Higher Fibrations, Adjunctions, and Indexed Algebra

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Abstract A cartesian fibration encodes a family of categories together with coherent substitution along maps of contexts. Adjunctions provide two kinds of additional structure on that family. Adjoints to substitution supply dependent sums, products, and quantifiers, while indexed free–forgetful adjunctions supply monads for algebraic operations. We formulate an indexed monadicity criterion in terms of mates and apply it to coherent groups in the slices of an arbitrary ∞-topos. The free group construction is $\Omega_\Gamma \Sigma_\Gamma (A_+)$, and its algebras assemble into a cartesian fibration over the category of contexts. A second application treats sheafification as an indexed idempotent monad. Examples compare word substitution, homotopy base change, group-valued local systems, and the Hopf fibration. In particular, the Hopf fibration has inhabited fibers but no global section, so dependent product and propositional truncation can give different results in opposite orders. We also distinguish cartesian fibrations, internal fibrations, and fibrations over $(\infty, 2)$-categories.

Keywords cartesian fibration, straightening, monadicity, Beck–Chevalley, categorical logic, ∞-topos

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1 Introduction

The free group on X is determined by what a map out of it must do to the generators. Pullback along a map of contexts is determined in much the same way. Given $f\colon \Delta \to \Gamma$ and a type over Γ, a map into the pulled-back type over Δ is determined by a map into the original type lying over f. The two universal properties have different consequences. In the first, a composite of adjoint functors is a monad that evaluates nested terms. In the second, coherent pullbacks give a fibration, and adjoints to the pullback functors bind variables. This article studies where these structures meet.

There is a historical reason to begin with these two universal properties. Kan's account of adjoint functors made natural correspondences of maps into a general method [10]. Lawvere used adjunctions both to characterize algebraic constructions and to organize quantification in indexed logic [13, 12]. Grothendieck's treatment of descent in terms of fibered categories provided the setting in which objects and maps could vary over a base [5]. An adjunction fixes the direction of a universal construction by identifying its mapping problem. A fibration then asks how these mapping problems vary together as the base changes.

The central application is to the slice family $\Gamma \mapsto \mathcal{X}_{/\Gamma}$ of an ∞-topos. Its reindexing is pullback. In each slice, the free coherent group on $A \to \Gamma$ is obtained by adjoining a disjoint base point, suspending, and looping. These operations commute with reindexing. Thus the free–forgetful adjunction, the induced word monad in its homotopy-coherent form, and the comparison to the category of algebras are all compatible with substitution. The criterion in Theorem 5.1 is a direct application of the calculus of mates and fiberwise monadicity. We develop it so that it also applies to an indexed sheaf reflection and makes its hypotheses explicit.

Several worked examples concern what can be recovered. The free group on the empty set has the identity as its only element. The free coherent group on a circle also has higher homotopy in its identity component. A circle bundle over a simply connected base can have nontrivial coherent transport and no section. Finally, the entire category of cartesian fibrations over a point fails the universality of colimits required of an ∞-topos. These examples distinguish algebra structures, transport, and the logical structure of a topos.

We use ∞-category to mean $(\infty, 1)$-category until Section 8, where noninvertible 2-morphisms enter explicitly. Mapping objects Map are spaces, and limits, colimits, and fibers are homotopy-coherent. Fix universes so that base categories and fibers are small relative to a larger working universe. Slice diagrams of a large ∞-topos are interpreted after enlarging that universe. The treatment uses ordinary mathematical arguments and established straightening, monadicity, and recognition theorems.

2 Free groups and the two composites of an adjunction

2.1 The empty alphabet and word evaluation

Let $F\colon \mathsf{Set} \to \mathsf{Grp}$ send a set X to the group of reduced words in letters x and formal inverses x^{-1}, $x \in X$. Multiplication concatenates and reduces. The empty word is the identity. The group $F(\varnothing)$ therefore has one element. There is no group with empty carrier, since its unit is a map from a singleton. In any group, e is the unique idempotent, as $g^2 = g$ gives $g = e$ upon cancellation.

A function $u\colon X \to Y$ acts on words by renaming letters and reducing. Functoriality also follows from the fact that a homomorphism out of $F(X)$ is determined by its values on X. If $U\colon \mathsf{Grp} \to \mathsf{Set}$ forgets the operations, evaluation gives the natural bijection

$$\mathrm{Hom}_{\mathsf{Grp}}(F(X), G) \;\cong\; \mathrm{Hom}_{\mathsf{Set}}(X, UG).$$

For a function $h\colon X \to UG$ the extension evaluates a word $x_1^{\epsilon_1} \cdots x_n^{\epsilon_n}$ as $h(x_1)^{\epsilon_1} \cdots h(x_n)^{\epsilon_n}$. Cancellations preserve this value. The functor F is the left adjoint. The unit $\eta_X\colon X \to UF(X)$ inserts letters, and the counit $\varepsilon_G\colon F(UG) \to G$ evaluates a formal word whose letters are group elements [16, Chs. IV and VI].

The monad $T = UF$ on Set has multiplication

$$\mu_X = U\varepsilon_{F(X)}\colon T^2X \longrightarrow TX.$$

A letter of an outer word is now an element of $F(X)$. Multiplication substitutes each inner word, using its inverse when the outer exponent is negative, and reduces. Naturality is compatibility with renaming generators. The associativity law expresses independence of the order in which two levels of substitution are flattened. The unit laws express substitution of the names of the original generators.

Example 2.1 (Substitution as Kleisli composition)**.** The category $\mathrm{Kl}(T)$ has sets as objects and arrows $X \rightsquigarrow Y$ given by functions $X \to TY$. For $\alpha\colon X \to TY$ and $\beta\colon Y \to TZ$, their composite is $\mu_Z T(\beta)\alpha$. For example, send s to $xy^{-1}x$, then send x to ab and y to b. The composite sends s to

$$(ab)b^{-1}(ab) = a^2b.$$

A Kleisli arrow is thus an assignment of a word to every generator. Its extension is a homomorphism between free groups, which identifies $\mathrm{Kl}(T)$ with the full subcategory of Grp on the free groups (up to equivalence).

2.2 Algebras, comonads, and comparison

An Eilenberg–Moore algebra is a set A with a map $a\colon TA \to A$ such that $a\eta_A = \mathrm{id}_A$ and $aTa = a\mu_A$. Define multiplication, inversion, and unit by evaluating the words xy, x^{-1}, and the empty word. For instance, evaluating $(xy)z$ either as a word in three letters or first evaluating xy gives $a(xyz) = a(xy) \cdot z$. Group reduction and the algebra laws give the group axioms, and the two constructions, forming the evaluation map and extracting the group operations, are inverse. Maps of algebras are precisely the homomorphisms. In the language of an adjunction $F \dashv U$, this says that the comparison

$$\mathsf{Grp} \longrightarrow \mathrm{Alg}_T(\mathsf{Set}), \qquad G \longmapsto (UG, U\varepsilon_G)$$

is an equivalence.

The other composite $Q = FU$ is a comonad on Grp, with counit ε and comultiplication $F\eta U$. An arrow $G \to H$ of $\mathrm{CoKl}(Q)$ is a homomorphism $QG \to H$. By adjunction this is an arbitrary function $UG \to UH$. Under this correspondence the coKleisli composite is ordinary composition of functions. Indeed $F\eta_{UG}$ sends an outer name for g to a name for that one-letter word, so applying the two homomorphisms in succession evaluates first at g, then at its image. The Kleisli construction and the dual coKleisli construction use opposite composites of the original adjunction [11, 3, 16].

Every adjunction induces a monad and a comonad. Every monad has an Eilenberg–Moore adjunction in the usual categorical setting. The assertion that a particular right adjoint U is *monadic* says that the resulting comparison is an equivalence. The forgetful functor $\mathsf{Top} \to \mathsf{Set}$, with the discrete-space left adjoint, induces the identity monad. Its comparison is the forgetful functor itself. The continuous bijection from an infinite discrete space to the same set with the indiscrete topology is sent to an isomorphism, but is itself noninvertible. The comparison therefore cannot be an equivalence. The failure is loss of topology under the forgetful functor.

2.3 A formula for syntax and semantics

For the ordinary Lawvere theory of groups $\mathbb{T}_{\mathsf{Grp}}$, the object n is the n-fold product of the generating object 1. A map $n \to m$ is an m-tuple of group terms in n variables modulo the group equations. Explicitly, these terms can be represented by reduced words. Composition is substitution. Product-preserving functors $\mathbb{T}_{\mathsf{Grp}} \to \mathsf{Set}$ are group models, and evaluation at 1 is their underlying-set functor [13]. With finite sets as arities, a map $n \to m$ in the theory assigns m output terms to n inputs, while a Kleisli map $m \rightsquigarrow n$ assigns a word in n variables to each of m generators. Thus the theory is the opposite of the finite-arity part of the Kleisli category, up to choice of skeleton. This gives a precise syntax–semantics passage whose adjunction is monadic. A logic with predicates, variable contexts, and quantification will require further data, described in Section 4.

3 Fibrations and their indexed families

3.1 The Grothendieck construction

In ordinary category theory a contravariant pseudofunctor $C\colon B^{\mathrm{op}} \to \mathsf{Cat}$ specifies categories C_b and reindexing functors $f^*_C\colon C_c \to C_b$ for $f\colon b \to c$. It also specifies coherent isomorphisms $(gf)^*_C \cong f^*_C g^*_C$ and $(\mathrm{id}_b)^*_C \cong \mathrm{id}_{C_b}$. Its Grothendieck construction $\int_B C$ has objects (b, x) with

$x \in C_b$. A morphism $(b, x) \to (c, y)$ consists of a base arrow $f\colon b \to c$ and an arrow $x \to f_C^* y$ in C_b. If the next morphism lies over $g\colon c \to d$ and is represented by $y \to g_C^* z$, composition is

$$x \longrightarrow f_C^* y \longrightarrow f_C^* g_C^* z \xrightarrow{\cong} (gf)_C^* z.$$

The coherence of the pseudofunctor gives associativity. The arrow $(b, f_C^* y) \to (c, y)$ corresponding to the identity of $f_C^* y$ is cartesian. Every arrow over f factors through it by a unique arrow in the fiber. Conversely, in a Grothendieck fibration one may choose such arrows over every base map and recover a pseudofunctor. The canonical isomorphisms between composites of chosen lifts come from their common universal property [5, 9].

For example, the fiber of the codomain functor $\mathsf{Set}^{[1]} \to \mathsf{Set}$ over a set I is the category of families $I \ni i \mapsto A_i$. The object in the total category is $\coprod_{i \in I} A_i \to I$. A morphism over $f\colon I \to J$ consists of maps $A_i \to B_{f(i)}$, and its cartesian lift at B uses the family $B_{f(i)}$ itself. For this reason change of context is contravariant even though the arrow in the total category points over f.

3.2 Cartesian lifts and straightening

Let $p\colon E \to B$ be an inner fibration of quasicategories, a simplicial presentation of an ∞-categorical functor. An arrow $u\colon e' \to e$ over $f\colon b' \to b$ is p-cartesian if for every $z \in E$ the square of spaces

$$\begin{array}{ccc} \mathrm{Map}_E(z, e') & \xrightarrow{u \circ (-)} & \mathrm{Map}_E(z, e) \\ {\scriptstyle p}\downarrow & & \downarrow{\scriptstyle p} \\ \mathrm{Map}_B(pz, b') & \xrightarrow[f \circ (-)]{} & \mathrm{Map}_B(pz, b) \end{array} \tag{1}$$

is a homotopy pullback. A cartesian fibration has such a lift ending at each e over every arrow into $p(e)$. A cocartesian fibration is defined dually by universal outgoing lifts. The cartesian lift of $y \in E_c$ over $f\colon b \to c$ gives an equivalence

$$\mathrm{Map}_E(x, y)_f \simeq \mathrm{Map}_{E_b}(x, f^* y), \qquad x \in E_b, \tag{2}$$

where the left side is the homotopy fiber of the mapping-space projection over f. A universal lift is unique through a contractible space of choices in the corresponding lifting problem. The higher simplices encode compatibility among lifts, their composites, and homotopies of arrows in the base.

Straightening and unstraightening give equivalences natural in B

$$\begin{aligned} \mathrm{CartFib}(B) &\simeq \mathrm{Fun}(B^{\mathrm{op}}, \mathsf{Cat}_\infty), & \mathrm{CocartFib}(B) &\simeq \mathrm{Fun}(B, \mathsf{Cat}_\infty), \\ \mathrm{RFib}(B) &\simeq \mathrm{Fun}(B^{\mathrm{op}}, \mathcal{S}), & \mathrm{LFib}(B) &\simeq \mathrm{Fun}(B, \mathcal{S}). \end{aligned} \tag{3}$$

Here morphisms in the first row preserve the indicated lifts. Right and left fibrations are the cartesian and cocartesian cases, respectively, with space-valued fibers [14, §§2.1, 2.4 and Thm. 3.2.0.1]. The cartesian equivalence takes p to the indexed family $b \mapsto E_b$, with the lifts supplying its contravariant transport. For $v\colon B' \to B$ and $C\colon B^{\mathrm{op}} \to \mathsf{Cat}_\infty$, naturality gives

$$\mathrm{Un}_{B'}(C \circ v^{\mathrm{op}}) \simeq B' \times_B \mathrm{Un}_B(C).$$

Thus the correspondence recovers the indexed family up to coherent equivalence, from the fibration together with its projection. It also gives a concrete form of Yoneda. The domain projection $B_{/b} \to B$ represents the right fibration $\mathrm{Map}_B(-, b)$. For a right fibration $R \to B$,

$$\mathrm{Map}_{\mathrm{RFib}(B)}(B_{/b}, R) \simeq R_b.$$

A map from that representable right fibration is specified by the point it chooses in the fiber over b, with the entire space of choices retained [19].

3.3 Maps over the base and their transformations

Suppose $C, D \colon B^{\mathrm{op}} \to \mathrm{Cat}_\infty$ classify cartesian fibrations and $H \colon \int_B C \to \int_B D$ is a functor over B. Restricting H gives $H_b \colon C_b \to D_b$. For $f \colon b \to c$, the image of the cartesian lift of $y \in C_c$ factors through the cartesian lift of $H_c(y)$. It induces a comparison

$$H_b f_C^* \longrightarrow f_D^* H_c. \tag{4}$$

These comparisons are natural in y and compatible with composition in B. They are equivalences exactly when H preserves cartesian lifts. This is the class of maps corresponding to natural transformations of the indexed families in (3). Choices of terminology for lax and oplax transformations can vary with variance conventions, so (4) specifies the direction.

For a concrete noninvertible comparison, take $B = [1]$ and both families constant at $\mathcal{S}$, so the total category is $\mathcal{S} \times [1]$. Define H to be the identity on the fiber over 0 and the constant terminal functor on the fiber over 1. On an arrow over $0 \to 1$, it uses the unique map from the source space to the terminal space. The comparison is $Y \to *$ for $Y \in \mathcal{S}$, which fails to be invertible for $Y = S^1$.

A transformation between two functors over B is vertical when its components project to identities. Between cartesian functors such a transformation induces a family of natural transformations on the fibers compatible with transport. The mapping *space* in $\mathrm{CartFib}(B)$ retains the invertible transformations between cartesian functors. To retain noninvertible vertical transformations as 2-morphisms, one uses mapping ∞-categories and works at the $(\infty, 2)$-categorical level. This distinction also matters when making adjunctions of indexed functors [8, 7].

4 Dependent operations in a higher topos

4.1 The slice fibration and its adjunctions

Let $\mathcal{X}$ be an ∞-topos. The codomain projection $\mathrm{Fun}([1], \mathcal{X}) \to \mathcal{X}$ is cartesian, with fiber $\mathcal{X}_{/\Gamma}$ at Γ. A cartesian lift over $f \colon \Delta \to \Gamma$ is a pullback square. Each slice is itself an ∞-topos. Types in context Γ are objects of the slice. A term of $A \to \Gamma$ is a section, and a term after substitution along $\sigma \colon \Delta \to \Gamma$ is a section of $\sigma^* A$. The natural equivalence

$$\mathrm{Map}_{\mathcal{X}_{/\Delta}}(\mathrm{id}_\Delta, \sigma^* A) \simeq \mathrm{Map}_{\mathcal{X}_{/\Gamma}}(\sigma, A)$$

is the semantic comprehension property. In particular $A \to \Gamma$ serves as the extended context. The diagonal $A \to A \times_\Gamma A$ interprets the family of identifications between the two generic terms, whose fibers are path spaces in the basic case $\mathcal{X} = \mathcal{S}$.

For every $f \colon \Delta \to \Gamma$ the substitution functor sits in a triple

$$\mathcal{X}_{/\Delta} \underset{\Pi_f}{\overset{\Sigma_f}{\overset{\longleftarrow}{\underset{\longrightarrow}{\overset{f^*}{\longleftarrow}}}}} \mathcal{X}_{/\Gamma} \qquad \Sigma_f \dashv f^* \dashv \Pi_f. \tag{5}$$

Here Σ_f and Π_f go from $\mathcal{X}_{/\Delta}$ to $\mathcal{X}_{/\Gamma}$. The first sends $A \to \Delta$ to its composite with f, and the pullback property gives its adjunction. The second exists by local cartesian closure. Equivalently, since colimits in an ∞-topos are universal, f^* preserves colimits, so presentability and the adjoint functor theorem give the right adjoint [14, §§5.5.2 and 6.1]. The Σ_f are cocartesian transport for the same codomain projection. A map over f out of A is a vertical map out of $\Sigma_f A$. The right adjoint Π_f represents families of sections along f.

In the slice of sets over a set Γ, the values at $\gamma \in \Gamma$ are

$$(\Sigma_f A)_\gamma = \coprod_{\delta \in f^{-1}(\gamma)} A_\delta, \qquad (\Pi_f A)_\gamma = \prod_{\delta \in f^{-1}(\gamma)} A_\delta.$$

In spaces, the corresponding index at γ is the homotopy fiber Δ_γ. The object $A|_{\Delta_\gamma} \to \Delta_\gamma$ classifies a functor $\Delta_\gamma \to \mathcal{S}$. Its colimit is the sum fiber and its limit is the product fiber. The limit takes compatible sections with all their coherence. The reason is that the slice of spaces over a space B identifies, by left straightening, with $\mathrm{Fun}(B, \mathcal{S})$. In this identification pulling back along f is restriction. Its adjoints are left and right Kan extension [14, §4.3]. At a base point the two comma categories for these Kan extensions are equivalent to Δ_γ, with a path and its inverse giving the two orientations.

4.2 Base change and the calculus of mates

Write a pullback square of contexts as

$$\begin{array}{ccc} \Delta' & \xrightarrow{u} & \Delta \\ {\scriptstyle f'}\downarrow & \lrcorner & \downarrow{\scriptstyle f} \\ \Gamma' & \xrightarrow[v]{} & \Gamma. \end{array} \tag{6}$$

Substitution gives $u^* f^* \simeq f'^* v^*$. The two Beck–Chevalley comparisons obtained by adjunction are equivalences

$$\Sigma_{f'} u^* \xrightarrow{\simeq} v^* \Sigma_f, \qquad v^* \Pi_f \xrightarrow{\simeq} \Pi_{f'} u^*. \tag{7}$$

For the sum, both sides applied to $A \to \Delta$ are the projection $A \times_\Gamma \Gamma' \to \Gamma'$. The displayed map is its canonical pullback identification. For the product, let $K \to \Gamma'$ be a test object. Apply the sum comparison for the same square with its roles interchanged and then take adjoints:

$$\begin{aligned} \mathrm{Map}_{\mathcal{X}_{/\Gamma'}}(K, v^* \Pi_f A) &\simeq \mathrm{Map}_{\mathcal{X}_{/\Gamma}}(\Sigma_v K, \Pi_f A) \\ &\simeq \mathrm{Map}_{\mathcal{X}_{/\Delta}}(f^* \Sigma_v K, A) \\ &\simeq \mathrm{Map}_{\mathcal{X}_{/\Delta}}(\Sigma_u f'^* K, A) \\ &\simeq \mathrm{Map}_{\mathcal{X}_{/\Delta'}}(f'^* K, u^* A) \\ &\simeq \mathrm{Map}_{\mathcal{X}_{/\Gamma'}}(K, \Pi_{f'} u^* A). \end{aligned}$$

The calculation identifies the canonical mate because each equivalence comes from the corresponding unit, counit, or pullback. These comparisons compose under pasting of pullback squares. For example, a span $\Gamma \xleftarrow{\ell} S \xrightarrow{r} \Lambda$ acts by $\Sigma_r \ell^*$. The composite for two spans is the operator for their pullback composite, using the sum comparison to move the middle pushforward past the middle pullback. Associativity follows from pasting. A map of spans $h \colon S \to S'$ over the

endpoints acts from the operator for S to that for S' by the counit $\Sigma_h h^* \to \mathrm{id}$. This expresses a part of the calculus of correspondences in the language of indexed categories [8, 4].

The existence of adjoints to transport does not itself imply Beck–Chevalley. Take a base which is the diamond poset with two incomparable middle objects. Interpret its pullback square as (6). Assign the terminal category to Γ, the category $\mathcal{S}$ to each other vertex, and reindex from Γ by selecting the terminal space. The other reindexing functors are identities. All reindexing functors have left adjoints and the substitution square commutes. Nevertheless the sum comparison at $A \in \mathcal{S}$ is $A \to *$. That comparison is invertible for the slice family because it compares two presentations of the same pullback.

Example 4.1 (A base point pulled back along itself)**.** Let $i \colon * \to S^1$ be a base point. Its pullback against itself has upper left corner $\Omega S^1 \simeq \mathbb{Z}$ as a space. Thus, for $A \in \mathcal{S}$, the two comparisons give

$$i^* \Sigma_i A \simeq \Omega S^1 \times A, \qquad i^* \Pi_i A \simeq \mathrm{Map}(\Omega S^1, A).$$

Taking $A = S^1$ gives the disjoint union of circles indexed by $\mathbb{Z}$ on the left, and a product of circles indexed by $\mathbb{Z}$ on the right. In the first calculation the n-th component records a loop of winding number n together with an element of A. In the second, a section chooses a value of A over every such component. Pullback in the homotopy theory of spaces therefore gives different answers from pullback of the underlying point sets.

4.3 Propositions and contexts

A proposition in context Γ is a subterminal object of $\mathcal{X}_{/\Gamma}$, equivalently a subobject of Γ. Write $\|A\|_{-1,\Gamma}$ for the subterminal reflection of $A \to \Gamma$, the image of that map in the effective-epimorphism–monomorphism factorization. The subobject classifier represents predicates as $\Gamma \to \Omega_{\mathcal{X}}$. For $P \in \mathrm{Sub}(\Delta)$, the adjunctions on propositions are

$$\exists_f P = \left\| \Sigma_f P \right\|_{-1,\Gamma}, \qquad \forall_f P = \Pi_f P, \qquad \exists_f \dashv f^* \dashv \forall_f.$$

The first formula follows by factoring a map to a subterminal object through the reflection. The second holds because Π_f preserves limits, hence terminal objects and the diagonal condition for a subterminal object. Images are stable under pullback, so (7) also gives the corresponding quantifier laws. Frobenius reciprocity is the equality

$$\exists_f (P \wedge f^* Q) \simeq (\exists_f P) \wedge Q.$$

Before truncation, the analogous formula uses fiber products and Σ_f. Both sides are $P \times_\Gamma Q$ over Γ. Taking images then gives the propositional formula. This is the higher slice version of Lawvere's quantification-as-adjunction, and its underlying ordered semantics is a hyperdoctrine of predicates [12, 9].

The subobjects of each context form a frame. Arbitrary joins are images of coproducts, and pullback stability of images and colimits implies that binary meets distribute over them. This gives the full Heyting implication by right adjoint to meet. In $\mathrm{Shv}(\mathbb{R})$, the ∞-topos of space-valued sheaves on the ordinary open-set site, the subobjects of the terminal object are the opens of $\mathbb{R}$. If $U = (0, \infty)$, then $\neg U = (-\infty, 0)$, and their join omits 0. The frame here is the topology of the space. In an arbitrary ∞-topos the frame describes propositional structure in a context, while the entire slice also carries untruncated types and their mapping spaces.

4.4 The monad of descent

The two adjunctions in (5) give monads on different fibers. The monad for $\Sigma_f \dashv f^*$ has a direct interpretation in terms of transport between the points of Δ over the same base point.

Proposition 4.2 (Descent along an effective epimorphism)**.** *For $f\colon \Delta \to \Gamma$ in an ∞-topos, f^* is monadic as the right adjoint of Σ_f if and only if f is an effective epimorphism. The associated monad on $\mathcal{X}_{/\Delta}$ is*

$$T_f(A) = \big(\Delta \times_\Gamma A \longrightarrow \Delta\big).$$

Proof. An effective epimorphism has a Čech nerve with realization Γ. Given any $A \to \Gamma$, pulling this realization back along $A \to \Gamma$ exhibits A as the realization of the base-changed nerve, by universality of colimits. If f^* takes a morphism $A \to A'$ in the slice to an equivalence, then its pullback to every degree of the Čech nerve is an equivalence. Taking realizations proves that the original morphism was an equivalence. Pullback preserves geometric realizations, so Barr–Beck applies. Conversely the image of f is a subobject of Γ whose inclusion becomes an equivalence after pullback along f. If f^* is monadic it is conservative, so the image must be all of Γ. This is equivalent to f being an effective epimorphism [14, §6.2.3]. □

The unit inserts the given point of Δ over Γ. Multiplication forgets the intermediate Δ-coordinate in $\Delta \times_\Gamma \Delta \times_\Gamma A$, composing the path identifications in Γ. An algebra map carries a over one lift of the base point to an element over another lift. Its coherences supply transport along the Čech groupoid. For $i\colon * \to S^1$, the monad is $\Omega S^1 \times (-)$, with multiplication given by loop composition. This is the monad for an action of $\mathbb{Z}$ on a space. Thus the local system over the circle is recovered from its based fiber together with that action.

5 Indexed monadicity and coherent free groups

5.1 Adjunctions that respect substitution

Consider $C, D\colon B^{\mathrm{op}} \to \mathrm{Cat}_\infty$ and a cartesian functor $U\colon \int_B D \to \int_B C$. For $f\colon b \to c$, its comparison identifies $f_C^* U_c \simeq U_b f_D^*$. Suppose $F_b \dashv U_b$ in each fiber. For $x \in C_c$ define the left mate by the composite

$$\begin{aligned} F_b f_C^* x &\xrightarrow{F_b f_C^* \eta_c} F_b f_C^* U_c F_c x \\ &\xrightarrow{\simeq} F_b U_b f_D^* F_c x \xrightarrow{\varepsilon_b} f_D^* F_c x. \end{aligned} \tag{8}$$

Invertibility of the comparison for U is part of cartesianness. Invertibility of its mate is a separate condition on the left adjoints.

Theorem 5.1 (Indexed monadicity under invertible mates)**.** *If the mates (8) are equivalences for all f, the F_b assemble to a cartesian functor $F\colon \int_B C \to \int_B D$ left adjoint to U, with unit and counit over the identity of B. The induced indexed monad $T = UF$ has coherent equivalences $T_b f_C^* \simeq f_C^* T_c$ compatible with unit and multiplication. If each U_b is monadic, the comparison gives an equivalence of cartesian fibrations over B*

$$\int_B D \;\simeq\; \int_{b \in B} \mathrm{Alg}_{T_b}(C_b). \tag{9}$$

On the right, transport of a T_c-algebra $a\colon T_c x \to x$ acts by

$$T_b f_C^* x \xrightarrow{\simeq} f_C^* T_c x \xrightarrow{f_C^* a} f_C^* x,$$

with its transported algebra coherences.

Proof. Apply the coherent calculus of mates to the squares for U. Mates of pasted squares are pastings of mates, with the unit and counit satisfying the coherent triangle identities. Thus the family F_b and the invertible comparisons in (8) define a natural transformation of indexed families and hence the cartesian F [8, 7]. To see the total adjunction, let $x \in C_b$ and $y \in D_c$. Over $f\colon b \to c$ there is a chain of equivalences

$$\begin{aligned}\mathrm{Map}_{\int_B D}(Fx, y)_f &\simeq \mathrm{Map}_{D_b}(F_b x, f_D^* y)\\ &\simeq \mathrm{Map}_{C_b}(x, U_b f_D^* y)\\ &\simeq \mathrm{Map}_{C_b}(x, f_C^* U_c y) \simeq \mathrm{Map}_{\int_B C}(x, Uy)_f.\end{aligned}$$

These equivalences are natural over the mapping space of base arrows, using the adjunction and its mates. They give the total mapping-space adjunction. Its unit and counit are the indexed maps obtained from the corresponding vertical maps in the fibers. Composing the coherent comparisons for U and F gives the claimed compatibility for T, whose multiplication is $U\varepsilon F$.

The comparison in each fiber sends y to $U_b y$ with structure map $U_b \varepsilon_b(y)$. This construction commutes with transport by compatibility of the counit. It is therefore a natural transformation of indexed families, and under fiberwise monadicity it is an objectwise equivalence. Straightening takes it to (9). □

The theorem concerns coherent monads. In an ∞-category C_b, such a monad is an associative algebra in $\mathrm{Fun}(C_b, C_b)$ under composition. An algebra for it is a module for the evaluation action of that monoidal category on C_b. A description by unit and multiplication on the homotopy category would lose the higher coherences that the transport must preserve [20, 7, 15].

There is a concrete case where a cartesian right adjoint is fiberwise monadic yet its free functors fail to commute with substitution. Let $B = [1]$, C be constant at sets, and D have Grp at 1 and Mon at 0, with transport forgetting inverses. The underlying-set square commutes. The mate of (8) is the natural map from the free monoid to the underlying monoid of the free group. On a nonempty alphabet the latter contains inverse letters outside its image. The indexed monad with invertible reindexing comparison does not exist for these fiberwise free functors. The compatible restriction of algebras still exists and is carried by a map of monads, as discussed in Section 8.

5.2 The free group in an ∞-topos

We write $\mathsf{Grp}(\mathcal{C})$ for coherent group objects in an ∞-category $\mathcal{C}$ with finite products. These can be presented by simplicial objects $G_\bullet$ with $G_0 \simeq \mathbf{1}$, Segal equivalences $G_n \simeq G_1^n$, and the invertibility condition for a groupoid object. Equivalently, after the Segal identifications, the shear $(x, y) \mapsto (x, xy)$ on $G_1 \times G_1$ is an equivalence. The underlying object is G_1. If $\mathcal{C} = \mathcal{X}_{/\Gamma}$ for an ∞-topos, realization of the one-object groupoid gives delooping $G \mapsto BG$. Effectivity of groupoid objects identifies G_1 with the loop object of BG. The loop and delooping constructions give the recognition equivalence

$$\mathsf{Grp}(\mathcal{X}_{/\Gamma}) \simeq (\mathcal{X}_{/\Gamma})_*^{\mathrm{conn}},$$

where the right side consists of pointed, 0-connected objects of the slice [15, Thm. 5.2.6.15]. Connectedness here is internal to the slice. For example, a space over Γ is connected in the slice when its homotopy fibers are connected.

For $A \in \mathcal{X}_{/\Gamma}$, let $A_+ = A \amalg \mathbf{1}_\Gamma$ with its disjoint base point. Write Σ_Γ and Ω_Γ for reduced suspension and looping of pointed objects *in the slice*, to distinguish Σ_Γ from the dependent sum

Σ_f. In particular $\Sigma_\Gamma A_+$ is the pointed pushout of $\mathbf{1}_\Gamma \leftarrow A_+ \rightarrow \mathbf{1}_\Gamma$. Its 0-truncation is terminal. Indeed, testing the pushout against a 0-truncated object asks for two sections of that object whose restrictions to A_+ agree. Since A_+ has a section, this is equivalent to choosing one section. It follows that $\Sigma_\Gamma A_+$ lies in the connected part of the pointed slice.

Corollary 5.2 (Free groups commute with change of context)**.** *Over the contexts of any ∞-topos $\mathcal{X}$, let $U_\Gamma \colon \mathsf{Grp}(\mathcal{X}_{/\Gamma}) \to \mathcal{X}_{/\Gamma}$ be the underlying-object functor. Its indexed left adjoint is the coherent group*

$$F_\Gamma(A) = \Omega_\Gamma \Sigma_\Gamma (A_+).$$

The indexed right adjoint U is fiberwise monadic. For $T_\Gamma = U_\Gamma F_\Gamma$, the cartesian fibration of groups over $\mathcal{X}$ is equivalent to the cartesian fibration of T_Γ-algebras over $\mathcal{X}$.

Proof. For $G \in \mathsf{Grp}(\mathcal{X}_{/\Gamma})$, use recognition, the suspension–loop adjunction, and the disjoint base point:

$$\begin{aligned} \mathrm{Map}_{\mathsf{Grp}(\mathcal{X}_{/\Gamma})}(\Omega_\Gamma \Sigma_\Gamma A_+, G) &\simeq \mathrm{Map}_{(\mathcal{X}_{/\Gamma})_*}(\Sigma_\Gamma A_+, BG) \\ &\simeq \mathrm{Map}_{(\mathcal{X}_{/\Gamma})_*}(A_+, \Omega_\Gamma BG) \\ &\simeq \mathrm{Map}_{\mathcal{X}_{/\Gamma}}(A, U_\Gamma G). \end{aligned}$$

This gives $F_\Gamma \dashv U_\Gamma$. Pullback f^* preserves finite limits and all colimits. It commutes with adjoining the base point, the suspension pushout, the loop pullback, and the delooping realization. The adjunction formulas are natural in these comparisons. Taking mates therefore gives the equivalence

$$F_\Delta(f^* A) \simeq f^* F_\Gamma(A)$$

for $f \colon \Delta \to \Gamma$. In particular the equivalence commutes with the free generator maps into the underlying groups.

For monadicity, evaluation at G_1 is conservative since a map of group objects is an equivalence if it is an equivalence in simplicial degree 1. Sifted colimits in an ∞-topos commute with finite products. This holds pointwise in presheaves of spaces and passes to a left-exact localization. Thus levelwise geometric realization of a simplicial diagram of group objects preserves the Segal conditions and the shear equivalence. It computes realization in group objects, and evaluation at G_1 preserves it. The ∞-categorical Barr–Beck theorem implies monadicity [15, Thm. 4.7.0.3 and Cor. 5.2.6.18]. The result now follows from Theorem 5.1. □

Example 5.3 (Discrete and circular generators)**.** In spaces, for a discrete alphabet X, suspension of X_+ gives a wedge $\bigvee_X S^1$. This is a classifying space for the ordinary free group on X. Its universal cover is the tree of reduced words. Its loop space is accordingly discrete up to equivalence and recovers the original free group. In particular the empty wedge gives the terminal group.

For a space of generators $A = S^1$, the reduced suspension of A_+ identifies the two tips of the unreduced suspension of a circle. That unreduced suspension is S^2. Identifying its tips gives $S^2 \vee S^1$ up to homotopy, seen by attaching an interval between them before contracting that interval. Consequently

$$F_*(S^1) \simeq \Omega(S^1 \vee S^2), \qquad \pi_0 F_*(S^1) \cong \mathbb{Z}, \qquad \pi_1(F_*(S^1), e) \cong \bigoplus_{n \in \mathbb{Z}} \mathbb{Z}.$$

For the last calculation, the universal cover of the wedge is a line of 2-spheres, with one sphere attached at each integer. It is simply connected, so its second homotopy group is its second

homology. Looping moves that group to π_1 at the identity. The resulting coherent free group contains both an ordinary group of components and nontrivial higher paths within each component [6, §§1.A and 4.2].

6 Groups, torsors, and higher transport

6.1 Group objects among fibrations

Over a fixed base B, cartesian fibrations are multiplied by fiber product over B. On right fibrations this corresponds to pointwise product of space-valued presheaves. The terminal right fibration is $B \to B$. Thus a group object in $\mathrm{RFib}(B)$ has multiplication $E \times_B E \to E$ and unit $B \to E$ over B, with coherent group laws. Straightening gives

$$\mathsf{Grp}(\mathrm{RFib}(B)) \simeq \mathrm{Fun}(B^{\mathrm{op}}, \mathsf{Grp}(\mathcal{S})). \tag{10}$$

Indeed, both the Segal conditions and the group-invertibility condition are checked pointwise. The same construction for all cartesian fibrations uses $\mathsf{Grp}(\mathsf{Cat}_\infty)$ in place of $\mathsf{Grp}(\mathcal{S})$. For a right fibration, groupoidal structure concerns paths between objects, while group-object structure supplies a multiplication over B and a section giving the identity. Transport preserves both of those operations.

For a base space B every functor $B \to \mathcal{S}$ takes paths to equivalences. The slice $\mathcal{S}_{/B}$ is the category of these space-valued local systems. This follows either by left straightening or by replacing a map of spaces by a Kan fibration. Right straightening gives the opposite path convention, identified by reversing paths. For $B = BG$ the local systems are spaces with coherent G-action. The dependent sum and product along $! \colon BG \to *$ are the homotopy orbits and the homotopy fixed points:

$$\Sigma_! A \simeq A_{hG}, \qquad \Pi_! A \simeq A^{hG}.$$

The colimit statement follows by mapping the total space into a test space. The limit statement follows by taking sections. For discrete G acting on a discrete set A, the limit is the fixed-point set viewed as a space, whereas the colimit is the realization of the action groupoid. In that groupoid a point with stabilizer H lies in a component of type BH.

Example 6.1 (Free groups in a family with no chosen generator)**.** Consider the covering $\mathbb{R} \to S^1$, $t \mapsto e^{2\pi i t}$. Its fiber over 1 is $\mathbb{Z}$, and its local system uses translation $n \mapsto n+1$ around the positively oriented loop. This is a torsor for the constant group $\mathbb{Z}$. Every fiber is inhabited, but the translation action has no fixed point, so the covering has no section.

Apply the free group functor in the slice $\mathcal{S}_{/S^1}$. By Corollary 5.2, its fiber is the discrete free group with generators x_n for $n \in \mathbb{Z}$, and the monodromy is the automorphism $x_n \mapsto x_{n+1}$. The identity word is a global section. It is the only one. Any nonempty reduced word uses a finite nonempty set of indices, which cannot be invariant under translation by 1. Thus the construction supplies a global group identity while retaining the nontrivial transport of its generators.

6.2 A simply connected base with a nontrivial family

A torsor for a group object G in a slice has an action on P, with the induced map $P \times_\Gamma G \to P \times_\Gamma P$ an equivalence and $P \to \Gamma$ an effective epimorphism. An inhabited fiber of P can be identified with the group after choosing an origin. A global section chooses such an origin coherently and

trivializes the torsor. This makes torsors a direct source of examples where fiberwise existence differs from existence in context.

Example 6.2 (The Hopf fibration)**.** Give $S^3 \subset \mathbb{C}^2$ the right action of S^1 by simultaneous multiplication of its two coordinates. The quotient map

$$q\colon S^3 \longrightarrow \mathbb{CP}^1 \simeq S^2, \qquad (z_0, z_1) \longmapsto [z_0 : z_1]$$

is an S^1-torsor in $\mathcal{S}_{/S^2}$. On the chart $z_0 \neq 0$, with coordinate $w = z_1/z_0$, a section is $s_0(w) = (1, w)/\sqrt{1+|w|^2}$. On the chart $z_1 \neq 0$, with $v = z_0/z_1$, use $s_1(v) = (v, 1)/\sqrt{1+|v|^2}$. On their overlap at $|w| = 1$, these satisfy $s_1(1/w) = s_0(w)w^{-1}$. The transition on the equator has winding number -1 in these conventions.

There can be no homotopy section. If $q \circ s \simeq \mathrm{id}_{S^2}$, the induced identity on $\pi_2(S^2) \cong \mathbb{Z}$ would factor through $\pi_2(S^3) = 0$. Moreover, the boundary map $\pi_2(S^2) \to \pi_1(S^1)$ in the fiber sequence is an isomorphism. The base has trivial fundamental group. In its straightened local system, ordinary monodromy on $\pi_1(S^2)$ therefore records no twisting. The boundary is seen by the higher transport, in the same degree as the winding of the transition function. Factoring the local system through the 1-truncation of S^2 would make it constant and give a section [6, §4.2].

In the logic of Section 4, let P denote q and let $!\colon S^2 \to *$. Since every homotopy fiber is inhabited, $\|P\|_{-1,S^2} \simeq \mathrm{id}_{S^2}$. But $\Pi_! P$ is the empty space of sections. Thus the natural map

$$\|\Pi_! P\|_{-1,*} \longrightarrow \Pi_! \, \|P\|_{-1,S^2}$$

is $\varnothing \to *$. Propositional truncation before the dependent product asks only for fiberwise inhabitation. Truncation after the product asks for a coherent global choice.

7 Topos structure and sheaf reflection

7.1 When a category of fibrations is a topos

Right fibrations over a small base B form the presheaf ∞-topos $\mathcal{P}(B)$ under (3). For the codomain fibration over an ∞-topos $\mathcal{X}$, every fiber $\mathcal{X}_{/\Gamma}$ is an ∞-topos. In this case even its total category $\mathrm{Fun}([1], \mathcal{X})$ is an ∞-topos. Indeed, if $\mathcal{X}$ is an accessible left-exact localization of $\mathcal{P}(C)$, apply the localization pointwise to $\mathrm{Fun}([1], \mathcal{P}(C))$. Before localization the functor category is $\mathcal{P}([1]^{\mathrm{op}} \times C)$, and the pointwise localization remains accessible and left exact [14, §6.1].

For general cartesian fibrations, universality of colimits gives a counterexample. In Cat_∞ the pushout of two copies of $[1]$ joined at a target and a source is $[2]$. This is the Segal property. Mapping into any ∞-category specifies a pair of composable arrows, with the space of composites and composition triangles contractible after fixing the pair. Regard the pushout as a colimit in $(\mathsf{Cat}_\infty)_{/[2]}$. Pull it back along the long edge $j\colon [1] \to [2]$ selecting 0 and 2. The two short edges pull back to the two endpoints of $[1]$, and their common vertex 1 pulls back to the empty category. The homotopy pullbacks here are ordinary pullbacks of the ordinal categories, whose only isomorphisms are identities. The top arrow in the following square is the base-change comparison.

$$\begin{array}{ccc} [0] \amalg_{\varnothing} [0] & \longrightarrow & [1] \\ \downarrow & & \downarrow{\scriptstyle j} \\ [1] \amalg_{[0]} [1] & \xrightarrow[\simeq]{} & [2]. \end{array}$$

It is not an equivalence, since it omits the nonidentity arrow. Hence colimits in Cat_∞ are not universal. Since $\mathrm{CartFib}(*) \simeq \mathsf{Cat}_\infty$, a category of all cartesian fibrations can fail to be an ∞-topos.

An arbitrary indexed family with topos fibers also needs hypotheses on its reindexing if it is to model substitution in topos logic. For example, over $[1]$ one can assign $\mathcal{S}$ to both vertices and reindex by the constant empty-space functor. This destroys the terminal object. In the slice family it is the combination of the topos axioms and pullback transport that accounts for the logical operations of Section 4.

7.2 The idempotent monad of a left-exact localization

Sheafification is a second kind of monadic construction, one which enforces a locality condition on a presheaf. Let $L\colon \mathcal{X} \rightleftarrows \mathcal{Y} : j$ be an accessible left-exact localization of ∞-topoi, with j fully faithful. Its monad jL is idempotent. The counit $Lj \simeq \mathrm{id}_{\mathcal{Y}}$ makes $jLjL \to jL$ invertible. Accordingly its algebras are the local objects, those A for which the unit $A \to jLA$ is an equivalence, with a contractible space of algebra structures on every local object [14, §§5.2.7 and 6.2]. In the usual geometric setting L can be sheafification from presheaves of spaces on a site. More general left-exact localizations have an analogous descent interpretation [2].

Proposition 7.1 (Localization in context)**.** *For $\Gamma \in \mathcal{X}$, the induced functor*

$$L_\Gamma\colon \mathcal{X}_{/\Gamma} \longrightarrow \mathcal{Y}_{/L\Gamma}, \qquad (A \to \Gamma) \longmapsto (LA \to L\Gamma)$$

has a fully faithful right adjoint

$$j_\Gamma(B \to L\Gamma) = \big(\Gamma \times_{jL\Gamma} jB \longrightarrow \Gamma\big).$$

The two functors commute with reindexing, in $\mathcal{Y}$ along Lf for f a context map in $\mathcal{X}$. Thus $j_\Gamma L_\Gamma$ is an indexed idempotent monad with algebras forming the slice family $\Gamma \mapsto \mathcal{Y}_{/L\Gamma}$ over $\mathcal{X}$.

Proof. A map in $\mathcal{X}_{/\Gamma}$ from A to the displayed pullback is a map from A to jB compatible with the map to $jL\Gamma$. Taking its adjoint gives exactly the space of maps $LA \to B$ over $L\Gamma$. Left exactness and $Lj \simeq \mathrm{id}_{\mathcal{Y}}$ give $L_\Gamma j_\Gamma \simeq \mathrm{id}$, so j_Γ is fully faithful. Left exactness also gives the base-change comparison for L_Γ. For j_Γ, the comparison is the canonical identification of the two iterated pullbacks, using preservation of pullbacks by j and naturality of the unit. These equivalences preserve the adjunction, and the comparison theorem applies since a reflective localization has precisely the algebras for its idempotent monad. □

When restricted to $\mathrm{Sub}(\Gamma)$, $j_\Gamma L_\Gamma$ is an extensive, idempotent operation preserving finite meets, hence a nucleus on that frame. Its fixed points describe the closed subobjects for this localization in that context. This is the propositional part of the sheaf reflection. The free group monad has different behavior: on the terminal space it gives the discrete group $\mathbb{Z}$, and its multiplication on a one-letter alphabet evaluates words in an alphabet indexed by all integer powers of that letter. Its unit fails to exhibit a left-exact localization, since the endofunctor fails to preserve the terminal object and multiplication has a nontrivial kernel. The two examples exhibit topology and algebra as different kinds of coherent monadic structure.

Example 7.2 (Sheafifying a covering sieve)**.** Use the open-set site of $I = (-2, 2)$. Let $R \subseteq y(I)$ be the sieve generated by the opens $U = (-2, 1)$ and $V = (-1, 2)$. As a presheaf of spaces it is terminal on W when $W \subseteq U$ or $W \subseteq V$, and empty otherwise. In particular $R(I) = \varnothing$. Sheafification sends $R \to y(I)$ to an equivalence. On every W the restrictions of its two local sections over

$W \cap U$ and $W \cap V$ agree where both exist, so the sheaf condition glues them uniquely up to a contractible choice. Equivalently, R is the pushout of $y(U)$ and $y(V)$ along $y(U \cap V)$, and after sheafification their union is $y(I)$. The locality constraint changes the global sections.

8 Changing the categorical level

8.1 Monads as objects of an indexed category

For a fixed ∞-category $\mathcal{C}$, one can form the category

$$\mathrm{Mnd}(\mathcal{C}) = \mathrm{Alg}_{\mathbb{E}_1}\big(\mathrm{Fun}(\mathcal{C},\mathcal{C}), \circ\big).$$

A map $\theta\colon T \to S$ of these associative algebras lets an S-algebra act as a T-algebra. Its action is $Tx \xrightarrow{\theta_x} Sx \to x$. This restriction is coherent under composition and gives an indexed family $T \mapsto \mathrm{Alg}_T(\mathcal{C})$ over $\mathrm{Mnd}(\mathcal{C})$, with contravariance in the monad. Its unstraightening is a cartesian fibration whose objects are pairs consisting of a monad and an algebra for it. An example on Set is the map from the free monoid monad to the free group monad, sending a list of generators to their positive word. Restricting an algebra along it gives the underlying monoid of a group. This is the restriction occurring in the counterexample to invertible mates in Section 5.

If the underlying category itself varies by equivalences, transport of monads is by conjugation. For an equivalence $e\colon \mathcal{C}' \to \mathcal{C}$, the transported monad is eTe^{-1}, with unit and multiplication transported along the same equivalence. A general reindexing functor $v^*\colon \mathcal{C}' \to \mathcal{C}$ need not have an inverse. Compatible monads can instead be specified by a map $Tv^* \to v^*T'$ respecting the units and multiplications. That direction is exactly what is required to restrict a T'-action to a T-action. The invertible comparisons in Theorem 5.1 are a special case. With variable underlying categories the choice of direction of the comparison is part of the definition of a morphism of monads [7].

8.2 Internal fibrations and fibrations over bicategories

There are two distinct extensions beyond the external theory (3). First, an ∞-category can be internal to an ∞-topos $\mathcal{X}$, for instance as a complete Segal object $C_\bullet$ in $\mathcal{X}$. Its category of generalized elements over $V \in \mathcal{X}$ is presented by the complete Segal space $\mathrm{Map}_{\mathcal{X}}(V, C_\bullet)$. Internal fibrations are maps between such category objects with lift properties in the internal sense. The ambient topos supplies descent and change of context in addition to the indexing along the internal base. Martini develops the corresponding internal (co)cartesian straightening equivalences [17], and Martini and Wolf study the extra structure for an internal topos [18].

For an elementary internal example, take the open-set topos of a space. The ordered set of integers gives an internal category whose n-simplices are the locally constant families of weakly increasing $n+1$-tuples of integers. Deleting or repeating entries gives the simplicial operations. Over a disconnected open set, the integers in such a tuple can vary between components. The Segal identities and the absence of nonidentity isomorphisms hold locally, so this is a complete Segal object. This example uses a sheaf of categories with varying generalized elements even though the order is fixed.

Second, the base itself can retain noninvertible 2-cells. Consider the ordinary 2-category of categories, functors, and natural transformations. For $a, b\colon C \to D$ and $\alpha\colon a \Rightarrow b$, precomposition on set-valued diagrams takes α to

$$a^* \implies b^* \quad \text{on} \quad \mathrm{Fun}(D, \mathsf{Set}) \longrightarrow \mathrm{Fun}(C, \mathsf{Set}), \qquad A \longmapsto A\alpha.$$

In this formula the opposite on the indexing 2-category reverses 1-morphisms while keeping the local direction of 2-morphisms. Taking $C = [0]$, $D = [1]$, and a diagram sending the nonidentity arrow to $\varnothing \to \mathbb{N}$ makes the transported transformation noninvertible. The underlying $(\infty, 1)$-category of categories does not encode α as a homotopy of base arrows. A higher Grothendieck construction over an $(\infty, 2)$-base keeps the mapping categories and their transport. Abellán and Stern establish a 2-cartesian straightening theorem with fibers that are themselves ∞-bicategories [1].

Dependent type theory is a language for the slice operations in Section 4. Its semantics distinguish substitution, context extension, terms, and their identifications. A choice of a precise type theory with universes calls for further coherence results. For example, Shulman's theorem supplies presentations of Grothendieck ∞-topoi that interpret homotopy type theory with strict univalent universes [21]. Calculi of mates and internal straightening describe additional structure for reasoning about the fibrations supporting those interpretations.

9 Conclusion

An indexed category has a dual description as a cartesian fibration with universal lifts. For the codomain fibration of an ∞-topos, those lifts implement substitution. Their adjoints implement dependent sums and products, and their mates implement the base-change laws. The same calculus orders a second layer of constructions over the contexts: an algebraic free–forgetful adjunction passes to a cartesian family of algebras when the free functors commute with substitution through their canonical mates. The free coherent group and the slice localization give two such families, one by adding group operations and the other by imposing locality.

The examples measure what these constructions preserve. Pulling a point back along itself distinguishes all winding numbers. A free group in a local system preserves the transport of the alphabet and coherently adds an identity. A torsor can have inhabited fibers without a coherent choice of origins. These are different consequences of keeping the base, the transport, and its levels of coherence in the mathematical description.

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