injective.

The source and target orders are 10 and 20. The image has order 10, the kernel has order 1, and the residual has 2 components.

The image IDs are $\{0, 2, 4, 6, 8, 10, 12, 14, 16, 18\}$. The kernel IDs are $\{0\}$.

The table gives the orbit representative r of each target element h and all source arrows carrying h to r. The least arrow gives an explicit equivalence to the representative.

ID	Element	r	Least	Count	Transporters
0	$(0,0)$	0	0	1	$\{0\}$
2	$(1,0)$	0	9	1	$\{9\}$
4	$(2,0)$	0	8	1	$\{8\}$
6	$(3,0)$	0	7	1	$\{7\}$
8	$(4,0)$	0	6	1	$\{6\}$
10	$(5,0)$	0	5	1	$\{5\}$
12	$(6,0)$	0	4	1	$\{4\}$
14	$(7,0)$	0	3	1	$\{3\}$
16	$(8,0)$	0	2	1	$\{2\}$
18	$(9,0)$	0	1	1	$\{1\}$
1	$(0,1)$	1	0	1	$\{0\}$
3	$(1,1)$	1	9	1	$\{9\}$
5	$(2,1)$	1	8	1	$\{8\}$
7	$(3,1)$	1	7	1	$\{7\}$
9	$(4,1)$	1	6	1	$\{6\}$

ID	Element	r	Least	Count	Transporters
11	$(5,1)$	1	5	1	$\{5\}$
13	$(6,1)$	1	4	1	$\{4\}$
15	$(7,1)$	1	3	1	$\{3\}$
17	$(8,1)$	1	2	1	$\{2\}$
19	$(9,1)$	1	1	1	$\{1\}$

32.24 H1.24 $C_{12} \to D_{12}$

The image is the rotations. Its two orbits consist of rotations and reflections. Every isotropy group is trivial because the map is injective.

The source and target orders are 12 and 24. The image has order 12, the kernel has order 1, and the residual has 2 components.

The image IDs are $\{0, 2, 4, 6, 8, 10, 12, 14, 16, 18, 20, 22\}$. The kernel IDs are $\{0\}$.

The table gives the orbit representative r of each target element h and all source arrows carrying h to r. The least arrow gives an explicit equivalence to the representative.

ID	Element	r	Least	Count	Transporters
0	$(0,0)$	0	0	1	$\{0\}$
2	$(1,0)$	0	11	1	$\{11\}$
4	$(2,0)$	0	10	1	$\{10\}$
6	$(3,0)$	0	9	1	$\{9\}$
8	$(4,0)$	0	8	1	$\{8\}$
10	$(5,0)$	0	7	1	$\{7\}$
12	$(6,0)$	0	6	1	$\{6\}$
14	$(7,0)$	0	5	1	$\{5\}$
16	$(8,0)$	0	4	1	$\{4\}$
18	$(9,0)$	0	3	1	$\{3\}$
20	$(10,0)$	0	2	1	$\{2\}$
22	$(11,0)$	0	1	1	$\{1\}$
1	$(0,1)$	1	0	1	$\{0\}$
3	$(1,1)$	1	11	1	$\{11\}$
5	$(2,1)$	1	10	1	$\{10\}$

ID	Element	r	Least	Count	Transporters
7	$(3,1)$	1	9	1	$\{9\}$
9	$(4,1)$	1	8	1	$\{8\}$
11	$(5,1)$	1	7	1	$\{7\}$
13	$(6,1)$	1	6	1	$\{6\}$
15	$(7,1)$	1	5	1	$\{5\}$
17	$(8,1)$	1	4	1	$\{4\}$
19	$(9,1)$	1	3	1	$\{3\}$
21	$(10,1)$	1	2	1	$\{2\}$
23	$(11,1)$	1	1	1	$\{1\}$

33 Double cosets and homotopy gluing

For subgroup inclusions $K, L \subseteq G$, the homotopy pullback $BK \times^h_{BG} BL$ is the action groupoid of $K \times L$ acting on G by $(k, l) \cdot g = kgl^{-1}$. The pullback remembers an identification of the images in BG. Its connected components are indexed by the double cosets $K\backslash G/L$.

At g, the stabilizer projects isomorphically onto $K \cap gLg^{-1}$. The forward map sends (k, l) to k, and the inverse sends k to $(k, g^{-1}kg)$. Each table verifies the orbit size by the orbit–stabilizer formula and independently checks the subgroup intersection. A pullback of the three connected-component sets would have one point for every example.

33.1 H2.1 Subgroups of the dihedral group of order 8

The left and right generating IDs are $\{1\}$ and $\{1\}$.

The corresponding subgroup IDs are $\{0, 1\}$ and $\{0, 1\}$.

The pullback has 3 connected components. For each component the following table displays its full orbit and the projected isotropy subgroup. The representative is its least element ID.

Rep.	Orbit IDs	Isotropy IDs	$\lvert O\rvert$	$\lvert G_r\rvert$
0	$\{0, 1\}$	$\{0, 1\}$	2	2
2	$\{2, 3, 6, 7\}$	$\{0\}$	4	1
4	$\{4, 5\}$	$\{0, 1\}$	2	2

The next rows give a concrete pair of subgroup arrows from each object to its representative.

ID	Element	Rep.	k	l
0	$(0,0)$	0	0	0
1	$(0,1)$	0	0	1
2	$(1,0)$	2	0	0
3	$(1,1)$	2	0	1
4	$(2,0)$	4	0	0
5	$(2,1)$	4	0	1
6	$(3,0)$	2	1	1
7	$(3,1)$	2	1	0

33.2 H2.2 Subgroups of the dihedral group of order 8

The left and right generating IDs are $\{1\}$ and $\{3\}$.

The corresponding subgroup IDs are $\{0,1\}$ and $\{0,3\}$.

The pullback has 2 connected components. For each component the following table displays its full orbit and the projected isotropy subgroup. The representative is its least element ID.

Rep.	Orbit IDs	Isotropy IDs	$\lvert O\rvert$	$\lvert G_r\rvert$
0	$\{0,1,3,6\}$	$\{0\}$	4	1
2	$\{2,4,5,7\}$	$\{0\}$	4	1

The next rows give a concrete pair of subgroup arrows from each object to its representative.

ID	Element	Rep.	k	l
0	$(0,0)$	0	0	0
1	$(0,1)$	0	1	0
2	$(1,0)$	2	0	0
3	$(1,1)$	0	0	3
4	$(2,0)$	2	1	3
5	$(2,1)$	2	0	3
6	$(3,0)$	0	1	3
7	$(3,1)$	2	1	0

33.3 H2.3 Subgroups of the dihedral group of order 10

The left and right generating IDs are $\{1\}$ and $\{3\}$.

The corresponding subgroup IDs are $\{0,1\}$ and $\{0,3\}$.

The pullback has 3 connected components. For each component the following table displays its full orbit and the projected isotropy subgroup. The representative is its least element ID.

Rep.	Orbit IDs	Isotropy IDs	$\lvert O\rvert$	$\lvert G_r\rvert$
0	$\{0,1,3,8\}$	$\{0\}$	4	1
2	$\{2,5,6,9\}$	$\{0\}$	4	1
4	$\{4,7\}$	$\{0,1\}$	2	2

The next rows give a concrete pair of subgroup arrows from each object to its representative.

ID	Element	Rep.	k	l
0	$(0,0)$	0	0	0
1	$(0,1)$	0	1	0
2	$(1,0)$	2	0	0
3	$(1,1)$	0	0	3
4	$(2,0)$	4	0	0
5	$(2,1)$	2	0	3
6	$(3,0)$	2	1	3
7	$(3,1)$	4	0	3
8	$(4,0)$	0	1	3
9	$(4,1)$	2	1	0

33.4 H2.4 Subgroups of the dihedral group of order 12

The left and right generating IDs are $\{1\}$ and $\{3\}$.

The corresponding subgroup IDs are $\{0,1\}$ and $\{0,3\}$.

The pullback has 3 connected components. For each component the following table displays its full orbit and the projected isotropy subgroup. The representative is its least element ID.

Rep.	Orbit IDs	Isotropy IDs	$\lvert O\rvert$	$\lvert G_r\rvert$
0	$\{0,1,3,10\}$	$\{0\}$	4	1
2	$\{2,5,8,11\}$	$\{0\}$	4	1
4	$\{4,6,7,9\}$	$\{0\}$	4	1

The next rows give a concrete pair of subgroup arrows from each object to its representative.

ID	Element	Rep.	k	l
0	$(0,0)$	0	0	0
1	$(0,1)$	0	1	0
2	$(1,0)$	2	0	0
3	$(1,1)$	0	0	3
4	$(2,0)$	4	0	0
5	$(2,1)$	2	0	3
6	$(3,0)$	4	1	3
7	$(3,1)$	4	0	3
8	$(4,0)$	2	1	3
9	$(4,1)$	4	1	0
10	$(5,0)$	0	1	3
11	$(5,1)$	2	1	0

33.5 H2.5 Subgroups of the dihedral group of order 12

The left and right generating IDs are $\{1,4\}$ and $\{3,6\}$.

The corresponding subgroup IDs are $\{0,1,4,5,8,9\}$ and $\{0,3,6,9\}$.

The pullback has 1 connected components. For each component the following table displays its full orbit and the projected isotropy subgroup. The representative is its least element ID.

Rep.	Orbit IDs	Isotropy IDs	$\lvert O\rvert$	$\lvert G_r\rvert$
0	$\{0,1,2,3,4,5,6,7,8,9,10,11\}$	$\{0,9\}$	12	2

The next rows give a concrete pair of subgroup arrows from each object to its representative.

ID	Element	Rep.	k	l
0	$(0,0)$	0	0	0
1	$(0,1)$	0	1	0
2	$(1,0)$	0	4	6
3	$(1,1)$	0	0	3
4	$(2,0)$	0	1	9
5	$(2,1)$	0	4	9
6	$(3,0)$	0	0	6
7	$(3,1)$	0	1	6
8	$(4,0)$	0	4	0
9	$(4,1)$	0	0	9
10	$(5,0)$	0	1	3
11	$(5,1)$	0	4	3

33.6 H2.6 Subgroups of the dihedral group of order 16

The left and right generating IDs are $\{1,4\}$ and $\{1,8\}$.

The corresponding subgroup IDs are $\{0,1,4,5,8,9,12,13\}$ and $\{0,1,8,9\}$.

The pullback has 2 connected components. For each component the following table displays its full orbit and the projected isotropy subgroup. The representative is its least element ID.

Rep.	Orbit IDs	Isotropy IDs	$\lvert O\rvert$	$\lvert G_r\rvert$
0	$\{0,1,4,5,8,9,12,13\}$	$\{0,1,8,9\}$	8	4
2	$\{2,3,6,7,10,11,14,15\}$	$\{0,5,8,13\}$	8	4

The next rows give a concrete pair of subgroup arrows from each object to its representative.

ID	Element	Rep.	k	l
0	$(0,0)$	0	0	0
1	$(0,1)$	0	0	1
2	$(1,0)$	2	0	0
3	$(1,1)$	2	0	1
4	$(2,0)$	0	4	8
5	$(2,1)$	0	4	9
6	$(3,0)$	2	1	9
7	$(3,1)$	2	1	8
8	$(4,0)$	0	0	8
9	$(4,1)$	0	0	9
10	$(5,0)$	2	0	8
11	$(5,1)$	2	0	9
12	$(6,0)$	0	4	0
13	$(6,1)$	0	4	1
14	$(7,0)$	2	1	1

ID	Element	Rep.	k	l
15	$(7,1)$	2	1	0

33.7 H2.7 Subgroups of the dihedral group of order 16

The left and right generating IDs are $\{4\}$ and $\{3,4\}$.

The corresponding subgroup IDs are $\{0,4,8,12\}$ and $\{0,3,4,7,8,11,12,15\}$.

The pullback has 2 connected components. For each component the following table displays its full orbit and the projected isotropy subgroup. The representative is its least element ID.

Rep.	Orbit IDs	Isotropy IDs	$\lvert O\rvert$	$\lvert G_r\rvert$
0	$\{0,3,4,7,8,11,12,15\}$	$\{0,4,8,12\}$	8	4
1	$\{1,2,5,6,9,10,13,14\}$	$\{0,4,8,12\}$	8	4

The next rows give a concrete pair of subgroup arrows from each object to its representative.

ID	Element	Rep.	k	l
0	$(0,0)$	0	0	0
1	$(0,1)$	1	0	0
2	$(1,0)$	1	0	15
3	$(1,1)$	0	0	3
4	$(2,0)$	0	0	4
5	$(2,1)$	1	0	12
6	$(3,0)$	1	0	11
7	$(3,1)$	0	0	7
8	$(4,0)$	0	0	8
9	$(4,1)$	1	0	8
10	$(5,0)$	1	0	7
11	$(5,1)$	0	0	11
12	$(6,0)$	0	0	12
13	$(6,1)$	1	0	4
14	$(7,0)$	1	0	3

ID	Element	Rep.	k	l
15	$(7,1)$	0	0	15

33.8 H2.8 Subgroups of the dihedral group of order 18

The left and right generating IDs are $\{1,6\}$ and $\{3,6\}$.

The corresponding subgroup IDs are $\{0,1,6,7,12,13\}$ and $\{0,3,6,9,12,15\}$.

The pullback has 2 connected components. For each component the following table displays its full orbit and the projected isotropy subgroup. The representative is its least element ID.

Rep.	Orbit IDs	Isotropy IDs	$\lvert O\rvert$	$\lvert G_r\rvert$
0	$\{0,1,3,4,6,7,9,10,12,13,15,16\}$	$\{0,6,12\}$	12	3
2	$\{2,5,8,11,14,17\}$	$\{0,1,6,7,12,13\}$	6	6

The next rows give a concrete pair of subgroup arrows from each object to its representative.

ID	Element	Rep.	k	l
0	$(0,0)$	0	0	0
1	$(0,1)$	0	1	0
2	$(1,0)$	2	0	0
3	$(1,1)$	0	0	3
4	$(2,0)$	0	1	15
5	$(2,1)$	2	0	3
6	$(3,0)$	0	0	6
7	$(3,1)$	0	1	12
8	$(4,0)$	2	0	6
9	$(4,1)$	0	0	9
10	$(5,0)$	0	1	9
11	$(5,1)$	2	0	9
12	$(6,0)$	0	0	12
13	$(6,1)$	0	1	6
14	$(7,0)$	2	0	12

ID	Element	Rep.	k	l
15	$(7,1)$	0	0	15
16	$(8,0)$	0	1	3
17	$(8,1)$	2	0	15

33.9 H2.9 Subgroups of the dihedral group of order 20

The left and right generating IDs are $\{1\}$ and $\{5\}$.

The corresponding subgroup IDs are $\{0,1\}$ and $\{0,5\}$.

The pullback has 6 connected components. For each component the following table displays its full orbit and the projected isotropy subgroup. The representative is its least element ID.

Rep.	Orbit IDs	Isotropy IDs	$\lvert O\rvert$	$\lvert G_r\rvert$
0	$\{0,1,5,16\}$	$\{0\}$	4	1
2	$\{2,7,14,19\}$	$\{0\}$	4	1
3	$\{3,18\}$	$\{0,1\}$	2	2
4	$\{4,9,12,17\}$	$\{0\}$	4	1
6	$\{6,10,11,15\}$	$\{0\}$	4	1
8	$\{8,13\}$	$\{0,1\}$	2	2

The next rows give a concrete pair of subgroup arrows from each object to its representative.

ID	Element	Rep.	k	l
0	$(0,0)$	0	0	0
1	$(0,1)$	0	1	0
2	$(1,0)$	2	0	0
3	$(1,1)$	3	0	0
4	$(2,0)$	4	0	0
5	$(2,1)$	0	0	5
6	$(3,0)$	6	0	0
7	$(3,1)$	2	0	5
8	$(4,0)$	8	0	0
9	$(4,1)$	4	0	5
10	$(5,0)$	6	1	5
11	$(5,1)$	6	0	5
12	$(6,0)$	4	1	5
13	$(6,1)$	8	0	5
14	$(7,0)$	2	1	5

ID	Element	Rep.	k	l
15	$(7,1)$	6	1	0
16	$(8,0)$	0	1	5
17	$(8,1)$	4	1	0
18	$(9,0)$	3	0	5
19	$(9,1)$	2	1	0

33.10 H2.10 Subgroups of the dihedral group of order 20

The left and right generating IDs are $\{4\}$ and $\{3,10\}$.

The corresponding subgroup IDs are $\{0,4,8,12,16\}$ and $\{0,3,10,13\}$.

The pullback has 1 connected components. For each component the following table displays its full orbit and the projected isotropy subgroup. The representative is its least element ID.

Rep.	Orbit IDs	Isotropy IDs	$\lvert O\rvert$	$\lvert G_r\rvert$
0	$\{0,1,2,3,4,5,6,7,$ $8,9,10,11,12,13,$ $14,15,16,17,18,19\}$	$\{0\}$	20	1

The next rows give a concrete pair of subgroup arrows from each object to its representative.

ID	Element	Rep.	k	l
0	$(0,0)$	0	0	0
1	$(0,1)$	0	12	13
2	$(1,0)$	0	8	10
3	$(1,1)$	0	0	3
4	$(2,0)$	0	16	0
5	$(2,1)$	0	8	13
6	$(3,0)$	0	4	10
7	$(3,1)$	0	16	3
8	$(4,0)$	0	12	0
9	$(4,1)$	0	4	13
10	$(5,0)$	0	0	10
11	$(5,1)$	0	12	3
12	$(6,0)$	0	8	0
13	$(6,1)$	0	0	13
14	$(7,0)$	0	16	10

ID	Element	Rep.	k	l
15	$(7,1)$	0	8	3
16	$(8,0)$	0	4	0
17	$(8,1)$	0	16	13
18	$(9,0)$	0	12	10
19	$(9,1)$	0	4	3

33.11 H2.11 Subgroups of the dihedral group of order 24

The left and right generating IDs are $\{1,6\}$ and $\{3,8\}$.

The corresponding subgroup IDs are $\{0,1,6,7,12,13,18,19\}$ and $\{0,3,8,11,16,19\}$.

The pullback has 1 connected components. For each component the following table displays its full orbit and the projected isotropy subgroup. The representative is its least element ID.

Rep.	Orbit IDs	Isotropy IDs	$\lvert O\rvert$	$\lvert G_r\rvert$
0	$\{0,1,2,3,4,5,6,7,8,9,10,11,12,13,14,15,16,17,18,19,20,21,22,23\}$	$\{0,19\}$	24	2

The next rows give a concrete pair of subgroup arrows from each object to its representative.

ID	Element	Rep.	k	l
0	$(0,0)$	0	0	0
1	$(0,1)$	0	1	0
2	$(1,0)$	0	6	8
3	$(1,1)$	0	0	3
4	$(2,0)$	0	7	3
5	$(2,1)$	0	6	11
6	$(3,0)$	0	1	19
7	$(3,1)$	0	7	0
8	$(4,0)$	0	0	8
9	$(4,1)$	0	1	16
10	$(5,0)$	0	6	16
11	$(5,1)$	0	0	11
12	$(6,0)$	0	7	19
13	$(6,1)$	0	6	19
14	$(7,0)$	0	1	11

ID	Element	Rep.	k	l
15	$(7,1)$	0	7	16
16	$(8,0)$	0	0	16
17	$(8,1)$	0	1	8
18	$(9,0)$	0	6	0
19	$(9,1)$	0	0	19
20	$(10,0)$	0	7	11
21	$(10,1)$	0	6	3
22	$(11,0)$	0	1	3
23	$(11,1)$	0	7	8

33.12 H2.12 Subgroups of the dihedral group of order 24

The left and right generating IDs are $\{1,4\}$ and $\{5,6\}$.

The corresponding subgroup IDs are $\{0,1,4,5,8,9,12,13,16,17,20,21\}$ and $\{0,5,6,11,12,17,18,23\}$.

The pullback has 1 connected components. For each component the following table displays its full orbit and the projected isotropy subgroup. The representative is its least element ID.

Rep.	Orbit IDs	Isotropy IDs	$\lvert O\rvert$	$\lvert G_r\rvert$
0	$\{0,1,2,3,4,5,6,7,8,9,10,11,12,13,14,15,16,17,18,19,20,21,22,23\}$	$\{0,5,12,17\}$	24	4

The next rows give a concrete pair of subgroup arrows from each object to its representative.

ID	Element	Rep.	k	l
0	$(0,0)$	0	0	0
1	$(0,1)$	0	1	0
2	$(1,0)$	0	1	23
3	$(1,1)$	0	8	11
4	$(2,0)$	0	8	12
5	$(2,1)$	0	0	5
6	$(3,0)$	0	0	6
7	$(3,1)$	0	1	18
8	$(4,0)$	0	1	17
9	$(4,1)$	0	8	17
10	$(5,0)$	0	8	18
11	$(5,1)$	0	0	11
12	$(6,0)$	0	0	12
13	$(6,1)$	0	1	12
14	$(7,0)$	0	1	11

ID	Element	Rep.	k	l
15	$(7,1)$	0	8	23
16	$(8,0)$	0	8	0
17	$(8,1)$	0	0	17
18	$(9,0)$	0	0	18
19	$(9,1)$	0	1	6
20	$(10,0)$	0	1	5
21	$(10,1)$	0	8	5
22	$(11,0)$	0	8	6
23	$(11,1)$	0	0	23

33.13 H2.13 Subgroups of the dihedral group of order 28

The left and right generating IDs are $\{4\}$ and $\{3,14\}$.

The corresponding subgroup IDs are $\{0,4,8,12,16,20,24\}$ and $\{0,3,14,17\}$.

The pullback has 1 connected components. For each component the following table displays its full orbit and the projected isotropy subgroup. The representative is its least element ID.

Rep.	Orbit IDs	Isotropy IDs	$\lvert O\rvert$	$\lvert G_r\rvert$
0	$\{0,1,2,3,4,5,6,7,8,9,10,11,12,13,14,15,16,17,18,19,20,21,22,23,24,25,26,27\}$	$\{0\}$	28	1

The next rows give a concrete pair of subgroup arrows from each object to its representative.

ID	Element	Rep.	k	l
0	$(0,0)$	0	0	0
1	$(0,1)$	0	16	17
2	$(1,0)$	0	12	14
3	$(1,1)$	0	0	3
4	$(2,0)$	0	24	0
5	$(2,1)$	0	12	17
6	$(3,0)$	0	8	14
7	$(3,1)$	0	24	3
8	$(4,0)$	0	20	0
9	$(4,1)$	0	8	17
10	$(5,0)$	0	4	14
11	$(5,1)$	0	20	3
12	$(6,0)$	0	16	0
13	$(6,1)$	0	4	17
14	$(7,0)$	0	0	14

ID	Element	Rep.	k	l
15	$(7,1)$	0	16	3
16	$(8,0)$	0	12	0
17	$(8,1)$	0	0	17
18	$(9,0)$	0	24	14
19	$(9,1)$	0	12	3
20	$(10,0)$	0	8	0
21	$(10,1)$	0	24	17
22	$(11,0)$	0	20	14
23	$(11,1)$	0	8	3
24	$(12,0)$	0	4	0
25	$(12,1)$	0	20	17
26	$(13,0)$	0	16	14
27	$(13,1)$	0	4	3

33.14 H2.14 Subgroups of the dihedral group of order 30

The left and right generating IDs are $\{1,6\}$ and $\{3,10\}$.

The corresponding subgroup IDs are $\{0,1,6,7,12,13,18,19,24,25\}$ and $\{0,3,10,13,20,23\}$.

The pullback has 1 connected components. For each component the following table displays its full orbit and the projected isotropy subgroup. The representative is its least element ID.

Rep.	Orbit IDs	Isotropy IDs	$\lvert O\rvert$	$\lvert G_r\rvert$
0	$\{0,1,2,3,4,5,6,7,8,9,10,11,12,13,14,15,16,17,18,19,20,21,22,23,24,25,26,27,28,29\}$	$\{0,13\}$	30	2

The next rows give a concrete pair of subgroup arrows from each object to its representative.

ID	Element	Rep.	k	l
0	$(0,0)$	0	0	0
1	$(0,1)$	0	1	0
2	$(1,0)$	0	18	20
3	$(1,1)$	0	0	3
4	$(2,0)$	0	6	10
5	$(2,1)$	0	18	23
6	$(3,0)$	0	19	13
7	$(3,1)$	0	6	13
8	$(4,0)$	0	1	23
9	$(4,1)$	0	19	10
10	$(5,0)$	0	0	10
11	$(5,1)$	0	1	20
12	$(6,0)$	0	18	0
13	$(6,1)$	0	0	13
14	$(7,0)$	0	6	20

ID	Element	Rep.	k	l
15	$(7,1)$	0	18	3
16	$(8,0)$	0	19	3
17	$(8,1)$	0	6	23
18	$(9,0)$	0	1	13
19	$(9,1)$	0	19	0
20	$(10,0)$	0	0	20
21	$(10,1)$	0	1	10
22	$(11,0)$	0	18	10
23	$(11,1)$	0	0	23
24	$(12,0)$	0	6	0
25	$(12,1)$	0	18	13
26	$(13,0)$	0	19	23
27	$(13,1)$	0	6	3
28	$(14,0)$	0	1	3
29	$(14,1)$	0	19	20

33.15 H2.15 Subgroups of the dihedral group of order 32

The left and right generating IDs are $\{1,4\}$ and $\{3,8\}$.

The corresponding subgroup IDs are $\{0,1,4,5,8,9,12,13,16,17,20,21,24,25,28,29\}$ and $\{0,3,8,11,16,19,24,27\}$.

The pullback has 1 connected components. For each component the following table displays its full orbit and the projected isotropy subgroup. The representative is its least element ID.

Rep.	Orbit IDs	Isotropy IDs	$\lvert O\rvert$	$\lvert G_r\rvert$
0	$\{0,1,2,3,4,5,6,7,8,9,10,11,12,13,14,15,16,17,18,19,20,21,22,23,24,25,26,27,28,29,30,31\}$	$\{0,8,16,24\}$	32	4

The next rows give a concrete pair of subgroup arrows from each object to its representative.

ID	Element	Rep.	k	l
0	$(0,0)$	0	0	0
1	$(0,1)$	0	1	0
2	$(1,0)$	0	5	3
3	$(1,1)$	0	0	3
4	$(2,0)$	0	4	8
5	$(2,1)$	0	5	0
6	$(3,0)$	0	1	27
7	$(3,1)$	0	4	11
8	$(4,0)$	0	0	8
9	$(4,1)$	0	1	24
10	$(5,0)$	0	5	27
11	$(5,1)$	0	0	11
12	$(6,0)$	0	4	16
13	$(6,1)$	0	5	24
14	$(7,0)$	0	1	19

ID	Element	Rep.	k	l
15	$(7,1)$	0	4	19
16	$(8,0)$	0	0	16
17	$(8,1)$	0	1	16
18	$(9,0)$	0	5	19
19	$(9,1)$	0	0	19
20	$(10,0)$	0	4	24
21	$(10,1)$	0	5	16
22	$(11,0)$	0	1	11
23	$(11,1)$	0	4	27
24	$(12,0)$	0	0	24
25	$(12,1)$	0	1	8
26	$(13,0)$	0	5	11
27	$(13,1)$	0	0	27
28	$(14,0)$	0	4	0
29	$(14,1)$	0	5	8

ID	Element	Rep.	k	l
30	$(15,0)$	0	1	3
31	$(15,1)$	0	4	3

33.16 H2.16 Subgroups of the dihedral group of order 36

The left and right generating IDs are $\{1,6\}$ and $\{3,12\}$.

The corresponding subgroup IDs are $\{0,1,6,7,12,13,18,19,24,25,30,31\}$ and $\{0,3,12,15,24,27\}$.

The pullback has 2 connected components. For each component the following table displays its full orbit and the projected isotropy subgroup. The representative is its least element ID.

Rep.	Orbit IDs	Isotropy IDs	$\lvert O\rvert$	$\lvert G_r\rvert$
0	$\{0,1,3,4,6,7,9,10,12,13,15,16,18,19,21,22,24,25,27,28,30,31,33,34\}$	$\{0,12,24\}$	24	3
2	$\{2,5,8,11,14,17,20,23,26,29,32,35\}$	$\{0,7,12,19,24,31\}$	12	6

The next rows give a concrete pair of subgroup arrows from each object to its representative.

ID	Element	Rep.	k	l
0	$(0,0)$	0	0	0
1	$(0,1)$	0	1	0
2	$(1,0)$	2	0	0
3	$(1,1)$	0	0	3
4	$(2,0)$	0	7	3
5	$(2,1)$	2	0	3
6	$(3,0)$	0	6	12
7	$(3,1)$	0	7	0
8	$(4,0)$	2	1	27
9	$(4,1)$	0	6	15
10	$(5,0)$	0	1	27
11	$(5,1)$	2	1	24
12	$(6,0)$	0	0	12
13	$(6,1)$	0	1	24
14	$(7,0)$	2	0	12

ID	Element	Rep.	k	l
15	$(7,1)$	0	0	15
16	$(8,0)$	0	7	27
17	$(8,1)$	2	0	15
18	$(9,0)$	0	6	24
19	$(9,1)$	0	7	24
20	$(10,0)$	2	1	15
21	$(10,1)$	0	6	27
22	$(11,0)$	0	1	15
23	$(11,1)$	2	1	12
24	$(12,0)$	0	0	24
25	$(12,1)$	0	1	12
26	$(13,0)$	2	0	24
27	$(13,1)$	0	0	27
28	$(14,0)$	0	7	15
29	$(14,1)$	2	0	27

ID	Element	Rep.	k	l
30	$(15,0)$	0	6	0
31	$(15,1)$	0	7	12
32	$(16,0)$	2	1	3
33	$(16,1)$	0	6	3
34	$(17,0)$	0	1	3
35	$(17,1)$	2	1	0

34 Equivariant sections and restrictions

Let $G = C_n$ act on a disjoint union of cycles X. A cycle of length $d \mid n$ has its points permuted by addition of one modulo d. In the topos of G-spaces the support of $X \to 1$ is terminal when X is inhabited. A global chosen point is a fixed point. Restriction to the subgroup of order $h \mid n$ changes the generator to the translation by n/h, so a whole cycle is fixed exactly when $d \mid n/h$.

The G-map $G/H \to 1$ is an effective epimorphism. A section of the residual after reindexing to G/H corresponds to an H-fixed point of X. Its value at the coset H determines the entire equivariant map, and the rule $gH \mapsto g \cdot x$ is well-defined exactly when H fixes x. The

table tests these local sections against global fixedness. The boundary graph construction of Section 5 realizes each X as a path residual.

34.1 H3.1 Cyclic action of order 4

The cycle lengths in index order are $2, 4$.

There are 6 points in the residual and 0 global sections. For each subgroup, the last column counts the local sections whose value at its identity coset extends to a global fixed point.

Order	Step	Fixed cycle IDs	Local	Global
1	4	$\{0, 1\}$	6	0
2	2	$\{0\}$	2	0
4	1	$\varnothing$	0	0

The next table gives the induced permutation on each cycle for all subgroup generators. A shift s on a d-cycle has $\gcd(d, s)$ orbits, each of size $d/\gcd(d, s)$.

Order	Cycle ID	Size	Shift	Orbits	Orbit size
1	0	2	0	2	1
1	1	4	0	4	1
2	0	2	0	2	1
2	1	4	2	2	2
4	0	2	1	1	2
4	1	4	1	1	4

34.2 H3.2 Cyclic action of order 6

The cycle lengths in index order are $2, 3, 6$.

There are 11 points in the residual and 0 global sections. For each subgroup, the last column counts the local sections whose value at its identity coset extends to a global fixed point.

Order	Step	Fixed cycle IDs	Local	Global
1	6	$\{0, 1, 2\}$	11	0
2	3	$\{1\}$	3	0
3	2	$\{0\}$	2	0
6	1	$\varnothing$	0	0

The next table gives the induced permutation on each cycle for all subgroup generators. A shift s on a d-cycle has $\gcd(d, s)$ orbits, each of size $d/\gcd(d, s)$.

Order	Cycle ID	Size	Shift	Orbits	Orbit size
1	0	2	0	2	1
1	1	3	0	3	1
1	2	6	0	6	1
2	0	2	1	1	2
2	1	3	0	3	1
2	2	6	3	3	2
3	0	2	0	2	1
3	1	3	2	1	3
3	2	6	2	2	3
6	0	2	1	1	2
6	1	3	1	1	3
6	2	6	1	1	6

34.3 H3.3 Cyclic action of order 8

The cycle lengths in index order are $1, 2, 4$.

There are 7 points in the residual and 1 global sections. For each subgroup, the last column counts the local sections whose value at its identity coset extends to a global fixed point.

Order	Step	Fixed cycle IDs	Local	Global
1	8	$\{0, 1, 2\}$	7	1
2	4	$\{0, 1, 2\}$	7	1
4	2	$\{0, 1\}$	3	1
8	1	$\{0\}$	1	1

The next table gives the induced permutation on each cycle for all subgroup generators. A shift s on a d-cycle has $\gcd(d, s)$ orbits, each of size $d/\gcd(d, s)$.

Order	Cycle ID	Size	Shift	Orbits	Orbit size
1	0	1	0	1	1
1	1	2	0	2	1
1	2	4	0	4	1
2	0	1	0	1	1
2	1	2	0	2	1
2	2	4	0	4	1
4	0	1	0	1	1
4	1	2	0	2	1
4	2	4	2	2	2
8	0	1	0	1	1
8	1	2	1	1	2
8	2	4	1	1	4

34.4 H3.4 Cyclic action of order 9

The cycle lengths in index order are $3, 3, 9$.

There are 15 points in the residual and 0 global sections. For each subgroup, the last column counts the local sections whose value at its identity coset extends to a global fixed point.

Order	Step	Fixed cycle IDs	Local	Global
1	9	$\{0, 1, 2\}$	15	0
3	3	$\{0, 1\}$	6	0
9	1	$\varnothing$	0	0

The next table gives the induced permutation on each cycle for all subgroup generators. A shift s on a d-cycle has $\gcd(d,s)$ orbits, each of size $d/\gcd(d,s)$.

Order	Cycle ID	Size	Shift	Orbits	Orbit size
1	0	3	0	3	1
1	1	3	0	3	1
1	2	9	0	9	1
3	0	3	0	3	1
3	1	3	0	3	1
3	2	9	3	3	3
9	0	3	1	1	3
9	1	3	1	1	3
9	2	9	1	1	9

34.5 H3.5 Cyclic action of order 10

The cycle lengths in index order are $2, 5, 10$.

There are 17 points in the residual and 0 global sections. For each subgroup, the last column counts the local sections whose value at its identity coset extends to a global fixed point.

Order	Step	Fixed cycle IDs	Local	Global
1	10	$\{0,1,2\}$	17	0
2	5	$\{1\}$	5	0
5	2	$\{0\}$	2	0
10	1	$\varnothing$	0	0

The next table gives the induced permutation on each cycle for all subgroup generators. A shift s on a d-cycle has $\gcd(d,s)$ orbits, each of size $d/\gcd(d,s)$.

Order	Cycle ID	Size	Shift	Orbits	Orbit size
1	0	2	0	2	1
1	1	5	0	5	1
1	2	10	0	10	1
2	0	2	1	1	2
2	1	5	0	5	1
2	2	10	5	5	2
5	0	2	0	2	1
5	1	5	2	1	5
5	2	10	2	2	5
10	0	2	1	1	2
10	1	5	1	1	5
10	2	10	1	1	10

34.6 H3.6 Cyclic action of order 12

The cycle lengths in index order are $2, 3, 4, 6$.

There are 15 points in the residual and 0 global sections. For each subgroup, the last column counts the local sections whose value at its identity coset extends to a global fixed point.

Order	Step	Fixed cycle IDs	Local	Global
1	12	$\{0,1,2,3\}$	15	0
2	6	$\{0,1,3\}$	11	0
3	4	$\{0,2\}$	6	0
4	3	$\{1\}$	3	0
6	2	$\{0\}$	2	0
12	1	$\varnothing$	0	0

The next table gives the induced permutation on each cycle for all subgroup generators. A shift s on a d-cycle has $\gcd(d,s)$ orbits, each of size $d/\gcd(d,s)$.

Order	Cycle ID	Size	Shift	Orbits	Orbit size
1	0	2	0	2	1
1	1	3	0	3	1
1	2	4	0	4	1
1	3	6	0	6	1
2	0	2	0	2	1
2	1	3	0	3	1
2	2	4	2	2	2
2	3	6	0	6	1
3	0	2	0	2	1
3	1	3	1	1	3
3	2	4	0	4	1
3	3	6	4	2	3
4	0	2	1	1	2
4	1	3	0	3	1
4	2	4	3	1	4
4	3	6	3	3	2
6	0	2	0	2	1
6	1	3	2	1	3
6	2	4	2	2	2
6	3	6	2	2	3
12	0	2	1	1	2
12	1	3	1	1	3
12	2	4	1	1	4
12	3	6	1	1	6

34.7 H3.7 Cyclic action of order 15

The cycle lengths in index order are $3, 5, 15$.

There are 23 points in the residual and 0 global sections. For each subgroup, the last column counts the local sections whose value at its identity coset extends to a global fixed point.

Order	Step	Fixed cycle IDs	Local	Global
1	15	$\{0,1,2\}$	23	0
3	5	$\{1\}$	5	0
5	3	$\{0\}$	3	0
15	1	$\varnothing$	0	0

The next table gives the induced permutation on each cycle for all subgroup generators. A shift s on a d-cycle has $\gcd(d,s)$ orbits, each of size $d/\gcd(d,s)$.

Order	Cycle ID	Size	Shift	Orbits	Orbit size
1	0	3	0	3	1
1	1	5	0	5	1
1	2	15	0	15	1
3	0	3	2	1	3
3	1	5	0	5	1
3	2	15	5	5	3
5	0	3	0	3	1
5	1	5	3	1	5
5	2	15	3	3	5
15	0	3	1	1	3
15	1	5	1	1	5
15	2	15	1	1	15

34.8 H3.8 Cyclic action of order 16

The cycle lengths in index order are $4, 8, 16$.

There are 28 points in the residual and 0 global sections. For each subgroup, the last column counts the local sections whose value at its identity coset extends to a global fixed point.

Order	Step	Fixed cycle IDs	Local	Global
1	16	$\{0, 1, 2\}$	28	0
2	8	$\{0, 1\}$	12	0
4	4	$\{0\}$	4	0
8	2	$\varnothing$	0	0
16	1	$\varnothing$	0	0

The next table gives the induced permutation on each cycle for all subgroup generators. A shift s on a d-cycle has $\gcd(d, s)$ orbits, each of size $d/\gcd(d, s)$.

Order	Cycle ID	Size	Shift	Orbits	Orbit size
1	0	4	0	4	1
1	1	8	0	8	1
1	2	16	0	16	1
2	0	4	0	4	1
2	1	8	0	8	1
2	2	16	8	8	2
4	0	4	0	4	1
4	1	8	4	4	2
4	2	16	4	4	4
8	0	4	2	2	2
8	1	8	2	2	4
8	2	16	2	2	8
16	0	4	1	1	4
16	1	8	1	1	8

Order	Cycle ID	Size	Shift	Orbits	Orbit size
16	2	16	1	1	16

34.9 H3.9 Cyclic action of order 18

The cycle lengths in index order are $2, 3, 6, 9$.

There are 20 points in the residual and 0 global sections. For each subgroup, the last column counts the local sections whose value at its identity coset extends to a global fixed point.

Order	Step	Fixed cycle IDs	Local	Global
1	18	$\{0, 1, 2, 3\}$	20	0
2	9	$\{1, 3\}$	12	0
3	6	$\{0, 1, 2\}$	11	0
6	3	$\{1\}$	3	0
9	2	$\{0\}$	2	0
18	1	$\varnothing$	0	0

The next table gives the induced permutation on each cycle for all subgroup generators. A shift s on a d-cycle has $\gcd(d, s)$ orbits, each of size $d/\gcd(d, s)$.

Order	Cycle ID	Size	Shift	Orbits	Orbit size
1	0	2	0	2	1
1	1	3	0	3	1
1	2	6	0	6	1
1	3	9	0	9	1
2	0	2	1	1	2
2	1	3	0	3	1
2	2	6	3	3	2
2	3	9	0	9	1
3	0	2	0	2	1
3	1	3	0	3	1
3	2	6	0	6	1
3	3	9	6	3	3
6	0	2	1	1	2
6	1	3	0	3	1

Order	Cycle ID	Size	Shift	Orbits	Orbit size
6	2	6	3	3	2
6	3	9	3	3	3
9	0	2	0	2	1
9	1	3	2	1	3
9	2	6	2	2	3
9	3	9	2	1	9
18	0	2	1	1	2
18	1	3	1	1	3
18	2	6	1	1	6
18	3	9	1	1	9

34.10 H3.10 Cyclic action of order 20

The cycle lengths in index order are $4, 5, 10$.

There are 19 points in the residual and 0 global sections. For each subgroup, the last column counts the local sections whose value at its identity coset extends to a global fixed point.

Order	Step	Fixed cycle IDs	Local	Global
1	20	$\{0, 1, 2\}$	19	0
2	10	$\{1, 2\}$	15	0
4	5	$\{1\}$	5	0
5	4	$\{0\}$	4	0
10	2	$\varnothing$	0	0
20	1	$\varnothing$	0	0

The next table gives the induced permutation on each cycle for all subgroup generators. A shift s on a d-cycle has $\gcd(d,s)$ orbits, each of size $d/\gcd(d,s)$.

Order	Cycle ID	Size	Shift	Orbits	Orbit size
1	0	4	0	4	1
1	1	5	0	5	1
1	2	10	0	10	1
2	0	4	2	2	2
2	1	5	0	5	1
2	2	10	0	10	1
4	0	4	1	1	4
4	1	5	0	5	1
4	2	10	5	5	2
5	0	4	0	4	1
5	1	5	4	1	5
5	2	10	4	2	5
10	0	4	2	2	2
10	1	5	2	1	5
10	2	10	2	2	5
20	0	4	1	1	4
20	1	5	1	1	5
20	2	10	1	1	10

34.11 H3.11 Cyclic action of order 24

The cycle lengths in index order are $2, 3, 4, 8, 12$.

There are 29 points in the residual and 0 global sections. For each subgroup, the last column counts the local sections whose value at its identity coset extends to a global fixed point.

Order	Step	Fixed cycle IDs	Local	Global
1	24	$\{0,1,2,3,4\}$	29	0
2	12	$\{0,1,2,4\}$	21	0
3	8	$\{0,2,3\}$	14	0
4	6	$\{0,1\}$	5	0
6	4	$\{0,2\}$	6	0
8	3	$\{1\}$	3	0
12	2	$\{0\}$	2	0
24	1	$\varnothing$	0	0

The next table gives the induced permutation on each cycle for all subgroup generators. A shift s on a d-cycle has $\gcd(d,s)$ orbits, each of size $d/\gcd(d,s)$.

Order	Cycle ID	Size	Shift	Orbits	Orbit size
1	0	2	0	2	1
1	1	3	0	3	1
1	2	4	0	4	1
1	3	8	0	8	1
1	4	12	0	12	1
2	0	2	0	2	1
2	1	3	0	3	1
2	2	4	0	4	1
2	3	8	4	4	2
2	4	12	0	12	1
3	0	2	0	2	1
3	1	3	2	1	3
3	2	4	0	4	1
3	3	8	0	8	1
3	4	12	8	4	3
4	0	2	0	2	1
4	1	3	0	3	1
4	2	4	2	2	2
4	3	8	6	2	4
4	4	12	6	6	2
6	0	2	0	2	1
6	1	3	1	1	3
6	2	4	0	4	1
6	3	8	4	4	2
6	4	12	4	4	3
8	0	2	1	1	2
8	1	3	0	3	1
8	2	4	3	1	4
8	3	8	3	1	8
8	4	12	3	3	4
12	0	2	0	2	1
12	1	3	2	1	3
12	2	4	2	2	2
12	3	8	2	2	4
12	4	12	2	2	6
24	0	2	1	1	2
24	1	3	1	1	3
24	2	4	1	1	4
24	3	8	1	1	8
24	4	12	1	1	12

34.12 H3.12 Cyclic action of order 30

The cycle lengths in index order are $1, 2, 3, 5, 15$.

There are 26 points in the residual and 1 global sections. For each subgroup, the last column counts the local sections whose value at its identity coset extends to a global fixed point.

Order	Step	Fixed cycle IDs	Local	Global
1	30	$\{0,1,2,3,4\}$	26	1
2	15	$\{0,2,3,4\}$	24	1
3	10	$\{0,1,3\}$	8	1
5	6	$\{0,1,2\}$	6	1
6	5	$\{0,3\}$	6	1
10	3	$\{0,2\}$	4	1
15	2	$\{0,1\}$	3	1
30	1	$\{0\}$	1	1

The next table gives the induced permutation on each cycle for all subgroup generators. A shift s on a d-cycle has $\gcd(d,s)$ orbits, each of size $d/\gcd(d,s)$.

Order	Cycle ID	Size	Shift	Orbits	Orbit size
1	0	1	0	1	1
1	1	2	0	2	1
1	2	3	0	3	1
1	3	5	0	5	1
1	4	15	0	15	1
2	0	1	0	1	1
2	1	2	1	1	2
2	2	3	0	3	1
2	3	5	0	5	1
2	4	15	0	15	1
3	0	1	0	1	1
3	1	2	0	2	1
3	2	3	1	1	3
3	3	5	0	5	1

Order	Cycle ID	Size	Shift	Orbits	Orbit size
3	4	15	10	5	3
5	0	1	0	1	1
5	1	2	0	2	1
5	2	3	0	3	1
5	3	5	1	1	5
5	4	15	6	3	5
6	0	1	0	1	1
6	1	2	1	1	2
6	2	3	2	1	3
6	3	5	0	5	1
6	4	15	5	5	3
10	0	1	0	1	1
10	1	2	1	1	2
10	2	3	0	3	1

Order	Cycle ID	Size	Shift	Orbits	Orbit size
10	3	5	3	1	5
10	4	15	3	3	5
15	0	1	0	1	1
15	1	2	0	2	1
15	2	3	2	1	3
15	3	5	2	1	5
15	4	15	2	1	15
30	0	1	0	1	1
30	1	2	1	1	2
30	2	3	1	1	3
30	3	5	1	1	5
30	4	15	1	1	15

34.13 H3.13 Cyclic action of order 36

The cycle lengths in index order are $2,3,4,6,9,12$.

There are 36 points in the residual and 0 global sections. For each subgroup, the last column counts the local sections whose value at its identity coset extends to a global fixed point.

Order	Step	Fixed cycle IDs	Local	Global
1	36	$\{0,1,2,3,4,5\}$	36	0
2	18	$\{0,1,3,4\}$	20	0
3	12	$\{0,1,2,3,5\}$	27	0
4	9	$\{1,4\}$	12	0
6	6	$\{0,1,3\}$	11	0
9	4	$\{0,2\}$	6	0
12	3	$\{1\}$	3	0
18	2	$\{0\}$	2	0
36	1	$\varnothing$	0	0

The next table gives the induced permutation on each cycle for all subgroup generators. A shift s on a d-cycle has $\gcd(d,s)$ orbits, each of size $d/\gcd(d,s)$.

Order	Cycle ID	Size	Shift	Orbits	Orbit size
1	0	2	0	2	1
1	1	3	0	3	1
1	2	4	0	4	1
1	3	6	0	6	1
1	4	9	0	9	1
1	5	12	0	12	1
2	0	2	0	2	1
2	1	3	0	3	1
2	2	4	2	2	2
2	3	6	0	6	1
2	4	9	0	9	1
2	5	12	6	6	2
3	0	2	0	2	1
3	1	3	0	3	1

Order	Cycle ID	Size	Shift	Orbits	Orbit size
3	2	4	0	4	1
3	3	6	0	6	1
3	4	9	3	3	3
3	5	12	0	12	1
4	0	2	1	1	2
4	1	3	0	3	1
4	2	4	1	1	4
4	3	6	3	3	2
4	4	9	0	9	1
4	5	12	9	3	4
6	0	2	0	2	1
6	1	3	0	3	1
6	2	4	2	2	2
6	3	6	0	6	1

Order	Cycle ID	Size	Shift	Orbits	Orbit size
6	4	9	6	3	3
6	5	12	6	6	2
9	0	2	0	2	1
9	1	3	1	1	3
9	2	4	0	4	1
9	3	6	4	2	3
9	4	9	4	1	9
9	5	12	4	4	3
12	0	2	1	1	2
12	1	3	0	3	1
12	2	4	3	1	4
12	3	6	3	3	2
12	4	9	3	3	3
12	5	12	3	3	4

Order	Cycle ID	Size	Shift	Orbits	Orbit size
18	0	2	0	2	1
18	1	3	2	1	3
18	2	4	2	2	2
18	3	6	2	2	3
18	4	9	2	1	9
18	5	12	2	2	6
36	0	2	1	1	2
36	1	3	1	1	3
36	2	4	1	1	4
36	3	6	1	1	6
36	4	9	1	1	9
36	5	12	1	1	12

34.14 H3.14 Cyclic action of order 42

The cycle lengths in index order are $2, 3, 6, 7, 14$.

There are 32 points in the residual and 0 global sections. For each subgroup, the last column counts the local sections whose value at its identity coset extends to a global fixed point.

Order	Step	Fixed cycle IDs	Local	Global
1	42	$\{0, 1, 2, 3, 4\}$	32	0
2	21	$\{1, 3\}$	10	0
3	14	$\{0, 3, 4\}$	23	0
6	7	$\{3\}$	7	0
7	6	$\{0, 1, 2\}$	11	0
14	3	$\{1\}$	3	0
21	2	$\{0\}$	2	0
42	1	$\varnothing$	0	0

The next table gives the induced permutation on each cycle for all subgroup generators. A shift s on a d-cycle has $\gcd(d, s)$ orbits, each of size $d/\gcd(d, s)$.

Order	Cycle ID	Size	Shift	Orbits	Orbit size
1	0	2	0	2	1
1	1	3	0	3	1
1	2	6	0	6	1
1	3	7	0	7	1
1	4	14	0	14	1
2	0	2	1	1	2
2	1	3	0	3	1
2	2	6	3	3	2
2	3	7	0	7	1
2	4	14	7	7	2
3	0	2	0	2	1
3	1	3	2	1	3
3	2	6	2	2	3
3	3	7	0	7	1

Order	Cycle ID	Size	Shift	Orbits	Orbit size
3	4	14	0	14	1
6	0	2	1	1	2
6	1	3	1	1	3
6	2	6	1	1	6
6	3	7	0	7	1
6	4	14	7	7	2
7	0	2	0	2	1
7	1	3	0	3	1
7	2	6	0	6	1
7	3	7	6	1	7
7	4	14	6	2	7
14	0	2	1	1	2
14	1	3	0	3	1
14	2	6	3	3	2

Order	Cycle ID	Size	Shift	Orbits	Orbit size
14	3	7	3	1	7
14	4	14	3	1	14
21	0	2	0	2	1
21	1	3	2	1	3
21	2	6	2	2	3
21	3	7	2	1	7
21	4	14	2	2	7
42	0	2	1	1	2
42	1	3	1	1	3
42	2	6	1	1	6
42	3	7	1	1	7
42	4	14	1	1	14

VII. Reproducibility

Source correspondence, verification commands, and notes on the evidential scope of the case studies.

35 Reproducing the checked claims

The archive accompanying the manuscript contains the LaTeX source, the independent finite calculations, the Lean 4 program, and the native HoTT program. Its readme gives commands for building each component. The case inputs and result records are retained as JSON. The scripts reproduce the full table data.

35.1 Lean 4 correspondence

The file `lean/Audit.lean` asks the kernel to report the dependencies of the central declarations. The following table maps the paper's claims to code. The two names in the last row specify the reflection theorem and its reindexing consequence.

Claim	Declaration
Finite saturation decides safety	`solve_safe_iff`
May safety transfers	`safe_of_may_safe`
Exact path projection	`Lift.nodes`, `lift_of_nodes`
Complete prefix table	`tabulate`
Exact fixed-path decision	`checkBad`
Factorization and termination	`cut_refines`, `cut_shorter`
All descendants eliminated	`cut_eliminates`
CEGAR decides finite safety	`run_safe_iff`
Layered certificate validation	`Layered.check_sound`
Finite logic and renaming	`Logic.reflects`, `Logic.rename_denote`

35.2 HoTT correspondence

The native HoTT code uses a separate Lean kernel. The formal theorems use the HoTT library's fiber and truncation constructions. The archive pins the upstream version and records the rechecking commands. The proof-relevant identity type is part of that kernel's own type theory.

Claim	Declaration
Reindexing fibers and support	`reindex_fiber`, `reindex_support`
Composite fiber	`residual_compose`
Transport and identity	`transport_reindex`, `residual_identity`
Distinct circle lifts	`lifts_are_distinct`
Logical reflection	`reflect`
Fixed-path reflection	`bad_reflect`
Closed witness and refutation	`reachable_path_lifts`, `spurious_path_refuted`

The code appendix prints the proof companion with mathematical symbols rendered for a narrow column. The UTF-8 files in the archive are the source of record for compilation.

36 Source listings

Each listing preserves the source line numbers. ASCII transliterations are used for symbols such as the dependent sum, arrow, and membership relation. Use the original UTF-8 file from the companion when running Lean.

36.1 Proof-carrying saturation

`lean/Cegar/Core.lean`

```
1  import Std
2
3  namespace Cegar
4
5  universe u v w
6
7  structure Graph (A : Type u) where
8    initial : A -> Bool
9    bad : A -> Bool
10   edge : A -> A -> Bool
11
12 inductive Walk {A : Type u} (G : Graph A) : A -> Type
       u
13   | start (a : A) (initial : G.initial a = true) :
       Walk G a
14   | step {a : A} (previous : Walk G a) (b : A) (edge
       : G.edge a b = true) : Walk G b
15
16 def Walk.nodes {A : Type u} {G : Graph A} {a : A} :
       Walk G a -> List A
17   | .start a _ => [a]
18   | .step p b _ => p.nodes ++ [b]
19
20 theorem Walk.last_mem {A : Type u} {G : Graph A} {a :
       A} (p : Walk G a) :
21     a  in  p.nodes := by
22   cases p <;> simp [Walk.nodes]
23
24 def BadWalk {A : Type u} (G : Graph A) :=
25   Sigma  a, { _p : Walk G a // G.bad a = true }
26
27 def Safe {A : Type u} (G : Graph A) : Prop := BadWalk
       G -> False
28
29 inductive Verdict {A : Type u} (G : Graph A) : Type u
30   | safe : Safe G -> Verdict G
```

```
  | counterexample : BadWalk G -> Verdict G

structure Enumeration (A : Type u) where
  items : List A
  complete : forall  a, a  in  items

def finEnumeration (n : Nat) : Enumeration (Fin n)
    where
  items := List.finRange n
  complete := fun a => List.mem_ofFn.mpr <a, rfl>

def sumEnumeration {A : Type u} {B : Type v}
    (left : Enumeration A) (right : Enumeration B) :
    Enumeration (Sum A B) where
  items := left.items.map Sum.inl ++ right.items.map
    Sum.inr
  complete := by
    intro a
    cases a with
    | inl a => exact List.mem_append_left _
    (List.mem_map.mpr <a, left.complete a, rfl>)
    | inr b => exact List.mem_append_right _
    (List.mem_map.mpr <b, right.complete b, rfl>)

inductive Pick {A : Type u} {B : A -> Type v} (f : (a
    : A) -> Option (B a)) :
    List A -> Type (max u v)
  | missing {xs} : (forall  a  in  xs, f a = none) ->
    Pick f xs
  | found {xs} (a : A) (value : B a) (rest : List A) :
      f a = some value ->
      (forall  b  in  xs, b = a \/ b  in  rest) ->
      rest.length < xs.length ->
      Pick f xs

def pick {A : Type u} {B : A -> Type v} (f : (a : A)
    -> Option (B a)) :
    (xs : List A) -> Pick f xs
  | [] => .missing (by simp)
  | x :: xs => match h : f x with
    | some value => .found x value xs h (by simp) (by
    simp)
    | none => match pick f xs with
      | .missing miss => .missing (by
          intro a ha
          rcases List.mem_cons.mp ha with rfl | hm
          . exact h
          . exact miss a hm)
      | .found a value rest hv cover shorter =>
          .found a value (x :: rest) hv (by
            intro b hb
            rcases List.mem_cons.mp hb with rfl | hm
            . exact Or.inr (by simp)
            . rcases cover b hm with he | hr
              . exact Or.inl he
              . exact Or.inr (List.mem_cons_of_mem _
    hr)) (by simpa using shorter)

inductive Checked (A : Type u) : Type u
  | yes : A -> Checked A
  | no : (A -> False) -> Checked A

def choose {A : Type u} {B : A -> Type v}
    (enum : Enumeration A) (f : (a : A) -> Checked (B
    a)) : Checked (Sigma  a, B a) :=
  let option (a : A) : Option (B a) := match f a with
    | .yes b => some b
    | .no _ => none
  match pick option enum.items with
  | .found a b _ _ _ _ => .yes <a, b>
  | .missing miss => .no (fun <a, b> => by
      have h := miss a (enum.complete a)
      cases ha : f a with
      | yes proof => simp [option, ha] at h
      | no refute => exact refute b)

def Prefix {A : Type u} (G : Graph A) := Sigma  a,
    Walk G a

def extendFrom {A : Type u} (G : Graph A) (b : A) (p
    : Prefix G) :
    Option (Walk G b) :=
  if h : G.edge p.1 b = true then some (.step p.2 b
    h) else none

def discover {A : Type u} (G : Graph A) (seen : List
    (Prefix G)) (b : A) :
    Option (Walk G b) :=
  if h : G.initial b = true then some (.start b h)
  else match pick (extendFrom G b) seen with
    | .found _ p _ _ _ _ => some p
    | .missing _ => none

theorem discover_initial {A : Type u} {G : Graph A}
    {seen : List (Prefix G)}
    {b : A} (missing : discover G seen b = none) :
    G.initial b != true := by
  intro h
  simp [discover, h] at missing

theorem discover_closed {A : Type u} {G : Graph A}
    {seen : List (Prefix G)}
    {b : A} (missing : discover G seen b = none) (p :
    Prefix G) (hp : p  in  seen) :
    G.edge p.1 b != true := by
  intro h
  unfold discover at missing
  split at missing
  . contradiction
  . split at missing
    . contradiction
    . rename_i hm _
      have impossible := hm p hp
      simp [extendFrom, h] at impossible

def Seen {A : Type u} {G : Graph A} (seen : List
    (Prefix G)) (a : A) : Prop :=
  exists  p  in  seen, p.1 = a

theorem closedWalk {A : Type u} {G : Graph A} {seen :
    List (Prefix G)}
    (roots : forall  a, G.initial a = true -> Seen
    seen a)
    (closed : forall  a b, Seen seen a -> G.edge a b
    = true -> Seen seen b)
    {a : A} (p : Walk G a) : Seen seen a := by
  induction p with
  | start a hi => exact roots a hi
  | step p b he ih => exact closed _ b ih he

def explore {A : Type u} (G : Graph A) (seen : List
    (Prefix G))
    (remaining : List A)
    (cover : forall  a, a  in  remaining \/ Seen seen
    a)
    (good : forall  p  in  seen, G.bad p.1 = false) :
    Verdict G :=
  match pick (discover G seen) remaining with
  | .missing miss =>
    .safe (fun <a, p, bad> => by
      have roots : forall  b, G.initial b = true ->
    Seen seen b := by
```

```
        intro b hi
        rcases cover b with hm | hs
        . exact (discover_initial (miss b hm) hi).elim
        . exact hs
      have closed : forall  a b, Seen seen a ->
     G.edge a b = true -> Seen seen b := by
        intro a b hs he
        rcases cover b with hm | found
        . obtain <q, hq, rfl> := hs
          exact (discover_closed (miss b hm) q hq
     he).elim
        . exact found
      obtain <q, hq, h> := closedWalk roots closed p
      have hb := good q hq
      rw [h, bad] at hb
      contradiction)
  | .found a p rest _ kept shorter =>
    if bad : G.bad a = true then .counterexample <a,
     p, bad>
    else explore G (<a, p> :: seen) rest (by
      intro b
      rcases cover b with hm | <q, hq, he>
      . rcases kept b hm with rfl | hr
        . exact Or.inr <<b, p>, by simp, rfl>
        . exact Or.inl hr
      . exact Or.inr <q, List.mem_cons_of_mem _ hq,
     he>) (by
      intro q hq
      rcases List.mem_cons.mp hq with rfl | hm
      . cases h : G.bad a with
        | false => rfl
        | true => exact (bad h).elim
      . exact good q hm)
termination_by remaining.length

def solve {A : Type u} (G : Graph A) (enum :
     Enumeration A) : Verdict G :=
  explore G [] enum.items (fun a => Or.inl
     (enum.complete a)) (by simp)

def verdictCode {A : Type u} {G : Graph A} : Verdict
     G -> Option (List A)
  | .safe _ => none
  | .counterexample <_, p, _> => some p.nodes

theorem solve_safe_iff {A : Type u} (G : Graph A)
     (enum : Enumeration A) :
    (verdictCode (solve G enum) = none) <-> Safe G :=
     by
  cases solve G enum with
  | safe hs => simp [verdictCode, hs]
  | counterexample p =>
    have hs : not  Safe G := fun h => h p
    obtain <a, p, hb> := p
    simp [verdictCode, hs]

end Cegar
```

36.2 Abstraction, tabulation, and refinement

lean/Cegar/Refinement.lean

```
import Cegar.Core

namespace Cegar

universe u v

section May

variable {S : Type u} {A : Type v} [DecidableEq A]

def may (G : Graph S) (states : Enumeration S) (q : S
     -> A) : Graph A where
  initial a := states.items.any fun x => decide (q x
     = a) && G.initial x
  bad a := states.items.any fun x => decide (q x = a)
     && G.bad x
  edge a b := states.items.any fun x =>
     states.items.any fun y =>
    decide (q x = a) && decide (q y = b) && G.edge x y

variable {G : Graph S} {states : Enumeration S} {q :
     S -> A}

theorem may_initial {a : A} :
    (may G states q).initial a = true <-> exists  x,
     q x = a /\ G.initial x = true := by
  simp [may, states.complete]

theorem may_bad {a : A} :
    (may G states q).bad a = true <-> exists  x, q x
     = a /\ G.bad x = true := by
  simp [may, states.complete]

theorem may_edge {a b : A} :
    (may G states q).edge a b = true <->
      exists  x y, q x = a /\ q y = b /\ G.edge x y =
     true := by
  simp [may, states.complete, and_assoc]

def mapWalk {x : S} : Walk G x -> Walk (may G states
     q) (q x)
  | .start x h => .start (q x) (may_initial.mpr <x,
     rfl, h>)
  | .step p y h => .step (mapWalk p) (q y)
     (may_edge.mpr <_, y, rfl, rfl, h>)

theorem safe_of_may_safe (h : Safe (may G states q))
     : Safe G :=
  fun <x, p, hb> => h <q x, mapWalk p, may_bad.mpr
     <x, rfl, hb>>

inductive Lift (G : Graph S) (states : Enumeration S)
     (q : S -> A) :
    {a : A} -> Walk (may G states q) a -> S -> Type
     (max u v)
  | start {a : A} {h : (may G states q).initial a =
     true} {x : S} :
    q x = a -> G.initial x = true -> Lift G states q
     (.start a h) x
  | step {a b : A} {w : Walk (may G states q) a}
      {h : (may G states q).edge a b = true} {x y :
     S} :
    Lift G states q w x -> q y = b -> G.edge x y =
     true ->
      Lift G states q (.step w b h) y

def Lift.walk {a : A} {w : Walk (may G states q) a}
     {x : S} :
    Lift G states q w x -> Walk G x
  | .start _ h => .start _ h
  | .step p _ h => .step p.walk _ h

theorem Lift.endpoint {a : A} {w : Walk (may G states
     q) a} {x : S}
    (l : Lift G states q w x) : q x = a := by
  cases l with
  | start e _ => exact e
  | step _ e _ => exact e
```

```
theorem Lift.nodes {a : A} {w : Walk (may G states q) a} {x : S}
    (l : Lift G states q w x) : l.walk.nodes.map q = w.nodes := by
  induction l with
  | start e _ => simp [Lift.walk, Walk.nodes, e]
  | step _ e _ ih => simp [Lift.walk, Walk.nodes, e, ih]

theorem lift_of_nodes {a : A} {x : S}
    (w : Walk (may G states q) a) (p : Walk G x)
    (same : p.nodes.map q = w.nodes) : Nonempty (Lift G states q w x) := by
  induction w generalizing x with
  | start a hi =>
    cases p with
    | start x hx =>
      have h : q x = a := by simpa [Walk.nodes] using same
      exact <.start h hx>
    | step previous x hx =>
      have h : previous.nodes.map q ++ [q x] = [] ++ [a] := by
        simpa [Walk.nodes] using same
      have empty := List.append_inj_left' h (by simp)
      simpa using (show (previous.nodes.map q).length != 0 from by
        intro he
        have hz : previous.nodes = [] := List.length_eq_zero_iff.mp (by simpa using he)
        have hm := previous.last_mem
        rw [hz] at hm
        contradiction) (congrArg List.length empty)
  | step w b he ih =>
    cases p with
    | start x hx =>
      have h : [] ++ [q x] = w.nodes ++ [b] := by
        simpa [Walk.nodes] using same
      have empty := List.append_inj_left' h (by simp)
      have hm := w.last_mem
      rw [<- empty] at hm
      contradiction
    | step previous x hx =>
      have h : previous.nodes.map q ++ [q x] = w.nodes ++ [b] := by
        simpa [Walk.nodes] using same
      obtain <hl, hr> := List.append_inj' h (by simp)
      obtain <l> := ih previous hl
      exact <.step l (by simpa using hr) hx>

def LiftedBad {a : A} (w : Walk (may G states q) a) :=
  Sigma  x, { _l : Lift G states q w x // G.bad x = true }

def LiftedBad.walk {a : A} {w : Walk (may G states q) a}
    (l : LiftedBad w) : BadWalk G :=
  <l.1, l.2.1.walk, l.2.2>

structure Table {a : A} (w : Walk (may G states q) a) where
  rows : List (Sigma  x, Lift G states q w x)
  complete : forall  x, Lift G states q w x -> exists  r  in  rows, r.1 = x

def extendRow {a b : A} {w : Walk (may G states q) a}
    (h : (may G states q).edge a b = true) (old : Table w) (y : S) :
    Option (Sigma  x, Lift G states q (.step w b h) x) :=
  if hy : q y = b then
    match pick (fun (r : Sigma  x, Lift G states q w x) => if he : G.edge r.1 y = true then
        some (show Lift G states q (.step w b h) y from .step r.2 hy he)
        else none) old.rows with
    | .found _ p _ _ _ _ => some <y, p>
    | .missing _ => none
  else none

theorem extendRow_complete {a b : A} {w : Walk (may G states q) a}
    (h : (may G states q).edge a b = true) (old : Table w) (y : S)
    (l : Lift G states q (.step w b h) y) :
    exists  r, extendRow h old y = some r /\ r.1 = y := by
  cases l with
  | step p hy he =>
    obtain <r, hr, rfl> := old.complete _ p
    unfold extendRow
    simp only [hy, dif_pos]
    split
    . exact <_, rfl, rfl>
    . rename_i miss _
      have contradictionAt := miss r hr
      simp [he] at contradictionAt

def startRow {a : A} (h : (may G states q).initial a = true) (x : S) :
    Option (Sigma  y, Lift G states q (.start a h) y) :=
  if hq : q x = a then
    if hi : G.initial x = true then some <x, .start hq hi> else none
  else none

def tabulate {a : A} (w : Walk (may G states q) a) : Table w :=
  match w with
  | .start _ h =>
    <states.items.filterMap (startRow h), by
      intro x l
      cases l with
      | start hq hi =>
        refine <<x, .start hq hi>, List.mem_filterMap.mpr <x, states.complete x, ?_>, rfl>
        simp [startRow, hq, hi]>
  | .step w _ h =>
    let old := tabulate w
    <states.items.filterMap (extendRow h old), by
      intro x l
      obtain <r, hr, hx> := extendRow_complete h old x l
      exact <r, List.mem_filterMap.mpr <x, states.complete x, hr>, hx>>

def checkBad {a : A} (w : Walk (may G states q) a) : Checked (LiftedBad w) :=
  let table := tabulate w
  let candidate (r : Sigma  x, Lift G states q w x) : Option (LiftedBad w) :=
    if hb : G.bad r.1 = true then some <r.1, r.2, hb> else none
  match pick candidate table.rows with
  | .found _ l _ _ _ _ => .yes l
  | .missing miss => .no (fun <x, l, hb> => by
      obtain <r, hr, rfl> := table.complete x l
      have contradictionAt := miss r hr
      simp [candidate, hb] at contradictionAt)
```

```

def UniqueAlong {a : A} (w : Walk (may G states q) a)
    : Prop :=
  forall  x y, q x = q y -> q x  in  w.nodes -> x = y

theorem lift_of_unique {a : A} (w : Walk (may G
    states q) a)
    (unique : UniqueAlong w) : Nonempty (Sigma  x,
    Lift G states q w x) := by
  induction w with
  | start a hi =>
    obtain <x, hx, h> := may_initial.mp hi
    exact <<x, .start hx h>>
  | @step a w b he ih =>
    have up : UniqueAlong w := fun x y eq hx =>
      unique x y eq (List.mem_append_left _ hx)
    obtain <z, l> := ih up
    obtain <x, y, hx, hy, h> := may_edge.mp he
    have e : z = x := unique z x (l.endpoint.trans
    hx.symm)
      (l.endpoint.symm  |>  List.mem_append_left [b]
    w.last_mem)
    exact <<y, .step l hy (e.symm  |>  h)>>

theorem liftBad_of_unique {a : A} {w : Walk (may G
    states q) a}
    (bad : (may G states q).bad a = true) (unique :
    UniqueAlong w) :
    Nonempty (LiftedBad w) := by
  obtain <x, l> := lift_of_unique w unique
  obtain <y, hy, hb> := may_bad.mp bad
  have e : x = y := unique x y (l.endpoint.trans
    hy.symm) (l.endpoint.symm  |>  w.last_mem)
  exact <<x, l, e.symm  |>  hb>>

end May

section Loop

variable {S : Type u} {B : Type v} [DecidableEq S]
    [DecidableEq B]

local instance : DecidableEq (Sum S B) :=
  fun x y => match x, y with
    | .inl a, .inl b => if h : a = b then isTrue
    (congrArg Sum.inl h)
      else isFalse (fun e => h (Sum.inl.inj e))
    | .inr a, .inr b => if h : a = b then isTrue
    (congrArg Sum.inr h)
      else isFalse (fun e => h (Sum.inr.inj e))
    | .inl _, .inr _ => isFalse (fun e => nomatch e)
    | .inr _, .inl _ => isFalse (fun e => nomatch e)

def key (seed : S -> B) (remaining : List S) (x : S)
    : Sum S B :=
  if x  in  remaining then .inr (seed x) else .inl x

def visits (path : List (Sum S B)) (k : Sum S B) :
    Bool :=
  path.any fun a => decide (a = k)

@[simp] theorem visits_iff (path : List (Sum S B)) (k
    : Sum S B) :
    visits path k = true <-> k  in  path := by
  simp [visits]

def cut (seed : S -> B) (remaining : List S) (path :
    List (Sum S B)) : List S :=
  remaining.filter fun x => !(visits path (key seed
    remaining x))

theorem cut_sub {seed : S -> B} {remaining : List S}
    {path : List (Sum S B)}
    {x : S} (h : x  in  cut seed remaining path) :
    x  in  remaining /\ key seed remaining x  notin
    path := by
  simpa [cut, List.mem_filter, Bool.eq_false_iff]
    using h

def coarsen (seed : S -> B) (remaining : List S) :
    Sum S B -> Sum S B
  | .inl x => key seed remaining x
  | .inr b => .inr b

theorem cut_refines (seed : S -> B) (remaining : List
    S) (path : List (Sum S B)) (x : S) :
    coarsen seed remaining (key seed (cut seed
    remaining path) x) =
      key seed remaining x := by
  by_cases h : x  in  cut seed remaining path
  . simp [key, coarsen, h, (cut_sub h).1]
  . simp [key, coarsen, h]

theorem cut_unique (seed : S -> B) (remaining : List
    S) (path : List (Sum S B))
    (x y : S) (same : key seed (cut seed remaining
    path) x =
      key seed (cut seed remaining path) y)
    (visit : key seed remaining x  in  path) : x = y
    := by
  have hx : x  notin  cut seed remaining path := fun
    h => (cut_sub h).2 visit
  rw [key, if_neg hx] at same
  by_cases hy : y  in  cut seed remaining path
  . simp [key, hy] at same
  . simpa [key, hy] using same

theorem old_unique_of_no_cut {G : Graph S} {states :
    Enumeration S}
    {seed : S -> B} {remaining : List S} {a : Sum S B}
    {w : Walk (may G states (key seed remaining)) a}
    (noCut : forall  x  in  remaining, key seed
    remaining x  notin  w.nodes) : UniqueAlong w :=
    by
  intro x y same visit
  have hx : x  notin  remaining := fun h => noCut x h
    visit
  rw [key, if_neg hx] at same
  by_cases hy : y  in  remaining
  . simp [key, hy] at same
  . simpa [key, hy] using same

theorem cut_shorter {G : Graph S} {states :
    Enumeration S}
    {seed : S -> B} {remaining : List S} {a : Sum S B}
    {w : Walk (may G states (key seed remaining)) a}
    (bad : (may G states (key seed remaining)).bad a
    = true)
    (refute : LiftedBad w -> False) :
    (cut seed remaining w.nodes).length <
    remaining.length := by
  by_cases h : remaining.any (fun x => visits w.nodes
    (key seed remaining x)) = true
  . obtain <x, hx, visit> := List.any_eq_true.mp h
    apply List.length_filter_lt_length_iff_exists.mpr
    exact <x, hx, by simp [visit]>
  . have noCut : forall  x  in  remaining, key seed
    remaining x  notin  w.nodes :=
      fun x hx hv => h (List.any_eq_true.mpr <x, hx,
    (visits_iff _ _).2 hv>)
    obtain <l> := liftBad_of_unique bad
    (old_unique_of_no_cut noCut)
```

```
    exact (refute l).elim

theorem cut_eliminates {G : Graph S} {states : Enumeration S}
    {seed : S -> B} {remaining : List S} {a : Sum S B}
    (w : Walk (may G states (key seed remaining)) a)
    (refute : LiftedBad w -> False)
    {b : Sum S B}
    (fine : Walk (may G states (key seed (cut seed remaining w.nodes))) b)
    (bad : (may G states (key seed (cut seed remaining w.nodes))).bad b = true)
    (projects : fine.nodes.map (coarsen seed remaining) = w.nodes) : False := by
  have unique : UniqueAlong fine := by
    intro x y same visit
    apply cut_unique seed remaining w.nodes x y same
    let oldKey := coarsen seed remaining (key seed (cut seed remaining w.nodes) x)
    have hv : oldKey  in  fine.nodes.map (coarsen seed remaining) :=
      List.mem_map.mpr <_, visit, rfl>
    have hw : oldKey  in  w.nodes :=
      (congrArg (fun nodes => oldKey  in  nodes) projects).mp hv
    simpa only [oldKey, cut_refines] using hw
  obtain <<x, l, hx>> := liftBad_of_unique bad unique
  have matched : l.walk.nodes.map (key seed remaining) = w.nodes := by
    let c := coarsen seed remaining
    let q' := key seed (cut seed remaining w.nodes)
    calc
      l.walk.nodes.map (key seed remaining) =
      l.walk.nodes.map (c  o  q') :=
        List.map_congr_left (fun y _ => (cut_refines seed remaining w.nodes y).symm)
      _ = (l.walk.nodes.map q').map c := List.map_map.symm
      _ = fine.nodes.map c := congrArg (List.map c) l.nodes
      _ = w.nodes := projects
  obtain <old> := lift_of_nodes w l.walk matched
  exact refute <x, old, hx>

structure Run {S : Type u} (G : Graph S) where
  verdict : Verdict G
  rounds : Nat

def runFrom (G : Graph S) (states : Enumeration S)
    (seeds : Enumeration B) (seed : S -> B)
    (remaining : List S) : Run G :=
  match solve (may G states (key seed remaining))
      (sumEnumeration states seeds) with
  | .safe h => <.safe (safe_of_may_safe h), 1>
  | .counterexample <a, p, _bad> =>
    match checkBad p with
    | .yes l => <.counterexample l.walk, 1>
    | .no refute =>
      let next := runFrom G states seeds seed (cut seed remaining p.nodes)
      <next.verdict, next.rounds + 1>
termination_by remaining.length
decreasing_by exact cut_shorter _bad refute

def run (G : Graph S) (states : Enumeration S) (seeds : Enumeration B) (seed : S -> B) :
    Run G :=
  runFrom G states seeds seed states.items

theorem run_safe_iff (G : Graph S) (states : Enumeration S) (seeds : Enumeration B)
    (seed : S -> B) : verdictCode (run G states seeds seed).verdict = none <-> Safe G := by
  cases (run G states seeds seed).verdict with
  | safe h => simp [verdictCode, h]
  | counterexample p =>
    have hs : not  Safe G := fun h => h p
    obtain <a, p, hb> := p
    simp [verdictCode, hs]

end Loop

end Cegar
```

36.3 Scoped finite logic

lean/Cegar/Logic.lean

```
import Cegar.Core

namespace Cegar.Logic

universe u

inductive Term (S : Type u) (n : Nat) where
  | var : Fin n -> Term S n
  | val : S -> Term S n

inductive Formula (S : Type u) (k : Nat) : Nat -> Type u
  | top {n} : Formula S k n
  | bot {n} : Formula S k n
  | atom {n} : Fin k -> Term S n -> Formula S k n
  | equal {n} : Term S n -> Term S n -> Formula S k n
  | edge {n} : Term S n -> Term S n -> Formula S k n
  | both {n} : Formula S k n -> Formula S k n -> Formula S k n
  | either {n} : Formula S k n -> Formula S k n -> Formula S k n
  | impl {n} : Formula S k n -> Formula S k n -> Formula S k n
  | someState {n} : Formula S k (n + 1) -> Formula S k n
  | everyState {n} : Formula S k (n + 1) -> Formula S k n

def Term.eval {S : Type u} {n : Nat} (env : Fin n -> S) : Term S n -> S
  | .var i => env i
  | .val x => x

def extend {S : Type u} {n : Nat} (x : S) (env : Fin n -> S) : Fin (n + 1) -> S :=
  Fin.cases x env

structure Model (S : Type u) (k : Nat) where
  states : Enumeration S
  atom : Fin k -> S -> Bool
  edge : S -> S -> Bool

def Denote {S : Type u} {k : Nat} (M : Model S k) :
    {n : Nat} -> Formula S k n -> (Fin n -> S) -> Prop
  | _, .top, _ => True
  | _, .bot, _ => False
  | _, .atom i t, rho => M.atom i (t.eval rho) = true
  | _, .equal s t, rho => s.eval rho = t.eval rho
  | _, .edge s t, rho => M.edge (s.eval rho) (t.eval rho) = true
  | _, .both p q, rho => Denote M p rho /\ Denote M q rho
```

```
  | _, .either p q, rho => Denote M p rho \/ Denote M q rho
  | _, .impl p q, rho => Denote M p rho -> Denote M q rho
  | _, .someState p, rho => exists  x, Denote M p (extend x rho)
  | _, .everyState p, rho => forall  x, Denote M p (extend x rho)

def eval {S : Type u} [DecidableEq S] {k : Nat} (M : Model S k) :
    {n : Nat} -> Formula S k n -> (Fin n -> S) -> Bool
  | _, .top, _ => true
  | _, .bot, _ => false
  | _, .atom i t, rho => M.atom i (t.eval rho)
  | _, .equal s t, rho => decide (s.eval rho = t.eval rho)
  | _, .edge s t, rho => M.edge (s.eval rho) (t.eval rho)
  | _, .both p q, rho => eval M p rho && eval M q rho
  | _, .either p q, rho => eval M p rho || eval M q rho
  | _, .impl p q, rho => !(eval M p rho) || eval M q rho
  | _, .someState p, rho => M.states.items.any fun x => eval M p (extend x rho)
  | _, .everyState p, rho => M.states.items.all fun x => eval M p (extend x rho)

theorem reflects {S : Type u} [DecidableEq S] {k n : Nat} (M : Model S k)
    (p : Formula S k n) (rho : Fin n -> S) : eval M p rho = true <-> Denote M p rho := by
  induction p with
  | top => simp [eval, Denote]
  | bot => simp [eval, Denote]
  | atom => simp [eval, Denote]
  | equal => simp [eval, Denote]
  | edge => simp [eval, Denote]
  | both _ _ hp hq => simp [eval, Denote, hp, hq]
  | either _ _ hp hq => simp [eval, Denote, hp, hq]
  | impl p _ hp hq =>
    simp only [eval, Denote, <- hp, <- hq]
    cases h : eval M p rho <;> simp
  | someState _ hp => simp [eval, Denote, M.states.complete, hp]
  | everyState _ hp => simp [eval, Denote, M.states.complete, hp]

def Term.rename {S : Type u} {n m : Nat} (f : Fin n -> Fin m) :
    Term S n -> Term S m
  | .var i => .var (f i)
  | .val x => .val x

def under {n m : Nat} (f : Fin n -> Fin m) : Fin (n + 1) -> Fin (m + 1) :=
  Fin.cases 0 (fun i => (f i).succ)

def Formula.rename {S : Type u} {k : Nat} :
    {n m : Nat} -> (Fin n -> Fin m) -> Formula S k n -> Formula S k m
  | _, _, _, .top => .top
  | _, _, _, .bot => .bot
  | _, _, f, .atom i t => .atom i (t.rename f)
  | _, _, f, .equal s t => .equal (s.rename f) (t.rename f)
  | _, _, f, .edge s t => .edge (s.rename f) (t.rename f)
  | _, _, f, .both p q => .both (p.rename f) (q.rename f)
  | _, _, f, .either p q => .either (p.rename f) (q.rename f)
  | _, _, f, .impl p q => .impl (p.rename f) (q.rename f)
  | _, _, f, .someState p => .someState (p.rename (under f))
  | _, _, f, .everyState p => .everyState (p.rename (under f))

@[simp] theorem rename_term {S : Type u} {n m : Nat} (f : Fin n -> Fin m)
    (rho : Fin m -> S) (t : Term S n) :
    (t.rename f).eval rho = t.eval (rho  o  f) := by
  cases t <;> rfl

theorem extend_under {S : Type u} {n m : Nat} (f : Fin n -> Fin m)
    (rho : Fin m -> S) (x : S) : extend x rho  o  under f = extend x (rho  o  f) := by
  funext i
  refine Fin.cases ?_ (fun j => ?_) i
  . simp [extend, under, Function.comp_def]
  . simp [extend, under, Function.comp_def]

theorem rename_denote {S : Type u} {k n : Nat} (M : Model S k)
    (p : Formula S k n) {m : Nat} (f : Fin n -> Fin m) (rho : Fin m -> S) :
    Denote M (p.rename f) rho <-> Denote M p (rho  o  f) := by
  induction p generalizing m with
  | top => rfl
  | bot => rfl
  | atom => simp [Formula.rename, Denote]
  | equal => simp [Formula.rename, Denote]
  | edge => simp [Formula.rename, Denote]
  | both _ _ hp hq => simp [Formula.rename, Denote, hp, hq]
  | either _ _ hp hq => simp [Formula.rename, Denote, hp, hq]
  | impl _ _ hp hq => simp [Formula.rename, Denote, hp, hq]
  | someState _ hp => simp [Formula.rename, Denote, hp, extend_under]
  | everyState _ hp => simp [Formula.rename, Denote, hp, extend_under]

theorem rename_eval {S : Type u} [DecidableEq S] {k n m : Nat} (M : Model S k)
    (p : Formula S k n) (f : Fin n -> Fin m) (rho : Fin m -> S) :
    eval M (p.rename f) rho = eval M p (rho  o  f) :=
  Bool.eq_iff_iff.mpr ((reflects _ _ _).trans ((rename_denote M p f rho).trans (reflects _ _ _).symm))

end Cegar.Logic
```

36.4 Layered certificate checker

lean/Cegar/Layered.lean

```
import Cegar.Core

namespace Cegar.Layered

universe u

inductive Layout (S : Type u) where
  | first (claim : S -> Bool)
```

```
  | next (previous : Layout S) (edge : S -> S -> Bool)
      (allowed : S -> Bool) (claim : S -> Bool)

def Layout.claim {S : Type u} : Layout S -> S -> Bool
  | .first p => p
  | .next _ _ _ p => p

inductive Travel {S : Type u} (initial : S -> Bool) :
    Layout S -> S -> Type u
  | start (p : S -> Bool) (x : S) :
      initial x = true -> Travel initial (.first p) x
  | step {previous : Layout S} {x : S}
      (earlier : Travel initial previous x) (edge : S
    -> S -> Bool)
      (allowed claim : S -> Bool) (y : S) :
      edge x y = true -> allowed y = true ->
      Travel initial (.next previous edge allowed
    claim) y

def Conditions {S : Type u} (initial : S -> Bool) :
    Layout S -> Prop
  | .first p => forall  x, initial x = true -> p x =
    true
  | .next previous edge allowed p =>
    Conditions initial previous /\
      forall  x y, previous.claim x = true -> edge x
    y = true ->
        allowed y = true -> p y = true

def checkConditions {S : Type u} (enum : Enumeration
    S)
    (initial : S -> Bool) : Layout S -> Bool
  | .first p => enum.items.all fun x => !(initial x)
    || p x
  | .next previous edge allowed p =>
    checkConditions enum initial previous &&
      enum.items.all (fun x => enum.items.all fun y =>
        !(previous.claim x && edge x y && allowed y)
    || p y)

private theorem impliesBool (p q : Bool) : ((!p || q)
    = true) <-> (p = true -> q = true) := by
  cases p <;> simp

theorem conditions_reflect {S : Type u} (enum :
    Enumeration S)
    (initial : S -> Bool) (layout : Layout S) :
    checkConditions enum initial layout = true <->
    Conditions initial layout := by
  induction layout with
  | first p => simp only [checkConditions,
    Conditions, List.all_eq_true, enum.complete,
    true_implies,
      impliesBool]
  | next previous edge allowed p ih =>
    simp only [checkConditions, Conditions,
    Bool.and_eq_true, List.all_eq_true,
      enum.complete, true_implies, impliesBool,
    and_imp, ih]

def check {S : Type u} (enum : Enumeration S)
    (initial bad : S -> Bool)
    (layout : Layout S) : Bool :=
  checkConditions enum initial layout &&
    enum.items.all (fun x => !(layout.claim x && bad
    x))

private theorem incompatible (a b : Bool) :
    (Bool.not (a && b) = true) <-> (a = true -> b !=
    true) := by
  cases a <;> cases b <;> decide

theorem check_reflect {S : Type u} (enum :
    Enumeration S)
    (initial bad : S -> Bool) (layout : Layout S) :
    check enum initial bad layout = true <->
      Conditions initial layout /\ forall  x,
    layout.claim x = true -> bad x != true := by
  simp only [check, Bool.and_eq_true,
    conditions_reflect, List.all_eq_true,
    enum.complete, true_implies]
  constructor
  . rintro <h, final>
    exact <h, fun x => (incompatible _ _).mp (final
    x)>
  . rintro <h, final>
    exact <h, fun x => (incompatible _ _).mpr (final
    x)>

theorem invariant {S : Type u} {initial : S -> Bool}
    {layout : Layout S}
    {x : S} (p : Travel initial layout x) (conditions
    : Conditions initial layout) :
    layout.claim x = true := by
  induction p with
  | start p x hi => exact conditions x hi
  | step earlier edge allowed claim y he ha ih =>
    exact conditions.2 _ y (ih conditions.1) he ha

theorem check_sound {S : Type u} (enum : Enumeration
    S)
    {initial bad : S -> Bool} {layout : Layout S}
    (verified : check enum initial bad layout = true)
    :
    (Sigma  x, { _p : Travel initial layout x // bad
    x = true }) -> False := by
  intro <x, p, hb>
  obtain <hc, hf> := (check_reflect enum initial bad
    layout).mp verified
  exact hf x (invariant p hc) hb

end Cegar.Layered
```

36.5 Executable input and output

lean/Main.lean

```
import Lean
import Cegar

open Lean
open Cegar

structure Input where
  states : Nat
  keys : Nat
  initial : List Nat
  bad : List Nat
  edges : List (Nat * Nat)
  seed : Array Nat
  deriving FromJson

structure Output where
  badWalk : Option (List Nat)
  rounds : Nat
  deriving ToJson

def verify (input : Input) : Except String Output :=
    do
  if positive : 0 < input.keys then
```

```
    if input.seed.size != input.states ||
     !(input.seed.all (. < input.keys)) then
      throw "seed must give one valid key for each
     concrete state"
    if !(input.initial.all (. < input.states)) ||
     !(input.bad.all (. < input.states)) ||
        !(input.edges.all (fun (x, y) => x <
     input.states && y < input.states)) then
      throw "boundary and edge indices must belong to
     the concrete carrier"
    let graph : Graph (Fin input.states) :=
      <fun x => input.initial.contains x.val,
       fun x => input.bad.contains x.val,
       fun x y => input.edges.contains (x.val, y.val)>
    let seed (x : Fin input.states) : Fin input.keys
     :=
      <input.seed[x.val]?.getD 0 % input.keys,
     Nat.mod_lt _ positive>
    let result := run graph (finEnumeration
     input.states) (finEnumeration input.keys) seed
    pure <(verdictCode result.verdict).map (..map
     Fin.val), result.rounds>
  else
    throw "keys must be positive; one unused key
     suffices for the empty carrier"

def main (args : List String) : IO UInt32 := do
  let [file] := args | IO.eprintln "usage: cegar_cli
     <batch.json>"; return 2
  let raw <- IO.FS.readFile file
  let result := do
    let json <- Json.parse raw
    let inputs <- fromJson? (alpha := List Input) json
    inputs.mapM verify
  match result with
  | .ok rows => IO.println (toJson rows).compress;
     return 0
  | .error error => IO.eprintln error; return 1
```

36.6 Lean 4 dependency audit

lean/Audit.lean

```
import Cegar

#print axioms Cegar.solve_safe_iff
#print axioms Cegar.safe_of_may_safe
#print axioms Cegar.Lift.nodes
#print axioms Cegar.lift_of_nodes
#print axioms Cegar.tabulate
#print axioms Cegar.checkBad
#print axioms Cegar.cut_refines
#print axioms Cegar.cut_shorter
#print axioms Cegar.cut_eliminates
#print axioms Cegar.run
#print axioms Cegar.run_safe_iff
#print axioms Cegar.Logic.reflects
#print axioms Cegar.Logic.rename_denote
#print axioms Cegar.Layered.conditions_reflect
#print axioms Cegar.Layered.check_reflect
#print axioms Cegar.Layered.check_sound
```

36.7 Proof-relevant residuals

hott/Residual.hlean

```
import types.fiber hit.trunc homotopy.circle

open eq eq.ops sigma sigma.ops equiv trunc
    trunc_index is_trunc
open function unit

namespace cegar_hott

definition Residual {X A : Type} (p : X -> A) (a : A)
    : Type :=
  fiber p a

definition Support {X A : Type} (p : X -> A) (a : A)
    : Type :=
  trunc -1 (Residual p a)

definition lift_support {X A : Type} {p : X -> A} {a
    : A} :
  Residual p a -> Support p a :=
  tr

definition refute_support {X A : Type} {p : X -> A}
    {a : A}
  (h : Residual p a -> empty) : Support p a -> empty
    :=
  @trunc.elim -1 (Residual p a) empty is_prop_empty h

definition refute_residual {X A : Type} {p : X -> A}
    {a : A}
  (h : Support p a -> empty) : Residual p a -> empty
    :=
  fun  r, h (lift_support r)

definition reindex {A U : Type} (b : U -> A) (P : A
    -> Type) (u : U) : Type :=
  P (b u)

definition pullback_total {A U : Type} (b : U -> A)
    (P : A -> Type) : Type :=
  Sigma  u : U, reindex b P u

definition projection {A U : Type} {b : U -> A} {P :
    A -> Type} :
  pullback_total b P -> U := pr1

definition reindex_fiber {A U : Type} (b : U -> A) (P
    : A -> Type) (u : U) :
  fiber (@projection A U b P) u ~= P (b u) :=
  fiber.fiber_pr1 (reindex b P) u

definition reindex_support {X A U : Type} (p : X ->
    A) (b : U -> A) (u : U) :
  trunc -1 (fiber (@projection A U b (Residual p)) u)
    ~= Support p (b u) :=
  trunc_equiv_trunc (-1) (reindex_fiber b (Residual
    p) u)

definition residual_compose {X A B : Type} (p : X ->
    A) (r : A -> B) (b : B) :
  Residual (r  o  p) b ~= (Sigma  a : Residual r b,
    Residual p (fiber.point a)) :=
  fiber.fiber_compose_equiv r p b

definition compose_point {X A B : Type} {p : X -> A}
    {r : A -> B} {b : B}
  (a : Residual r b) (x : Residual p (fiber.point a))
    : Residual (r  o  p) b :=
  fiber.mk (fiber.point x) (ap r (fiber.point_eq x)
    @  fiber.point_eq a)

definition transport_reindex {A U : Type} (b : U ->
    A) (P : A -> Type)
  {u v : U} (e : u = v) (x : P (b u)) :
```

```
  transport (reindex b P) e x = transport P (ap b e)
    x :=
  by induction e; reflexivity

definition residual_identity {X A : Type} {p : X ->
    A} {a : A}
  (x y : Residual p a) :
  (x = y) ~=
  (Sigma  e : fiber.point x = fiber.point y,
    fiber.point_eq x = ap p e  @  fiber.point_eq y) :=
  fiber.fiber_eq_equiv x y

open circle

definition constant_circle : unit -> S1 := fun  x,
    base
definition reflexive_lift : Residual constant_circle
    base := fiber.mk star idp
definition winding_lift : Residual constant_circle
    base := fiber.mk star loop

definition lifts_are_distinct : winding_lift !=
    reflexive_lift :=
  fun  h, loop_neq_idp (ap (@fiber.point_eq unit S1
    constant_circle base) h)

definition circle_support : Support constant_circle
    base :=
  lift_support winding_lift

end cegar_hott
```

36.8 Native HoTT logical reflection

hott/Logic.hlean

```
import types.bool types.prod hit.trunc

open eq eq.ops is_trunc trunc unit prod prod.ops sum
    bool

namespace cegar_hott

inductive Formula (A : Type) : Type :=
| top : Formula A
| bot : Formula A
| atom : A -> Formula A
| both : Formula A -> Formula A -> Formula A
| either : Formula A -> Formula A -> Formula A
| impl : Formula A -> Formula A -> Formula A

open Formula

definition Holds {A : Type} (v : A -> bool) (f :
    Formula A) : Type0 :=
  @Formula.rec_on A (fun  f, Type0) f unit empty (fun
     a, v a = tt)
    (fun  p q P Q, P * Q)
    (fun  p q P Q, trunc -1 (P + Q))
    (fun  p q P Q, P -> Q)

definition test {A : Type} (v : A -> bool) (f :
    Formula A) : bool :=
  Formula.rec_on f tt ff v
    (fun  p q b c, band b c)
    (fun  p q b c, bor b c)
    (fun  p q b c, bor (bnot b) c)

structure Reflection (b : bool) (P : Type) :=
  (decode : b = tt -> P)
  (encode : P -> b = tt)

open Reflection

definition ref_both {b c : bool} {P Q : Type} (r :
    Reflection b P)
  (s : Reflection c Q) : Reflection (band b c) (P *
    Q) :=
  Reflection.mk
    (fun  h, (decode r (band_elim_left h), decode s
     (band_elim_right h)))
    (fun  h, band_intro (encode r h.1) (encode s h.2))

definition ref_either {b c : bool} {P Q : Type} (r :
    Reflection b P)
  (s : Reflection c Q) : Reflection (bor b c) (trunc
    -1 (P + Q)) :=
  Reflection.mk
    (fun  h, sum.elim (or_of_bor_eq h)
      (fun  p, tr (sum.inl (decode r p)))
      (fun  q, tr (sum.inr (decode s q))))
    (trunc.elim (fun  z, sum.elim z
      (fun  p, bor_inl (encode r p))
      (fun  q, bor_inr (encode s q))))

definition ref_impl {b c : bool} {P Q : Type} (r :
    Reflection b P)
  (s : Reflection c Q) : Reflection (bor (bnot b) c)
    (P -> Q) :=
begin
  induction b,
  { exact Reflection.mk (fun  h p, empty.elim
    (ff_ne_tt (encode r p))) (fun  f, idp) },
  { exact Reflection.mk (fun  h p, decode s ((ff_bor
    c)-1  @  h))
      (fun  f, ff_bor c  @  encode s (f (decode r
    idp))) }
end

definition reflect {A : Type} (v : A -> bool) (f :
    Formula A) :
  Reflection (test v f) (Holds v f) :=
  Formula.rec_on f
    (Reflection.mk (fun  h, star) (fun  h, idp))
    (Reflection.mk ff_ne_tt (fun  e, empty.elim e))
    (fun  a, Reflection.mk (fun  h, h) (fun  h, h))
    (fun  p q r s, ref_both r s)
    (fun  p q r s, ref_either r s)
    (fun  p q r s, ref_impl r s)

definition rename {A B : Type} (f : A -> B) (p :
    Formula A) : Formula B :=
  Formula.rec_on p (@top B) (@bot B) (fun  a, atom (f
    a))
    (fun  p q r s, both r s)
    (fun  p q r s, either r s)
    (fun  p q r s, impl r s)

definition check_rename {A B : Type} (f : A -> B) (v
    : B -> bool)
  (p : Formula A) : test v (rename f p) = test (fun
    a, v (f a)) p :=
begin
  induction p with a p q hp hq p q hp hq p q hp hq,
  { reflexivity },
  { reflexivity },
  { reflexivity },
  { exact ap011 band hp hq },
  { exact ap011 bor hp hq },
  { exact ap011 (fun  x y, bor (bnot x) y) hp hq }
end
```

```

definition positive_case : Holds (fun  a : bool, a)
      (impl (both (atom tt) (@top bool)) (atom tt)) :=
  decode (reflect (fun  a : bool, a) (impl (both
     (atom tt) (@top bool)) (atom tt))) idp

definition negative_case : Holds (fun  a : bool, a)
      (both (atom tt) (atom ff)) -> empty :=
  fun  p, ff_ne_tt (encode (reflect (fun  a : bool,
     a) (both (atom tt) (atom ff))) p)

end cegar_hott
```

36.9 Native HoTT finite lift checker

hott/FiniteLift.hlean

```
import .Logic .Residual

open eq eq.ops unit bool sum sigma sigma.ops prod
     prod.ops trunc trunc_index

namespace cegar_hott

structure BoolGraph :=
  (initial : bool -> bool)
  (edge : bool -> bool -> bool)
  (bad : bool -> bool)

inductive KeyPath : Type0 :=
| start : bool -> KeyPath
| next : bool -> KeyPath -> KeyPath

definition equal_bool (x y : bool) : bool :=
  bool.rec_on x (bnot y) y

definition eq_reflect (x y : bool) : Reflection
     (equal_bool x y) (x = y) :=
begin
  cases x,
  { cases y,
    { exact Reflection.mk (fun  h, idp) (fun  h, idp)
     },
    { exact Reflection.mk (fun  h, h) (fun  h, h) } },
  { cases y,
    { exact Reflection.mk (fun  h, h-1) (fun  h, h-1)
     },
    { exact Reflection.mk (fun  h, idp) (fun  h, idp)
     } }
end

definition ref_exists_bool {P : bool -> Type0} {f :
     bool -> bool}
  (rf : Reflection (f ff) (P ff)) (rt : Reflection (f
     tt) (P tt)) :
  Reflection (bor (f ff) (f tt)) (Sigma  x : bool, P
     x) :=
  Reflection.mk
    (fun  h, sum.elim (or_of_bor_eq h)
      (fun  p, <ff, Reflection.decode rf p>)
      (fun  p, <tt, Reflection.decode rt p>))
    (fun  w, (bool.rec_on w.1
      (fun  p, bor_inl (Reflection.encode rf p))
      (fun  p, bor_inr (Reflection.encode rt p))) w.2)

definition EndLift (g : BoolGraph) (q : bool -> bool)
     (w : KeyPath) :
  bool -> Type0 :=
  KeyPath.rec_on w
    (fun  a x, (q x = a) * (BoolGraph.initial g x =
     tt))
    (fun  a p L y, (q y = a) *
      (Sigma  x : bool, L x * (BoolGraph.edge g x y =
     tt)))

definition EndTest (g : BoolGraph) (q : bool -> bool)
     (w : KeyPath) :
  bool -> bool :=
  KeyPath.rec_on w
    (fun  a x, band (equal_bool (q x) a)
     (BoolGraph.initial g x))
    (fun  a p L y, band (equal_bool (q y) a)
      (bor (band (L ff) (BoolGraph.edge g ff y))
           (band (L tt) (BoolGraph.edge g tt y))))

definition ref_atom (b : bool) : Reflection b (b =
     tt) :=
  Reflection.mk (fun  h, h) (fun  h, h)

definition end_reflect (g : BoolGraph) (q : bool ->
     bool) (w : KeyPath) :
  Pi  x : bool, Reflection (EndTest g q w x) (EndLift
     g q w x) :=
  KeyPath.rec_on w
    (fun  a x, ref_both (eq_reflect (q x) a)
     (ref_atom (BoolGraph.initial g x)))
    (fun  a p R y, ref_both (eq_reflect (q y) a)
      (ref_exists_bool
        (ref_both (R ff) (ref_atom (BoolGraph.edge g
     ff y)))
        (ref_both (R tt) (ref_atom (BoolGraph.edge g
     tt y)))))

definition BadLift (g : BoolGraph) (q : bool -> bool)
     (w : KeyPath) : Type0 :=
  Sigma  x : bool, EndLift g q w x * (BoolGraph.bad g
     x = tt)

definition BadTest (g : BoolGraph) (q : bool -> bool)
     (w : KeyPath) : bool :=
  bor (band (EndTest g q w ff) (BoolGraph.bad g ff))
      (band (EndTest g q w tt) (BoolGraph.bad g tt))

definition bad_reflect (g : BoolGraph) (q : bool ->
     bool) (w : KeyPath) :
  Reflection (BadTest g q w) (BadLift g q w) :=
  ref_exists_bool
    (ref_both (end_reflect g q w ff) (ref_atom
     (BoolGraph.bad g ff)))
    (ref_both (end_reflect g q w tt) (ref_atom
     (BoolGraph.bad g tt)))

definition lift_from_test {g : BoolGraph} {q : bool
     -> bool} {w : KeyPath}
  (h : BadTest g q w = tt) : BadLift g q w :=
  Reflection.decode (bad_reflect g q w) h

definition no_lift_from_test {g : BoolGraph} {q :
     bool -> bool} {w : KeyPath}
  (h : BadTest g q w = ff) : BadLift g q w -> empty :=
  fun  z, ff_ne_tt (h-1  @  Reflection.encode
     (bad_reflect g q w) z)

definition support_from_test {g : BoolGraph} {q :
     bool -> bool} {w : KeyPath}
  (h : BadTest g q w = tt) : trunc -1 (BadLift g q w)
     :=
  tr (lift_from_test h)

definition spurious_graph : BoolGraph :=
```

```
  BoolGraph.mk bnot equal_bool (fun  x, x)

definition merge_keys : bool -> bool := fun  x, ff

definition spurious_path : KeyPath := KeyPath.start ff

definition spurious_start_may :
  (Sigma  x : bool, (merge_keys x = ff) *
     (BoolGraph.initial spurious_graph x = tt)) :=
  <ff, (idp, idp)>

definition spurious_bad_may :
  (Sigma  x : bool, (merge_keys x = ff) *
     (BoolGraph.bad spurious_graph x = tt)) :=
  <tt, (idp, idp)>

definition spurious_path_refuted :
  BadLift spurious_graph merge_keys spurious_path ->
     empty :=
  no_lift_from_test idp

definition reachable_graph : BoolGraph :=
  BoolGraph.mk bnot (fun  x y, band (bnot x) y) (fun
     x, x)

definition reachable_path : KeyPath := KeyPath.next
     ff spurious_path

definition reachable_path_lifts :
  BadLift reachable_graph merge_keys reachable_path :=
  lift_from_test idp

end cegar_hott
```

References

[1] B. Alpern and F. B. Schneider. Defining liveness. *Information Processing Letters*, 21(4), 181–185, 1985. Cornell repository.

[2] C. Baier and J.-P. Katoen. *Principles of Model Checking*. MIT Press, 2008. Publisher.

[3] C. Barrett, R. Sebastiani, S. A. Seshia, and C. Tinelli. Satisfiability modulo theories. In *Handbook of Satisfiability*, 825–885, IOS Press, 2009. Author copy.

[4] B. van den Berg and R. Garner. Types are weak ω-groupoids. *Proceedings of the London Mathematical Society*, 102(2), 370–394, 2011. arXiv.

[5] A. R. Bradley. SAT-based model checking without unrolling. In *VMCAI*, LNCS 6538, 70–87, 2011. Author copy.

[6] E. M. Clarke, O. Grumberg, S. Jha, Y. Lu, and H. Veith. Counterexample-guided abstraction refinement. In *CAV*, LNCS 1855, 154–169, 2000. Bibliographic record.

[7] E. M. Clarke, O. Grumberg, S. Jha, Y. Lu, and H. Veith. Counterexample-guided abstraction refinement for symbolic model checking. *Journal of the ACM*, 50(5), 752–794, 2003. Author copy.

[8] P. Cousot and R. Cousot. Abstract interpretation: a unified lattice model for static analysis of programs by construction or approximation of fixpoints. In *POPL*, 238–252, 1977. Author copy.

[9] W. Craig. Three uses of the Herbrand–Gentzen theorem in relating model theory and proof theory. *Journal of Symbolic Logic*, 22(3), 269–285, 1957. DOI.

[10] C. Cohen, T. Coquand, S. Huber, and A. Mörtberg. Cubical type theory: a constructive interpretation of the univalence axiom. In *TYPES 2015*, LIPIcs 69, 5:1–5:34, 2018. arXiv.

[11] R. Giacobazzi, F. Ranzato, and F. Scozzari. Making abstract interpretations complete. *Journal of the ACM*, 47(2), 361–416, 2000. Author page.

[12] S. Graf and H. Saïdi. Construction of abstract state graphs with PVS. In *CAV*, LNCS 1254, 72–83, 1997. Author copy.

[13] R. Goré and C. Kikkert. CEGAR-tableaux: improved modal satisfiability via modal clause-learning and SAT. In *TABLEAUX*, LNAI 12842, 74–91, 2021. Paper.

[14] T. A. Henzinger, R. Jhala, R. Majumdar, and G. Sutre. Lazy abstraction. In *POPL*, 58–70, 2002. Institutional record.

[15] M. Hofmann and T. Streicher. The groupoid interpretation of type theory. In *Twenty-Five Years of Constructive Type Theory*, 83–111, Oxford University Press, 1998. Bibliographic record.

[16] The Univalent Foundations Program. *Homotopy Type Theory: Univalent Foundations of Mathematics*. Institute for Advanced Study, 2013. Open book.

[17] M. Huth, R. Jagadeesan, and D. A. Schmidt. Modal transition systems: a foundation for three-valued program analysis. In *ESOP*, LNCS 2028, 155–169, 2001. Bibliographic record.

[18] B. Jacobs. *Categorical Logic and Type Theory*. Studies in Logic and the Foundations of Mathematics 141, Elsevier, 1999. Author page.

[19] K. Kapulkin and P. LeFanu Lumsdaine. The simplicial model of univalent foundations (after Voevodsky). *Journal of the European Mathematical Society*, 23(6), 2071–2126, 2021. arXiv.

[20] K. G. Larsen and B. Thomsen. A modal process logic. In *LICS*, 203–210, 1988. Bibliographic record.

[21] J.-M. Lagniez, D. Le Berre, T. de Lima, and V. Montmirail. A recursive shortcut for CEGAR: application to the modal logic K satisfiability problem. In *IJCAI*, 674–680, 2017. Proceedings.

[22] F. W. Lawvere. Adjointness in foundations. *Dialectica*, 23(3–4), 281–296, 1969. Author-commented reprint.

[23] The Lean developers. Lean 2 archive and HoTT-mode library. Pinned source.

[24] The Lean developers. Lean 4.19.0. Versioned release.

[25] P. LeFanu Lumsdaine and M. A. Warren. The local universes model: an overlooked coherence construction for dependent type theories. *ACM Transactions on Computational Logic*, 16(3), article 23, 2015. arXiv.

[26] J. Lurie. *Higher Topos Theory*. Annals of Mathematics Studies 170, Princeton University Press, 2009. arXiv.

[27] P. Martin-Löf. *Intuitionistic Type Theory*. Bibliopolis, 1984. Open scan.

[28] K. L. McMillan. Interpolation and SAT-based model checking. In *CAV*, LNCS 2725, 1–13, 2003. Paper.

[29] K. L. McMillan. Lazy abstraction with interpolants. In *CAV*, LNCS 4144, 123–136, 2006. Paper.

[30] F. Ranzato and F. Tapparo. Generalized strong preservation by abstract interpretation. *Journal of Logic and Computation*, 17(1), 157–197, 2007. arXiv.

[31] C. Rezk. A model for the homotopy theory of homotopy theory. *Transactions of the American Mathematical Society*, 353(3), 973–1007, 2001. arXiv.

[32] E. Riehl and D. Verity. *Elements of ∞-Category Theory*. Cambridge University Press, 2022. Publisher.

[33] E. Rijke, M. Shulman, and B. Spitters. Modalities in homotopy type theory. *Logical Methods in Computer Science*, 16(1), 2020. arXiv.

[34] M. Shulman. All $(\infty, 1)$-toposes have strict univalent universes. 2019. arXiv.

[35] A. M. Turing. On computable numbers, with an application to the Entscheidungsproblem. *Proceedings of the London Mathematical Society*, s2-42(1), 230–265, 1937. Open scan.

[36] Wikipedia contributors. Counterexample-guided abstraction refinement. Accessed 11 November 2026. Overview.