Connections, Curvature, and Higher Geometry

From covariant differentiation
to differential cohomology

Anonymous

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Part I Metric and tangent geometry

CHAPTER 1

Metrics and tangent geometry

A tangent vector describes the first-order motion of a point, and a metric assigns a length to that motion. This assignment gives norms on the fibers of the tangent bundle. Its smooth dependence on the base point is the input for connection and curvature. We begin with the way the assignment transforms under a change of coordinates, as in Tu's treatment of metrics and curves [1, §§ 1–4].

1.1 The tangent map and pullback

For a smooth manifold M, a tangent vector at p can be defined as a derivation on germs of smooth functions at p. In coordinates $x^1, \dots, x^n$, any such derivation is $\sum_i v^i \partial_i|_p$, where $v^i = v(x^i)$. To see the uniqueness, expand a function near p as its value at p plus

$$\sum_i (x^i - x^i(p)) a_i, \qquad a_i(p) = \partial_i f(p).$$

One obtains the expansion by integrating the gradient along coordinate line segments. The product rule then gives $v(f) = \sum_i v^i a_i(p)$. This identifies T_pM with $\mathbb{R}^n$ after a chart is chosen. If $f : M \to N$, set $(df_p v)(h) = v(h \circ f)$. It is a derivation at $f(p)$, and this definition proves the chain rule $d(g \circ f)_p = dg_{f(p)}\, df_p$ directly.

Definition 1.1. A Riemannian metric on M is a smooth section g of $\operatorname{Sym}^2 T^*M$ such that g_p is positive definite for every $p \in M$. Write $|v|_g = \sqrt{g_p(v,v)}$ for $v \in T_pM$.

A metric has the local expression $g_{ij}\, dx^i\, dx^j$, with summation understood and the product denoting the corresponding symmetric tensor. If $x = x(y)$, the matrix in y-coordinates equals $J^{\mathsf{T}} g J$, where $J = (\partial x^i / \partial y^a)$. This formula follows by evaluating g on the vectors $\partial/\partial y^a$. In particular, positive definiteness does not depend on the chart.

Proposition 1.2. *For every smooth $f : M \to N$ and metric h on N, the symmetric tensor*

$$(f^*h)_p(v,w) = h_{f(p)}(df_p v, df_p w)$$

is positive semidefinite. It is positive definite at p exactly when df_p is injective. If $j : N \to Q$ and h is a metric on Q, then $(j \circ f)^* h = f^*(j^* h)$.

Proof. The squared length of v under the tensor is $|df_p v|_h^2$, whose zero set is $\ker df_p$. Smoothness follows from the coordinate formula. The formula for composites follows by applying the chain rule to both inputs. □

For an immersion $M \to N$, the metric is the restriction of the ambient inner product to the immersed tangent directions. For a submersion with positive-dimensional fibers, its pullback metric vanishes along the vertical tangent bundle. This distinction will matter when taking quotient atlases. A tensor that is positive definite transversely to the fibers can encode a metric on a quotient even though it is degenerate on the atlas.

1.2 Existence by convex gluing

We use Hausdorff second-countable smooth manifolds, so smooth partitions of unity exist. On a coordinate chart U_i choose its Euclidean metric g_i. Take a locally finite partition of unity $\{\phi_i\}$ subordinate to the charts and put

$$g = \sum_i \phi_i g_i.$$

Each product extends smoothly by zero outside the chart because the support of ϕ_i lies in U_i. The sum is smooth by local finiteness. At each p and for each nonzero $v \in T_pM$, every summand in $g_p(v, v)$ is nonnegative, and at least one is positive since the coefficients sum to one. Thus g is a metric. The same argument for Euclidean forms in bundle frames gives a metric on any real vector bundle over M.

For any two metrics g_0, g_1, the tensor $g_t = (1 - t)g_0 + tg_1$ is positive definite for $0 \le t \le 1$. Hence every two metrics are joined by a smooth family. This interpolation is a statement about the convex space of metrics, with the underlying manifold and points fixed. Geodesics and curvature may vary along such an interpolation.

Example 1.3 (Rotational coordinates)**.** The map $(r, \theta) \mapsto (r\cos\theta, r\sin\theta)$ has Euclidean pullback

$$dx^2 + dy^2 = dr^2 + r^2 d\theta^2$$

on any angular chart with $r > 0$. This follows by squaring the two differentials and canceling the cross terms. The determinant of the metric matrix is r^2. At $r = 0$, the angular vector maps to zero, so the tensor written in these variables ceases to be positive definite. The degeneracy reflects the parametrization at the origin.

1.3 Length, speed, and oriented turning

For a piecewise smooth curve $\gamma : [a, b] \to M$, the metric defines its length by

$$L_g(\gamma) = \int_a^b |\dot\gamma(t)|_g \, dt.$$

Length is invariant under increasing changes of parameter by the change-of-variables formula. Reversing the parameter leaves length unchanged because the norm is even. On a regular smooth segment, the length coordinate

$$s(t) = \int_a^t |\dot\gamma(u)|_g \, du$$

has positive derivative. The inverse function theorem gives a smooth unit-speed parametrization on that segment.

In the standard oriented plane, let J be rotation by $+\pi/2$, write $T = d\gamma/ds$, and take the normal JT. Differentiation of $\langle T, T\rangle = 1$ shows that dT/ds is perpendicular to T. Hence

$$\frac{dT}{ds} = \kappa \, JT.$$

The real number κ is the signed curvature. Its value for an arbitrary regular parameter t is

$$\kappa(t) = \frac{\det(\dot\gamma(t), \ddot\gamma(t))}{|\dot\gamma(t)|^3}. \tag{1.1}$$

Indeed $\dot\gamma = vT$, with $v = ds/dt$, so $\ddot\gamma = \dot v T + v^2 \kappa JT$. The determinant of T, JT equals one. The term involving $\dot v$ drops out, and the result is $v^3\kappa$. If a local real angle θ satisfies $T = (\cos\theta, \sin\theta)$, then $\kappa = d\theta/ds$. The oriented tangent angle therefore records a one-dimensional form of curvature.

Example 1.4 (An ellipse and its osculating radius)**.** For the counterclockwise ellipse $\gamma(t) = (a\cos t, b\sin t)$, $a, b > 0$, the determinant in (1.1) is ab, and

$$\kappa(t) = \frac{ab}{(a^2\sin^2 t + b^2\cos^2 t)^{3/2}}.$$

At $(a, 0)$ the curvature is a/b^2. Its reciprocal b^2/a is the radius of the oriented circle tangent there with the same curvature. If $a = b$, the signed curvature is $1/a$ along the entire loop. Reversing the direction of traversal negates κ, while the norm of its curvature vector stays the same.

A tangent-vector field along a plane curve is an ordinary map into $\mathbb{R}^2$, once the ambient parallel coordinates have been chosen. A field along a curve in M is instead a section of γ^*TM. The vectors above distinct points then belong to distinct vector spaces. Chapter 2 constructs a derivative of such sections by choosing a connection; the metric provides a canonical connection for the tangent bundle.

1.4 Volume and coordinate changes

On an oriented n-manifold, the metric gives the positive form

$$dV_g = \sqrt{\det(g_{ij})}\, dx^1 \wedge \cdots \wedge dx^n$$

in an oriented chart. Here $\det(J^{\mathsf{T}} g J) = (\det J)^2 \det g$. Consequently the formula transforms as an n-form under orientation-preserving changes of chart. Forgetting orientation gives the well-defined density $|dV_g|$. For a map f that is an orientation-preserving local diffeomorphism, the same computation gives $dV_{f^*h} = f^* dV_h$.

This construction depends on the metric algebraically at each point. If $g' = e^{2u} g$ for a smooth real function u, the length of a tangent vector is multiplied by e^u and $dV_{g'} = e^{nu} dV_g$. Sectional curvature will depend on derivatives of u as well. The different orders of dependence, first for connections and second for curvature, reappear in the local description by differential forms.

The length metric on each connected component is defined by $d_g(p, q) = \inf_\gamma L_g(\gamma)$ over piecewise smooth paths from p to q. In a relatively compact coordinate ball about p, the positive metric matrix admits uniform bounds $cI \le (g_{ij}) \le CI$ for some $0 < c \le C$. Paths staying in the chart obey the same two-sided bounds for lengths up to factors $\sqrt{c}, \sqrt{C}$. Paths leaving a smaller ball incur a positive lower length bound before their first exit. Coordinate line segments supply the local upper estimate in the smaller ball. These facts prove nondegeneracy and show that d_g induces the manifold topology. Finding length minimizers is the next question. Its local equation is a connection equation, as treated in Chapter 4.

CHAPTER 2

Levi-Civita connection and torsion

A connection furnishes a derivative of tangent-vector fields. It compares vectors above adjacent points by specifying an infinitesimal rule with a product law for multiplying sections by functions. The construction recovers the directional derivative on Euclidean space and gives an intrinsic rule when the tangent bundle is presented only by local coordinates. We use the curvature convention $R(X,Y) = [\nabla_X, \nabla_Y] - \nabla_{[X,Y]}$ throughout this chapter and Chapter 3.

2.1 Affine connections and the derivative of a section

Definition 2.1. An affine connection on M is an $\mathbb{R}$-bilinear operation $(X,Y) \mapsto \nabla_X Y$ on smooth vector fields such that

$$\nabla_{fX} Y = f \nabla_X Y, \qquad \nabla_X (fY) = X(f)Y + f \nabla_X Y.$$

Its torsion is the operation

$$T(X,Y) = \nabla_X Y - \nabla_Y X - [X,Y].$$

These identities make all expressions local. For example, if Y vanishes near a given point p, choose a cutoff that is one near p with support in the vanishing neighborhood. Applying the product rule to the zero field obtained by multiplying by this cutoff gives $\nabla_X Y|_p = 0$. Smooth local formulas for ∇ therefore restrict to smaller coordinate domains. In a local frame $e_1, \dots, e_n$, put $\nabla_{e_i} e_j = \Gamma^k{}_{ij} e_k$. Expanding the connection rules gives

$$\nabla_X Y = \sum_k \left(X(Y^k) + \sum_{i,j} X^i Y^j \Gamma^k{}_{ij} \right) e_k.$$

The coefficients are arbitrary smooth functions for a local connection. A second connection differs from the first by a section of $T^*M \otimes T^*M \otimes TM$, since the terms differentiating a multiplier cancel. For torsion the bracket satisfies $[X, fY] = X(f)Y + f[X,Y]$. The same cancellation proves $C^\infty(M)$-bilinearity and skew symmetry. In a coordinate frame $T^k{}_{ij} = \Gamma^k{}_{ij} - \Gamma^k{}_{ji}$. In a general frame there is also the contribution from $[e_i, e_j]$.

Torsion can be changed without changing the equation for a curve. For any affine connection, put $\nabla^\circ_X Y = \nabla_X Y - \frac{1}{2} T(X,Y)$. The subtracted term is a tensor,

so the Leibniz rule for the connection is preserved. The antisymmetric part of that tensor has exactly the size required to cancel T. Thus ∇° is torsion-free. Since $T(v,v)=0$, the two connections have the same affine geodesics. This construction uses the antisymmetry of torsion and requires no metric compatibility.

2.2 The metric and symmetry determine the connection

Theorem 2.2 (Levi-Civita)**.** *For a Riemannian metric g, exactly one affine connection satisfies*

$$Xg(Y,Z) = g(\nabla_X Y, Z) + g(Y, \nabla_X Z), \qquad T(X,Y) = 0.$$

Proof. Write the compatibility identity in the cyclic orders (X,Y,Z), (Y,Z,X), and (Z,X,Y). Add the first two and subtract the third. Torsion-freeness replaces each reversed derivative by the given derivative and a bracket. The result is the Koszul formula

$$\begin{aligned} 2g(\nabla_X Y, Z) = {} & Xg(Y,Z) + Yg(Z,X) - Zg(X,Y) \\ & - g(X,[Y,Z]) + g(Y,[Z,X]) + g(Z,[X,Y]). \end{aligned} \tag{2.1}$$

Nondegeneracy gives uniqueness. To see existence, denote the right side by $K(X,Y,Z)$. The product rule and the bracket identity show that $K(X,Y,fZ) = fK(X,Y,Z)$: the coefficients of $X(f)$ and $Y(f)$ from differentiating the metric evaluations cancel the corresponding bracket terms. The same expansion gives $K(fX,Y,Z) = fK(X,Y,Z)$ and $K(X,fY,Z) = fK(X,Y,Z) + 2X(f)g(Y,Z)$. Thus (2.1) defines a smooth affine connection using g to recover a vector from a covector. Subtracting $K(Y,X,Z)$ from $K(X,Y,Z)$ leaves $2g([X,Y],Z)$, which proves zero torsion. Adding $K(X,Y,Z)$ to $K(X,Z,Y)$ leaves $2Xg(Y,Z)$, which proves metric compatibility. □

With coordinate fields all brackets vanish. Inverting the metric matrix gives the explicit formula

$$\Gamma^k{}_{ij} = \frac{1}{2} \sum_\ell g^{k\ell} (\partial_i g_{\ell j} + \partial_j g_{\ell i} - \partial_\ell g_{ij}). \tag{2.2}$$

In particular, the connection depends on the first derivatives of the metric. Formula (2.1) applies also to orthonormal frames with nonzero brackets, where the derivative terms of the metric evaluations are then zero. These are two complementary ways to compute the same connection [1, §§ 6,13].

Proposition 2.3. *Let $f : (M,g) \to (N,h)$ be an isometric local diffeomorphism. If X, Y on M correspond locally to $df(X), df(Y)$ on N, then $df(\nabla^g_X Y) = \nabla^h_{df(X)} df(Y)$ there.*

Proof. The derivative of a local diffeomorphism preserves the vector-field bracket because the conjugated derivations have conjugated commutators. Transport the connection on N to M in any neighborhood where f is a diffeomorphism. The transported connection has zero torsion. It is compatible with $f^*h = g$ by differentiation of the pulled-back inner products. Uniqueness gives the asserted identity. The local computations agree on overlaps. □

There is a closely related rule for an isometric immersion into Euclidean space. Decompose the ambient derivative $D_X Y$ into its tangential and normal parts. The tangential projection is compatible with the induced metric because normal fields are perpendicular to tangent fields. Its torsion is the tangential projection of $D_X Y - D_Y X - [X, Y] = 0$. Thus the tangential projection is the Levi-Civita connection. The normal projection defines the second fundamental form used in Chapter 3.

2.3 Covariant differentiation along a path

Let $\gamma : I \to M$ and let $V(t) \in T_{\gamma(t)}M$ be a smooth section of the pullback tangent bundle. In coordinates around a segment of γ define

$$(\nabla_t V)^k = \frac{dV^k}{dt} + \sum_{i,j} \Gamma^k{}_{ij}(\gamma)\dot\gamma^i V^j. \tag{2.3}$$

This can also be derived locally by expressing $V = \sum_j V^j(t) e_j(\gamma(t))$ in any frame and applying the product rule with $\nabla_t(e_j \circ \gamma) = (\nabla e_j)_{\gamma(t)}(\dot\gamma(t))$. If a frame is changed by a matrix $B(\gamma(t))$, differentiating B by the ordinary chain rule gives the same change as applying $\nabla_{\dot\gamma}$ to the new frame fields. Hence the result is independent of the frame. Formula (2.3) also applies when the path crosses itself or is momentarily stationary. It requires only the pulled-back section and no extension as a field on M.

A parallel section has $\nabla_t V = 0$. This is a linear differential equation in V, so an initial vector determines a unique solution over the path interval. The solution operators compose on concatenation, and reversing the path gives the inverse. Here $\gamma_{12} * \gamma_{01}$ denotes first γ_{01} and then γ_{12}.

$$T_{p_0}M \xrightarrow{P_{\gamma_{01}}} T_{p_1}M \xrightarrow{P_{\gamma_{12}}} T_{p_2}M, \qquad P_{\gamma_{12}*\gamma_{01}} = P_{\gamma_{12}} \circ P_{\gamma_{01}}$$

For a metric connection, compatibility gives

$$\frac{d}{dt} g(V, W) = g(\nabla_t V, W) + g(V, \nabla_t W).$$

In particular, parallel transport preserves the Euclidean structures on the initial and terminal fibers. Its dependence on the full path encodes curvature, as developed in Chapter 4 and in the subsequent treatment of holonomy.

A geodesic with affine parameter satisfies $\nabla_t \dot\gamma = 0$, or $\ddot\gamma^k + \Gamma^k{}_{ij}(\gamma)\dot\gamma^i\dot\gamma^j = 0$ in coordinates. Local existence and uniqueness follow by converting the equation into a smooth first-order system on TM. Metric compatibility makes the speed constant. The equation is invariant under affine reparametrization $t \mapsto at + b$. A smooth reparametrization with a nonzero second derivative introduces a tangential acceleration proportional to that second derivative.

Example 2.4 (A circle in a flat metric)**.** For the metric $dr^2 + r^2 d\theta^2$ in Chapter 1, the nonzero coordinate coefficients are

$$\Gamma^r{}_{\theta\theta} = -r, \qquad \Gamma^\theta{}_{r\theta} = \Gamma^\theta{}_{\theta r} = r^{-1}.$$

The unit field $e_\theta = r^{-1}\partial_\theta$ satisfies $\nabla_{e_\theta} e_\theta = -r^{-1}\partial_r$ on a circle of radius r. Its acceleration has length $1/r$ and points toward the center. This agrees with (1.1). Ambient Euclidean geodesics remain straight lines. In polar coordinates these same lines satisfy an equation with nonzero connection coefficients, since the frame varies with the point.

2.4 A connection on a Lie group

Let G carry a bi-invariant Riemannian metric, such as the metric induced by the negative trace pairing on the skew-Hermitian matrices of a unitary group. Identify the Lie algebra with the left-invariant fields. Invariance of the inner product at the identity under the adjoint action gives

$$g([X, Y], Z) + g(Y, [X, Z]) = 0$$

on such fields, by differentiating the adjoint action along a one-parameter subgroup. In the Koszul formula their pairwise inner products are constant, and interchanging arguments in the bracket terms using the displayed identity gives $\nabla_X Y = \frac{1}{2}[X, Y]$. A one-parameter subgroup $\gamma(t) = \exp(t\xi)$ therefore has $\nabla_t \dot\gamma = 0$. In left-translated coordinates $v(t) \in \mathfrak{g}$, a field along this curve is parallel when

$$\dot v(t) + \tfrac{1}{2}[\xi, v(t)] = 0.$$

Its solution is $v(t) = \exp(-t\ \mathrm{ad}_\xi\, /2)v(0)$. The first exponential above is the Lie group exponential, and the one in this formula is the exponential of a linear operator. The two exponentials describe the moving base point and the moving vector, respectively. The operator is skew for the invariant inner product, so transport is orthogonal. In the next chapter the same connection gives curvature through a single Lie bracket.

The connection is the structure governing motion of a single vector along a single curve. The intrinsic effect of commuting two such infinitesimal motions is the curvature tensor. Its definition subtracts the derivative along the commutator of the underlying vector fields. This subtraction is essential when the fields come from a noncoordinate frame.

CHAPTER 3

Intrinsic and extrinsic curvature

The acceleration of a curve in Euclidean space can be resolved into components tangent and normal to an immersed submanifold. The tangential component is governed by the induced metric. In dimension two the normal component also determines the intrinsic Gaussian curvature through a quadratic identity. We derive that identity from flatness of the ambient derivative and the splitting of the ambient tangent bundle. This gives the tensor form of the calculation underlying Tu's treatment [1, §§ 5,8–9,12].

3.1 The curvature of a connection

The curvature of ∇ is

$$R(X,Y)Z = \nabla_X \nabla_Y Z - \nabla_Y \nabla_X Z - \nabla_{[X,Y]} Z. \tag{3.1}$$

It is $C^\infty(M)$-linear in all three fields. In the first input, the term $-Y(f)\nabla_X Z$ arising from fX cancels the contribution of the bracket $[fX,Y] = f[X,Y] - Y(f)X$. Skew symmetry then handles the second input. In the third input, the terms differentiating f once cancel between the two compositions of ∇, and the scalar second derivatives cancel against $[X,Y](f)$. Thus R is a two-form with values in endomorphisms of TM.

For a metric connection, applying $XY - YX - [X,Y]$ to the scalar $g(Z,W)$ and expanding metric compatibility gives

$$g(R(X,Y)Z,W) = -g(Z,R(X,Y)W).$$

The mixed terms from differentiating both fields cancel in pairs. For a torsion-free connection, the cyclic sum in X,Y,Z of (3.1) is zero. To check this at a point, use coordinate fields, so that all brackets vanish. Pairing terms with the same outer derivative leaves $\nabla_X(\nabla_Y Z - \nabla_Z Y)$ and its cyclic permutations, each zero by symmetry of the Christoffel symbols. Tensoriality extends the statement to all fields.

For Levi-Civita curvature there is also pair exchange. Use the components

$$R_{ijk\ell} = g(R(\partial_i,\partial_j)\partial_k,\partial_\ell).$$

At a point p, start with a chart x centered there and introduce y by

$$x^a = y^a - \tfrac{1}{2}\Gamma^a{}_{bc}(p)y^b y^c.$$

This defines new coordinates near p by the inverse function theorem. The second derivatives of $x(y)$ cancel the old Christoffel symbols at p. Torsion-freeness ensures they are symmetric in b, c. In the new chart $\Gamma(p) = 0$, and metric compatibility also gives $\partial_i g_{jk}(p) = 0$. Evaluating (2.2) and its first derivatives at p gives

$$\begin{aligned} 2R_{ijk\ell} = {} & \partial_i\partial_k g_{\ell j} - \partial_i\partial_\ell g_{jk} \\ & - \partial_j\partial_k g_{\ell i} + \partial_j\partial_\ell g_{ik}. \end{aligned}$$

Since mixed derivatives commute and g is symmetric, this expression equals $2R_{k\ell ij}$. These identities characterize the symmetries of the metric curvature tensor [2].

For a two-plane $\Pi \subseteq T_pM$, choose a spanning pair X, Y. Its sectional curvature is

$$K(\Pi) = \frac{g(R(X,Y)Y,X)}{|X|_g^2|Y|_g^2 - g(X,Y)^2}.$$

In the numerator, skew symmetry in each pair gives a quadratic expression on $X \wedge Y$. Both numerator and denominator therefore scale by the square of the determinant of a change of basis of Π. This proves independence of the spanning pair. On a surface there is only one plane at each point, giving its Gaussian curvature.

Example 3.1 (Bi-invariant metrics)**.** In the Lie group example of Chapter 2, let X, Y, Z be left-invariant fields. The commutator of their covariant derivatives acts on Z by $\frac{1}{4}[[X,Y],Z]$, since the Jacobi identity rewrites the difference of the two nested brackets. The last term of (3.1) acts by $\frac{1}{2}[[X,Y],Z]$. Thus

$$R(X,Y)Z = -\tfrac{1}{4}[[X,Y],Z].$$

Taking $Z = Y$, pairing with X, and using skew-adjointness of the bracket gives $g(R(X,Y)Y,X) = \frac{1}{4}|[X,Y]|^2$. Positive sectional curvature in a plane therefore occurs exactly when the two directions fail to commute. On a torus all brackets vanish, and this metric is flat.

3.2 The Gauss and Codazzi identities

Let M^n be immersed in $\mathbb{R}^{n+1}$ with induced metric. Work locally with a smooth unit normal N and write D for the ambient Euclidean derivative. Differentiating $\langle N, N\rangle = 1$ shows that $D_X N$ is tangent. Define the shape operator $S(X) =$

$-D_X N$ and the second fundamental form $\mathrm{II}(X,Y) = g(SX,Y)$. Differentiating $\langle Y, N\rangle = 0$ yields

$$D_X Y = \nabla_X Y + \mathrm{II}(X,Y)N.$$

Here ∇ is Levi-Civita by Chapter 2. Taking the normal component of $D_X Y - D_Y X = [X,Y]$ proves that II is symmetric. This decomposition can also be recorded by the splitting

$$\begin{array}{ccc} TM \oplus \mathbb{R}N & \xrightarrow{\simeq} & M \times \mathbb{R}^{n+1} \\ {\scriptstyle \mathrm{pr}_{TM}}\downarrow & \swarrow_{\mathrm{pr}_{TM}} & \\ TM. & & \end{array}$$

The derivative in the ambient bundle does not preserve either summand in general, and S and II measure its off-diagonal terms.

Theorem 3.2 (Gauss and Codazzi for a Euclidean hypersurface)**.** *One has*

$$R(X,Y)Z = \mathrm{II}(Y,Z)SX - \mathrm{II}(X,Z)SY,$$
$$(\nabla_X \mathrm{II})(Y,Z) = (\nabla_Y \mathrm{II})(X,Z).$$

Proof. Insert the decomposition of $D_Y Z$ into $D_X D_Y Z$, subtract the counterpart with X, Y interchanged, and subtract $D_{[X,Y]}Z$. Flatness of the Euclidean derivative makes the result zero. Its tangential component is $R(X,Y)Z - \mathrm{II}(Y,Z)SX + \mathrm{II}(X,Z)SY$. Its normal component is the coefficient

$$\begin{aligned} X\mathrm{II}(Y,Z) - Y\mathrm{II}(X,Z) + \mathrm{II}(X,\nabla_Y Z) - \mathrm{II}(Y,\nabla_X Z) \\ - \mathrm{II}([X,Y],Z). \end{aligned}$$

Use $[X,Y] = \nabla_X Y - \nabla_Y X$ and the product rule for the covariant derivative of a covariant tensor. The coefficient is the difference of the two derivatives in the second identity. □

When $n = 2$, choosing an orthonormal basis gives $K = \det S$. Changing N to $-N$ negates S while preserving its determinant in dimension two. Because the right-hand definition of K uses only the metric and its Levi-Civita connection, two isometric surfaces have equal Gaussian curvature at corresponding points. This is the intrinsic content of the Theorema Egregium.

Example 3.3 (Graph, cylinder, and sphere)**.** For a graph $z = f(x,y)$ in $\mathbb{R}^3$, put $w = (1 + |\nabla f|^2)^{1/2}$ and use the upward normal $N = (-f_x, -f_y, 1)/w$. Its first fundamental form has determinant w^2. The second fundamental form has matrix (f_{ij}/w). Thus $K = \det(f_{ij})/w^4$. Taking $f = xy$ gives negative curvature at every point of its graph. On the cylinder of radius a, the two eigenvalues of S with outward normal are $-1/a$ in the circular direction and 0 along its axis. Hence $K = 0$. The local parametrization $(s,t) \mapsto (a\cos(s/a), a\sin(s/a), t)$ exhibits the same result by pulling the metric back to $ds^2 + dt^2$. On the sphere of radius a one has $S = -a^{-1}\,\mathrm{id}$, so $K = a^{-2}$.

3.3 A curvature form on an oriented surface

Take a positive orthonormal frame e_1, e_2 and define a one-form ω by $\nabla e_1 = \omega e_2$. Metric compatibility gives $\nabla e_2 = -\omega e_1$. Insert these two equations into (3.1). The product terms in $\omega(X)\omega(Y)$ cancel, yielding

$$R(X,Y)e_1 = (d\omega)(X,Y)e_2, \qquad d\omega = -K\, dV_g.$$

The minus sign follows from $K = g(R(e_1,e_2)e_2,e_1)$. For the positive coframe (θ^1,θ^2), the first structure equations are $d\theta^1 = \omega \wedge \theta^2$ and $d\theta^2 = -\omega \wedge \theta^1$. They follow by evaluating $d\theta^i$ on two fields and replacing their bracket by the difference of the covariant derivatives. Rotating the frame through an angle ϕ changes ω to $\omega + d\phi$, by differentiation of the rotation. The curvature form is thus unchanged. When local angles have only circle-valued transitions, the differentials $d\phi$ still agree for different real lifts. In Chapter 4, integration of this identity turns those frame changes into a topological integer.

Example 3.4 (The effect of conformally scaling a plane metric)**.** Take $g = e^{2u}(dx^2+dy^2)$ and the positive coframe $(e^u dx, e^u dy)$. The first structure equations give $\omega = -u_y\, dx + u_x\, dy$. Consequently

$$K = -e^{-2u}(u_{xx} + u_{yy}).$$

For $u = -\log y$ on the upper half-plane, this is -1. For $u = \log(2a^2) - \log(a^2 + x^2 + y^2)$, the Laplacian is $-4a^2/(a^2+x^2+y^2)^2$, giving $K = a^{-2}$. The latter metric is the usual stereographic expression for the round sphere of radius a. More generally, on a conformal chart the curvature is controlled by a scalar second-order equation. The sign of its Laplacian determines the sign of the Gaussian curvature pointwise.

CHAPTER 4

Geodesics, Jacobi fields, and surfaces

The covariant derivative turns geodesic motion into a differential equation. A variation of one geodesic through other geodesics satisfies a new, linear equation containing curvature. On surfaces the same curvature has a global integral constrained by the topology. The local motion and the global identity thus give two ways of detecting the curvature defined in Chapter 3. We use Tu's conventions for the geodesic equation and his presentation of the surface theorem as a comparison [1, §§ 13–17].

4.1 Energy variation and the exponential map

Let $\Gamma(s,t)$ be a smooth variation of a path $\gamma(t) = \Gamma(0,t)$ on $[a,b]$. Put $T(s,t) = \partial_t\Gamma(s,t)$ and $V(s,t) = \partial_s\Gamma(s,t)$. The energy of the path $t \mapsto \Gamma(s,t)$ is

$$E(s) = \frac{1}{2}\int_a^b g_{\Gamma(s,t)}\big(T(s,t), T(s,t)\big)\,dt,$$

with s held fixed during integration. Both fields are sections of Γ^*TM. The torsion-free identity $\nabla_s T = \nabla_t V$ holds because the mixed partials of Γ agree and the Christoffel contributions are symmetric. Therefore

$$E'(s) = \big[g(V,T)\big]_a^b - \int_a^b g(V, \nabla_t T)\,dt. \tag{4.1}$$

Here metric compatibility first replaces the derivative of the squared speed by $2g(\nabla_s T, T)$, and then gives the integration-by-parts formula. For variations fixing both endpoints, the boundary term vanishes. Every smooth field supported in a small segment of the interior serves as a variation vector there, using coordinate displacement multiplied by a cutoff. Thus $E'(0) = 0$ for every fixed-endpoint variation implies the geodesic equation along γ by the fundamental lemma of the calculus of variations. The reverse implication follows from (4.1).

Let γ_v start at p with velocity v. Put $\exp_p(v) = \gamma_v(1)$ when the solution exists on $[0,1]$. The solution depends smoothly on the initial conditions. The rescaling relation $\gamma_{tv}(1) = \gamma_v(t)$ on its domain gives $(d\exp_p)_0 = \mathrm{id}_{T_pM}$, under the canonical identification of the tangent space of T_pM at zero. Thus $\exp_p$ maps a sufficiently small open ball diffeomorphically onto a normal neighborhood. The coordinates from a linear orthonormal identification $T_pM \simeq \mathbb{R}^n$ then have radial lines as geodesics.

Proposition 4.1 (Radial orthogonality)**.** *At a point v in a normal ball and for $w \in T_pM$, one has*

$$g_{\exp_p(v)}((d\exp_p)_v v, (d\exp_p)_v w) = g_p(v, w).$$

Proof. Use $\Gamma(s,t) = \exp_p(t(v+sw))$ for small s and $0 \le t \le 1$. Along a radial geodesic $\nabla_t T = 0$, so

$$\partial_t g(V,T) = g(\nabla_s T, T) = \tfrac{1}{2}\partial_s |v+sw|^2_{g_p}.$$

As $V(s,0) = 0$, integration to $t = 1$ and evaluation at $s = 0$ give the formula. The vector $T(s,1)$ at that endpoint is $(d\exp_p)_{v+sw}(v+sw)$. □

Consequently radial directions have unit length under the radial distance coordinate and are perpendicular to variations tangent to its spheres. For a path in the normal neighborhood the speed is at least the absolute derivative of the radius $|\exp_p^{-1}(\gamma(t))|$ wherever the radius is positive. Integrating, with limiting endpoints at zero allowed, proves minimality of the radial segment among paths in the neighborhood. If the endpoint has radius r smaller than the radius R of a compactly contained normal ball, any path leaving that ball has initial length at least R by the same estimate up to its first exit. Thus the radial segment minimizes among all paths to the sufficiently near endpoint. These arguments are local and do not assume geodesic completeness.

4.2 The linearized geodesic equation

Suppose now that every $t \mapsto \Gamma(s,t)$ is geodesic. The curvature commutator on Γ^*TM gives

$$\nabla_t^2 V = \nabla_t \nabla_s T = \nabla_s \nabla_t T + R(T,V)T = -R(V,T)T.$$

At $s = 0$, the field $J = V|_{s=0}$ thus satisfies

$$\nabla_t^2 J + R(J, \dot\gamma)\dot\gamma = 0. \tag{4.2}$$

Conversely, the linear equation has unique solutions prescribed by $J(0), \nabla_t J(0)$. Choose a smooth curve of initial positions with derivative $J(0)$, and vary the initial velocities along it with covariant derivative $\nabla_t J(0)$. To construct the varied velocities, use parallel transport along the curve of positions and add the desired first-order term in that trivialization. Smooth dependence of the geodesic flow gives a variation, at least locally on the geodesic interval, whose variation field has those two initial values. Uniqueness of the linear equation identifies it with J. This exhibits Jacobi fields as the linear response of geodesic motion [2].

For a unit-speed geodesic in constant sectional curvature K, the operator $J \mapsto R(J, \dot\gamma)\dot\gamma$ on the orthogonal complement of $\dot\gamma$ is $K\,\mathrm{id}$. Indeed it is self-adjoint by the curvature symmetries, and its quadratic form on perpendicular vectors is $K|J|^2$. In a parallel orthonormal frame perpendicular to $\dot\gamma$, a field with $J(0) = 0$ has coefficients proportional to

$$s_K(t) = \begin{cases} \sin(\sqrt{K}t)/\sqrt{K}, & K > 0, \\ t, & K = 0, \\ \sinh(\sqrt{-K}t)/\sqrt{-K}, & K < 0. \end{cases}$$

The tangential component with zero initial value is linear in t. On the round n-sphere of radius a with $n \geq 2$, the first positive time for a nonzero such field to vanish is πa, so the antipode is the first conjugate point. Varying the initial velocity in a geodesic with initial point held fixed identifies these vanishing fields with the kernel of the derivative of the exponential map at that time. For $K \leq 0$ the displayed solutions give no positive conjugate time in a constant-curvature model.

4.3 Second variation and local stability

Let γ be a geodesic and suppose the variation fixes both endpoints. Differentiate (4.1) at $s = 0$. The term involving $\nabla_s V$ vanishes there because $\nabla_t T = 0$, so

$$E''(0) = -\int_a^b g(V, \nabla_s \nabla_t T)\, dt.$$

The curvature commutator and $\nabla_s T = \nabla_t V$ give $\nabla_s \nabla_t T = \nabla_t^2 V + R(V, T)T$ at $s = 0$. Integration by parts, using $V(a) = V(b) = 0$, yields the quadratic form

$$I(V, V) = \int_a^b \left(|\nabla_t V|^2 - g(R(V, T)T, V) \right) dt.$$

Its bilinear polarization is symmetric by the metric curvature identities. A Jacobi field J vanishing at the endpoints pairs to zero with every such V, by (4.2) and integration by parts. The converse follows by testing against fields supported on interior segments. Thus conjugate endpoint data are detected by the kernel of the second variation.

For a unit-speed segment of length L on the round sphere, choose a perpendicular parallel unit field e, and set $V(t) = \sin(\pi t/L)e(t)$ for $0 \leq t \leq L$. The two terms in I integrate explicitly to

$$I(V, V) = \frac{L}{2}\left((\pi/L)^2 - a^{-2} \right).$$

The variation $(s, t) \mapsto \exp_{\gamma(t)}(sV(t))$ is defined for sufficiently small s, by compactness of the path interval and the local existence theorem. When $L > \pi a$

the quadratic form is negative on V, giving a decrease in energy through that variation. At $L = \pi a$, the displayed field is the conjugate Jacobi field already computed.

Example 4.2 (The upper half-plane)**.** On $y > 0$, the metric $g = y^{-2}(dx^2 + dy^2)$ has oriented orthonormal frame $e_1 = y\partial_x$, $e_2 = y\partial_y$. Its bracket is $[e_1, e_2] = -e_1$, so the Koszul formula yields $\nabla_{e_1} e_1 = e_2$ and $\nabla_{e_2} e_1 = 0$. Hence $\omega = dx/y$ and $d\omega = dx \wedge dy/y^2 = dV_g$, giving $K = -1$. Curves $t \mapsto x_0 + iy_0 e^t$ have unit velocity e_2 and are geodesics. For $a, b, c, d \in \mathbb{R}$ with $ad - bc > 0$, the map $z \mapsto (az+b)/(cz+d)$ multiplies the imaginary part and the norm of the Euclidean differential by the same factor $(ad - bc)/|cz + d|^2$. It therefore preserves g. The images of the imaginary axis under $z \mapsto x_0 + r(z - 1)/(z + 1)$, $r > 0$, are the semicircles centered at x_0. These are geodesics as well, by the naturality result of Chapter 2.

4.4 From boundary turning to Gauss–Bonnet

For a unit oriented boundary tangent $T = \cos\theta\, e_1 + \sin\theta\, e_2$ on a smooth arc, define $k_g = g(\nabla_T T, JT)$, where J is positive rotation by $\pi/2$. In a local positive frame the derivative of the expression for T gives $k_g = d\theta/ds + \omega(T)$. On a positively oriented embedded triangular disk D in an oriented chart, the boundary tangent, with signed exterior turns inserted at its three corners, turns by 2π. One may compute this using the planar turning theorem in the chart, since a positive frame change extending over a disk contributes zero winding. The turning identity and Stokes' theorem, with $d\omega = -K\, dV_g$, give

$$\int_D K\, dV_g + \int_{\partial D} k_g\, ds + \sum_{\text{corners}} (\pi - \alpha) = 2\pi, \tag{4.3}$$

where α is the interior angle at that corner [1, § 17.4].

For a compact oriented surface without boundary, choose a smooth triangulation fine enough that each triangle lies in a coordinate neighborhood. Sum (4.3) over triangles. The contributions along a shared edge cancel: reversing its orientation negates k_g and preserves ds. Interior angles around each vertex sum to 2π, because the incident tangent sectors fill its tangent circle. As each triangle has three corners, the sum of exterior turns is $3\pi F - 2\pi V$, where F, V count faces and vertices. With $3F = 2E$ for the edges, this yields

$$\int_M K\, dV_g = 2\pi(V - E + F) = 2\pi\chi(M).$$

For the round sphere the calculation gives $a^{-2}(4\pi a^2) = 4\pi$. The independence from a expresses a general fact: a curvature form can vary with the metric while the cohomology class it represents stays fixed. This is the basic surface example of the characteristic-class construction developed later in the book.

Part II Bundles, connections, and transport

CHAPTER 5

Vector bundles and their descent

A frame identifies vectors over nearby points with a single model vector space. Two frames identify that space in different ways, and their change of coordinates is part of the geometry. The descent description retains the change of coordinates and the equations it satisfies. It thus applies to sections, to bundle maps and to connections. We use smooth manifolds that are Hausdorff and second countable, with real or complex vector bundles of finite rank. The local frame and section descriptions follow the setting of Tu [1, §7].

5.1 Pullback, local frames, and sections

A rank-r vector bundle over M is a smooth map $p\colon E \to M$ with a vector space structure on each fiber and local product identifications

$$E|_U \simeq U \times \mathbb{K}^r, \qquad \mathbb{K} = \mathbb{R} \text{ or } \mathbb{C},$$

which are linear on fibers. A frame on U is an ordered tuple of sections whose values are a basis at every point of U. Write e_i for the row of basis vectors in a frame on U_i, so $e_i v_i$ denotes the vector with column of coordinates v_i. On an overlap put $e_j = e_i g_{ij}$. Then

$$g_{ij}\colon U_i \cap U_j \longrightarrow \mathrm{GL}_r(\mathbb{K}), \qquad v_i = g_{ij} v_j.$$

The matrix entries of g_{ij} are smooth because a bundle chart smoothly extracts the coordinates of the smooth sections of the second frame. On a triple overlap, $g_{ij} g_{jk} = g_{ik}$, by expressing e_k in frame e_i in two ways.

For $f\colon N \to M$ the pullback bundle is

$$f^*E = \{(y, e) \in N \times E \mid f(y) = p(e)\}.$$

It is smooth and locally trivial since the inverse images of bundle charts give product charts for it. The commutative square

$$\begin{array}{ccc} f^*E & \longrightarrow & E \\ \downarrow & & \downarrow{\scriptstyle p} \\ N & \xrightarrow{f} & M \end{array}$$

has the pullback property in smooth manifolds. A smooth map into the upper left corner consists of maps into N and E satisfying the compatibility at M. Smoothness of the resulting map can be checked in the product charts above, so no assumption that arbitrary fiber products of manifolds exist is needed. On frames, pullback replaces g_{ij} by $g_{ij} \circ f$. For a further map $h\colon P \to N$, the comparison $h^*f^*E \simeq (fh)^*E$ forgets the redundant intermediate base point. The comparisons for three successive maps agree since both delete the same intermediate points.

Sections over an open set U form a $C^\infty(U, \mathbb{K})$-module $\Gamma(U, E)$. Their restrictions form a sheaf, because smoothness of a section can be tested in local frames. Concretely a section over M is a family of smooth $v_i\colon U_i \to \mathbb{K}^r$ with $v_i = g_{ij}v_j$. A bundle map $F\colon E \to E'$ over M is recorded on a shared trivializing cover by matrices F_i with

$$F_i g_{ij} = g'_{ij} F_j.$$

This equation holds for rectangular matrices as well. It states that converting a vector to the i-frame and applying the map agrees with applying the map in the j-frame and converting the result.

Proposition 5.1. *Every $C^\infty(M, \mathbb{K})$-linear map $\Gamma(M, E) \to \Gamma(M, E')$ arises from a unique bundle map.*

Proof. Let T be such a map. If a section s vanishes near x, choose a bump function χ equal to one in a smaller neighborhood and supported where $s = 0$. Then $0 = T(\chi s) = \chi T(s)$, so $T(s)(x) = 0$. Thus T depends at x only on the germ of s. Multiply a local frame by a cutoff equal to one on a smaller open set and extend it by zero. The resulting global sections have images under T that supply the columns of a smooth local matrix. For any s, write it in that frame near x and extend its local coefficient functions by another cutoff. Germ locality and linearity over functions show that $T(s)(x)$ is the action of the local matrix on $s(x)$. Every vector in E_x extends to a section by a local frame and a cutoff. This proves uniqueness and forces the local matrices to agree as bundle maps. □

5.2 Effective gluing

Theorem 5.2 (Descent for vector bundles)**.** *Let (U_i) be an open cover of M. Bundles over M are equivalent to families of bundles $E_i \to U_i$ with isomorphisms $\varphi_{ij}\colon E_j|_{U_i \cap U_j} \to E_i|_{U_i \cap U_j}$ such that $\varphi_{ii} = 1$ and $\varphi_{ij}\varphi_{jk} = \varphi_{ik}$ on triple intersections. Maps of these families are bundle maps commuting with the isomorphisms.*

Proof. Identify the point $e \in E_j$ over $x \in U_i \cap U_j$ with $\varphi_{ij}(e) \in E_i$. The cocycle law makes these identifications an equivalence relation since it ensures both

inverses and closure under composition. The quotient is a set over M, and over U_i every class has a unique representative in E_i. The quotient topology restricts to the original topology on E_i, since each overlap map is a homeomorphism. The transported smooth bundle charts agree on overlaps because the overlap isomorphisms are smooth. Different fibers can be separated by base neighborhoods, and two vectors in the same fiber lie in a common Hausdorff chart. Hence the total space is Hausdorff. A countable subcover of local charts exists because the base is second countable, and a countable union of their countable bases gives a countable base upstairs. The linear structures in the model charts glue under the linear isomorphisms.

A compatible family of maps is constant on equivalence classes after taking the target quotient, and so gives a unique bundle map. It is smooth on the covering charts. Restriction and gluing are inverse up to the canonical isomorphisms supplied by the identifications over U_i. □

For trivial local bundles this construction is the quotient by the matrix transitions g_{ij}. Replacing e_i by $e_i h_i$ changes the transition to $h_i^{-1} g_{ij} h_j$. Thus a cocycle is an object in a local presentation and its changes of frame are arrows. This matters when forming a moduli object. A bundle can possess many automorphisms even when its isomorphism class is fixed.

Example 5.3 (Two real line bundles on a circle)**.** Use two open arcs covering S^1, with two components in their intersection. The transition can be $+1$ on both components or $+1$ on one and -1 on the other. A nowhere-zero section in the latter case would give a continuous nonzero function on each arc. Each function has one sign on its entire arc. The transition would require their signs to agree on one overlap component and to disagree on the other, which is impossible. The first case has the constant unit section, so the two gluings already differ as bundles. Their local frames present the same model fiber.

Example 5.4 (Winding in a clutching presentation)**.** On the two disc-like charts of S^2, use a unitary line transition restricting on the equator to $g(e^{i\theta}) = e^{im\theta}$. A change of frame on either disc has zero winding on its boundary because it extends across the disc. Thus the winding of the transition is unchanged by any bundle isomorphism, written as changes of frame over the given charts. In particular a nonzero m yields a nontrivial bundle. This topological obstruction will also be detectable by integration of curvature once a connection has been chosen.

5.3 Descent along a surjective submersion

Open inclusions suffice to construct each bundle, but descent under maps with local sections will be useful for quotient presentations. Let $q\colon Y \to M$ be a

surjective submersion and write $Y^{[j]} = Y \times_M \cdots \times_M Y$ with j factors. A descent datum is a bundle $D \to Y$ with compatible isomorphisms

$$\alpha(y_1, y_2)\colon D_{y_2} \longrightarrow D_{y_1}, \qquad \alpha(y_1, y_2)\alpha(y_2, y_3) = \alpha(y_1, y_3).$$

The first formula is a smooth bundle isomorphism on $Y^{[2]}$ and the second is an equality over $Y^{[3]}$. A bundle pulled back from M has this datum because both fibers above (y_1, y_2) are copies of the same fiber over $q(y_1)$.

Conversely, the submersion theorem supplies a cover by open sets U_i with sections $s_i\colon U_i \to Y$ of q. Use $D_i = s_i^* D$ and the transition $\alpha(s_i(x), s_j(x))$ in Theorem 5.2. The resulting E is a bundle over M. Its pullback to Y identifies with D over $q^{-1}(U_i)$ by applying $\alpha(y, s_i(q(y)))$ in the fiber. The cocycle law makes the identifications agree on overlaps and intertwine the given descent datum. A map commuting with α restricts to maps of the D_i that glue by the same theorem. This gives effectivity on objects and on maps. Using a common refinement compares two choices of local sections, so the isomorphisms furnished by this construction are canonical for the given descent datum.

5.4 Frames as a torsor

The fiber of the frame bundle $\mathrm{Fr}(E)$ at x is the set of linear isomorphisms $\mathbb{K}^r \to E_x$. Right composition by $\mathrm{GL}_r(\mathbb{K})$ acts freely and transitively. Here *transitive* means that any two frames are related by some change of basis, and *free* means that change of basis is unique. The quotient of $\mathrm{Fr}(E) \times \mathbb{K}^r$ by

$$(eg, v) \sim (e, gv)$$

recovers E by $[e, v] \mapsto e(v)$, with the local charts above. An action of another group G can replace the action of GL_r when a representation of G on the model fiber is specified. Chapter 7 develops the connection on such a principal bundle and the connection it induces on the associated bundle.

CHAPTER 6

Connections on vector bundles

Sections of a vector bundle live in different fibers at different points. A connection specifies how to take a derivative of a section while accounting for that variation. Tu develops the covariant derivative, its matrices and their curvature in §§10–11 and 22 [1]. The gluing theorem of Chapter 5 allows the derivative to be specified entirely in local frames.

6.1 Leibniz rule and existence

Definition 6.1. A connection on a vector bundle $E \to M$ is an $\mathbb{R}$-linear operator $\nabla\colon \Gamma(E) \to \Omega^1(M;E)$, complex-linear for a complex bundle, such that

$$\nabla(fs) = df \otimes s + f\nabla s.$$

It acts on a vector field X by contraction, $\nabla_X s = (\nabla s)(X)$.

If s vanishes near x, multiply by a cutoff χ supported in this neighborhood and equal to one near x. Applying the Leibniz rule to $\chi s = 0$ gives $(\nabla s)(x) = 0$. This locality induces connection operators on restrictions to open sets. In a frame e_i write $\nabla e_{i,b} = \sum_a e_{i,a}(A_i)_{ab}$. The local formula on a section $e_i v_i$ is then

$$\nabla(e_i v_i) = e_i(dv_i + A_i v_i).$$

Every smooth matrix A_i of one-forms defines a connection on this product chart. Choose local product connections $\nabla^{(i)}$ and a locally finite smooth partition of unity (ρ_i) subordinate to their domains. The products $\rho_i \nabla^{(i)} s$ extend by zero and their sum is smooth. The coefficient of $df \otimes s$ in its Leibniz identity is $\sum_i \rho_i = 1$, which proves existence.

The difference of two connections satisfies $(\nabla' - \nabla)(fs) = f(\nabla' - \nabla)(s)$. The local matrix argument of Chapter 5 therefore identifies it with a section of $T^*M \otimes \operatorname{End} E$, and adding any such section to a connection gives another. The set of connections is consequently an affine space modeled on $\Omega^1(M;\operatorname{End} E)$. For a fiber metric, the compatibility condition is

$$d\langle s,t\rangle = \langle \nabla s, t\rangle + \langle s, \nabla t\rangle.$$

In an orthonormal frame this is $A_i^* = -A_i$, with transpose in the real case and Hermitian adjoint in the complex case. Local orthonormal frames supply local

metric connections. The partition of unity construction preserves this metric identity, and two metric connections differ by a one-form valued in skew-adjoint endomorphisms.

6.2 Changing the frame and pulling back

The frame $e_j = e_i g_{ij}$ gives the coordinate relation $v_i = g_{ij} v_j$. Substituting into the Leibniz rule yields

$$A_j = g_{ij}^{-1} A_i g_{ij} + g_{ij}^{-1} dg_{ij}. \tag{6.1}$$

Indeed $(d + A_i)(g_{ij} v_j) = g_{ij} dv_j + (dg_{ij} + A_i g_{ij}) v_j$. The compatibility of local derivatives is the square

$$\begin{array}{ccc} C^\infty(U_{ij}, \mathbb{K}^r) & \xrightarrow{d+A_j} & \Omega^1(U_{ij}; \mathbb{K}^r) \\ \downarrow{\scriptstyle g_{ij}} & & \downarrow{\scriptstyle g_{ij}} \\ C^\infty(U_{ij}, \mathbb{K}^r) & \xrightarrow{d+A_i} & \Omega^1(U_{ij}; \mathbb{K}^r). \end{array}$$

Conversely, matrices satisfying (6.1) give compatible operators in this square, hence a connection on the glued bundle by the sheaf property. The term $g_{ij}^{-1} dg_{ij}$ records differentiation of the frame itself. A bundle isomorphism which identifies two connections is accordingly governed by the same formula.

For $f\colon N \to M$, define the pullback connection by matrices $f^* A_i$ in the pullback frames. Pullback commutes with d on functions, so these matrices satisfy (6.1) for the pulled-back transitions. This definition works for any smooth map and commutes with the canonical comparison for two successive pullbacks.

6.3 Curvature and the Bianchi identity

Extend ∇ to E-valued differential forms by

$$\nabla(\alpha \otimes s) = d\alpha \otimes s + (-1)^k \alpha \wedge \nabla s \qquad (\alpha \in \Omega^k(M)).$$

The Leibniz rule gives the same result after moving a smooth function between the two factors. In a frame the extension is $d + A\wedge$, where matrix products are combined with wedge products of entries. Applying it twice gives $(d + A\wedge)^2 = F\wedge$, with

$$F = dA + A \wedge A. \tag{6.2}$$

The d-terms hitting the argument cancel by the graded product rule. Thus the square is tensorial and defines an $\operatorname{End} E$-valued two-form. In the j-frame its matrix is $F_j = g_{ij}^{-1} F_i g_{ij}$ by applying the compatible operators twice. On vector fields the same square reads

$$F(X, Y)s = \nabla_X \nabla_Y s - \nabla_Y \nabla_X s - \nabla_{[X,Y]} s.$$

The matrix formula shows that $F_{f^*\nabla} = f^* F_\nabla$.

The induced covariant derivative on endomorphism-valued k-forms is

$$D\eta = d\eta + A \wedge \eta - (-1)^k \eta \wedge A.$$

Its Bianchi identity is $DF = 0$. Expanding gives $dF = dA \wedge A - A \wedge dA$, while $A \wedge F - F \wedge A = A \wedge dA - dA \wedge A$. The cubic terms cancel by associativity. The two-form F therefore satisfies a differential constraint involving the same connection that produced it. In particular $d \operatorname{tr} F = 0$, since the graded commutator has zero trace. Closed traces of higher curvature products will appear in Chern–Weil theory.

6.4 Tensor operations and a matrix example

For bundles E, E' with connections, the connection on their tensor product is

$$\nabla(s \otimes t) = (\nabla s) \otimes t + s \otimes (\nabla' t).$$

The Leibniz rules make this well-defined on the tensor product over functions. Its curvature is $F \otimes 1 + 1 \otimes F'$, as the mixed terms have opposite signs when the derivative is applied twice. The dual connection is fixed by requiring $d(\lambda(s)) = (\nabla^\vee \lambda)(s) + \lambda(\nabla s)$. In dual frames its matrix is $-A^{\mathsf{T}}$ and its curvature is $-F^{\mathsf{T}}$. The connection on the determinant $\bigwedge^r E$ is obtained by differentiating each wedge factor. Only the diagonal entries contribute to the connection of the top exterior frame, so its local one-form is $\operatorname{tr} A$ and its curvature is $\operatorname{tr} F = d \operatorname{tr} A$. For unitary connections we normalize the determinant curvature by $i/(2\pi)$. The scalar resulting two-form is real.

Example 6.2 (Constant coefficients with nonzero curvature)**.** On the trivial Hermitian bundle $\mathbb{R}^2 \times \mathbb{C}^2$, take $A = X\,dx + Y\,dy$ with

$$X = \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix}, \qquad Y = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}.$$

The entries of A are constant and both matrices are skew-Hermitian. Its curvature is

$$F = [X, Y]\, dx \wedge dy = \begin{pmatrix} 0 & 2i \\ 2i & 0 \end{pmatrix} dx \wedge dy.$$

Thus the connection has curvature even though the determinant connection is the product connection. The commutator is also the leading difference between the two orders of transport along short coordinate segments. Chapter 8 makes this comparison explicit.

6.5 Normal frames and varying the connection

At a prescribed point x_0, choose coordinates x^μ and write $A = \sum_\mu A_\mu\, dx^\mu$. The frame change

$$h(x) = \exp\Big(-\sum_\mu (x^\mu - x_0^\mu) A_\mu(x_0)\Big)$$

has $h(x_0) = I$ and $(dh)_{x_0} = -A_{x_0}$. Equation (6.1) then gives $A'_{x_0} = 0$. If the initial frame is orthonormal for a metric connection, each exponent is skew-adjoint and h remains orthogonal or unitary. Such a frame simplifies the derivative at a point. The curvature there is dA'_{x_0}, which can be nonzero. Vanishing on an entire neighborhood would imply $F = 0$ there by (6.2). This distinguishes pointwise normalization from a local parallel frame.

For a global one-form α valued in $\operatorname{End} E$, the connection $\nabla + \alpha$ has curvature

$$F_{\nabla+\alpha} = F_\nabla + D\alpha + \alpha \wedge \alpha.$$

The identity follows by expanding $d(A+\alpha)+(A+\alpha)^2$ and collecting terms of each degree in α. For the path of connections $\nabla_t = \nabla + t\alpha$, it follows that $\frac{d}{dt} F_{\nabla_t} = D_t \alpha$, where D_t is the derivative induced by ∇_t. The first variation of the curvature is linear in the direction, whereas the finite change includes the quadratic term. This distinction will produce a transgression between characteristic forms of two connections.

Example 6.3 (A projected connection)**.** Let $P\colon M \to \operatorname{End}(\mathbb{C}^N)$ be a smooth family of orthogonal projections of constant rank. The images form a vector bundle E because projected fixed vectors give local frames wherever an appropriate minor is invertible. Define $\nabla s = P\, ds$. The relation $Ps = s$ gives the Leibniz rule, and orthogonality shows that pairing with a vector in E is unchanged when projecting. Hence the connection preserves the induced metric. Its extension to E-valued forms is obtained by applying d componentwise and projecting. Differentiating $P^2 = P$ gives $P(dP)P = 0$. For a section s one has $ds = (dP)s + P ds$, so

$$\nabla^2 s = P\, d(P ds) = P\, dP \wedge ds = P(dP \wedge dP)s.$$

The projection can therefore produce a curved connection on a subbundle of a flat product. In the tautological line of Chapter 7, the projection is $P = zz^\dagger$ for a local unit column z. The induced one-form in that frame is $z^\dagger dz$, which provides a second construction of the same connection.

CHAPTER 7

Principal connections

A frame bundle is a bundle of choices of coordinates. Its fibers have a free transitive group action, so a local coordinate choice can be changed by a unique group element. Principal bundles keep this structure for any Lie group G. Connections become invariant horizontal distributions and can then be applied to several representations of the same group. The constructions below correspond to Tu's treatment of horizontal distributions, connection forms and associated bundles [1, §§27–31].

7.1 Right torsors and horizontal directions

A principal G-bundle is a smooth locally trivial bundle $\pi\colon P \to M$ whose fibers are right G-torsors. Local trivializations intertwine the given action with multiplication on the right of the second factor of $U \times G$. Equivalently, with the bundle charts in place, the action gives the diffeomorphism $P \times G \simeq P \times_M P$, $(p, g) \mapsto (p, pg)$. For each p, the map

$$\mathfrak{g} \longrightarrow V_pP = \ker(d\pi_p), \qquad \xi \longmapsto \xi_p^{\#} = \left.\frac{d}{dt}\right|_0 p\exp(t\xi)$$

is an isomorphism, as seen in a product chart. Right translation R_h sends $\xi_p^{\#}$ to $(\operatorname{Ad}_{h^{-1}}\xi)_{ph}^{\#}$. This follows from $\exp(t\xi)h = h\exp(t\operatorname{Ad}_{h^{-1}}\xi)$.

A horizontal distribution is a smooth complement H to VP in TP, invariant under right translation. The restriction $d\pi|_{H_p}$ is an isomorphism onto $T_{\pi(p)}M$, so a tangent vector downstairs has one horizontal lift at each point of its fiber. Map any tangent vector v to the Lie algebra element corresponding to its vertical component. This gives the connection form $\omega \in \Omega^1(P;\mathfrak{g})$ satisfying

$$\omega(\xi^{\#}) = \xi, \qquad R_h^*\omega = \operatorname{Ad}_{h^{-1}}\omega. \tag{7.1}$$

Conversely, (7.1) makes $v \mapsto (\omega(v))^{\#}$ an equivariant projection onto VP. Its kernel is the invariant complement. Thus the form and the horizontal distribution determine each other.

7.2 Local potentials and curvature

A local section $s_i\colon U_i \to P$ gives the potential $A_i = s_i^*\omega$. In coordinates $p = s_i(x)g$,

$$\omega = \mathrm{Ad}_{g^{-1}}\,\pi^* A_i + g^{-1}dg. \tag{7.2}$$

For the base tangent this is equivariance with g fixed. For the group tangent it is the left Maurer–Cartan form $g^{-1}dg$, meaning the tangent translated back to the identity by left multiplication by g^{-1}. On an overlap write $s_j = s_i g_{ij}$. Pulling (7.2) along s_j gives

$$A_j = \mathrm{Ad}_{g_{ij}^{-1}}\,A_i + g_{ij}^{-1}dg_{ij}.$$

Local forms with these transitions give a global connection by (7.2). To check compatibility, substitute $g = g_{ij}g'$ and use the product rule for the Maurer–Cartan form. Local product connections can also be combined using a partition of unity pulled back from M. Their weighted sum reproduces each vertical generator because the weights add to one, and equivariance survives because the weights are constant on the fibers. In particular every smooth principal bundle in our setting admits a connection.

Use the graded Lie bracket convention $[\omega,\omega](v,w) = 2[\omega(v),\omega(w)]$. Define the curvature by

$$\Omega = d\omega + \tfrac{1}{2}[\omega,\omega].$$

It is equivariant by (7.1). It is horizontal as well. Indeed the flow of $\xi^{\#}$ is $R_{\exp(t\xi)}$. Differentiating equivariance gives $\mathcal{L}_{\xi^{\#}}\omega = -[\xi,\omega]$. Cartan's formula and $\omega(\xi^{\#}) = \xi$, which is constant on P, give $\iota_{\xi^{\#}}d\omega = -[\xi,\omega]$. The contraction of the bracket term is $[\xi,\omega]$, so the sum vanishes. In local sections the curvature is $F_i = dA_i + \tfrac{1}{2}[A_i,A_i]$. In matrix representations this becomes $dA_i + A_i \wedge A_i$. The motion of horizontal directions is measured by the same form. For horizontal fields X^H, Y^H,

$$\Omega(X^H,Y^H) = -\omega([X^H,Y^H]). \tag{7.3}$$

This is the exterior derivative on a pair of fields in $\ker\omega$. Thus the horizontal subbundle is involutive precisely when $\Omega = 0$. This identifies flatness with an integrability condition before constructing any transport.

7.3 Flat gauges and smaller structure groups

When $\Omega = 0$, the Frobenius theorem supplies a local integral submanifold of H through any point of P. At that point the restriction of $d\pi$ to its tangent space is an isomorphism. After shrinking, the restriction of π is a diffeomorphism to an open neighborhood in M. Its inverse gives a horizontal local section, so the potential in that section is zero. Conversely a zero potential gives zero curvature on its domain. If two horizontal sections coexist on a connected open set, then

$g_{ij}^{-1} dg_{ij} = 0$ in their change of frame law. Thus they differ by a single constant group element. Flat bundles are consequently locally assembled by constant transitions, although their global gluing can be nontrivial.

A reduction of the structure group to a closed Lie subgroup $K \subset G$ is a principal K-subbundle $Q \subset P$. The connection restricts to a principal K-connection if $\omega(TQ) \subset \mathfrak{k}$. This condition is equivalent to $H_q \subset T_q Q$ at every q. If the image is in $\mathfrak{k}$, subtracting the corresponding vertical K-vector makes any tangent vector in $T_q Q$ horizontal. Since $T_q Q$ surjects onto $T_{\pi(q)} M$, it contains all horizontal lifts there. In the other direction, the decomposition of $T_q Q$ into its horizontal vectors and its vertical K-directions shows that its image is in $\mathfrak{k}$. The reproduction and equivariance axioms restrict as well. For instance, an orthogonal reduction of a real frame bundle arises from a metric, and requiring the connection to restrict to that reduction is precisely the skew matrix condition of Chapter 6.

7.4 Associated bundles and pullback

For a smooth linear representation $\rho \colon G \to \mathrm{GL}(W)$, define $E = P \times_G W$ by the relation $(pg, w) \sim (p, \rho(g)w)$. The chart on U_i identifies $[s_i(x), w]$ with w, giving transition $\rho(g_{ij})$. A section is equivalently a smooth map $u \colon P \to W$ such that $u(pg) = \rho(g)^{-1} u(p)$. Restrict du to horizontal vectors to define a connection on E. Equivariance of H ensures this definition is independent of the representative of a point of E. The Leibniz rule follows from differentiating $(f \circ \pi)u$.

In local coordinates let $v_i = u \circ s_i$. Subtracting $(A_i(X))^{\#}$ from $ds_i(X)$ gives the horizontal lift of X at $s_i(x)$. Differentiating u in that vertical direction gives $-\rho_*(A_i(X))v_i$. The local derivative is consequently

$$dv_i + \rho_*(A_i) v_i.$$

As ρ_* preserves brackets, the curvature is $\rho_*(F_i)$. In the adjoint representation the horizontal equivariant curvature Ω thus gives a form with values in $\operatorname{ad} P = P \times_G \mathfrak{g}$. Here one uses equivariance to transport the value of Ω between representatives and horizontality to choose tangent lifts. The curvature endomorphism of the associated adjoint connection acts by the Lie bracket with the descended value of Ω.

A smooth map $f \colon N \to M$ gives a pullback principal bundle and connection by pulling ω back along the projection to P. Fibers are mapped equivariantly, so the reproduction and equivariance conditions still hold. Associated bundles commute with this construction, as both sides identify $[((y, p), w)]$ with $(y, [p, w])$. The local potential formula then shows that their induced connections agree. For the frame bundle of a vector bundle, using its defining representation recovers the original bundle connection from the matrices of Chapter 6.

Example 7.1 (The unit circle bundle on the projective line)**.** Give $P = S^3 \subset \mathbb{C}^2$ the right scalar action of $U(1)$, so its quotient is $\mathbb{CP}^1$. The form $\omega = z^\dagger dz$ is imaginary on S^3, since the derivative of $z^\dagger z = 1$ has zero real part. It evaluates to ξ on $z\xi$, and unitary scalar multiplication preserves it. It therefore defines a principal connection. On the chart of lines spanned by $(1, w)$, take the unit section $(1, w)/\sqrt{1 + |w|^2}$. The potential and curvature are

$$A = \frac{\bar{w}\, dw - w\, d\bar{w}}{2(1 + |w|^2)}, \qquad F = dA = \frac{d\bar{w} \wedge dw}{(1 + |w|^2)^2}.$$

For example, differentiating the numerator gives $2\, d\bar{w} \wedge dw$, and differentiating the denominator subtracts the term proportional to $|w|^2$. The associated line for the standard representation has fiber the given line in $\mathbb{C}^2$, hence is the tautological line. In the complex orientation, writing $w = x + iy$ gives

$$\int_{\mathbb{CP}^1} \frac{i}{2\pi} F = -\frac{1}{\pi} \int_{\mathbb{R}^2} \frac{dx\, dy}{(1 + x^2 + y^2)^2} = -1.$$

The omitted point has zero area. In polar coordinates the real integral is

$$2\pi \int_0^\infty \frac{r}{(1 + r^2)^2}\, dr = \pi.$$

The normalization records both the orientation and the choice of line in this principal bundle construction.

CHAPTER 8

Paths, thin homotopy, and transport

A connection can be tested by transporting vectors and frames along paths. Composition of paths is then a composition of invertible maps of fibers. Curvature controls the dependence on a homotopy of paths. The appropriate functor for a general connection uses homotopies that sweep out no infinitesimal area.

8.1 Transport equations

Write a section along $\gamma\colon [0,1] \to M$ in a frame as a column $v(t)$. It is parallel when

$$\dot{v}(t) = -A_{\gamma(t)}(\dot{\gamma}(t))v(t).$$

Existence and uniqueness for linear ordinary differential equations give a solution on each closed segment contained in a chart. Along chart changes the relation $v_i = g_{ij}v_j$ and equation (6.1) give the same equation in the new frame. Subdivision and uniqueness therefore give a well-defined transport isomorphism $T_\gamma\colon E_{\gamma(0)} \to E_{\gamma(1)}$. On a principal bundle, a lift $\gamma^H(t) = s_i(\gamma(t))g(t)$ is horizontal when

$$\dot{g}(t) = -(dR_{g(t)})_1 A_i(\dot{\gamma}(t)).$$

This is (7.2) evaluated on the lifted tangent. It has a solution over each compact chart segment. To see the extension in the group variable, take a short solution starting at 1 and right translate it to the required initial value. The Lie algebra input is bounded on the segment, so local existence near 1 gives a uniform short time for these restarts. For a matrix group the right-hand side is simply $-A_i(\dot{\gamma}(t))g(t)$. The uniqueness of the horizontal lift gives equivariance, $T_\gamma(pg) = T_\gamma(p)g$. In a representation, the horizontal lift induces exactly the parallel equation with potential $\rho_*(A_i)$. This agrees with the frame interpretation in Tu [1, §29] and the principal parallel displacement of Kobayashi–Nomizu [3, Ch. II].

We use smooth paths with sitting instants, meaning constant segments near both endpoints. They can be joined smoothly at an endpoint. Ordinary changes of speed preserving the endpoint order leave transport unchanged by the chain rule. If γ_1 ends where γ_2 begins, uniqueness on the two pieces gives

$$T_{\gamma_2 \star \gamma_1} = T_{\gamma_2} \circ T_{\gamma_1}.$$

The constant path gives the identity. The transport of a reversed path is the inverse, since reversing the solution of the parallel equation gives the solution with reversed endpoints.

8.2 Thin homotopy and curvature variation

A thin homotopy of paths with fixed endpoints is a smooth map $H(s,t)$ interpolating between them, with constant ends in the path variable and rank $dH \leq 1$ everywhere. Homotopies can be reparameterized to sit near their boundaries before concatenating them. Reparameterizations of a path are thin because their interpolation factors through the path interval. A path followed by its reverse contracts by shortening the portion of the path traversed in both directions. This homotopy likewise factors through an interval. These facts give associativity, units and inverses for the thin homotopy classes of sitting paths. Write $\mathcal{P}_1(M)$ for this groupoid, with the points of M as objects [4, §2.1].

Proposition 8.1 (Thin invariance)**.** *Parallel transport of a principal connection is invariant under thin homotopies. For a flat connection it is invariant under all smooth homotopies relative to the endpoints.*

Proof. Lift $t \mapsto H(s,t)$ horizontally, starting for every s at the same point p_0 above the initial endpoint. Smooth dependence of the differential equation on parameters gives a smooth family $p(s,t)$. On the square let $a_s = \omega(\partial_s p)$ and $a_t = \omega(\partial_t p)$. Then $a_t = 0$, and the structure equation gives

$$(p^*\Omega)(\partial_s, \partial_t) = \partial_s a_t - \partial_t a_s + [a_s, a_t] = -\partial_t a_s.$$

The curvature is horizontal, so its value on a pair of tangent vectors is determined by their projections to the base. Under a thin homotopy those projections are linearly dependent. Thus the left side vanishes. It also vanishes for an arbitrary homotopy if the connection is flat. It follows that a_s is constant in t. At $t = 0$ it is zero because the lifted initial point is fixed. At $t = 1$ the field $\partial_s p$ is vertical because the terminal base point is fixed, and $a_s = 0$ implies that this vertical vector is zero. Hence the lifted terminal point is independent of s. □

It follows that transport assigns a functor

$$\mathcal{P}_1(M) \longrightarrow G\text{-Tor}, \qquad x \longmapsto P_x.$$

The target consists of right G-torsors and their equivariant isomorphisms. A connection-preserving bundle map induces a natural transformation, since it carries horizontal lifts with specified initial values to the corresponding horizontal

lifts. In particular the following square commutes for a path $\gamma\colon x \to y$ and such a bundle map Φ.

$$\begin{array}{ccc} P_x & \xrightarrow{T_\gamma} & P_y \\ \downarrow{\scriptstyle \Phi_x} & & \downarrow{\scriptstyle \Phi_y} \\ P'_x & \xrightarrow{T'_\gamma} & P'_y. \end{array}$$

8.3 Small rectangles and global holonomy

Consider a vector bundle and a coordinate rectangle of side ε in the x, y plane, beginning at a point taken as the origin and following $+x, +y, -x, -y$ in that order. Take a frame with $A(0) = 0$, as in Chapter 6. The integral of A along each edge is $O(\varepsilon^2)$. The integral equation for parallel transport on an edge consequently gives $I - \int_{\text{edge}} A + O(\varepsilon^4)$, with the bound from the second iterated integral and higher terms. Multiplying the four edge matrices gives

$$T_{\partial R_\varepsilon} = I - \oint_{\partial R_\varepsilon} A + O(\varepsilon^4) = I - \varepsilon^2 F_0(\partial_x, \partial_y) + O(\varepsilon^3).$$

The last equality uses Stokes entry by entry and $F_0 = (dA)_0$. The same leading formula holds in any frame at the base point, by conjugation. For constant $A = X\,dx + Y\,dy$ one can also multiply the exact transports in reverse traversal order,

$$T_{\partial R_\varepsilon} = e^{\varepsilon Y} e^{\varepsilon X} e^{-\varepsilon Y} e^{-\varepsilon X}.$$

Their second-order term is $-\varepsilon^2[X, Y]$, in agreement with curvature. In particular if vector transport descends to the ordinary fundamental groupoid, every small rectangle has trivial transport. Dividing by ε^2 and taking the limit proves $F = 0$. For a general Lie group, use coordinates on the base centered at the initial point. The gauge $h(x) = \exp(-\sum_\mu x^\mu A_\mu(0))$ makes $A(0) = 0$ by the same first-derivative argument. Use a group chart at 1 with differential equal to the identity. Each edge solution starts at 1 and stays $O(\varepsilon^2)$ from it in that chart, because its right-invariant driving term is $O(\varepsilon)$ on an edge of length ε. Translating its tangent back to the group chart changes the driving term by $O(\varepsilon^3)$, so its displacement is $-\int_{\text{edge}} A + O(\varepsilon^4)$. Group multiplication likewise differs from addition by $O(\varepsilon^4)$ for these displacements. This proves the same curvature limit in the Lie algebra. Thus flatness is exactly the condition for factoring through ordinary path homotopy. This formulation uses the usual smooth relative path model of the fundamental groupoid.

Example 8.2 (Holonomy distinguishes connections)**.** For a unitary line connection $A = -2\pi i\, a$ in a global frame, the parallel equation integrates to $T_\gamma = \exp(2\pi i \int_\gamma a)$. On $\mathbb{R}^2$ take $a = x\,dy$. The loop around a rectangle of

oriented area b has holonomy $e^{2\pi i b}$ even though it is contractible. Its phase is determined by the curvature integral. On $S^1 = \mathbb{R}/\mathbb{Z}$, take $a = \lambda\, d\theta$. This connection is flat and transport around a loop of winding m multiplies by $e^{2\pi i \lambda m}$. The gauge $g = e^{-2\pi i k\theta}$ for integer k replaces λ by $\lambda + k$. Conversely, two such constants with equal holonomy differ by an integer. Here curvature vanishes and the global holonomy still detects different connections up to gauge.

8.4 Flat transport and local systems

A functor from the ordinary fundamental groupoid to right G-torsors determines a flat bundle by gluing. Choose a cover by coordinate balls U_i with distinguished points x_i. On U_i use the product of U_i with the torsor assigned to x_i, with zero potential in a chosen model frame. A path inside U_i from x_i to x identifies that model torsor with the torsor assigned to x. All such paths have the same relative homotopy class. On the overlap, compose one of these identifications with the inverse of the other. The resulting transition is constant on each connected piece of the overlap. To verify constancy, join nearby points of the overlap by a path inside it and append it to both coordinate-ball paths. The induced map in the target groupoid cancels between the second identification and the inverse first one.

The transitions satisfy the cocycle law because they all use the torsor assigned to the point in M. Gluing the local products uses the same quotient as in Theorem 5.2, with multiplication on the G-coordinate in place of linear maps. The local zero potentials agree under the constant transitions, so the bundle is flat. Its transport recovers the original functor by subdivision into the coordinate balls. Conversely, a flat connection has horizontal trivializations by Chapter 7, and the products above recover its transition maps. Natural transformations give compatible maps between these products, locally constant in the chosen frames, and therefore produce the connection-preserving isomorphisms. In a connected manifold, a choice of base point and a point in its torsor describes the resulting groupoid by representations of the fundamental group up to conjugation. The description without a chosen base point retains the conjugating maps as morphisms.

8.5 Recovering a connection from transport

The smooth information in transport is essential for recovery. In a locally chosen frame, suppose a family of short paths from x to $x + \varepsilon v$ is obtained by smoothly reparameterizing the straight segment to have sitting instants. Let $U(\varepsilon)$ be its arrow in G, written in that frame at both endpoints. The transport equation

yields

$$A_x(v) = -U'(0) \in \mathfrak{g}.$$

Indeed the infinitesimal path velocity is proportional to v, and the integral of the time derivative of the scalar reparameterization is one. The slope thus recovers the local one-form. Under a change of frame g, a path arrow from x to y changes to $g(y)^{-1}Ug(x)$. Its derivative along a short segment recovers precisely the gauge law for the one-form.

The reconstruction theorem of Schreiber and Waldorf places this local argument in an equivalence of categories. A transport functor with BG-structure admits local trivializations by smooth BG-valued functors on a surjective submersion, with smooth descent data. Under those conditions it comes from a principal bundle with connection, and its natural morphisms arise from maps of the corresponding bundles with connection [4, Theorem 5.4]. The local hypotheses give the smooth dependence needed for the derivative and the transition data needed to glue the resulting forms. Thus the base groupoid and the smooth descent condition together retain both the local curvature and the global bundle.

Part III Chern–Weil theory and transgression

CHAPTER 9

Invariant polynomials and Chern–Weil forms

Curvature transforms by conjugation when a frame changes. This transformation law suggests a way to construct ordinary differential forms from it. A polynomial on the Lie algebra invariant under conjugation eliminates dependence on the frame. Closedness and connection independence then follow from two distinct differential identities. Tu develops this passage first for vector bundles and then for principal bundles [1, §§23,32].

9.1 Encoding a polynomial by polarization

Let G be a Lie group with real Lie algebra $\mathfrak{g}$. For a homogeneous polynomial $p\colon \mathfrak{g} \to \mathbb{R}$ of degree m, define

$$P(X_1, \dots, X_m) = \frac{1}{m!} \frac{\partial^m}{\partial t_1 \cdots \partial t_m} p\left(\sum_i t_i X_i\right)\Big|_{t=0}.$$

Then P is symmetric and multilinear, and $p(X) = P(X, \dots, X)$. The formula follows by extracting the coefficient of $t_1 \cdots t_m$, which also proves uniqueness. If p is invariant under $\mathrm{Ad}(G)$, simultaneous application of Ad_g to all inputs of P leaves it unchanged. Differentiating at $g = \exp(sZ)$ gives

$$\sum_{i=1}^{m} P(X_1, \dots, [Z, X_i], \dots, X_m) = 0. \tag{9.1}$$

For disconnected G, the differentiated equation tests only the identity component. The statement about descent uses invariance under the whole group.

The notation $P(\beta_1, \dots, \beta_m)$ also applies to $\mathfrak{g}$-valued differential forms. In a fixed basis, insert their coefficients in the displayed order and wedge the scalar forms. This description is independent of the basis by multilinearity. Permuting two inputs of degrees a, b gives the sign $(-1)^{ab}$. In particular, degree-two forms can be interchanged inside P. Write $P(F^m)$ for $P(F, \dots, F)$. For example, when $\mathfrak{g}$ is a matrix Lie algebra and $p(X) = \operatorname{tr}(X^2)$, one gets

$$P(F^2) = \operatorname{tr}(F \wedge F).$$

The symmetric polarization is useful even for matrix polynomials. The inputs are multiplied as matrices with form coefficients before taking the trace.

9.2 Descent and the Bianchi identity

Let $Q \to X$ be a smooth right principal G-bundle. Give it a connection with local potentials A_i in sections s_i, where $s_j = s_i g_{ij}$. Write

$$F_i = dA_i + \tfrac{1}{2}[A_i, A_i].$$

For matrix-valued forms the bracket is the graded commutator of their wedge products, so this formula reads $dA_i + A_i \wedge A_i$. If $D_A = d + [A, -]$ is the covariant exterior derivative, then $D_A F = 0$. In a matrix Lie algebra the cancellation is explicit:

$$\begin{aligned} dF &= dA \wedge A - A \wedge dA, \\ [A, F] &= A \wedge dA - dA \wedge A. \end{aligned}$$

The intrinsic Lie-algebra calculation replaces associativity of the matrix product by the graded Jacobi identity.

Theorem 9.1 (Chern–Weil forms)**.** *For every* $\mathrm{Ad}(G)$*-invariant polynomial* p *of homogeneous degree* m*, the expressions* $P(F_i^m)$ *patch to a closed* $2m$*-form* $p(F)$ *on* X*. The cohomology class* $[p(F)]_{\mathrm{dR}}$ *depends only on the principal bundle. It is natural under pullback, and the assignment from invariant polynomials to de Rham cohomology is a graded algebra homomorphism when polynomial degree is doubled.*

Proof. Since $F_j = \mathrm{Ad}_{g_{ij}^{-1}} F_i$, invariance identifies the two expressions on each overlap. Equivalently the curvature on Q is horizontal and equivariant. Applying P gives a horizontal invariant form on Q, which is the pullback of the form on X. To see the uniqueness of such a descended form, evaluate on lifts of tangent vectors at a point in a fiber. Horizontality makes the lifts immaterial, and invariance makes the fiber point immaterial. Local sections give smoothness.

The derivative of $P(F_i^m)$ is $mP(dF_i, F_i^{m-1})$ because F_i has even degree. The coefficientwise version of (9.1), applied with the coefficients of A_i and with its scalar one-form placed first, gives

$$mP([A_i, F_i], F_i^{m-1}) = 0.$$

Consequently $dP(F_i^m) = mP(D_{A_i} F_i, F_i^{m-1}) = 0$.

For independence take connections A_0, A_1. Their difference on Q is horizontal and equivariant, so their affine combination $A_t = (1-t)A_0 + tA_1$ satisfies the connection identities for every t. Use A_t as a connection on the pullback of Q to $[0,1] \times X$, with zero potential in the t-direction. Its curvature in frames constant in t is

$$\widetilde{F} = F_t + dt \wedge \dot{A}_t.$$

Inserting it into P and dropping terms with two copies of dt gives

$$P(\widetilde{F}^m) = P(F_t^m) + dt \wedge mP(\dot{A}_t, F_t^{m-1}).$$

Everything on the right descends. In particular $\dot{A}_t$ transforms homogeneously since the transition functions have no t-dependence. If a closed form is written $\alpha_t + dt \wedge \beta_t$, closedness in the coefficient of dt gives $\partial_t \alpha_t = d_X \beta_t$. Integrating this identity shows

$$p(F_1) - p(F_0) = d_X \left(m \int_0^1 P(\dot{A}_t, F_t^{m-1})\, dt \right). \tag{9.2}$$

The proof for constant polynomials uses the constant form. Pullback commutes both with curvature and with applying a polynomial. Multiplicativity holds already at the level of forms because the entries being substituted have even degree and commute in the exterior algebra. These properties give the assertion about classes. □

The central gluing step of the proof can be expressed on an overlap by the diagram

$$\begin{array}{ccc} F_i & \xrightarrow{\mathrm{Ad}_{g_{ij}^{-1}}} & F_j \\ & \searrow^{p} & \downarrow{\scriptstyle p} \\ & & p(F)|_{U_i \cap U_j}. \end{array}$$

Here each upper entry is a local section of the curvature-form bundle expressed in its own frame. The lower entry is a genuine scalar form.

9.3 Two distinctions visible on a torus

For $U(1)$, use a real local potential a with unitary connection matrix $A = -2\pi i\, a$. Then $F = -2\pi i\, da$ and $p(F) = iF/(2\pi) = da$ for the normalized degree-one polynomial. Take the trivial Hermitian line over $\mathbb{R}^2/\mathbb{Z}^2$ with

$$a = \sin(2\pi x)\, dy.$$

Its normalized curvature is

$$p(F) = 2\pi \cos(2\pi x)\, dx \wedge dy.$$

This form varies pointwise and is nonzero on open subsets, while its integral is zero. The trivial line admits the zero connection as well, and (9.2) exhibits the exact difference as da. Thus a fixed characteristic class allows different local curvature forms.

On the same torus, a closed real one-form such as $a = \theta\, dx$ gives curvature zero. A parallel section along a path solves $ds - 2\pi i\, a\, s = 0$, so a loop in the first circle direction transports by multiplication by $\exp(2\pi i\theta)$. These flat connections share zero curvature and a trivial topological line. Their parallel transport distinguishes them when the parameters differ in $\mathbb{R}/\mathbb{Z}$. Differential characteristic classes will retain that information.

9.4 Change of group

For a Lie-group homomorphism $\rho\colon G \to H$, extension of structure group constructs $Q \times_G H$ and its induced connection. Its local potentials and curvature are $\rho_* A_i$ and $\rho_* F_i$, since ρ_* preserves brackets. For an invariant polynomial q on $\mathfrak{h}$, the polynomial $q \circ \rho_*$ on $\mathfrak{g}$ is invariant, and

$$q(F_{Q\times_G H}) = (q \circ \rho_*)(F_Q).$$

This identity on forms explains how a representation of a principal bundle leads to the characteristic classes of an associated vector bundle. Replacing a representation by a direct sum replaces its curvature by a block sum. The determinant and trace extract different algebraic expressions from that block sum, as the next chapter develops.

9.5 A degree-four matrix polynomial

For a unitary matrix connection let $H = iF/(2\pi)$. Eliminating the eigenvalues from the degree-two elementary symmetric polynomial gives

$$c_2(\nabla) = \tfrac{1}{2}\big((\operatorname{tr} H)^2 - \operatorname{tr}(H \wedge H)\big) = \frac{\operatorname{tr}(F \wedge F) - \operatorname{tr} F \wedge \operatorname{tr} F}{8\pi^2}.$$

The first equality is a polynomial identity, so it holds over the even scalar forms even where diagonalization would make no sense. On an $SU(r)$-bundle the connection and curvature are traceless and only $\operatorname{tr}(F \wedge F)/(8\pi^2)$ remains. On a compact oriented four-manifold the normalization therefore turns an integral of a quadratic local expression into an integral topological number. Chapter 10 justifies the normalization through line bundles. Chapter 11 compares two such local representatives by an explicit three-form.

CHAPTER 10

Chern, Pontryagin, and Euler classes

The invariant polynomials of Chapter 9 yield many closed forms. A normalization selects those corresponding to the named topological classes. The comparison uses a complex line, where the period can be computed, and a flag construction, where the cohomology pullback is injective. Integral classes also contain torsion, which the computation with real forms cannot see [1, §25].

10.1 Unitary curvature and Chern forms

Let E be a complex vector bundle of rank r with Hermitian metric and unitary connection. In a unitary frame its curvature F is skew-Hermitian as an endomorphism-valued two-form. Define the total Chern form by

$$c(\nabla) = \det\left(I + \frac{i}{2\pi}F\right) = 1 + c_1(\nabla) + \cdots + c_r(\nabla).$$

The determinant is taken over the commutative algebra of scalar even-degree forms. Its homogeneous term containing j curvature factors has degree $2j$. Conjugation invariance allows Chapter 9 to be applied. The coefficients are real because the matrix iF equals its conjugate transpose, and the determinant is unchanged by transpose. All higher Chern forms vanish by rank.

For two bundles, the direct-sum connection has block diagonal curvature. Consequently its Chern form is the product of the two original Chern forms. A metric on a short exact sequence of smooth complex bundles identifies the middle bundle with the direct sum of the ends by taking the orthogonal complement of the subbundle. Together with connection independence this yields the Whitney product identity for the real classes. The corresponding integral Whitney identity is the standard topological one [5, §21].

The Chern character uses an additive invariant polynomial in each degree:

$$\operatorname{ch}(\nabla) = \operatorname{tr}\exp\left(\frac{i}{2\pi}F\right).$$

Only finitely many terms survive on a finite-dimensional base. The constant term is r; the terms of degrees two and four are $c_1(\nabla)$ and $\frac{1}{2}(c_1(\nabla)^2 - 2c_2(\nabla))$. These formulas follow from the Newton identities over the algebra of forms. On $E \otimes E'$, the induced curvature is $F \otimes I + I \otimes F'$, with commuting summands.

The exponential factors, and the trace of the tensor product is the product of the traces. This gives multiplicativity for the Chern character under tensor products and additivity under direct sums.

10.2 The normalization on a projective line

Let $L = \mathcal{O}(-1) \to \mathbb{CP}^1$ be the tautological line with its metric induced from $\mathbb{C}^2$. On the affine chart $[1 : z]$ the holomorphic frame $(1, z)$ has squared norm $h = 1 + |z|^2$. The Chern connection in this frame has potential $A = \partial \log h$. This formula follows from two conditions: its $(0, 1)$ part is ordinary $\bar{\partial}$ in a holomorphic frame, and preserving the metric requires $dh = h(A + \overline{A})$. Taking the $(1, 0)$ part gives the claimed expression. Although the potential in this frame need not be imaginary, it becomes imaginary in the corresponding unitary frame. Its curvature is independent of the frame:

$$A = \frac{\overline{z}\, dz}{1 + |z|^2}, \qquad F = \frac{d\overline{z} \wedge dz}{(1 + |z|^2)^2}.$$

The normalization integrates to

$$\int_{\mathbb{CP}^1} \frac{iF}{2\pi} = -\frac{1}{\pi} \int_{\mathbb{R}^2} \frac{dx\, dy}{(1 + x^2 + y^2)^2} = -1.$$

The integral is computed in polar coordinates, since the omitted point has measure zero for a smooth top form. The bundle's topological first Chern class has the same sign: the dual line $\mathcal{O}(1)$ has a section defined by a nonzero linear functional on $\mathbb{C}^2$, with one positively oriented transverse zero. Its Euler number and hence its degree are 1. The duality of lines reverses the Chern class. One can also recover the sign by using the transition z^{-1} from the frame $(1, z)$ to the frame $(z^{-1}, 1)$.

The standard comparison with integral Chern classes reduces to this calculation [5, Chapter IV]. For a line bundle, naturality applied to the universal line over projective space reduces normalization to the projective-line inclusion, which detects its degree-two generator. For a bundle E, pass to the full flag bundle $\pi\colon \operatorname{Flag}(E) \to X$. The successive projectivizations have cohomology free over the cohomology of their bases by the projective bundle theorem. The unit is one of the module generators, so each pullback is injective. In particular π^* is injective in real cohomology. The pullback of E has a flag with line-bundle quotients; choosing a metric splits the filtration as a direct sum of these lines. On the flag bundle, both the real image of the topological total Chern class and the curvature-defined total class are the product of $1 + c_1$ of the line factors. Injectivity proves equality downstairs.

10.3 Real bundles and Pontryagin classes

For a real bundle V, set

$$p_j(V) = (-1)^j c_{2j}(V \otimes_{\mathbb{R}} \mathbb{C}) \in H^{4j}(X;\mathbb{Z}).$$

A metric connection on V complexifies to a unitary connection, so the formula gives a Chern–Weil representative for the real image. In a real orthonormal frame $F^{\mathsf{T}} = -F$, and transposing the matrix in $\det(I + iF/(2\pi))$ shows that this polynomial is even in F. Thus the odd-index Chern forms of the complexification vanish. The integral odd-index Chern classes may still be two-torsion. Indeed conjugation identifies the complexification with its conjugate, and $c_k(\overline{E}) = (-1)^k c_k(E)$. It follows that the Pontryagin total class satisfies the product formula after inverting 2. Over the integers, products of two odd-index classes in the Whitney formula can contribute two-torsion [6, §15].

For an explicit torsion calculation, let λ be the tautological real line on $\mathbb{RP}^4$ and let $u = w_1(\lambda)$. The real bundle underlying $\lambda_{\mathbb{C}}$ is $\lambda \oplus \lambda$, whose second Stiefel–Whitney class is u^2. Thus $a = c_1(\lambda_{\mathbb{C}})$ reduces to u^2, so a^2 reduces to $u^4 \neq 0$. For $V = \lambda \oplus \lambda$ the integral $p_1(V) = -a^2$ is nonzero, whereas $p_1(\lambda) = 0$ by rank. The transition functions can be chosen to take values in $\{\pm 1\}$, and the two line summands admit flat connections. Their real characteristic forms vanish. This example exhibits both the two-primary qualification on the Whitney identity and the failure of real forms to detect torsion.

10.4 Orientation and the Pfaffian

For an oriented real bundle of rank $2m$, take an oriented orthonormal frame. Define the Euler form

$$e(\nabla) = \operatorname{Pf}\left(\frac{F}{2\pi}\right).$$

We use the Pfaffian with $\operatorname{Pf}\left(\begin{smallmatrix} 0 & a \\ -a & 0 \end{smallmatrix}\right) = a$. One definition is the coefficient of the ordered top exterior basis vector in the m-th divided power of $\sum_{i<j} F_{ij} e_i \wedge e_j/(2\pi)$. Conjugation by an orthogonal matrix multiplies this coefficient by its determinant. The polynomial is therefore invariant under $SO(2m)$, and reversing orientation changes its sign. On a complex line with its complex orientation, the skew matrix for multiplication by i is $\left(\begin{smallmatrix} 0 & -1 \\ 1 & 0 \end{smallmatrix}\right)$. For $F = i\,d\alpha$ this gives $e(\nabla) = -d\alpha/(2\pi) = c_1(\nabla)$. The Pfaffian product on oriented block sums then implies equality of the Euler form and the top Chern form for the realification of a unitary bundle. This can be checked on block diagonal matrices with real scalar entries, which determine the restricted invariant polynomial on $\mathfrak{u}(m)$ by unitary diagonalization. Polarization extends the equality to matrices of two-forms.

The comparison of the Pfaffian class with the real image of the integral Euler class is the Euler Chern–Weil theorem. The latter class is obtained by pulling the

Thom class back along the zero section [1, §25]. In odd rank, negation of each fiber reverses orientation and fixes the zero section. It follows that the integral Euler class is killed by 2. Its real image is zero.

Take $E = \mathcal{O}(1) \oplus \mathcal{O}(1)$ on $\mathbb{CP}^2$ and let $h = c_1(\mathcal{O}(1))$. The top Chern class is h^2. Since $(\mathcal{O}(1))_{\mathbb{R}} \otimes \mathbb{C} \cong \mathcal{O}(1) \oplus \mathcal{O}(-1)$, the first Pontryagin class of $E_{\mathbb{R}}$ is $2h^2$. Two independent linear functionals give hyperplanes intersecting in one positively oriented point, so $\int_{\mathbb{CP}^2} h^2 = 1$. The resulting Euler and Pontryagin numbers are 1 and 2, respectively, for the same oriented real bundle. They detect different invariant polynomials of its curvature.

CHAPTER 11

Chern–Simons forms and coherent transgression

A form that witnesses the equality of two de Rham classes contains additional information. It records how the two curvature representatives are related, and it changes when the comparison is changed. Chern and Simons used this principle to construct secondary invariants [7]. Here we make the comparison depend on a simplex of connections, so comparisons between comparisons have their own differential identities.

11.1 An oriented path of connections

Let P be an invariant symmetric m-linear form with $m \geq 1$, and let A_0, A_1 be connections of a fixed principal bundle $Q \to X$. Set $B = A_1 - A_0$ and $A_t = A_0 + tB$. Define the relative transgression by

$$T_1(A_0, A_1) = m \int_0^1 P(B, F_t^{m-1})\, dt.$$

Although written in a frame, the integrand is a global form because B and F_t transform by simultaneous conjugation. Its exterior derivative is $P(F_1^m) - P(F_0^m)$ by (9.2). A second proof differentiates the curvature formula:

$$\dot{F}_t = D_{A_t} B.$$

The derivative of $P(F_t^m)$ is $mP(D_{A_t}B, F_t^{m-1})$. Invariance converts covariant differentiation to exterior differentiation inside P. Terms differentiating F_t vanish by Bianchi, so this derivative equals $m\, d_X P(B, F_t^{m-1})$.

For the quadratic polynomial $p(X) = \operatorname{tr}(X^2)$ on a matrix Lie algebra, expand

$$F_t = F_0 + tD_{A_0}B + t^2 B \wedge B.$$

Integration gives the three-form

$$T_1(A_0, A_1) = \operatorname{tr}\Big(2B \wedge F_0 + B \wedge D_{A_0}B + \tfrac{2}{3} B \wedge B \wedge B\Big).$$

On a trivial bundle with the reference connection $A_0 = 0$, this reduces to the usual local expression $\operatorname{tr}(A \wedge dA + \frac{2}{3} A^{\wedge 3})$. The definition of T_1 also makes sense when neither connection has a global potential, since it uses a reference connection on the same bundle.

11.2 The sign from integration over a simplex

Orient the simplex with ordered vertices $0, \dots, q$ by coordinates $t_1, \dots, t_q$, with $t_0 = 1 - \sum_{i=1}^{q} t_i$. On $\Delta^q \times X$ use the connection $\widetilde{A} = \sum_{i=0}^{q} t_i A_i$ in the X directions. It has zero components in the simplex directions in local frames pulled back from X. For a form ω on the product, let $I_q\omega$ mean the integral of the component with simplex degree q, with all simplex differentials written first. Its degree on X is $\deg(\omega) - q$. For a closed ω,

$$d_X I_q \omega = (-1)^{q+1} I_{\partial \Delta^q} \omega. \tag{11.1}$$

Indeed, write $\omega = \sum_p \omega_p$ by simplex degree. In degree q, the closedness equation is the sum of the simplex derivative of ω_{q-1} and $(-1)^q$ times the X-differential of the coefficients of ω_q. Stokes on the parameter simplex gives (11.1).

Apply this to the closed form $P(\widetilde{F}^m)$ and define, for $q \geq 0$,

$$T_q(A_0, \dots, A_q) = (-1)^{q(q-1)/2} I_q P(\widetilde{F}^m). \tag{11.2}$$

At $q = 0$ this is $P(F_0^m)$. Each oriented face pulls the connection back to the affine family defined by the surviving vertices. Since $(-1)^{q(q-1)/2}(-1)^{q+1} = (-1)^{(q-1)(q-2)/2}$, the sign convention gives

$$d_X T_q(A_0, \dots, A_q) = \sum_{i=0}^{q} (-1)^i T_{q-1}(A_0, \dots, \widehat{A}_i, \dots, A_q). \tag{11.3}$$

The hat omits the indicated connection. The case $q = 2$ is the relation

$$T_1(A_0, A_1) + T_1(A_1, A_2) - T_1(A_0, A_2) = d_X T_2(A_0, A_1, A_2).$$

It says that making the comparison in two steps and taking the direct comparison differ by an exact form. For a polynomial of degree m, the form T_q vanishes for $q > m$, since each curvature factor carries at most one simplex differential. The relations still hold at those levels because the boundary sum in (11.3) then vanishes.

11.3 A two-parameter calculation

For $q = 2$, put $B_i = A_i - A_0$ for $i = 1, 2$. The curvature of the affine family consists of $F_{\boldsymbol{t}}$ and the terms $dt_i \wedge B_i$. Each product of the two mixed curvature terms contributes a minus sign when dt_2 is moved past B_1. The prefactor in (11.2) is also negative, so for $m \geq 2$,

$$T_2(A_0, A_1, A_2) = m(m-1) \int_{\Delta^2} P(B_1, B_2, F_{\boldsymbol{t}}^{m-2})\, dt_1\, dt_2.$$

If $m = 2$ the integral is over a coefficient independent of the parameter, and $\operatorname{area}(\Delta^2) = 1/2$ in these coordinates. Thus $T_2 = P(B_1, B_2)$. With $p = c_1^2$ and $A_i = -2\pi i\, a_i$ for a unitary line, write $b_i = a_i - a_0$, which is a globally defined real one-form. The identity becomes $T_2 = b_1 \wedge b_2$. Expanding the closed curvature expressions directly gives

$$T_1(A_0, A_1) + T_1(A_1, A_2) - T_1(A_0, A_2) = db_1 \wedge b_2 - b_1 \wedge db_2.$$

The two sides agree with $d(b_1 \wedge b_2)$, confirming the orientation choice in a case where the two-form can be nonzero.

11.4 Gauge transformations and periods

Let X be closed and oriented of dimension $2m-1$. Suppose $P(F^m)$ is normalized so that it represents the real image of an integral class on every principal G-bundle under consideration. A gauge automorphism $u\colon Q \to Q$ produces $A_1 = u^*A_0$. Choose a path from A_0 to A_1 constant near the endpoints. Gluing the product bundle along $(1, p) \sim (0, u(p))$ gives a principal bundle on $S^1 \times X$. The endpoint conditions make the family of connections descend smoothly. Orient $S^1 \times X$ with the parameter first. Integrating its characteristic form yields

$$\int_X T_1(A_0, u^*A_0) = \int_{S^1 \times X} P(\widetilde{F}^m) \in \mathbb{Z}.$$

Here T_1 can be computed using the path with constant ends. A monotone reparametrization of the affine path gives one such path and leaves its integral unchanged. More generally the transgression for any two smooth paths with the same endpoints differs by an exact form, by Stokes applied to an interpolating square in the affine space of connections. This period identity uses both the closedness of X and the integral normalization of the polynomial. A choice of an arbitrary multiple of a real polynomial changes the period group.

For $P = c_1^2$, the comparison of a line connection under a gauge map $u\colon X \to U(1)$ can also be computed on X itself. Use the normalized forms

$$f = \frac{iF}{2\pi}, \qquad \beta = \frac{i}{2\pi} u^{-1} du.$$

The comparison gives $T_1 = 2\beta \wedge f$. For $X = S^1 \times S^2$, take $u(t, x) = e^{-2\pi i t}$ and let the line be pulled back from a degree-one line on S^2. Then $\beta = dt$ and the period is 2. The mapping-torus argument explains its integrality without any choice of a global potential. Later, additive differential cohomology will package curvature and secondary period data in a single object.

11.5 A secondary value from a filling

Let $X = \partial Z$ with its boundary orientation, and assume that a bundle with connection on X extends over the compact oriented $2m$-manifold Z. For the same integral normalization, define a number in $\mathbb{R}/\mathbb{Z}$ by

$$\mathcal{S}(X, Q, A) = \left[\int_Z P(F^m)\right] \pmod{\mathbb{Z}}.$$

Two choices of connections on a fixed extension with equal boundary restriction yield the same integral: their difference is the integral of dT_1, and the restricted T_1 is zero. By this argument, one may arrange that the connection is constant in the normal direction in a collar. Two different fillings can then be glued using the boundary bundle isomorphism, with the second filling oppositely oriented. The difference of integrals is the characteristic number of the glued closed manifold, an integer. This proves independence of the filling whenever at least one filling exists.

This construction determines a change formula. Suppose that both boundary connections A_0, A_1 occur as ends of a cylinder on the same bundle and the first bounds with its connection. Attach the cylinder with its product orientation, with the interval first, to obtain a filling of the second. The two values then differ by $\int_X T_1(A_0, A_1)$ modulo integers. For the normalized first Chern form on a disk carrying a trivialized line connection $A = -2\pi i\, a$, the filling rule gives $\exp(2\pi i \int_D da) = \exp(2\pi i \int_{\partial D} a)$, the actual boundary holonomy by Stokes. A differential character extends this relationship to cycles without requiring a chosen filling.

CHAPTER 12

The Weil algebra and equivariant curvature

A principal connection assigns forms to its potential and curvature, subject to the structure equation and Bianchi's equation. The Weil algebra encodes those two equations before a bundle or a base has been chosen. To describe geometry with an extra group action, one combines this algebra with the forms of the space acted upon. The Cartan model results from eliminating the formal connection variables. The comparison is classical [8, Chapters 3–5].

12.1 Formal generators and their basic subalgebra

Let K be a Lie group with finite-dimensional real Lie algebra $\mathfrak{k}$. The Weil algebra as a graded commutative algebra is

$$W(\mathfrak{k}) = \Lambda(\mathfrak{k}^*[1]) \otimes \mathrm{Sym}(\mathfrak{k}^*[2]).$$

Write θ and u for the tautological $\mathfrak{k}$-valued expressions in its generators of degrees 1 and 2, respectively. Define the differential by

$$d\theta = u - \tfrac{1}{2}[\theta,\theta], \qquad du = -[\theta,u].$$

The structure constants are those of $\mathfrak{k}$, with graded signs from the exterior product. Extending d by the Leibniz rule gives $d^2 = 0$ by the graded Jacobi identity. For example,

$$d^2u = -[u,u] + \tfrac{1}{2}[[\theta,\theta],u] - [\theta,[\theta,u]] = 0.$$

The first term vanishes because u has even degree, and Jacobi identifies the other two terms.

For $\xi \in \mathfrak{k}$ the contraction ι_ξ is the degree -1 derivation with $\iota_\xi\theta = \xi,\ \iota_\xi u = 0$. Its commutator with d is the infinitesimal coadjoint action on the generators, expressed as $L_\xi\theta = -[\xi,\theta]$ and $L_\xi u = -[\xi,u]$ in the tautological expressions. A form is horizontal if every contraction annihilates it, and basic if it is horizontal and invariant under K. The horizontal subalgebra consists of polynomials in u: the contractions are all derivatives with respect to the exterior generators. Thus

$$W(\mathfrak{k})_{\mathrm{bas}} = \big(\mathrm{Sym}(\mathfrak{k}^*)\big)^K, \qquad d|_{\mathrm{bas}} = 0.$$

The differential vanishes there by the infinitesimal invariance identity of Chapter 9. Full invariance under the group accounts for disconnected components.

The cohomology of the entire Weil algebra is $\mathbb{R}$ in degree zero. Indeed, replacing the degree-two generators u^a by $v^a = d\theta^a$ is an invertible polynomial change of variables. In the new generators $d\theta^a = v^a$ and $dv^a = 0$. The degree -1 operator $h = \sum_a \theta^a \partial/\partial v^a$ has $dh + hd$ equal to the operator that counts all factors of θ and v in a monomial. On the positive-count part, divide h by the factor count to obtain a contraction. This explains how the basic algebra can carry nontrivial classes although the unqualified algebra is acyclic in positive degrees. Primitives in the whole algebra can fail the basic condition.

Given a right principal K-bundle with connection, evaluation $\theta \mapsto A,\ u \mapsto F$ is a morphism of differential graded algebras into forms on the total space. It preserves contractions by the reproduction property for A and the horizontality of F. It also preserves group actions. Its restriction to basic forms recovers the Chern–Weil homomorphism on representatives discussed in Chapter 9.

12.2 Forms on a space with a group action

Let X have a smooth right K-action. Use $\xi_X(x) = \frac{d}{dt}(x \cdot \exp(t\xi))|_0$ for a fundamental vector field. On $W(\mathfrak{k}) \otimes \Omega(X)$ take diagonal contractions and the diagonal group action, using the graded tensor signs. The basic subcomplex is the Weil model. The alternative Cartan model is

$$\Omega_K(X) = \big(\operatorname{Sym}(\mathfrak{k}^*) \otimes \Omega(X)\big)^K, \qquad (d_K\alpha)(\xi) = d(\alpha(\xi)) - \iota_{\xi_X}\alpha(\xi),$$

with polynomial generators in degree two. The equivariance convention is

$$R_k^*\alpha(\xi) = \alpha(\operatorname{Ad}_k \xi).$$

Cartan's identity gives $d_K^2\alpha(\xi) = -L_{\xi_X}\alpha(\xi) = 0$, since the one-parameter subgroup generated by ξ fixes ξ under the adjoint action.

Here is the comparison at the level of complexes. Choose dual bases e_a of $\mathfrak{k}$ and θ^a of degree-one generators. Put $T = \sum_a \theta^a \iota^X_{e_a}$ on the tensor product, with contractions on the second factor carrying the usual Koszul sign. This degree-zero derivation is nilpotent because it adds one exterior generator at each application. Its exponential is finite. One checks

$$[\iota_\xi^{\mathrm{tot}}, T] = \iota_\xi^X, \qquad [\iota_\xi^X, T] = 0.$$

The first identity follows by differentiating θ^a and the second uses anticommutation of contractions. It follows that $\exp(-T)$ sends elements independent of θ to horizontally constrained elements, and its inverse sends horizontal elements back to ones independent of θ. The operator T is invariant by the pairing of a basis

with its dual. Conjugating the differential and then setting $\theta = 0$, its only terms that survive are the coefficient differential d_X and $-\sum_a u^a \iota_{e_a}^X$. The latter comes from differentiating the first-order term of $\exp(-T)$, since the degree-two term of $d\theta$ is u. This proves the Cartan differential formula.

For compact K the equivariant de Rham theorem identifies the cohomology of this complex with $H^*(EK \times_K X; \mathbb{R})$ [8, Chapters 1,4–5]. This comparison invokes the compact-group hypotheses. The algebraic identification of the Weil and Cartan complexes above is a separate calculation. A useful geometric evaluation is obtained from any right principal K-bundle $R \to S$ with connection and an equivariant map $R \to X$. Evaluate its Weil generators on the bundle and its X-forms by pullback. The resulting forms on R are basic and descend to S. Thus an equivariant cohomology class yields ordinary cohomology classes for these bundles equipped with a map to X. This pattern will reappear as a characteristic map on a quotient stack.

12.3 The moment term of an invariant connection

Let $Q \to X$ be a right principal G-bundle and suppose the right K-action on X lifts to Q commuting with the G-action. Choose a K-invariant connection A; when K is compact one can average any connection using normalized Haar measure. For $\xi \in \mathfrak{k}$ put

$$\mu(\xi) = \iota_{\xi_Q} A, \qquad F_K(\xi) = F - \mu(\xi).$$

The term μ is an equivariant $\mathfrak{g}$-valued function on Q, linear in the parameter, so both summands of F_K have total degree two. Our sign follows from Cartan's identity and $L_{\xi_Q} A = 0$:

$$\iota_{\xi_Q} F = -d\mu(\xi) + [\mu(\xi), A] = -D_A \mu(\xi).$$

Consequently $(D_A - \iota_{\xi_Q}) F_K(\xi) = 0$, by the ordinary Bianchi identity. Each $P(F_K^m)$ is G-basic, so it descends to an equivariant form on X. Invariance of P converts D_A to d after polarization. Therefore the equivariant form is d_K-closed. Independence of the invariant connection follows by taking an affine path A_t and differentiating:

$$\partial_t F_{K,t} = (D_{A_t} - \iota_{\xi_Q}) \dot{A}_t.$$

The same polarization argument as for (9.2) expresses the derivative of $P(F_{K,t}^m)$ as a Cartan coboundary.

For a concrete calculation take $X = *$ and $K = G = U(1)$. On $Q = U(1)$ use $A = z^{-1} dz$, and lift the K-action by $z \cdot k = z k^m$ for an integer m. Write the real Lie-algebra coordinate as $\xi = 2\pi i\, s$. Then $F = 0$ and $\mu(\xi) = 2\pi i\, m s$. The

equivariant first Chern polynomial is

$$\frac{i}{2\pi}(F - \mu(\xi)) = ms.$$

The degree-two generator of the Cartan model of the point is normalized here by the weight-one action. Taking higher weights multiplies this generator even though the ordinary curvature on the point is zero. The moment term records the isotropy action that ordinary forms on the base cannot record.

12.4 Free actions and the sign of the universal generator

For a free compact K-action with smooth principal quotient $X \to X/K$, the projection $EK \times_K X \to X/K$ has contractible fiber and is a homotopy equivalence. The Cartan map can be evaluated using a connection on the principal quotient and the identity equivariant map $X \to X$. In the Weil description its evaluation sends u to the curvature and uses the horizontal part of the coefficient forms. The horizontal projection arises from $\exp(-T)$: its degree-one effect on a one-form α is $\alpha - \sum_a A^a \alpha((e_a)_X)$, and multiplicativity gives the projection on higher forms. Changing the connection gives the same cohomology map by applying the path construction on the quotient times an interval.

For the Hopf circle bundle $S^3 \to \mathbb{CP}^1$, a point over a line is a unit vector in that line. The right circle action is multiplication of the vector by the circle coordinate. Its associated line for the standard representation is $\mathcal{O}(-1)$, using $(qg, v) \sim (q, gv)$. If A is the corresponding circle connection, the variable s in the preceding weight example evaluates to $F/(2\pi i) = -iF/(2\pi)$. Its integral over $\mathbb{CP}^1$ is therefore 1 by Chapter 10. This also explains the sign for right actions in the quotient construction. A separate K-action $q \cdot k = qk^m$ on a circle torsor over a point, pulled back along the principal Hopf bundle, gives the diagonal relation $(rk, qk^m) \sim (r, q)$. The associated character on the underlying Hopf bundle is k^{-m}, so its first Chern number is m, exactly the evaluation of ms.

Part IV Smooth stacks and characteristic maps

CHAPTER 13

The smooth site and its higher sheaves

A connection can be described in every coordinate chart of its base. The description includes transition functions and their equations. Keeping the transitions is essential. If one retains only the isomorphism class of the object on each chart, the information needed to assemble the object over the whole base can disappear. We now make the difference between these two procedures precise.

13.1 Test manifolds and local data

Let **Mfd** denote a category of smooth Hausdorff second-countable finite-dimensional manifolds without boundary, in a fixed universe. We use open covers to define a Grothendieck topology. The later arguments about bundles also use surjective submersions. Such a map has local smooth sections by the submersion theorem, so it is a cover for the topology generated by open covers. When integrating a family, a boundary hypothesis will be stated explicitly.

For a manifold X, put $yX(U) = C^\infty(U, X)$. This is a sheaf of sets because smoothness is local on the source. If $\mathcal{F}$ is a presheaf, the action of a map $v : V \to U$ goes in the direction $\mathcal{F}(U) \to \mathcal{F}(V)$. For differential forms it is the ordinary pullback v^*. The sheaf Ω^q assigns to U its real q-forms, and Ω^q_{cl} assigns the kernel of $d : \Omega^q(U) \to \Omega^{q+1}(U)$. A family of forms on U_i glues uniquely when its two restrictions to each $U_i \cap U_j$ agree. Checking the exterior derivative locally shows that the glued form is closed when the given forms are closed.

On a cover $\{U_i \to U\}$, write $U_{i_0 \cdots i_p} = U_{i_0} \cap \cdots \cap U_{i_p}$. The Čech nerve has at level p the union of the $U_{i_0 \cdots i_p}$, with a separate copy for every tuple. Its face maps omit one index and its degeneracies repeat an index. If necessary we use formal coproducts of representable sheaves for this construction. At the first two levels this is the diagram

$$\coprod_{i,j} yU_{ij} \;\rightrightarrows\; \coprod_{i} yU_i \;\longrightarrow\; yU.$$

For a sheaf of sets the limit test amounts to the equalizer at levels zero and one. The higher levels record identities already forced by equality of restrictions.

13.2 Descent with isomorphisms

A presheaf of groupoids has objects and their isomorphisms over each U. We allow restriction functors associative up to specified coherent natural isomorphisms. This is how pullback of bundles behaves without choosing a strict model. The descent groupoid over a cover has as objects $a_i \in \mathcal{F}(U_i)$ and isomorphisms $\phi_{ij} : a_j|_{U_{ij}} \to a_i|_{U_{ij}}$. We suppress the canonical pullback isomorphisms from the notation. The equations are

$$\phi_{ii} = 1, \qquad \phi_{ij}\phi_{jk} = \phi_{ik}.$$

A morphism is a family $f_i : a_i \to b_i$ with $f_i\phi_{ij} = \psi_{ij}f_j$. Composition and inversion are componentwise. The restriction functor from $\mathcal{F}(U)$ to this groupoid is fully faithful exactly when isomorphisms between global objects glue uniquely. It is essentially surjective exactly when compatible local objects can be glued, up to isomorphism. A stack satisfies both tests.

Proposition 13.1 (Vector bundles satisfy descent)**.** *Let $\mathcal{V}_r(U)$ be the groupoid of real rank r smooth vector bundles over U and bundle isomorphisms over its identity. Then $\mathcal{V}_r$ is a stack.*

Proof. Given descent data, take the disjoint union of the local total spaces E_i. Over a base point in U_{ij}, identify v_j with $\phi_{ij}(v_j)$. The equations make this an equivalence relation. Every equivalence class over a point of U_i has exactly one representative in E_i. Give the quotient the topology for which the quotient charts from E_i over U_i are open, and use the vector bundle charts of E_i as smooth trivializations. The identifications on overlaps are smooth and linear, so they supply the change of coordinates. Points over different base points have disjoint neighborhoods by Hausdorffness of U. Points over the same base point lie in one of the open bundle charts and are separated there. With a countable refinement of the base cover and countable trivializations this gives the usual second-countable total space. Applying a compatible family of local isomorphisms to each representative gives the unique global isomorphism. This verifies both parts of descent. □

This proof does not identify nonisomorphic gluing choices. For example, cover a circle by two connected arcs whose intersection has two components. Take trivial real lines on the two arcs. Gluing by the sign $+1$ on one component and -1 on the other produces the Möbius line. A change of local frames consists of two continuous nonzero functions, each with a constant sign on its arc. Their sign ratio is therefore the same on both components. It cannot convert the chosen transition to the identity on both. Consequently a presheaf keeping just isomorphism classes is not separated: the Möbius line and the trivial line give

different classes on the circle and identical classes on all chart domains. The different transition signs are precisely the information missing from the local isomorphism classes.

13.3 From groupoids to spaces

The nerve of a groupoid is a space with no homotopy above degree one. Its points are objects, paths represent isomorphisms, and the composition law determines the higher simplices. The groupoid of descent data above models a homotopy limit of the Čech diagram of nerves. One way to see this is to read the data of a coherent cone over the diagram. Level zero specifies the a_i. Level one supplies paths between their restrictions. A triangle filling in level two is the displayed composition equation, since the mapping spaces between fixed objects of a groupoid nerve are discrete. Higher coherence contains no new independent choices once the equations hold.

For sheaves of arbitrary spaces we use the corresponding homotopy-limit condition

$$\mathcal{F}(U) \simeq \operatorname*{holim}_{[p]\in\mathbf{\Delta}} \prod_{i_0,\dots,i_p} \mathcal{F}(U_{i_0\cdots i_p}).$$

In this case an agreement may itself have nontrivial higher paths. A weakly associative system of transition maps can therefore carry additional information in the fillers and their compatibilities. We use $\mathcal{H}$ for this ∞-category of smooth sheaves of spaces. It is an ∞-topos by the general theory of sheaves of spaces on a site [9, Chapter 6]. In these chapters descent refers to cover descent unless a hypercover is specified. Degree-one descriptions only require groupoid descent. For the bounded differential complexes used later, the constructions can be checked on refinements by good covers of manifolds.

Representables remain sheaves, and the Yoneda equivalence has the form

$$\operatorname{Map}_{\mathcal{H}}(yU, \mathcal{F}) \simeq \mathcal{F}(U).$$

In particular, a natural operation on families can be written as a morphism of sheaves of spaces. Pullback along a map of test spaces is built into the morphism.

Example 13.2 (A homotopy fiber retains an identification)**.** Let $F : \mathcal{X} \to \mathcal{Y}$ be a functor between groupoids and y an object of $\mathcal{Y}$. A model of the homotopy fiber has objects $(x, \alpha : F(x) \cong y)$ and arrows $u : x \to x'$ satisfying $\alpha' F(u) = \alpha$. The square of nerves is a homotopy pullback. To check the assertion, replace the target point by the groupoid of arrows with target y, with morphisms the commuting triangles. This groupoid is contractible, and its source projection to $\mathcal{Y}$ lifts isomorphisms. The ordinary pullback along that projection is precisely the indicated groupoid. An automorphism of x survives in the fiber if $F(u) = 1$.

We will use this test to fix the underlying bundle while allowing its connection to vary.

13.4 Local presentations of a moduli object

A presentation by an atlas starts with a map $yX \to \mathcal{X}$ admitting local lifts of points in the sense of sheaves. Its relation object is the homotopy fiber product $yX \times^h_{\mathcal{X}} yX$. A point of this relation has two lifts and an identification between their images. Iterating the construction gives the Čech nerve of the atlas. When the map is an effective epimorphism, the geometric realization of that nerve is $\mathcal{X}$. The local lifting condition is the ordinary criterion for effective epimorphisms of sheaves of spaces in terms of locally surjective maps on components. In the groupoid case the assertion follows directly by gluing the chosen lifts and the isomorphisms on their overlaps.

The atlas by itself supplies a collection of local representatives. Curvature and characteristic forms computed on those representatives must be compatible with the full relation object in order to define natural operations. This is one reason to distinguish basic forms on a principal bundle, differential forms on a chosen chart, and a cohomology class on the stack. The next chapter works out the relation object for torsor classification.

CHAPTER 14

The classifying stack of smooth torsors

The familiar classifying space BG provides information about the isomorphism class of a topological G-bundle. A smooth classifying stack records bundles, their isomorphisms, and smooth pullback along test maps. This refinement lets us express a construction such as extension of structure group by an actual map of moduli objects.

14.1 The stack BG

Let G be a Lie group, regarded as the sheaf $U \mapsto C^\infty(U, G)$. The one-object groupoid with automorphism sheaf G gives a presheaf of groupoids $\mathcal{P}_G$. Its stackification is denoted $\mathbf{B}G$. Stackification permits local objects with descent isomorphisms to define global objects. Thus, for a cover $\{U_i\}$, an object is represented by smooth functions $g_{ij} : U_{ij} \to G$ satisfying

$$g_{ii} = 1, \qquad g_{ij}g_{jk} = g_{ik}.$$

A choice of right-principal local sections s_i relates them by $s_j = s_i g_{ij}$. With this convention the overlap gluing identifies $(x, h)_j$ with $(x, g_{ij}(x)h)_i$.

Theorem 14.1 (Classification on a test manifold)**.** *For every U, $\mathbf{B}G(U)$ is equivalent to the groupoid of smooth right principal G-bundles over U, and these equivalences commute with pullback up to the canonical isomorphisms.*

Proof. A principal bundle is locally a product $U_i \times G$. Choose local sections to express each change of frame by g_{ij}. Multiplying changes of frame establishes the cocycle identity. Conversely the indicated gluing of the local products produces a smooth locally trivial right G-space with base U. The equivalence relation commutes with the right action and a local product identifies each fiber with G acting on the right. Given an equivariant isomorphism between such products over U_i, it is left multiplication by a smooth function on U_i. The compatibility condition on the overlaps is exactly the condition that these local maps respect the identifications defining the quotient. They therefore glue to the given global isomorphisms, and descriptions from different covers agree on a common refinement. Pullback replaces every local product and every transition

function by their pullbacks. This identifies the stackification with the indicated groupoid on each test. □

The terminal smooth sheaf $*$ maps to $\mathbf{B}G$ by picking the trivial torsor. Its relation object

$$* \times^h_{\mathbf{B}G} * \simeq G$$

is the sheaf of its automorphisms. On a manifold U, an isomorphism between the two trivial right bundles has the form $(x, h) \mapsto (x, k(x)h)$, for a unique $k : U \to G$. This computes the fiber product directly, including its functorial action on tests. Iteration gives the nerve with G^n at level n, with faces given by multiplication and deletion and degeneracies by identities. In particular, the smooth group is recovered by looping the stack at the trivial torsor.

14.2 Fibers over a given torsor

A map $yU \to \mathbf{B}G$ classifying P has a homotopy pullback square

$$\begin{array}{ccc} yP & \longrightarrow & * \\ \downarrow{\scriptstyle \pi} & & \downarrow \\ yU & \xrightarrow{[P]} & \mathbf{B}G. \end{array}$$

For a test map $v : V \to U$, a point in the upper left consists of an isomorphism from the trivial bundle over V to v^*P. Evaluation at the identity picks a section of v^*P, and that section extends equivariantly to the isomorphism. Such sections are the lifts $V \to P$. There are no further automorphisms of a given identification in this fiber. Thus the square reconstructs the original bundle, including its projection, without a choice of coordinates. Right multiplication of the evaluation point corresponds to precomposing the isomorphism by the appropriate automorphism of the trivial right bundle.

For a homomorphism $\rho : G \to K$, extension of structure group sends P to $P \times_G K$, the quotient of $P \times K$ by $(pg, k) \sim (p, \rho(g)k)$. It gives a map $\mathbf{B}G \to \mathbf{B}K$. The transition functions become $\rho(g_{ij})$, which already proves compatibility with any base pullback and with iterated extension. A linear representation $\rho : G \to GL(V)$ works in the same way with V in place of K, giving the associated vector bundle $P \times_G V$. Its section over U is an equivariant map $P \to V$ satisfying $t(pg) = \rho(g)^{-1}t(p)$. Evaluation in the local sections s_i gives $t(s_j) = \rho(g_{ij})^{-1}t(s_i)$, which is exactly the change of columns in the associated frames.

Proposition 14.2 (Reductions as sections)**.** *Let H be a closed Lie subgroup of G, and let P be a smooth right G-bundle over U. The groupoid of H-bundles*

together with an identification of their extended G-bundle with P is equivalent to the set of smooth sections of $P/H \to U$, viewed as a discrete groupoid.

Proof. An identification $Q \times_H G \cong P$ sends the canonical copy of Q to an H-subbundle of P. Taking its cosets gives a section of P/H. Conversely a section selects one right H-coset in each fiber. Pulling back $P \to P/H$ along the section gives an H-bundle. Here $G \to G/H$ is a locally trivial smooth H-bundle, so the pullback gives exactly the indicated smooth subbundle. Multiplication extends it to P. Constructions in both directions are inverse up to unique isomorphism, since a map between reductions is fixed by its required extension to P. Thus no additional automorphisms remain when the identification is specified. □

For $H = O(r) \subset GL_r(\mathbb{R})$, this is the familiar relation between a positive definite inner product and its orthonormal frame bundle. In a real vector bundle E, a frame p transports the Euclidean inner product onto the fiber. Replacing p by ph, with $h \in O(r)$, leaves that inner product unchanged. Conversely frames defining the same inner product differ by an orthogonal matrix. Therefore P/H is the bundle of positive definite forms and the section in the proposition is a metric. A metric is a reduction of structure group. A connection is different data and is treated by a separate moduli stack in Chapter 15.

14.3 Stacks and underlying homotopy types

A classifying stack is functorial on every smooth test. Its set of connected components on a test gives isomorphism classes of torsors, but it also retains their gauge groups. For a bundle over a point all objects of $\mathbf{B}G(*)$ are isomorphic, while its automorphisms are the elements of G. Over a manifold the group of gauge automorphisms of P is the group of sections of the conjugation bundle $P \times_G G$. Indeed an automorphism can be written $p \mapsto p\,u(p)$, with $u(pg) = g^{-1}u(p)g$. This formula also exhibits the group law by pointwise multiplication in the conjugation bundle.

One can apply a realization or shape construction that forgets the smooth structure of the sheaf and obtains an ordinary homotopy type. For the Čech presentation above this recovers the classifying homotopy type of the topological Lie group, with the usual model obtained by realizing the nerve of the topological group. It answers a different question from evaluation on a fixed smooth test. A smooth-family map into $\mathbf{B}G$ specifies a bundle over the product of the parameter manifold with the base, together with its descent data when no global trivialization is available. The same distinction will be needed for connections, differential cohomology, and higher bundles.

CHAPTER 15

Connections as a smooth moduli stack

To a bundle with connection we can assign its transport, its curvature, or a characteristic form. Each assignment is compatible with pullback, but they retain different amounts of the original data. A stack makes the domain of these assignments explicit. Its automorphisms are the gauge transformations that preserve the given connection.

15.1 Gluing connective data

In a local right-principal section, the connective data for a Lie group G are a form $A \in \Omega^1(U; \mathfrak{g})$. Define

$$A^g = \mathrm{Ad}_{g^{-1}} A + g^* \theta_L,$$

where θ_L is the left Maurer–Cartan form on G. For a matrix group $g^*\theta_L = g^{-1}dg$. The product rule for gh gives $(A^g)^h = A^{gh}$. Thus there is an action groupoid of these forms and smooth gauge changes on each U. Its stackification is written $\mathbf{B}G_\nabla$.

An object of $\mathbf{B}G_\nabla(U)$ has local data (g_{ij}, A_i) satisfying

$$g_{ij} g_{jk} = g_{ik}, \qquad A_j = A_i^{g_{ij}}.$$

Replacing frames by $s_i k_i$ sends A_i to $A_i^{k_i}$ and sends g_{ij} to $k_i^{-1} g_{ij} k_j$. When the local descriptions come from different covers one first restricts to a common refinement. This presentation of the connection stack is used in higher Chern–Weil theory [11, 10].

Theorem 15.1 (Classification with connection)**.** *For a test manifold U, $\mathbf{B}G_\nabla(U)$ is equivalent to the groupoid of smooth principal G-bundles with connection and connection-preserving isomorphisms.*

Proof. A global connection ω on a right bundle P gives $A_i = s_i^* \omega$. Right equivariance for a constant group element and reproduction of fundamental vector fields for the derivative of a varying one give $A_j = A_i^{g_{ij}}$. Conversely, glue

the local trivial bundles using g_{ij}, as in Theorem 14.1. On the local product with group coordinate h set

$$\omega_i = \mathrm{Ad}_{h^{-1}} A_i + \theta_L(h).$$

The form reproduces vertical fundamental vectors and has the required behavior under right translation. The chart change from j to i replaces h by $g_{ij}(x)h$. Pulling ω_i back along it gives $\mathrm{Ad}_{h^{-1}} A_i^{g_{ij}} + \theta_L(h)$, which is ω_j. Hence the forms glue to a connection. Local gauge isomorphisms between two presentations preserve the local connection forms by the corresponding formula for A^g, and they glue by the compatibility of the underlying bundle maps. All operations commute with pullback along maps of test manifolds. □

The pointwise curvature of the local connection is $F_i = dA_i + \frac{1}{2}[A_i, A_i]$, using the graded bracket on forms. The change of frame gives $F_j = \mathrm{Ad}_{g_{ij}^{-1}} F_i$ by the Maurer–Cartan equation. Consequently the F_i glue to a two-form with values in $\mathrm{ad}\, P = P \times_G \mathfrak{g}$. One cannot in general use a single $\mathfrak{g}$-valued form on the whole base without choosing a gauge. An invariant polynomial on $\mathfrak{g}$ eliminates this dependence and produces a scalar form.

15.2 Forgetting the connection

The map $u : \mathbf{B}G_\nabla \to \mathbf{B}G$ forgets the connective data. Take its homotopy fiber on U over a bundle P. A point consists of (Q, ω) and an isomorphism $Q \cong P$. Transport of ω across the isomorphism gives a connection on P. Every connection arises this way. An isomorphism over a prescribed identification of the underlying bundle is forced to be the identity after transport. Thus the fiber on U is equivalent to the discrete groupoid of connections on P. When nonempty it is an affine space under $\Omega^1(U; \mathrm{ad}\, P)$: the difference of two principal connections is horizontal and equivariant, and adding an adjoint-valued one-form preserves the defining conditions. In the present manifold setting, local connections patch by a partition of unity, so the affine space is nonempty.

Discreteness here refers to automorphisms in a groupoid over a fixed test. On mathematical families it still has smooth variation. A map $y(S \times U) \to \mathbf{B}G_\nabla$ specifies a connection on the whole product. In a chosen trivialization it may include one-form components in the parameter directions of S. A family of connections only along fibers of $S \times U \to S$ uses relative one-forms and a relative exterior derivative. A relative connection by itself gives no chosen components along S. The difference can be seen on $U = *$: a connection along the point fibers is empty data, whereas an ordinary connection on S has nontrivial curvature in general.

Extension of structure group induces a square

$$\begin{array}{ccc} \mathbf{B}G_\nabla & \xrightarrow{\rho_\nabla} & \mathbf{B}K_\nabla \\ \downarrow{\scriptstyle u} & & \downarrow{\scriptstyle u} \\ \mathbf{B}G & \xrightarrow{\rho_*} & \mathbf{B}K \end{array}$$

for a Lie group homomorphism $\rho : G \to K$. In local data the upper map uses $\rho(g_{ij})$ and the Lie algebra map $d\rho(A_i)$. The naturality of θ_L under ρ verifies the gauge law, and $d\rho(F_i)$ gives the resulting curvature. For a representation this recovers the connection on the associated vector bundle.

15.3 Infinitesimal gauge data

Choose a model P for the underlying bundle and a connection A. In the first-order problem below we retain gauge transformations of this model that reduce to the identity. In the marked fiber of Section 15.2, the identification of the underlying bundle removes these transformations and the variations form $\Omega^1(U; \operatorname{ad} P)$ alone. A change is $A + \epsilon a$, where $\epsilon^2 = 0$ and $a \in \Omega^1(U; \operatorname{ad} P)$. An infinitesimal change of frame with parameter $\epsilon\xi$, for $\xi \in \Omega^0(U; \operatorname{ad} P)$, adds $d_A\xi$ to a, with

$$d_A\xi = d\xi + [A, \xi].$$

This follows by expanding the gauge formula to first order. It is independent of the chosen matrix realization because the derivative of the adjoint action is the Lie bracket. The first-order gauge action groupoid is therefore encoded by

$$\Omega^0(U; \operatorname{ad} P) \xrightarrow{d_A} \Omega^1(U; \operatorname{ad} P).$$

Place the two terms in degrees $-1, 0$. Infinitesimal automorphisms lie in the kernel, and isomorphism classes of changes form the cokernel.

The derivative of curvature is $d_A a$. It follows by expanding $F(A + \epsilon a) = F(A) + \epsilon(da + [A, a])$. A direct calculation with the graded Jacobi identity gives $d_A^2\eta = [F(A), \eta]$. At a flat connection, imposing flatness to first order requires $d_A a = 0$, and the two arrows extend to the ordinary covariant de Rham complex. At a curved connection the linearized curvature of a gauge change is $[F(A), \xi]$, which is the infinitesimal conjugation law. The flat deformation complex thus has a flatness hypothesis as part of its definition.

Example 15.2 (A flat circle connection on the two-torus)**.** On $U = \mathbb{R}^2/\mathbb{Z}^2$ consider $A = -2\pi i(r\,dx + s\,dy)$ on the trivial circle bundle. Its curvature vanishes. The two fundamental loops have holonomies $\exp(2\pi i r)$ and $\exp(2\pi i s)$. A gauge change $g = \exp(-2\pi i(mx + ny))$ for $m, n \in \mathbb{Z}$ adds the integer periods

(m, n). Conversely the logarithmic derivative of any smooth circle-valued gauge has integral periods in this normalization. Hence two members of this family are gauge equivalent exactly when their two periods differ by integers. Constant circle gauges preserve every member. This example exhibits curvature, holonomy and automorphisms as three separate pieces of data of the connection stack.

CHAPTER 16

Characteristic maps and their differential lifts

Classical Chern–Weil theory produces a closed form and identifies its cohomology class. The two constructions, the form itself and its cohomology class, have different domains of naturality. There is a map on bundles with connection that produces the form. Integrality concerns the corresponding bundle after forgetting the connection. The space of differential refinements is a way to keep both together with their comparison.

16.1 The characteristic form as a natural operation

Let P_m be an invariant symmetric m-linear form on $\mathfrak{g}$. Applying it to the curvatures of a connection gives

$$\omega_{P_m} = P_m(F, \dots, F) \in \Omega^{2m}(U).$$

Invariance implies that the local expressions agree under changes of frame. Their exterior derivative is the sum of the terms obtained by replacing one input by $d_A F$. To see why covariant differentiation can replace ordinary differentiation in this expression, differentiate the invariance identity along a one-parameter subgroup. The resulting sum with $[\xi, F]$ in each position vanishes. In local coordinates this cancels the terms $[A, F]$ in the graded derivative. The Bianchi identity then proves $d\omega_{P_m} = 0$. This repeats the defining calculation in a form suitable for descent.

The assignment commutes with smooth pullback, since exterior differentiation and substitution into P_m commute with pullback. A connection-preserving isomorphism fixes the resulting form, so this is a morphism

$$\mathbf{B}G_\nabla \longrightarrow \Omega^{2m}_{\mathrm{cl}}$$

of smooth sheaves of spaces. Its values are actual closed forms. If two connections on the same bundle are joined by their affine interpolation, the transgression calculation of Chapter 11 gives an exact difference between their characteristic forms. Equality of classes here uses the interpolation, whereas equality of forms would require further hypotheses.

16.2 Representing the real comparison

For $n \geq 1$, write $K(C, n)$ for the n-fold delooping of the locally constant sheaf of an abelian group C. On a test manifold its space of sections represents cohomology with those coefficients, with π_0 given by $H^n(U; C)$. One model uses the sheafified cochain complex of C placed in degree $-n$, followed by the Dold–Kan construction. Equivalently, cover U by a good cover, apply the cochain construction on its nerve, and keep the homotopies between cocycles. This computes the same space because the good cover is acyclic for the locally constant coefficients.

The Poincaré lemma resolves the locally constant reals by differential forms. In fact

$$\Omega^0 \xrightarrow{d} \Omega^1 \xrightarrow{d} \cdots \xrightarrow{d} \Omega^{n-1} \xrightarrow{d} \Omega^n_{\mathrm{cl}}$$

has only degree-zero local cohomology, the locally constant functions, when displayed starting in cohomological degree zero. Placing its rightmost term in degree zero instead realizes $K(\mathbb{R}, n)$, after Dold–Kan and sheafification. Inclusion of the rightmost term gives the morphism $\mathrm{dR} : \Omega^n_{\mathrm{cl}} \to K(\mathbb{R}, n)$. On connected components over U it sends a closed form to its de Rham class. The comparison uses sheaves because locally exact potentials need not be globally exact.

The map $\mathrm{real} : K(\mathbb{Z}, n) \to K(\mathbb{R}, n)$ comes from changing coefficients. Form the pullback

$$\begin{array}{ccc} \widehat{\mathbf{H}}^n & \xrightarrow{I} & K(\mathbb{Z}, n) \\ \downarrow{\scriptstyle R} & & \downarrow{\scriptstyle \mathrm{real}} \\ \Omega^n_{\mathrm{cl}} & \xrightarrow{\mathrm{dR}} & K(\mathbb{R}, n). \end{array}$$

All pullbacks in this square are homotopy pullbacks. Sections of $\widehat{\mathbf{H}}^n$ therefore include an integral cocycle, a closed form, and a path comparing their real images. The path can be represented by a cochain with precisely the prescribed differential. Replacing a path by a homotopic one, or altering the chosen cocycle representatives with their compatibilities, gives the usual equivalence in this space. Its set of components is the degree-n differential cohomology group modeled in Chapter 17. This is the discrete-coefficient specialization of the differential refinement constructions in [13, 14].

Proposition 16.1 (A lift supplies the real comparison)**.** *Let X be a smooth sheaf of spaces and let $c : X \to K(\mathbb{Z}, n)$ and $\omega : X \to \Omega^n_{\mathrm{cl}}$ be given. The space of lifts to $\widehat{\mathbf{H}}^n$ with these components fixed is the space of paths from* $\mathrm{real} \circ c$ *to* $\mathrm{dR} \circ \omega$ *in* $\mathrm{Map}(X, K(\mathbb{R}, n))$*.*

Proof. The universal property of the homotopy pullback identifies maps into it with the homotopy pullback of the three mapping spaces. Taking the fiber over

the pair (c, ω) gives a model with exactly the indicated path as its remaining datum. □

For a normalized Chern polynomial, the integral component is the ordinary characteristic map of the underlying bundle and the form component is its Chern–Weil form. A differential characteristic map records a *natural* comparison in the above sense. Existence of a refinement on each individual test does not itself produce the coherent map on all tests. The classical differential refinements have concrete cocycle constructions [12, 11]. The circle case gives an elementary illustration.

16.3 The differential first Chern class in coordinates

Take a unitary frame of a complex line over each U_i. Use $A_i = -2\pi i\, a_i$ and, on a good cover, write $g_{ij} = \exp(-2\pi i\, \lambda_{ij})$. The connection law gives $a_j - a_i = d\lambda_{ij}$, and $\delta\lambda$ is integer-valued. The ordinary exponential sequence with exponential $t \mapsto \exp(2\pi i t)$ computes the integral class from the logarithms $-\lambda_{ij}$. With those signs put

$$n = -\delta\lambda, \qquad \omega = da_i.$$

The da_i agree on overlaps. For total Čech degree p and form degree q, use the total differential $D = \delta + (-1)^p d$. Then

$$D(a + \lambda) = \omega - n.$$

Thus the cochain $a + \lambda$ gives the real comparison between the two displayed representatives. The curvature is exactly $iF/(2\pi)$, since $F = dA_i = -2\pi i\, da_i$.

For completeness, a lift of the comparison on a good cover is represented by n in integral Čech degree two, a closed global ω, and total real degree-one data $a + \lambda$ with the displayed differential. Reading its bidegrees gives $da_i = \omega$, $\delta a = d\lambda$, $\delta\lambda = -n$. Consequently the exponentials reconstruct the cocycle of a circle bundle and the a_i reconstruct its connection. Changing a local frame by $\exp(-2\pi i\mu_i)$ replaces a_i by $a_i + d\mu_i$ and λ by $\lambda + \delta\mu + k$, for an integral overlap cochain k. The comparison cochain changes by $D\mu + k$, and n by $-\delta k$, giving the equivalent triple. Changing μ by an integral local function ℓ and k by $-\delta\ell$ gives the same gauge transformation. These changes of representatives recover the Čech equivalences, retaining the circle-valued gauge automorphisms. Locally refinements of covers merely restrict the data. This yields the identification of the degree-two differential cocycle stack with $\mathbf{B}U(1)_\nabla$, sending a line and connection to its first differential Chern class.

In particular a flat line on a circle may have nontrivial holonomy although ω and n can both be taken zero. It is the comparison, modulo integral changes, that remembers the flat class. For a nontrivial bundle the same comparison

also records how the curvature periods agree with the integral class. Higher differential cohomology replaces these overlap logarithms by longer strings of cochains, as the next part develops.

Part V Differential cohomology

CHAPTER 17

Differential cohomology as a pullback

A characteristic form has a real cohomology class. An integral characteristic class can contain torsion, and a connection can have nontrivial holonomy even when both that form and that class vanish. Differential cohomology records all three phenomena. We develop it in ordinary integral degrees, using smooth singular chains to specify exactly how differential forms and holonomy fit together.

17.1 Characters on cycles

Write $C_k(M;\mathbb{Z})$ for smooth singular chains, $Z_k(M;\mathbb{Z}) = \ker\partial$, and $C^k(M;A) = \mathrm{Hom}(C_k(M;\mathbb{Z}),A)$. Smooth means smooth up to the faces of each simplex. The cochain differential is $(\delta a)(c) = a(\partial c)$. Integration gives a cochain map $J:\Omega^\bullet(M)\to C^\bullet(M;\mathbb{R})$. The smooth singular and ordinary singular cohomologies agree for manifolds. Write $\Omega^q_{\mathbb{Z}}(M)$ for closed q-forms whose integrals on smooth integral q-cycles lie in $\mathbb{Z}$.

Definition 17.1. A degree $q\geq 1$ differential character on M is a homomorphism $h: Z_{q-1}(M;\mathbb{Z})\to\mathbb{R}/\mathbb{Z}$ for which there is a q-form ω satisfying

$$h(\partial c) = \int_c \omega \pmod{\mathbb{Z}} \qquad (c\in C_q(M;\mathbb{Z})). \tag{17.1}$$

The group under pointwise addition is denoted $\widehat{H}^q(M;\mathbb{Z})$.

Pullback along $f:N\to M$ sends h to $h\circ f_*$ on cycles and sends ω to $f^*\omega$. Both conclusions follow from the corresponding statements for smooth singular chains and forms. This is the contravariant functoriality of differential characters [12].

Lemma 17.2. *The form ω in (17.1) is unique and belongs to $\Omega^q_{\mathbb{Z}}(M)$.*

Proof. The difference of two such forms integrates to an integer over every smooth q-simplex. Take a shrinking affine simplex in a chart, with edge vectors $tv_1,\dots,tv_q$. The difference of the integrals tends to zero, so it is zero for small t. The limit after division by $t^q/q!$ is evaluation of the difference of the forms on $v_1,\dots,v_q$. This proves uniqueness. Replacing c in (17.1) by a cycle proves integral periods. If b is a smooth $(q+1)$-simplex, the same equation for $c=\partial b$ gives $\int_b d\omega\in\mathbb{Z}$ by Stokes. The shrinking-simplex argument now proves $d\omega=0$. □

Write $R(h) = \omega$ for the curvature. When $q = 1$, a character is precisely a smooth map $f : M \to \mathbb{R}/\mathbb{Z}$, regarded as a homomorphism on the free group of zero-chains. Indeed the boundary of a path gives the change in f as the integral of the curvature. On a small coordinate ball this expresses f as a constant plus a smooth real primitive of the curvature, reduced modulo $\mathbb{Z}$. Conversely, the normalized differential of a circle-valued function satisfies the boundary rule on paths.

17.2 Differential cocycles and integral classes

A real cochain $\widetilde{h} \in C^{q-1}(M;\mathbb{R})$ lifting h on cycles exists. The cycles and boundaries in each degree are free abelian as subgroups of free abelian groups. The surjection $C_{q-1} \to B_{q-2}$ therefore splits, and one may lift the values on a basis of Z_{q-1} and extend by zero on a complement. In degree zero the complement can be taken to be zero. The equation for curvature implies that $c = J\omega - \delta\widetilde{h}$ takes integral values. Since $d\omega = 0$, it satisfies $\delta c = 0$.

Proposition 17.3. *A character is represented by a triple $(c, \widetilde{h}, \omega)$ with $c \in C^q(M;\mathbb{Z})$, $\delta c = 0$, $\omega \in \Omega^q(M)$, $d\omega = 0$, and*

$$\delta\widetilde{h} = J\omega - c.$$

Two triples represent the same character precisely when they differ by

$$(\delta b, -b - \delta u, 0), \quad b \in C^{q-1}(M;\mathbb{Z}), \quad u \in C^{q-2}(M;\mathbb{R}). \tag{17.2}$$

In particular $I(h) = [c] \in H^q(M;\mathbb{Z})$ is well defined and its realification is $[\omega]_{\mathrm{dR}}$.

Proof. A triple defines a character by evaluation of $\widetilde{h}$ on cycles modulo integers. The defining cochain equation gives (17.1). Terms of the form (17.2) evaluate to zero on cycles and leave the form unchanged. Conversely, two lifts of the same character differ on the cycle group by an integral homomorphism. Subtract a cochain extending that integral homomorphism using the splitting above. The remaining real difference vanishes on cycles, hence factors through B_{q-2}. Extend its real functional from B_{q-2} to C_{q-2}, using divisibility of $\mathbb{R}$. Thus the lift difference is an integral cochain plus a real coboundary, with the change in c then forced by the equation. Realifying the equation and applying de Rham's theorem gives the last assertion. □

Let $a(\eta)$ be the character $z \mapsto \int_z \eta \bmod \mathbb{Z}$ for a $(q-1)$-form η. Its representative is $(0, J\eta, d\eta)$. A flat class $\xi \in H^{q-1}(M;\mathbb{R}/\mathbb{Z})$ defines a character $j(\xi)$ by evaluation on cycles. Under the convention $\delta\widetilde{h} = J\omega - c$, the integral class of $j(\xi)$ is the negative of the Bockstein calculated by lifting ξ to a real cochain and taking its coboundary. The sign follows directly by taking $\omega = 0$.

Theorem 17.4. *There are natural short exact sequences*

$$0 \longrightarrow H^{q-1}(M;\mathbb{R}/\mathbb{Z}) \xrightarrow{j} \widehat{H}^q(M;\mathbb{Z}) \xrightarrow{R} \Omega^q_{\mathbb{Z}}(M) \longrightarrow 0,$$
$$0 \longrightarrow \Omega^{q-1}(M)/\Omega^{q-1}_{\mathbb{Z}}(M) \xrightarrow{a} \widehat{H}^q(M;\mathbb{Z}) \xrightarrow{I} H^q(M;\mathbb{Z}) \longrightarrow 0.$$

Here $\Omega^0_{\mathbb{Z}}$ consists of locally constant integer-valued functions.

Proof. A flat character factors through $H_{q-1}(M;\mathbb{Z})$, and every such homomorphism is a unique cohomology class by the universal coefficient theorem with injective coefficients $\mathbb{R}/\mathbb{Z}$. For an integral-period form ω, the integer homomorphism on $H_q(M;\mathbb{Z})$ given by periods lifts to integral cohomology by the universal coefficient theorem. Choose a cocycle c representing that lift. The real cochain $J\omega - c$ has zero real class and is a coboundary. This produces a character with curvature ω.

The condition $a(\eta) = 0$ says that the integrals of η on all $(q-1)$-cycles are integers. Integrating over boundaries and applying the shrinking-simplex argument gives $d\eta = 0$. Thus its kernel is the indicated subgroup. If $I(h) = 0$, the cochain representative can be chosen with $c = 0$. Then the curvature is exact by de Rham's theorem, so write $\omega = d\eta$. The real cochain $\widetilde{h} - J\eta$ is closed and can be represented up to coboundary by $J\beta$ for a closed $(q-1)$-form β. Evaluating on cycles gives $h = a(\eta + \beta)$. Finally, for any integral cocycle c one can choose a closed ω representing its real class, then solve $\delta\widetilde{h} = J\omega - c$. This proves surjectivity of I. □

For example, on $S^1 = \mathbb{R}/\mathbb{Z}$ the degree-two group is $H^1(S^1;\mathbb{R}/\mathbb{Z}) \cong \mathbb{R}/\mathbb{Z}$. All its classes have $R = I = 0$. The form $a_0 = \theta\, dx$ on the trivial line defines the character $a(a_0)$, whose value on the positively oriented circle is θ mod $\mathbb{Z}$. This is the phase of the flat connection with matrix $-2\pi i a_0$.

17.3 The space of cocycles

Fix q and use the form complex truncated by setting degrees less than q to zero. The data in a triple are cochains in the homotopy pullback

$$\begin{array}{ccc} D(q)(M) & \longrightarrow & \Omega^{\geq q}(M) \\ \downarrow & & \downarrow{\scriptstyle J} \\ C^\bullet(M;\mathbb{Z}) & \longrightarrow & C^\bullet(M;\mathbb{R}). \end{array} \tag{17.3}$$

The lower map is realification. Concretely the pullback complex has degree-n terms $(c, \widetilde{h}, \omega)$ with respective degrees $n, n-1, n$ and differential

$$(c, \widetilde{h}, \omega) \longmapsto (\delta c, J\omega - c - \delta\widetilde{h}, d\omega).$$

The middle component makes the comparison of the two real images part of the data. Its second differential vanishes because J commutes with the differentials. At degree q, cocycles and coboundaries are precisely those above. Dropping the comparison component before passing to cohomology would retain only the pair (I, R), which already loses the circle example. This mapping-cone model is the ordinary integral case of differential cocycles [13].

The space attached to (17.3) is obtained by shifting degree q to degree zero and taking the associated infinite loop space. Its connected components are $\widehat{H}^q(M;\mathbb{Z})$. For $i > 0$ its homotopy groups are $H^{q-i}(D(q)(M))$. In degrees less than q the form term does not enter the cohomology calculation, leaving the homotopy fiber of $\mathbb{Z} \to \mathbb{R}$ on cochains. Its cohomology in degree r is $H^{r-1}(M;\mathbb{R}/\mathbb{Z})$. Thus degree two has connection-preserving gauge automorphisms $H^0(M;\mathbb{R}/\mathbb{Z})$ at each object, and degree three has $H^1(M;\mathbb{R}/\mathbb{Z})$ and $H^0(M;\mathbb{R}/\mathbb{Z})$ in the first two homotopy degrees. The groups of components alone do not specify these automorphisms.

For a manifold of parameters S, evaluating the cocycle space on $S \times M$ gives families with forms in all directions of the product, including the parameter directions. Restrictions at points of S give individual characters. A collection of such pointwise restrictions can miss curvature components involving tangent vectors to S. The presheaf of spaces defined by (17.3) satisfies descent. Smooth singular cochains satisfy homotopy descent for covers by subdivision, and each sheaf of forms is fine on manifolds. These give descent for the constituents and hence for their homotopy pullback. This use of sheaves of complexes separates the family object from evaluation in a single degree [14].

CHAPTER 18

Products and differential characteristic classes

The Chern–Weil construction gives a closed form from a connection. To refine an integral characteristic class, that form must be supplied with its integral class and a comparison between them in real cochains. The comparison participates in products and remembers the secondary change under variation of the connection.

18.1 Multiplication at the cochain level

Use J and the cocycle model from Chapter 17. Fix the usual singular cup product, for instance the Alexander–Whitney product, and a natural real cochain homotopy B between integrating the wedge product and taking the cup product of the integrations. For forms ω, η with ω of degree p, the sign convention is

$$\delta B(\omega,\eta) + B(d\omega,\eta) + (-1)^p B(\omega, d\eta) = J(\omega\wedge\eta) - J\omega \smile J\eta. \tag{18.1}$$

Such a homotopy is a form of the multiplicative de Rham comparison. The construction and its coherent choices are treated in the differential cochain setting by Hopkins and Singer [13, §3.2].

Take cocycles $x = (c, h, \omega)$ of degree p and $y = (c', h', \eta)$ of degree q. Define their product by the triple

$$\left(c \smile c',\ h \smile J\eta + (-1)^p c \smile h' + B(\omega,\eta),\ \omega\wedge\eta\right). \tag{18.2}$$

Proposition 18.1. *Formula (18.2) defines a product on differential cohomology classes. Its integral class and curvature are the products of the corresponding integral classes and curvatures. For a form α of degree $p-1$,*

$$a(\alpha) \smile y = a\big(\alpha \wedge R(y)\big). \tag{18.3}$$

Proof. The coboundary of the middle component of (18.2) is

$$\begin{aligned}(J\omega - c) \smile J\eta + c \smile (J\eta - c') + J(\omega\wedge\eta) - J\omega \smile J\eta \\ = J(\omega\wedge\eta) - c \smile c'.\end{aligned}$$

This is the cocycle equation. The first and last components give the claimed product formulas for I and R.

For invariance under changing x, replace (c, h) by $(c + \delta b, h - b - \delta u)$. The change in the first component is $\delta(b \smile c')$. If ΔH is the change in the middle component, the coboundary rule for the cup product gives

$$\Delta H + b \smile c' = (-1)^p \delta(b \smile h') - \delta(u \smile J\eta).$$

Thus the change is an equivalence of the form (17.2). Changing y by $(\delta b', -b' - \delta u', 0)$ instead changes the first component by $\delta((-1)^p c \smile b')$ and the middle component by $-(-1)^p c \smile b' - \delta(c \smile u')$. This too is a cocycle equivalence.

For (18.3), substitute $c = 0$, $h = J\alpha$, and $\omega = d\alpha$. Equation (18.1) says that the resulting middle component and $J(\alpha \wedge \eta)$ differ by a coboundary. The first component is zero in both triples and the last component is $d(\alpha \wedge \eta)$. □

The coherent multiplicative comparison makes the operation associative and graded commutative on classes [13, §3.2]. The correction B has a concrete purpose. The two cochain products in (18.1) agree in real cohomology, and the term B supplies the comparison before passing to cohomology. Consequently multiplication preserves both the integral class and the curvature while controlling the character on cycles. Omitting this correction generally breaks the cocycle equation.

18.2 The cylinder and changes of connection

Lemma 18.2 (Homotopy formula)**.** *For $x \in \widehat{H}^q([0,1] \times M; \mathbb{Z})$ and the inclusions i_0, i_1,*

$$i_1^* x - i_0^* x = a\left(\int_{[0,1]} R(x)\right).$$

Here the interval direction is first in the integral. The smooth cochain and character constructions admit manifolds with boundary in this calculation.

Proof. For a cycle z, triangulate the prism $[0,1] \times z$ so that its boundary is $(i_1)_* z - (i_0)_* z$. This is the chain cross product with the interval as the first factor. The boundary formula for the cross product gives the two endpoint terms, since $\partial z = 0$. Apply (17.1) and integrate first over the interval. The equality on every cycle proves the formula. □

For two connections on one bundle, a smooth path of connections gives a connection on the pullback bundle over the cylinder. The curvature of a degree-$2j$ differential characteristic class there is the Chern–Weil form of the cylinder connection. Its interval integral is the transgression of that form. The homotopy formula says that the secondary change of the differential characteristic class is exactly the character a of that transgression. A constant curvature form on each slice does not determine the interval components of the cylinder curvature.

18.3 Characteristic refinements for unitary bundles

Let G be a Lie group with finitely many components, and let P be a principal G-bundle with connection ∇. Suppose an invariant polynomial p of degree j and a class $\xi \in H^{2j}(BG;\mathbb{Z})$ have the same real universal Chern–Weil class. The Cheeger–Simons characteristic refinement for this compatible pair is natural under pullback and satisfies

$$R(\widehat{\xi}(P,\nabla)) = p(F_\nabla), \qquad I(\widehat{\xi}(P,\nabla)) = \xi(P).$$

Existence in this setting is the characteristic-class construction of Cheeger and Simons [12, Theorem 2.2]. We use its specialization to $G = U(r)$, with the polynomial obtained as the degree-j coefficient of the determinant

$$\det\left(1 + t\,\frac{iF_\nabla}{2\pi}\right).$$

It refines the Chern class $c_j(E)$ of the associated Hermitian vector bundle. We denote the differential class by $\widehat{c}_j(E,\nabla)$. Compatibility between the determinant form and the real integral Chern class is the classical input [1].

Theorem 18.3 (Uniqueness for the unitary construction)**.** *On smooth manifolds, there is at most one natural rule on Hermitian rank-r bundles with unitary connection whose integral class is c_j and whose curvature is the corresponding determinant coefficient.*

Proof. Subtract two such rules. Their difference has zero curvature on every manifold, including the cylinder. Interpolate two unitary connections by their affine path on the cylinder. The homotopy formula applied to the difference shows that it is independent of the connection. Naturality then reduces it to a rule on Hermitian bundles.

First consider a bundle admitting a finite trivializing cover. Using a smooth partition of unity subordinate to that cover, it embeds isometrically into a finite-dimensional trivial Hermitian bundle. For example, choose smooth functions ρ_i supported in local trivializations with $\sum_i \rho_i^2 = 1$, and send a vector to the tuple of its coordinates weighted by ρ_i. Such functions are obtained by normalizing the squares of a finite partition of unity. The image gives a smooth classifying map into a finite complex Grassmannian, and the bundle is pulled back from its tautological bundle. Since the Grassmannian has only even-dimensional cells, its $H^{2j-1}(-;\mathbb{R}/\mathbb{Z})$ vanishes. The curvature exact sequence then forces the difference for its tautological bundle to be zero. Naturality forces the same for every bundle admitting a finite trivializing cover.

For an arbitrary base, the image of each smooth singular cycle is compact. Choose an open neighborhood of that image formed from finitely many trivializing

opens. Naturality and the finite-cover argument show that the difference evaluates to zero on that cycle. A character with zero value on every cycle is zero. □

Write $\widehat{c} = 1 + \sum_{j \geq 1} \widehat{c}_j$, extending degree zero by $H^0(M; \mathbb{Z})$. The proof also gives the differential Whitney sum formula by applying the same argument on products of Grassmannians. Both candidate total classes have the same curvature because the block-diagonal curvature of a direct sum gives a product of determinant forms. Their integral classes agree by the ordinary Whitney sum theorem. Each positive-degree coefficient of their difference is flat and becomes zero on a product of the finite Grassmannians used for the two bundles. This proves

$$\widehat{c}(E \oplus E', \nabla \oplus \nabla') = \widehat{c}(E, \nabla) \smile \widehat{c}(E', \nabla').$$

18.4 First Chern class as holonomy

For a unitary line, take the real local one-form a_i such that the frame connection is $A_i = -2\pi i a_i$. Parallel transport in a frame on an interval multiplies by $\exp(2\pi i \int a_i)$. Along a loop, include the changes of frame where the path crosses charts. The abelian holonomy defines a character h on smooth one-cycles. One can choose frames at the vertices of the one-chain and multiply the transport scalars of its oriented edges with their integer powers. Changing a vertex frame contributes opposite powers on incoming and outgoing edges, so the result is unchanged. On a smooth two-simplex the pulled back bundle is trivial. Stokes in a frame on that simplex gives the boundary rule with curvature $da_i = iF/(2\pi)$.

The induced integral class is c_1. In local frames $s_j = s_i g_{ij}$, write $g_{ij} = \exp(-2\pi i \lambda_{ij})$. Then $a_j - a_i = d\lambda_{ij}$ and the integral representative for c_1 is $-\delta\lambda$ using the exponential $t \mapsto \exp(2\pi i t)$. With $D = \delta + (-1)^p d$ in Čech degree p, one has $D(a + \lambda) = da + \delta\lambda$, exhibiting the real comparison with curvature. The higher-holonomy interpretation of $\widehat{c}_1$ thus recovers the actual phase of parallel transport. For a flat line its curvature is zero, but that phase can still detect either a nonzero torsion integral class or a continuous holonomy parameter.

CHAPTER 19

Integration, holonomy, and transgression

Integrating along an oriented fiber takes a differential cocycle on a total space to a cocycle on the space of parameters. The integration acts before curvature is taken. This lets the output retain the holonomy on the fibers even when integrating the curvature alone would lose it.

19.1 Coherent transfers of smooth chains

Let $\pi : E \to M$ be a smooth proper submersion with closed fibers of dimension d, equipped with an orientation of the vertical tangent bundle. One may also take an oriented smooth fiber bundle with a closed fiber as the starting point. For a smooth simplex $\sigma : \Delta^k \to M$, put $E_\sigma = \Delta^k \times_M E$, a compact smooth manifold with corners. Orient it with the simplex direction first and the vertical direction second. Proper local triviality of the oriented family and the contractibility of the simplex give E_σ the homotopy type of its fiber. The same conclusion follows by pulling back a smooth fiber bundle and taking a trivialization over the simplex. Homology here may be calculated by smooth singular chains, including for the manifolds with corners that occur in this construction.

Let E_{σ_i} be the pullback over face i and $j_i : E_{\sigma_i} \to E_\sigma$ its inclusion. We choose, for each simplex, a smooth relative fundamental chain s_σ of degree $k + d$, with prescribed boundary

$$\partial s_\sigma = \sum_{i=0}^{k} (-1)^i (j_i)_* s_{\sigma_i} \quad (k > 0). \tag{19.1}$$

For $k = 0$, choose an oriented smooth fundamental cycle of the fiber. Suppose lower degrees have been chosen. The right side of (19.1) is a cycle in ∂E_σ because the contributions from codimension-two faces cancel. It represents the boundary fundamental class. Indeed on the interior of each facet it has the corresponding local orientation, and those interiors meet every component of the boundary. The boundary of any relative fundamental representative also represents this class. Their difference bounds a chain in ∂E_σ. Adding such a chain to the representative gives the desired boundary. This argument requires orientations of the entire family, since the face choices have to agree under every identification of fibers.

Compose s_σ with the projection $E_\sigma \to E$ and extend linearly. The resulting degree-d transfer T on smooth singular chains satisfies $\partial T = T\partial$ in the base-first convention. For a $(k+d)$-form ω, relative fundamental chains compute its integral on E_σ after pullback. This holds for every top form on E_σ. A boundary change integrates to zero by Stokes, and a chain supported in the boundary has zero integral by dimension. Fiberwise Fubini therefore gives

$$\int_{T(\sigma)} \omega = \int_{E_\sigma} \omega|_{E_\sigma} = \int_\sigma \pi_! \omega, \tag{19.2}$$

where the first term uses the projection and the middle term uses its pullback. Here $\pi_!$ on forms is defined by evaluating first on horizontal lifts of base directions, then on the vertical directions. Changing a lift by a vertical vector gives a vanishing term by the fiber dimension, so this definition needs no horizontal connection. It gives, in particular, $\pi_!(\pi^*\alpha \wedge \beta) = \alpha \wedge \pi_!\beta$.

Lemma 19.1. *Two choices T, T' in this construction are chain homotopic by a homotopy K for which $K(\sigma)$ factors through E_σ.*

Proof. Induct on k, keeping the homotopy values as chains in the pullbacks before projecting to E. In degree zero the difference of the two fundamental cycles bounds in the fiber. Suppose chains k_{σ_i} giving the homotopy have been chosen on the faces. In E_σ, the chain

$$s'_\sigma - s_\sigma - \sum_i (-1)^i (j_i)_* k_{\sigma_i}$$

is a cycle because of (19.1) and the identities on the faces. For $k > 0$ its degree is $k + d > d$, so it bounds in E_σ by the fiber homotopy type. Let k_σ be a bounding chain there. Projecting to E gives $T' - T = \partial K + K\partial$. In each degree $K(\sigma)$ thus factors through a smooth space of dimension one less than the degree of that chain, a property stronger than the existence of an abstract chain homotopy. □

19.2 Pushforward of differential characters

Suppose $q > d$ and $h \in \widehat{H}^q(E; \mathbb{Z})$. For a smooth integral $(q-d-1)$-cycle z of M, define

$$(\widehat{\pi}_! h)(z) = h(T(z)). \tag{19.3}$$

By the chain equation $T(z)$ is a cycle. The boundary of a $(q-d)$-chain c is sent to the boundary of $T(c)$. The character boundary rule and (19.2) give

$$(\widehat{\pi}_! h)(\partial c) = \int_{T(c)} R(h) = \int_c \pi_! R(h) \pmod{\mathbb{Z}}.$$

Thus the pushforward is a character. For different choices of transfer its change on z is $h(\partial K(z))$. The integral of $R(h)$ on $K(z)$ vanishes because each summand factors through a pullback of dimension $q-1$. The character is consequently independent of the fundamental chains used to define it.

In an oriented pullback square

$$\begin{array}{ccc} E' & \xrightarrow{\widetilde{g}} & E \\ \downarrow{\scriptstyle \pi'} & & \downarrow{\scriptstyle \pi} \\ M' & \xrightarrow{g} & M \end{array}$$

the bundles over a simplex of M' are canonically the bundles over the composite simplex of M. Using corresponding fundamental chains and applying independence of choices gives

$$g^*\widehat{\pi}_!(h) = \widehat{\pi}'_!(\widetilde{g}^*h).$$

The curvature and integral class formulas are

$$R(\widehat{\pi}_!h) = \pi_!R(h), \qquad I(\widehat{\pi}_!h) = \pi_!I(h).$$

For the second formula, use a triple $(c, \widetilde{h}, \omega)$ on E and precompose the two cochains with T. Equation (19.2) identifies the resulting triple with curvature $\pi_!\omega$. The integral cochain operation is the ordinary oriented transfer dual to the relative fundamental chains of the bundle. This is a smooth singular description of the integral Gysin map. Fiber integration for characters has several equivalent constructions [15, §7].

A useful special case is $h = a(\eta)$. By (19.2) in degree $q-1$, $\widehat{\pi}_!a(\eta) = a(\pi_!\eta)$. For a product $M \times F \to M$, the transfer can be chosen as the chain cross product with a smooth fundamental cycle of F, with the base cycle written first. A relative fundamental chain of a fiber with boundary would instead contribute an additional term to $\partial T(z)$. The definition with absolute cycles above uses closed fibers. Relative characters describe integration when boundary data, such as a trivialization, are specified [15, §§7.2 and 8.3].

19.3 Loop families

Let B be a finite-dimensional smooth parameter manifold and let $\Gamma : B \times S^1 \to M$ be a smooth family of loops. For every $h \in \widehat{H}^q(M;\mathbb{Z})$ with $q \geq 2$, set

$$\tau_\Gamma(h) = \widehat{\pi}_!\Gamma^*h \in \widehat{H}^{q-1}(B;\mathbb{Z}), \qquad \pi : B \times S^1 \longrightarrow B.$$

For $q = 2$, this character is the smooth S^1-valued holonomy function on the family. Evaluating it at b evaluates h on the oriented loop $\Gamma(b, -)$. The differential of

the function is described in real angular units by the fiber integral of $\Gamma^* R(h)$, including its component with one base direction and one circle direction. The formula explains how holonomies vary in a family even when each individual loop sees only the phase.

For $q = 3$, the output is a degree-two class, so it is represented by a line with connection on B. Evaluate on a loop $b : S^1 \to B$. Equation (19.3) identifies the resulting phase with h evaluated on the mapped torus

$$S^1 \times S^1 \longrightarrow M, \qquad (s, t) \longmapsto \Gamma(b(s), t).$$

The orientation of the torus is s first, t second. This characterization lets one use loop-space transgression without choosing a particular calculus on the whole infinite-dimensional free loop space. Each smooth family gives a well-defined pullback datum, and the construction is natural in the parameter manifold.

Example 19.2. On $M = \mathbb{R}^2/\mathbb{Z}^2$ take $h = a(\theta\, dx \wedge dy)$ of degree three with $\theta \in \mathbb{R}$. Let $B = S^1$ and $\Gamma(s, t) = (s, t)$. The resulting degree-two class is $a(\theta\, ds)$. Its curvature is zero, and its phase on the positive generator is θ mod $\mathbb{Z}$. Transgressing the curvature alone would give zero for every θ, whereas transgressing the character distinguishes the phases.

CHAPTER 20

Bundle gerbes and surface phases

A unitary line with connection has one-form potentials on charts and circle-valued gluing functions on double intersections. A bundle gerbe with connection and curving has two-form potentials on charts, one-form connection data on double intersections, and circle-valued coherence on triple intersections. Its invariant is a degree-three character, so the boundary rule compares its phase on a surface with the integral of a three-form over a filling.

20.1 The Deligne complex and its signs

At the level of sheaves, the real cochain sheaf in the pullback model of Chapter 17 can be computed by the de Rham complex, using the Poincaré lemma. Write $\underline{\mathbb{Z}}$ for the locally constant integer sheaf. The homotopy pullback then identifies with the homotopy fiber of $\underline{\mathbb{Z}} \to \Omega^{<q}$, since $\Omega^{<q}$ is the quotient by the subcomplex $\Omega^{\geq q}$. A signed presentation of that fiber is the Deligne complex

$$\underline{\mathbb{Z}} \longrightarrow \Omega^0 \xrightarrow{d} \cdots \xrightarrow{d} \Omega^{q-1},$$

with $\underline{\mathbb{Z}}$ in degree zero and the first map the inclusion. The sign convention can be read directly from the middle component $-c - \delta\widetilde{h}$ of the fiber differential. In internal degree $n > 0$, multiply that component's cochain variable by $(-1)^n$ to obtain the displayed positive differentials. In degree q the curvature map on the top form is therefore $(-1)^q d$. Projection onto $\underline{\mathbb{Z}}$ gives the integral class. The hypercohomology in degree q is $\widehat{H}^q(M;\mathbb{Z})$, and the shifted complex retains the higher equivalences between cocycles [13, 14].

Choose a good open cover, so that every nonempty finite intersection is contractible and the Čech complex computes the required hypercohomology. Use δ for the alternating Čech differential. In Čech degree p the total differential is $D = \delta + (-1)^p d_{\mathrm{Del}}$, where d_{Del} includes the first map of the Deligne complex. In total degree three write the entries in order of increasing internal degree as

$$(n, \lambda, -A, -B).$$

The entries have Čech degrees $3, 2, 1, 0$, respectively, with n integer-valued, λ real-valued, A a one-form cochain and B a two-form cochain. The cocycle conditions become

$$\delta n = 0, \qquad \delta\lambda = n, \qquad \delta A = d\lambda, \qquad \delta B = dA. \tag{20.1}$$

The curvings B_i give a global three-form $H = dB_i$, because the difference on a double intersection is the differential of A_{ij}. This is the curvature of the class. For comparison, $D(B + A - \lambda) = H - n$ in the full real Čech–de Rham complex with $D = \delta + (-1)^p d$. Thus the real image of $[n]$ is $[H]$. The integer cochain remains part of the cocycle even when it represents torsion.

For an integer two-cochain m, a real one-cochain μ, and a zero-cochain Λ of one-forms, adding a total coboundary with components $(m, \mu, -\Lambda)$ in degree two gives

$$B' = B + d\Lambda, \qquad A' = A + \delta\Lambda + d\mu,$$
$$\lambda' = \lambda + \delta\mu + m, \qquad n' = n + \delta m.$$

These signs follow by applying D to each component. They preserve (20.1) and H. Higher gauge relations between the changes are obtained from the preceding degrees of the same total complex. This explains why the gerbe cocycle belongs to a space with nontrivial higher homotopy groups.

In degree two the analogous entries are (n, λ, a), and the equations are $\delta a = d\lambda$ and $n = -\delta\lambda$. Exponentiating with $g_{ij} = \exp(-2\pi i \lambda_{ij})$ gives precisely the frame changes for a line with $A_i = -2\pi i a_i$. Thus the Deligne model identifies the integral component of its holonomy character with the integral Chern cocycle, including its possible torsion. This also verifies the normalization in Chapter 18.

20.2 Geometric realization by bundle gerbes

Take a surjective submersion $p : Y \to M$ possessing local sections. Denote its repeated fiber products by $Y^{[k]}$. A unitary bundle gerbe has a Hermitian line $L \to Y^{[2]}$, with multiplication over $Y^{[3]}$

$$L_{y_1,y_2} \otimes L_{y_2,y_3} \longrightarrow L_{y_1,y_3}.$$

The map is a unitary isomorphism and the two composites on $Y^{[4]}$ agree. This is the line-bundle realization of the gerbe descent condition [16]. A connection is compatible when multiplication preserves the tensor product connection. Its normalized curvature $F^L_{\mathbb{R}} = iF_L/(2\pi)$ therefore has zero alternating sum of pullbacks to $Y^{[3]}$. A curving is a two-form B on Y with

$$F^L_{\mathbb{R}} = \mathrm{pr}_2^{*} B - \mathrm{pr}_1^{*} B.$$

A curving exists for a compatible line connection on a paracompact base. Choose local sections s_i and a subordinate smooth partition of unity ρ_i on M. On $p^{-1}(U_i)$ let B'_i be the pullback of $F^L_{\mathbb{R}}$ by $y \mapsto (s_i(p(y)), y)$. The cocycle identity for $F^L_{\mathbb{R}}$ gives $B'_i(y_2) - B'_i(y_1) = F^L_{\mathbb{R}}(y_1, y_2)$ as pullback maps on forms. Hence $B = \sum_i p^{*}\rho_i B'_i$, with terms extended by zero outside their supports, is a

curving. Taking d gives equality of the two pullbacks of dB. A form with this equality descends along p, since pullback to any two local sections gives the same local form on the overlap. Thus $dB = p^*H$.

Choose local sections of Y on a good cover and unit frames of the induced lines on double intersections. Write the connection form in each line frame as $-2\pi i A_{ij}$. With the frame multiplication scalar $\exp(-2\pi i \lambda_{ijk})$, compatibility of multiplication with the connection gives $\delta A = d\lambda$. Associativity gives $\delta\lambda \in \mathbb{Z}$. The curving identity gives $\delta B = dA$ for $B_i = s_i^* B$. In these conventions the integral Dixmier–Douady cocycle is the Bockstein of the inverse frame multiplication scalars, which is $\delta\lambda$ for the positive exponential. This assigns the sign so that the integral class realifies to the normalized three-curvature.

Conversely, on a good cover, (20.1) constructs a bundle gerbe with $Y = \coprod_i U_i$. Put a trivial line with connection $-2\pi i A_{ij}$ on each double intersection. Use the exponential of $-2\pi i\lambda$ for multiplication. The equations are exactly the compatibility, associativity and curving conditions. Coboundaries describe changes of local representatives and tensoring by the corresponding trivialization data. Up to refinement and stable equivalence with connection, this gives the gerbe classes described by degree-three differential cohomology. For the underlying topological gerbe the equivalence relation is stable isomorphism, and the class is the signed Dixmier–Douady class above [28]. Surface phases are evaluations of the associated differential character on smooth two-cycles.

20.3 A three-sphere and its equator

Let H be a closed three-form on an oriented S^3 with $\int_{S^3} H = N \in \mathbb{Z}$. Cover S^3 by U_+ and U_-, the complements of the south and north pole. Choose two-forms B_+ and B_- there with $dB_\pm = H$, since each open is contractible. The difference $K = B_- - B_+$ is closed on the overlap. Orient the equator as the boundary of the southern closed hemisphere. Stokes on the two hemispheres gives

$$\int_{S^2} K = \int_{S^3} H = N.$$

The overlap has the homotopy type of that S^2, so K has integral periods. Choose a line L on the overlap with c_1 pairing to N on the equator. Its normalized curvature can be chosen to equal K, because any initial normalized curvature represents the same real class and can be adjusted by an exact form. Use L on the ordered overlap $(+,-)$, its dual on $(-,+)$, and the trivial lines on the equal-index overlaps. Evaluation and tensor product multiplication give the associative gerbe multiplication. The two curvings yield exactly H, and the integral class of the gerbe has value N on the fundamental class of S^3.

Every integral two-cycle z on S^3 bounds a three-chain C, so the gerbe phase is

$$\exp\left(2\pi i \int_C H\right).$$

A different filling leaves the phase unchanged because periods are integral. If H is the round volume form normalized to have integral N, its integral over either hemisphere is $N/2$. The equatorial phase is $(-1)^N$. This calculation uses both the orientation of the fiberwise boundary and the period convention. A global curving on S^3 could not give this nonzero integral of H for $N \neq 0$, since the integral of its exterior derivative would vanish by Stokes.

Part VI Higher groups and surface transport

CHAPTER 21

Strict two-groups and crossed modules

An ordinary gauge transformation changes a frame. In higher gauge theory there are also transformations between frame changes. A useful local model is a groupoid whose objects and arrows have group structures and whose structure maps respect multiplication. It has a particularly explicit description by two groups, a boundary homomorphism, and an action. The distinction between an arrow and its boundary will become the distinction between a surface transport and the path transports on its edges.

21.1 The groupoid of higher arrows

Definition 21.1 (Crossed module)**.** A crossed module of Lie groups consists of a smooth homomorphism $\partial : H \to G$ and a smooth left action $g \triangleright h$ of G on H by group automorphisms, such that

$$\partial(g \triangleright h) = g\,\partial(h)\,g^{-1}, \tag{21.1}$$

$$\partial(h) \triangleright k = hkh^{-1}. \tag{21.2}$$

The second identity is the Peiffer identity.

The associated strict Lie two-group has objects $g \in G$. An arrow (h, g) starts at g and ends at $\partial(h)g$. Composition in the groupoid is multiplication of the H-labels in chronological order from right to left:

$$(k, \partial(h)g) \circ (h, g) = (kh, g).$$

The identity of g is $(1, g)$ and the inverse for groupoid composition is $(h^{-1}, \partial(h)g)$. There is a second multiplication, the tensor product, with

$$(h, g) \otimes (k, \ell) = \big(h(g \triangleright k), g\ell\big).$$

The object group multiplies by $g \otimes \ell = g\ell$. Equation (21.1) verifies the target of the tensor product:

$$\partial\big(h(g \triangleright k)\big)g\ell = (\partial(h)g)(\partial(k)\ell).$$

The group law on arrows is the semidirect product $H \rtimes G$. Its inverse uses action by g^{-1}, whereas the inverse of an arrow in the groupoid uses the target $\partial(h)g$.

Tensor and groupoid composition must satisfy interchange. If the upper arrows in two composable pairs carry labels h' and k' and the lower arrows carry h and k, respectively, multiplying each pair first gives the label

$$h'h(g \triangleright (k'k)).$$

Tensoring at each level first gives the label

$$h'\big((\partial(h)g) \triangleright k'\big)h(g \triangleright k).$$

Since $(\partial(h)g) \triangleright k' = h(g \triangleright k')h^{-1}$, the two labels agree. This calculation explains why equivariance and the Peiffer identity need separate hypotheses.

A strict monoidal groupoid with such an arrow group can also be read as a one-object strict two-category. Its one-morphisms are G, its two-morphisms are the arrows just described, and horizontal composition is the tensor product. In this reading, when one path runs after another, the corresponding one-morphism is multiplied on the left. This agrees with successive parallel transport.

21.2 Recovering a crossed module from a group object

Proposition 21.2. *The crossed module construction and passage to the kernel of the source map are inverse up to natural isomorphism. They identify crossed modules of Lie groups with internal groupoids in Lie groups.*

Proof. Let $\mathcal{G}_1 \rightrightarrows \mathcal{G}_0$ be an internal groupoid with source s, target t, and unit u. Put $G = \mathcal{G}_0$, $H = \ker s$, and $\partial = t|_H$. The split surjection s is a submersion and its kernel is a Lie subgroup. Conjugation by $u(g)$ defines the action on H and applying t verifies (21.1).

To prove the Peiffer identity, use that groupoid composition is a group homomorphism on composable pairs. If $a \in \ker s$ and $b \in \ker t$, then the composable pairs $(a, 1)$ and $(1, b)$ commute in the group of composable pairs. Their composites are a and b, so a commutes with b in $\mathcal{G}_1$. For $h \in H$, the element $h^{-1}u(\partial(h))$ belongs to $\ker t$. Its commutation with $k \in H$ is exactly

$$u(\partial(h))ku(\partial(h))^{-1} = hkh^{-1}.$$

The map $(h, g) \mapsto hu(g)$ identifies $H \rtimes G$ with $\mathcal{G}_1$, with inverse $a \mapsto (au(s(a))^{-1}, s(a))$. This identification respects source and target. For a composable pair with lower arrow $hu(g)$, the upper arrow has the form $ku(\partial(h)g)$. Factor the pair as

$$(k, 1) \cdot (u(\partial(h)g), hu(g))$$

in $\mathcal{G}_1 \times_G \mathcal{G}_1$. Applying the composition homomorphism gives $khu(g)$. This is the composition formula of the crossed module groupoid. The constructions on homomorphisms follow by restricting to H and applying the same identification. □

The result addresses a strict model. A weak two-group has coherent associators, and questions about smooth presentations must keep track of how it is presented. In particular a choice of crossed module as a smooth model is extra structure. Baez and Lauda develop the corresponding algebraic and internal notions and the role of coherence [17].

21.3 The nerve and the two homotopy groups

The groupoid nerve has an explicit n-simplex. It consists of n consecutive H-labels and an initial object:

$$(h_1, \dots, h_n; g) \longleftrightarrow g \xrightarrow{h_1} \partial(h_1)g \xrightarrow{h_2} \partial(h_2h_1)g \longrightarrow \cdots .$$

In degree two its middle face replaces h_1, h_2 by h_2h_1. The first face adjusts the initial object by $\partial(h_1)$. Degeneracies insert identity arrows. Since all structure maps of the groupoid respect group multiplication, the nerve is a simplicial group. This gives a nerve description of the same strict model, including the operation responsible for the triangular faces.

For the next interpretation take the underlying *discrete* groups. The natural components of the groupoid are cosets of $\partial(H)$, which is normal by (21.1). At each object the automorphism group is $\ker\partial$. A groupoid can be replaced, up to equivalence of categories, by one representative of each component together with all of its automorphisms. Indeed inclusion of that subcategory is fully faithful and essentially surjective. The groupoid's nerve therefore has components of the form $B(\ker\partial)$, with contractible universal cover. Delooping the simplicial group gives a connected two-type with

$$\pi_1 \cong G/\partial(H), \qquad \pi_2 \cong \ker\partial.$$

Here the delooping can be defined by the simplicial classifying construction, whose loop space is the realization of the simplicial group. Passing instead to smooth sheaves or keeping the topology of the Lie groups changes that interpretation. For example, the classifying space of the topological circle has its first nonzero homotopy group in degree two, while the classifying space of the discrete circle has fundamental group the circle regarded as an abstract group.

If $a \in \ker\partial$, then (21.2) gives $aka^{-1} = k$ for every $k \in H$. Thus $\ker\partial$ is abelian, and the action of $\partial(H)$ on it is trivial. This supplies the $G/\partial(H)$-action on the second homotopy group. The action is visible in the tensor formula as conjugation of a two-cell by the one-cell g.

21.4 Models and their geometric interpretations

Example 21.3 (Ordinary groups and abelian two-groups)**.** For $H = \{1\}$ the associated groupoid is discrete on G, and its delooping gives BG in the discrete interpretation. If $G = \{1\}$ and H is abelian, the groupoid has a single object and all its arrows are elements of H. The resulting two-category models a twice-delooped abelian group. When $H = U(1)$ is kept as a smooth group, the corresponding smooth bundle theory is the gerbe theory of Chapter 20.

Example 21.4 (An inner action)**.** Let H be a Lie group, and take G to be a Lie group of smooth automorphisms of H containing its inner automorphisms, with smooth evaluation and inner-automorphism map. The map $h \mapsto \mathrm{Ad}_h$ with the evaluation action gives a crossed module since $\alpha \,\mathrm{Ad}_h\, \alpha^{-1} = \mathrm{Ad}_{\alpha(h)}$. Its kernel is the center $Z(H)$, and its quotient of objects is the chosen group of automorphisms modulo inner automorphisms. A choice of finite-dimensional Lie group G is part of this example. For instance $H = U(n)$ and $G = PU(n)$ with conjugation work. This model keeps central two-automorphisms even when every object belongs to a single component.

Example 21.5 (An abelian kernel with a nontrivial action)**.** Let V be a finite-dimensional real representation of G, with $H = (V, +)$ and trivial boundary. The Peiffer identity holds because V is abelian. There are no arrows between distinct g's, but tensoring with g acts on the automorphism arrows by the given representation. This provides a model in which the two-form connection takes values in an abelian group and still depends on transport by a nonabelian one-form connection.

CHAPTER 22

Two-connections and their curvatures

A strict two-group assigns a boundary in G to each arrow label in H. On a small surface, the path transport around the boundary is controlled by the curvature of an ordinary connection. A two-form suitable for surface transport must have that curvature as its infinitesimal boundary. This yields a compatibility condition between the one- and two-form data.

22.1 Forms for a differential crossed module

Differentiate the crossed module of Chapter 21. Write $\mathfrak{h}, \mathfrak{g}$ for the Lie algebras, ∂_* for the differential of the boundary, and $X \triangleright b$ for the induced action of $\mathfrak{g}$ on $\mathfrak{h}$ by derivations. The differentiated identities are

$$\partial_*(X \triangleright b) = [X, \partial_* b], \qquad (\partial_* a) \triangleright b = [a, b].$$

They extend to forms by combining wedge of scalar forms with the indicated bracket or action on coefficients. In particular, brackets of forms are graded antisymmetric. When both inputs have degree one, the factor $\frac{1}{2}$ in $\frac{1}{2}[A, A]$ accounts for the two cross terms.

On a trivialization over U, a pair of potentials consists of $A \in \Omega^1(U; \mathfrak{g})$ and $B \in \Omega^2(U; \mathfrak{h})$. Write $D_A = d + A \triangleright (-)$ on $\mathfrak{h}$-valued forms and $D_A = d + [A, -]$ on $\mathfrak{g}$-valued forms. Our conventions are

$$F_A = dA + \tfrac{1}{2}[A, A], \tag{22.1}$$

$$K = F_A - \partial_* B, \tag{22.2}$$

$$\mathcal{H} = D_A B. \tag{22.3}$$

The form K is the fake curvature and $\mathcal{H}$ is the three-form curvature. We reserve the term *fake-flat* for $K = 0$. In the path two-functor model this condition is part of the local correspondence; a pair of forms can also be considered without imposing it [18, 19].

Proposition 22.1 (Higher Bianchi identities)**.** *For a pair of potentials with conventions* (22.1)–(22.3)*,*

$$D_A K = -\partial_* \mathcal{H}, \qquad D_A \mathcal{H} = K \triangleright B.$$

If $K = 0$, then $\mathcal{H}$ takes values in $\ker \partial_*$ and is D_A-closed there. The operator D_A is flat on $\ker \partial_*$.

Proof. Equivariance of ∂_* gives $D_A \partial_* B = \partial_* D_A B$. The ordinary Bianchi identity $D_A F_A = 0$, applied to (22.2), gives the first formula. On an $\mathfrak{h}$-valued form u, expansion gives

$$D_A^2 u = dA \triangleright u + A \triangleright (A \triangleright u) = F_A \triangleright u.$$

The two terms involving du cancel, and the representation identity makes $A \triangleright (A \triangleright u)$ equal to $\frac{1}{2}[A, A] \triangleright u$. In particular,

$$D_A \mathcal{H} = K \triangleright B + (\partial_* B) \triangleright B = K \triangleright B + [B, B].$$

Here $[B, B] = 0$ because B has even degree and the coefficients are over $\mathbb{R}$. When $K = 0$, the first identity gives $\partial_* \mathcal{H} = 0$. The differentiated Peiffer identity makes $\ker \partial_*$ central in $\mathfrak{h}$, and equivariance makes it stable under the G-action. On that subspace $D_A^2 u = (\partial_* B) \triangleright u = [B, u] = 0$. □

Even under fake flatness D_A-closedness is a statement about the flat local system on the kernel, rather than ordinary closedness of each coefficient in an arbitrary frame. In a parallel local frame for that system it becomes ordinary closedness. The distinction is meaningful when the $G/\partial(H)$-action on $\ker \partial_*$ is nontrivial.

22.2 Two elementary changes of potentials

The ordinary frame change by $g : U \to G$ acts by

$$A^g = \mathrm{Ad}(g^{-1})A + g^{-1} dg, \qquad B^g = g^{-1} \triangleright B.$$

Differentiating the representation sends $g^{-1} dg$ to the logarithmic derivative of the matrix acting on $\mathfrak{h}$. This cancels the derivative of g^{-1} in $D_{A^g}(g^{-1} \triangleright B)$. It follows that K transforms by $\mathrm{Ad}(g^{-1})$ and $\mathcal{H}$ by $g^{-1} \triangleright (-)$.

There is also a one-form change. For $a \in \Omega^1(U; \mathfrak{h})$, put

$$A' = A + \partial_* a, \qquad B' = B + D_A a + \tfrac{1}{2}[a, a].$$

Expanding the ordinary curvature gives

$$F_{A'} = F_A + \partial_*(D_A a + \tfrac{1}{2}[a, a]), \qquad K' = K.$$

By the Peiffer identity $D_{A'} = D_A + [a, -]$. Use the graded derivation rule $D_A[a, a] = -2[a, D_A a]$ and the graded Jacobi identity $[a, [a, a]] = 0$. Expanding the new three-form curvature leaves

$$\mathcal{H}' = \mathcal{H} + F_A \triangleright a + [a, B] = \mathcal{H} + K \triangleright a.$$

In the last equality the terms $[B, a]$ and $[a, B]$ cancel. Thus the one-form change preserves $\mathcal{H}$ in the fake-flat case. These formulas are local. Gluing a two-bundle requires transition data and coherence on overlaps in addition to these changes of representatives.

22.3 The flat quotient connection

Suppose the image $N = \partial(H)$ is a closed Lie subgroup of G with Lie algebra $\partial_*\mathfrak{h}$. Equivariance makes it normal. Write $Q = G/N$, and use q for the quotient map and q_* for its differential. In a fake-flat model

$$F_{q_*A} = q_*F_A = q_*\partial_*B = 0.$$

The equality uses the fact that q_* preserves brackets. The change $A \mapsto A + \partial_*a$ has no effect on q_*A.

This calculation also gives a descent statement with explicit hypotheses. Suppose local fake-flat pairs have changes of frame g_{ij} on double overlaps, and h_{ijk} on triple overlaps with $g_{ij}g_{jk} = \partial(h_{ijk})g_{ik}$. Suppose the local one-forms satisfy

$$A_j = A_i^{g_{ij}} + \partial_*a_{ij}$$

for one-forms a_{ij} valued in $\mathfrak{h}$. Then $q(g_{ij})q(g_{jk}) = q(g_{ik})$, so the quotient transition functions glue an ordinary principal Q-bundle. The projected forms satisfy the ordinary gauge relation with these transition functions. They define a flat connection on that bundle. This is the part of the higher connection visible after identifying two path transports joined by a two-arrow. A smooth Lie group quotient was an assumption in this argument. At the Lie-algebra level the projection to $\mathfrak{g}/\partial_*\mathfrak{h}$ and its local flatness still make sense.

22.4 Local models and transport compatibility

Example 22.2 (Abelian curving)**.** Let G be trivial and $H = U(1)$. In real coordinates on the Lie algebra, the two-form potential b has three-form curvature db. A one-form change is $b \mapsto b + d\lambda$. On $\mathbb{R}^3$, take $b = x\,dy \wedge dz$. Then $db = dx \wedge dy \wedge dz$, so a fake-flat potential can have nonzero curvature. The leading transport on a surface comes from integrating b, and its change along a three-dimensional deformation comes from integrating db.

Example 22.3 (A twisted abelian kernel)**.** For the crossed module $V \to G$ with zero boundary from Chapter 21, fake flatness forces $F_A = 0$. The form B is V-valued and its curvature is D_AB. Take $G = GL(V)$ and $U = (\mathbb{R}/\mathbb{Z}) \times S^2$. Let $T \in \mathrm{End}(V)$, $A = T\,d\theta$, and $B = v\,\omega$, with θ the circle coordinate, $v \in V$, and ω an area form on S^2 pulled back to U. Then $F_A = 0$ and

$$\mathcal{H} = Tv\,d\theta \wedge \omega.$$

The local system can have monodromy $\exp(-T)$ around the circle. The three-form uses the action even though its coefficients lie in an abelian Lie algebra.

Example 22.4 (Identity boundary)**.** For $\partial = \mathrm{id}_G$ with conjugation action, fake flatness sets $B = F_A$, hence $\mathcal{H} = D_A F_A = 0$. The underlying groupoid has a unique arrow between any pair of objects. It records a comparison between transports but has no nonidentity automorphism at a fixed transport.

For the relevance of K to surface integration, consider a small coordinate rectangle oriented by $dx \wedge dy$. Its bottom and right edges give a path γ_0 from the lower left to the upper right. The left and top edges give a parallel path γ_1. Use U_γ for parallel transport with equation $\dot{U} = -A(\dot{\gamma})U$. Comparing the two endpoint operators gives $U_{\gamma_1} U_{\gamma_0}^{-1}$. This product corresponds to the clockwise boundary based at the common endpoint. On a square of side length ϵ, the standard curvature expansion, in a smooth frame, reads

$$U_{\gamma_1} U_{\gamma_0}^{-1} = 1 + \epsilon^2 F_A(\partial_x, \partial_y) + O(\epsilon^3)$$

in coordinates at the identity. One can see the coefficient by choosing a frame with $A = 0$ at the lower left. The line integral around the rectangle gives the exterior derivative term to order two, and the bracket term at that point is zero in the chosen frame. Covariance gives the formula in the original frame. A surface arrow (h, U_{γ_0}) has target U_{γ_1} exactly when $\partial(h) = U_{\gamma_1} U_{\gamma_0}^{-1}$. The coefficient of area in h is $B(\partial_x, \partial_y)$. Equating coefficients gives $K = 0$. The next chapter describes the global composition law for such arrows.

CHAPTER 23

Thin homotopy and surface transport

A bundle with connection sends a path to an isomorphism between fibers. A higher connection sends a deformation between two paths to a comparison between their isomorphisms. The parameterization of the deformation should have no effect on this comparison. The rank conditions defining thin homotopy express the precise independence needed here. A homotopy whose image sweeps out a three-dimensional volume can change the comparison, and the three-form curvature measures that effect.

23.1 The path two-groupoid

Take paths with sitting instants, so a path is constant on collars of its endpoints. One-morphisms from x to y are paths modulo endpoint-preserving smooth homotopies of differential rank at most one. A *bigon* from γ_0 to γ_1 is a smooth map $\Sigma(t, s)$ on the square. The parameter t runs along a path and s varies it:

$$\Sigma(t,0) = \gamma_0(t), \qquad \Sigma(t,1) = \gamma_1(t), \qquad \Sigma(0,s) = x, \qquad \Sigma(1,s) = y.$$

Require collars on which the map is independent of the coordinate normal to each side. We orient its integration domain by $dt \wedge ds$, so its boundary is $\gamma_0 - \gamma_1$ after discarding the constant edges. Two bigons are equivalent if they are joined by a collared smooth homotopy of rank at most two, with fixed end objects and with rank at most one on the two side faces that vary the boundary paths. The two-morphisms are these classes of bigons.

Vertical composition stacks squares in s. If boundary paths have only the same thin class, a thin bigon joins the chosen representatives before stacking. Horizontal composition uses concatenation in t. The collars make concatenation smooth. Different choices of rescaling on the concatenated intervals give homotopic parameterizations through the same surface. Such a homotopy has rank at most two because it factors through that surface, and its restrictions to the moving path faces have rank at most one. In the full quotient these compositions give the strict path two-groupoid $\Pi_2^{\mathrm{thin}}(M)$. In particular the independence of the connecting thin bigon and the interchange law are part of the path two-groupoid construction in [19, §2.1]. Thin equivalence of two bigons is more restrictive than ordinary homotopy, even when the boundary paths agree pointwise.

23.2 Smooth two-functors and local potentials

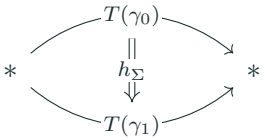
Let $\mathcal{G}$ be the finite-dimensional strict Lie two-group defined by a smooth crossed module. Its one-object two-category is denoted by $B\mathcal{G}$ in this chapter. A two-functor $T : \Pi_2^{\mathrm{thin}}(U) \to B\mathcal{G}$ sends the paths to elements of G. It associates to a bigon Σ with source γ_0 an arrow $(h_\Sigma, T(\gamma_0))$ as in Chapter 21. In particular

$$\partial(h_\Sigma)T(\gamma_0) = T(\gamma_1).$$

For successive vertical bigons with labels h, h', the composite has label $h'h$. For horizontal composition, if Σ_1 runs before Σ_2 and their sources have transports g_1, g_2, the label is

$$h_2(g_2 \triangleright h_1).$$

The action accounts for transporting the first surface label to the frame used by the second. Diagrammatically, the comparison belongs to the hom-groupoid of the target two-category:



Smoothness here means smoothness on smooth families in the path and bigon quotients with their quotient diffeologies. Use the smooth structure of G and $H \rtimes G$ on the values. Thus a set-theoretic two-functor need not have local differential-form data.

With a fixed trivialization on U, Schreiber and Waldorf prove that these smooth two-functors correspond to pairs $(A, B) \in \Omega^1(U; \mathfrak{g}) \times \Omega^2(U; \mathfrak{h})$ satisfying $F_A = \partial_* B$, in the conventions for orientation and infinitesimal extraction used there [19, Theorem 2.21]. One extracts the one-form from short paths and the two-form from shrinking rectangles. The converse uses ordinary differential equations for path transport and a second integration for the bigon label. The boundary calculation of Chapter 22 is the infinitesimal necessity of the equation in our conventions. The theorem supplies sufficiency and reconstructs the two-functor. Global transport for a nontrivial higher bundle uses local trivializations and their higher transformation data. The global formulation with transport two-functors and descent is developed in [20].

23.3 Abelian transport from a two-form

The abelian case has a direct construction. Take $G = \{1\}$ and $H = U(1)$. Let $b \in \Omega^2(M)$ be a global curving for a trivial gerbe. All path transports are the

sole G-value, and the surface assignment is

$$h_b(\Sigma) = \exp\left(2\pi i \int_{[0,1]^2} \Sigma^* b\right).$$

For either form of concatenation, the integral over the new square is the sum of the integrals over the two old squares after orientation-preserving changes of variable. Thus the assignment respects both compositions. The identity bigon factors through a path, so its integral is zero. Smoothness on parameter families follows by differentiating an integral over a fixed compact domain.

For thin invariance take a bigon homotopy V on the cube. At every point $\operatorname{rank}(dV) \leq 2$, so the pullback of db is zero. Two side faces are constant and the other two have rank at most one, so their pullbacks of b are zero. Using the cube orientation with the deformation parameter first and Stokes' theorem gives

$$\int \Sigma'^* b - \int \Sigma^* b = \int V^* db = 0.$$

This proves that the surface formula descends to the thin quotient even if db is nonzero. For an unrestricted homotopy with fixed boundary paths the same calculation gives

$$\frac{h_b(\Sigma')}{h_b(\Sigma)} = \exp\left(2\pi i \int V^* db\right).$$

In particular the local curvature is recoverable from infinitesimal three-dimensional changes of holonomy.

If $b' = b + d\lambda$ for a real one-form λ, then

$$\frac{h_{b'}(\Sigma)}{h_b(\Sigma)} = \exp\left(2\pi i \left(\textstyle\int_{\gamma_0} \lambda - \int_{\gamma_1} \lambda\right)\right).$$

The ratio is a boundary comparison for an open bigon and is one for a closed surface. On smooth singular two-cycles z, the rule $z \mapsto \int_z b \bmod \mathbb{Z}$ is a degree-three differential character. For any smooth singular three-chain c, its value on ∂c is $\int_c db \bmod \mathbb{Z}$. The underlying integral class is zero for this globally defined potential. A gerbe with nonzero integral class uses locally defined curvings with transition terms, as in Chapter 20.

For example, on $\mathbb{R}^3$ take $b = \frac{1}{2} x\, dy \wedge dz$ and let c be the unit cube with its standard orientation. The curvature integral is $1/2$. Equivalently the face at $x = 1$ contributes $1/2$ with outward orientation while the other faces contribute zero. Its surface holonomy is -1 even though ∂c is a boundary. This is consistent with the differential character law and with the invariance of holonomy under thin deformations.

23.4 Transgression and orientation

For a global b, a smooth family of loops carries the transgressed one-form

$$(\tau b)_\gamma(V) = \int_{S^1} b_{\gamma(t)}(\dot\gamma(t), V(t))\, dt.$$

Here the variation vector V is along the loop. The formula defines a form on families without putting a choice of infinite-dimensional manifold structure on the free loop space. A path of loops makes a cylinder. With the loop tangent ordered first, the integral of τb over the path equals the integral of b on this cylinder. Consequently the trivial line with connection form $-2\pi i\,\tau b$ has precisely the cylinder transport coming from h_b.

The sign in its curvature can be checked on any finite-dimensional parameter space S. Write the pulled-back two-form on $S^1 \times S$ as $dt \wedge \alpha(t) + \beta(t)$, with α, β forms on S. The coefficient of dt in its exterior derivative is $\partial_t \beta - d_S \alpha$. Integrating in t, the derivative of a periodic function integrates to zero. Thus $d\,\tau b = -\tau(db)$ for this fiber-first convention. One can instead choose a convention that reverses both transgression and cylinder orientation. The formula on a trivial gerbe explains why transgression on a global gerbe produces a line bundle with connection on loop families and why the resulting curvature is a transgressed three-form.

CHAPTER 24

Weil algebras and higher characteristic maps

For an ordinary Lie group a characteristic form is obtained by evaluating an invariant polynomial on curvature. Chapter 12 expressed that evaluation in the Weil algebra. A higher analogue starts from a graded algebra whose differential encodes brackets of several arities. The characteristic form, the Chern–Simons element, and the infinitesimal cocycle then appear as three parts of one calculation. Passing from that calculation to a differential characteristic class requires global integration and descent.

24.1 The algebra of connections and curvatures

Work over $\mathbb{R}$ with cohomological degrees. For the local models here, a finite-dimensional L_∞-algebra is presented by its Chevalley–Eilenberg algebra

$$\mathrm{CE} = (\mathrm{Sym}(V), q)$$

where V is a finite graded space in positive degrees, Sym is graded-commutative, and q is an augmentation-preserving derivation of degree one with $q^2 = 0$. The homogeneous components of q in each word length are dual to the brackets. For an ordinary Lie algebra the generators have degree one and q is quadratic. A degree-two generator can record a higher abelian direction. We only use free graded algebras with polynomial differential, so no completion is needed.

Write sV for a copy of V raised by one degree. As a graded algebra, the Weil algebra is

$$W = \mathrm{Sym}(V \oplus sV).$$

Extend $s(v) = sv$, $s(sv) = 0$ as a derivation of degree one. The differential is determined by

$$d_W v = qv + sv, \qquad d_W(sv) = -s(qv).$$

On the unshifted algebra, d_W is $q + s$. The identity $d_W s + s d_W = 0$ holds on the generators and hence everywhere. On v, $d_W^2 v = s(qv) - s(qv) = 0$. On sv it is $s\, d_W qv = s^2 qv = 0$, so this is indeed a differential graded algebra. The map $j^* : W \to \mathrm{CE}$ kills sV and is a chain map. It restricts the algebra of connections to the corresponding flat infinitesimal data.

Proposition 24.1 (Acyclicity of the Weil algebra)**.** *The cohomology of W is $\mathbb{R}$ in degree zero and zero in positive degrees.*

Proof. Use v and $t(v) = sv + qv$ as algebra generators; the change is invertible because qv belongs to the unshifted algebra. In these coordinates $d_W v = t(v)$ and $d_W t(v) = 0$. Define the degree-minus-one derivation h by $h(v) = 0$ and $h(t(v)) = v$. The derivation $d_W h + h d_W$ acts as the identity on generators, so on monomials of word length r it acts as multiplication by r. The differential preserves this word length. On positive word length define the contraction by h/r. It contracts the augmentation ideal, leaving the constants. □

A map of differential graded algebras $\Phi : W \to \Omega^\bullet(U)$ evaluates v as a potential of degree $|v|$, and sv as its raw curvature of degree $|v| + 1$. It is specified by the potentials: the shifted values must be $d\Phi(v) - \Phi(qv)$, and the equation for $d_W(sv)$ then follows from $d_W^2 = 0$. For a Lie algebra, write A for the collection of degree-one generators and F for their shifts. The universal identities are exactly

$$d_W A = F - \tfrac{1}{2}[A, A], \qquad d_W F = -[A, F].$$

They recover the structure equation and Bianchi identity after evaluation. A map that annihilates the shifted generators factors through CE.

24.2 An invariant form and its two primitives

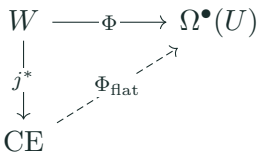
The subalgebra generated by sV is the algebra of horizontal polynomials. Call a horizontal polynomial P invariant if $d_W P = 0$. For ordinary reductive Lie algebras this gives the usual invariant polynomials on their curvature variables [11, Remark 4.1.17]. This elementary definition suffices for the examples below.

When P has positive degree $n + 1$, choose $C \in W^n$ with $d_W C = P$ using acyclicity. Since $j^* P = 0$, $\mu = j^* C$ is a cocycle of degree n. Two choices of C differ by a closed element. If $n > 0$, acyclicity gives an exact difference, so the class of μ is independent of that choice. This is the transgression relation in [11, Definition 4.1.20]. On U, evaluation gives

$$d\Phi(C) = \Phi(P).$$

Under flat evaluation it gives $\Phi(C) = \Phi(\mu)$, using the factorization through the Chevalley–Eilenberg algebra. Thus $\Phi(C)$ is a curvature primitive on a chart and μ records its flat restriction.

The diagram separates algebraic restriction from form evaluation:



The dashed factorization is defined when $\Phi(sV) = 0$. The algebra gives a local real form for P even without that factorization. Gluing this form and refining it to an integral differential class requires the global structure group, compatible descent data, and an integral normalization. For example, if $\Phi(P)$ is the curvature of a differential character, its integrals on integral cycles must be integers. The local Weil calculation by itself has no such period condition.

Example 24.2 (The degree-three abelian case)**.** Take $\mathrm{CE} = \mathbb{R}[b]$ with $|b| = 2$ and $qb = 0$. The Weil algebra adds $h = sb$ with $|h| = 3$ and $d_W b = h,\ d_W h = 0$. Thus $P = h,\ C = b,\ \mu = b$. A map of Weil algebras into forms assigns $B = \Phi(b)$ and $\mathcal{H} = \Phi(h) = dB$. Taking the smooth circle as the integrated coefficient group gives the curving of a gerbe on a trivializing open set. Integral Čech descent data can change its underlying degree-three class while leaving the same equations on each open set. The holonomy and curvature law for a single global B were computed in Chapter 23.

24.3 The string transgression

Let $\mathfrak{g}$ be an ordinary Lie algebra with an invariant symmetric bilinear form Q. Its degree-four characteristic polynomial is $P = Q(F, F)$. The connection interpolation $A_t = tA$ in the ordinary Weil algebra has curvature

$$F_t = tF + \tfrac{1}{2}(t^2 - t)[A, A].$$

Differentiating the polynomial with respect to t gives $2d_W Q(A, F_t)$ by invariance of Q and the Bianchi identity for A_t. Integrating from 0 to 1 gives

$$C = Q(A, F) - \tfrac{1}{6}Q(A, [A, A]), \qquad d_W C = P.$$

In these signs, $\mu = -\frac{1}{6}Q(A, [A, A])$ is the Chevalley–Eilenberg restriction. Reversing the sign of the new abelian generator gives the alternate convention for the corresponding central extension.

Adjoin a degree-two generator b to the Chevalley–Eilenberg algebra with $qb = -\mu$. This is a differential because $q\mu = 0$. In its Weil algebra, write $h = sb$. Then $d_W b = h - \mu$, $d_W h = s\mu$, and $d_W \mu = s\mu$. The modified element

$$\mathcal{K} = h + C - \mu$$

satisfies $j^*\mathcal{K} = 0$ and $d_W \mathcal{K} = P$. After evaluation this reads

$$H = dB + \Phi(C), \qquad dH = \Phi(P).$$

The new three-form H uses a Chern–Simons term. This Bianchi equation is different from the fake-flat strict crossed module equation in Chapter 22. It arises from an extension whose Chevalley–Eilenberg differential has the cubic term μ.

For the spin group in stable rank, take Q normalized so P represents the real image of the integral class $\lambda = \frac{1}{2}p_1$. The topological string lift is a lift through the homotopy fiber of

$$B\,\mathrm{Spin} \xrightarrow{\lambda} K(\mathbb{Z}, 4).$$

By the universal property of a homotopy fiber, a lift of a fixed spin classifying map is a chosen nullhomotopy of its composite with λ. Consequently existence is equivalent to vanishing of the integral class on the base. Vanishing of the de Rham image alone leaves a possible torsion obstruction. At the differential level, if the underlying integral class vanishes, the differential character is represented by a three-form H modulo closed forms of integral periods, and its curvature is dH. A geometric differential string structure keeps track of a choice of trivialization and its coherences. Treating it as a nullhomotopy to the zero *differential* character would impose zero curvature and would miss the three-form in this Bianchi identity.

Fiorenza, Schreiber, and Stasheff construct the global differential refinement of the fractional Pontryagin map at the level of smooth cocycle spaces and describe the corresponding differential trivializations [11, §§5.3 and 6]. Their construction uses integration of the infinitesimal model and compatible periods. The local equations above explain which forms such a refinement must recover. The global characteristic map also retains the integral class and the higher automorphisms that the form equations leave undetermined.

Part VII Obstructions, equivariance, and derived geometry

CHAPTER 25

The string obstruction and its differential lifts

A spin connection has a curvature polynomial of degree four. The same polynomial represents a distinguished integral class, whose vanishing is a lifting problem. A primitive of the curvature form can exist even when the integral obstruction survives. Conversely an integral lift can exist while a particular differential refinement is nonzero. Keeping track of the spaces in which those conditions live separates the choices.

25.1 The topological fiber

Let $E \to M$ be an oriented Euclidean real vector bundle of rank $n \geq 5$ over a smooth manifold, with a specified spin lift P of its oriented orthonormal frame bundle. A spin lift supplies a classifying map $c_P : M \to B\,\mathrm{Spin}(n)$, where the domain denotes the underlying homotopy type. The universal integral class $\lambda \in H^4(B\,\mathrm{Spin}(n); \mathbb{Z})$ satisfies $2\lambda = p_1$ pulled back along $B\,\mathrm{Spin}(n) \to B\,\mathrm{SO}(n)$. We use this universal class to define the half-Pontryagin class of P. An equation $2x = p_1(E)$ in the cohomology of a given M would not define it uniquely when $H^4(M; \mathbb{Z})$ has two-torsion. The integral assertion is part of the spin classification [21, Section 5].

Represent λ by a map to $K(\mathbb{Z}, 4)$. Define the topological classifying space for string lifts by the homotopy pullback

$$\begin{array}{ccc} B\,\mathrm{String}(n) & \longrightarrow & * \\ \downarrow & & \downarrow{\scriptstyle 0} \\ B\,\mathrm{Spin}(n) & \xrightarrow{\lambda} & K(\mathbb{Z}, 4). \end{array}$$

The right-hand arrow chooses the zero class. A lift is taken over c_P, so the choice includes the comparison with that specified spin bundle.

Proposition 25.1 (Lifts of a fixed spin bundle)**.** *The space of lifts of c_P is nonempty precisely when $\lambda(P) = 0$. When it is nonempty, its components form a torsor for $H^3(M; \mathbb{Z})$. After choosing a lift as basepoint, its homotopy groups in degrees 1 through 3 are $H^{3-j}(M; \mathbb{Z})$ in degree j, and the higher groups vanish.*

Proof. Mapping out of M preserves limits in the category of spaces. The lift space is therefore the space of paths from λc_P to the constant map inside $\mathrm{Map}(M, K(\mathbb{Z}, 4))$. The description of connected components by cohomology gives the obstruction. When a path exists, composing with it identifies every other path with a loop at the constant map, up to coherent homotopy. Looping commutes with taking this mapping space, giving $\mathrm{Map}(M, K(\mathbb{Z}, 3))$. Its components are $H^3(M; \mathbb{Z})$, and the action by path composition is simply transitive on components. For $j \geq 1$, a based j-sphere of maps to $K(\mathbb{Z}, 3)$ has adjoint a map from $S^j \wedge M_+$ to $K(\mathbb{Z}, 3)$, where M_+ adds a disjoint basepoint. The suspension isomorphism gives the formula. □

The proof distinguishes classification from existence. For instance, the trivial spin bundle over S^3 has infinitely many lift classes, indexed by $\mathbb{Z}$ after choosing one of them. Its spin curvature may be identically zero throughout this family. The same proof yields automorphisms and higher automorphisms, since a choice of lift class identifies a component without eliminating its higher homotopy.

25.2 Normalization and refinement

Take the curvature $F = dA + A \wedge A$ of a spin connection in the real defining representation of $\mathfrak{so}(n)$. In our Chern normalization, the real form for λ is

$$\lambda(F) = -\frac{1}{16\pi^2} \operatorname{tr}(F \wedge F).$$

Indeed the degree-four part of $\det(1 + iF/(2\pi))$ for the complexification of E is $\operatorname{tr}(F \wedge F)/(8\pi^2)$, since $\operatorname{tr} F = 0$. The relation $p_1(E) = -c_2(E \otimes \mathbb{C})$ supplies the minus sign. Trace invariance and the Bianchi identity make this form closed and independent of the frame. The universal integral normalization specifies which refinement is meant.

The differential characteristic class $\widehat{\lambda}(P, A)$ has integral class $\lambda(P)$ and curvature $\lambda(F)$. It can be realized by the Chern–Simons bundle 2-gerbe of the spin bundle, equipped with its induced connection [22, Theorems 1.1.3 and 1.3.3]. A geometric string structure in this model consists of a trivialization of that bundle 2-gerbe and compatible connective data. Such data produce a 3-form H with $dH = \lambda(F)$. Tensoring the trivialization by a bundle gerbe with connection adds that gerbe’s curvature to H. Waldorf’s existence theorem gives compatible connections on every topological trivialization [22, Theorem 1.3.4 and Section 3.2].

The exact sequence of differential cohomology makes the role of H explicit. Write $\Omega^3_{\mathbb{Z}}(M)$ for closed forms with integral periods. We have

$$\Omega^3(M)/\Omega^3_{\mathbb{Z}}(M) \xrightarrow{a} \widehat{H}^4(M; \mathbb{Z}) \xrightarrow{I} H^4(M; \mathbb{Z}) \longrightarrow 0.$$

Here $a(H)$ evaluates on a 3-cycle by integration modulo $\mathbb{Z}$. If $\lambda(P) = 0$, the refinement has the form $a(H)$. Its curvature is dH, which may be nonzero. Changing H by a closed form of integral periods keeps $a(H)$ fixed. The exact sequence describes the image of the differential class as a form modulo periods. A specified trivialization includes gluing data beyond that image. In particular the requirement $\widehat{\lambda}(P, A) = 0$ is stronger than existence of a geometric string structure, which allows a topologically trivial 2-gerbe carrying a nonzero global curving.

The difference of two such compatible trivializations, for the same spin connection, is a bundle gerbe with connection. One way to see this is to compare their local descent cochains. Their coboundaries represent the same obstruction cocycle, so the quotient of their data satisfies the unobstructed gerbe descent equations. Conversely multiplication by such a cocycle changes the trivialization and preserves its relation to the obstruction cocycle. On isomorphism classes this is the torsor action of $\widehat{H}^3(M; \mathbb{Z})$. Taking underlying integral classes recovers the H^3-action in Proposition 25.1, and taking curvature recovers addition to H. In the 2-gerbe description these operations are tensor product and inverse of gerbes with connection [22, Corollary 1.3.2 and Section 3.2].

25.3 Nonzero integral and differential obstructions

The quaternionic Hopf bundle Q on $S^4 = \mathbb{H}P^1$ has complex rank two and structure group $\mathrm{Sp}(1) = \mathrm{SU}(2)$. Its unit sphere bundle is $S^7 \to S^4$, so the Gysin sequence identifies its Euler class with a generator of $H^4(S^4; \mathbb{Z})$, up to orientation. This Euler class is $c_2(Q)$ for the complex orientation. The real five-plane bundle $Q_{\mathbb{R}} \oplus \mathbb{R}$ has a spin lift. The group $\mathrm{SU}(2)$ is simply connected, so its orthogonal representation on $\mathbb{H} \oplus \mathbb{R}$ lifts to $\mathrm{Spin}(5)$. For a complex two-plane bundle, $p_1(Q_{\mathbb{R}}) = c_1(Q)^2 - 2c_2(Q) = -2c_2(Q)$, as follows from the formal Chern roots. Here $H^4(S^4; \mathbb{Z})$ is torsion-free, so the spin λ is $-c_2(Q)$. Proposition 25.1 then obstructs a string structure.

There is also a flat model with a torsion obstruction. Let $Y = S^5/\mu_m$, where the m-th roots of unity act by common scalar multiplication on the unit sphere in $\mathbb{C}^3$ and $m \geq 2$. The map $p : Y \to \mathbb{C}P^2$ is the unit circle bundle of $L^{\otimes m}$, with L the tautological line. Indeed the fiberwise tensor-power map on the unit circle bundle of L is surjective with kernel μ_m. Put $x = c_1(L)$. The Gysin sequence for p contains

$$H^2(\mathbb{C}P^2; \mathbb{Z}) \xrightarrow{\smile mx} H^4(\mathbb{C}P^2; \mathbb{Z}) \xrightarrow{p^*} H^4(Y; \mathbb{Z}) \longrightarrow 0.$$

The final zero is $H^3(\mathbb{C}P^2; \mathbb{Z})$, and the first two groups are generated by x and x^2. It follows that $u = p^*(x^2)$ has order exactly m. The line p^*L admits a flat connection because its pullback to S^5 has the unit frame $q \mapsto q$, and

the deck group changes this frame by constant phases. The rank-two bundle $Q = p^*L \oplus (p^*L)^{-1}$ carries the direct sum of the flat connection and its dual. Its determinant reduction has structure group $\mathrm{SU}(2)$ and $c_2(Q) = -u$. For the associated spin five-plane bundle the identity $\lambda = -c_2$ follows by naturality from $B\,\mathrm{SU}(2)$, whose integral H^4 is free, by the complexification computation above. Thus $\lambda = u$ while $F = 0$. This obstruction comes entirely from the integral gluing data.

For the other direction take the trivial spin bundle on $T^3 = (\mathbb{R}/\mathbb{Z})^3$ and the connection $A = tX\alpha$ in a global frame. Here X is a nonzero real skew-symmetric matrix from $\mathfrak{spin}(n)$, $t \in \mathbb{R}$, and

$$\alpha = \cos(2\pi z)\,dx + \sin(2\pi z)\,dy.$$

The transgression from the trivial connection to A is $-t^2\,\mathrm{tr}(X^2)\alpha \wedge d\alpha/(16\pi^2)$. This follows by inserting sA into the curvature polynomial and integrating its derivative for $0 \le s \le 1$. The cubic term vanishes because A has just one form factor. Direct differentiation gives $\alpha \wedge d\alpha = -2\pi\,dx \wedge dy \wedge dz$, so the transgression integral is a nonzero multiple of t^2. The differential characteristic class changes by a of this transgression. For small nonzero t the resulting degree-four differential character has nonzero value on T^3, while its integral class and its 4-form curvature both vanish. It records the comparison of two spin connections on the same topologically trivial bundle. A geometric trivialization can retain that comparison as a curving.

CHAPTER 26

Connections on quotient stacks

A group action carries information at each point about its stabilizer. Taking an orbit space discards those groups. A quotient stack keeps them as automorphisms, and characteristic classes can detect representations of those automorphisms even when the base orbit space is a point.

26.1 The action groupoid and its test objects

Let a Lie group K act smoothly from the left on X. The action groupoid has objects $x \in X$ and arrows $x \to kx$ for $k \in K$. Given arrows labelled k and l in that order, their composite is labelled lk. Its nerve in degree q is $K^q \times X$. A sequence of composable arrows is determined by its initial point and its q group elements, and the inner face maps multiply adjacent elements in their composition order. The outer faces forget the first or last vertex. In particular the first face also moves the initial point.

The quotient stack $[X/K]$ is described on a test manifold T by pairs (P, u) with $P \to T$ a right principal K-bundle and $u : P \to X$ a smooth map satisfying $u(pk) = k^{-1}u(p)$. Arrows are principal bundle isomorphisms commuting with the maps to X. A frame $p_i : T_i \to P$ gives a map $x_i : T_i \to X$. On T_{ij} write $p_j = p_i k_{ij}$, so $x_j = k_{ij}^{-1} x_i$. On triples $k_{ij}k_{jk} = k_{ik}$. Conversely this local data glue the trivial principal bundles by their transition functions, and equivariance of the local maps makes them glue to u. The same argument on isomorphisms identifies the arrows. This exhibits the passage from the action groupoid to its stack as descent for locally trivial bundles.

Taking K trivial recovers X. Taking $X = *$ gives BK, because the map u is then forced. For a free proper action with smooth quotient and principal bundle $X \to X/K$, this stack is represented by X/K. Indeed the composition $P \to X \to X/K$ descends to T, and the resulting map to X/K makes P isomorphic to its pullback of X, with the right action on X obtained by inversion. This follows fiberwise since an equivariant map between torsors is an isomorphism, and local sections give smoothness of the inverse. In general a test map $* \to [X/K]$ determined by $x \in X$ has automorphisms identified with its stabilizer. The projection to the orbit set sends each such automorphism to the identity.

26.2 Bundles and the condition on a connection

Let G be another Lie group. A principal G-bundle on $[X/K]$ amounts to a K-equivariant principal G-bundle $Q \to X$. Pullback to the atlas $X \to [X/K]$ gives Q. The chosen descent isomorphism on $K \times X$ acts in the fiber over x by $Q_x \to Q_{kx}$. The cocycle condition on $K^2 \times X$ says this action respects multiplication. Conversely these isomorphisms give the bundle over any test (P, u), namely $(u^*Q)/K \to T$. The right action on u^*Q sends (p, q) to $(pk, k^{-1}q)$. Its quotient is a principal G-bundle, as can be checked after a local trivialization of P. This also proves the assertion for morphisms.

A connection form ω on Q descends with equality on the atlas nerve exactly when

$$b^*\omega = \mathrm{pr}_2^*\,\omega, \qquad b : K \times Q \longrightarrow Q, \quad b(k, q) = kq.$$

Evaluating on the Q-directions at each fixed k requires K-invariance. Evaluating on vectors tangent to K at the identity requires $\omega(\xi_Q) = 0$, where $\xi_Q(q) = \frac{d}{dt}|_0 \exp(t\xi)q$. Those conditions imply the equality on all tangent directions by their behavior under translations. Thus a K-invariant connection can fail this descent condition. An invariant connection still has a role in computing equivariant characteristic forms. The groupoid formulation and the comparison with the usual equivariant Chern–Weil map are developed in [23].

For a K-invariant connection write $\mu_\xi = \omega(\xi_Q)$, an equivariant $\mathfrak{g}$-valued function. If F is the ordinary curvature and $D_\omega = d + [\omega, -]$, then

$$\iota_{\xi_Q} F = -D_\omega \mu_\xi.$$

In fact Cartan's formula gives $\iota_{\xi_Q} d\omega = -d\mu_\xi$, and contraction of $\frac{1}{2}[\omega, \omega]$ is $[\mu_\xi, \omega]$. Consequently $F - \mu_\xi$ satisfies $(D_\omega - \iota_{\xi_Q})(F - \mu_\xi) = 0$, using the Bianchi identity and the fact that contraction of a function vanishes. For an invariant polynomial, evaluating on this equivariant curvature gives a closed expression in the Cartan complex with differential $d - \iota_{\xi_X}$. The calculation comes from the derivation rule for the invariant polynomial, which kills the adjoint-bracket terms. The ξ-polynomial degree is counted as twice its ordinary degree. For compact K the Cartan comparison computes the real cohomology of the Borel construction, or equivalently the simplicial de Rham model of the action groupoid. This use of an equivariant complex differs from the complex of invariant horizontal forms on X.

An example is $X = *$, $K = U(1)$, and $Q = U(1)$ with left action $kq = k^r q$ for some integer r. The connection $q^{-1}dq$ is invariant, but $\mu_\xi = ir\xi$ for the generator parameterization $\exp(it\xi)$. The ordinary curvature is zero. The equivariant Chern polynomial in degree two gives $r\xi/(2\pi)$. Thus the real first Chern class over $BU(1)$ can be nonzero, whereas the positive-degree ordinary forms on the atlas point vanish.

26.3 A flat torsion line on a stack

Let $\Gamma = \mathbb{Z}/m$ with $m \geq 2$. A complex line on $B\Gamma$ is supplied by the unitary character $\chi(j) = \exp(-2\pi ij/m)$. A trivial connection on the line over the point is preserved by each constant unitary isomorphism. Since Γ is discrete, the horizontal condition has no infinitesimal group directions to impose, so this is a flat line with connection on the stack. Its holonomy on a generator of the stabilizer is nontrivial.

Compute its integral class using

$$0 \longrightarrow \mathbb{Z} \longrightarrow \mathbb{R} \xrightarrow{t \mapsto e^{2\pi it}} U(1) \longrightarrow 0.$$

For homogeneous real group cochains in positive degree, averaging over the first group entry gives a contracting homotopy: insert a first entry h, sum over h, and divide by m. The alternating deletion differential and this homotopy have anticommutator the identity in positive degree because the term deleting h is the original cochain and the other terms cancel in pairs. Invariance is preserved by reindexing the average. Thus $H^q(B\Gamma; \mathbb{R}) = 0$ for $q > 0$. The connecting homomorphism gives an isomorphism

$$\mathrm{Hom}(\Gamma, U(1)) = H^1(B\Gamma; U(1)) \xrightarrow{\sim} H^2(B\Gamma; \mathbb{Z}),$$

where the coefficients on the left are constant coefficients on the realized groupoid. Lifting the character to the real cochain $j \mapsto -j/m$ for $0 \leq j < m$ gives the integer cocycle $(j, k) \mapsto -\lfloor (j + k)/m \rfloor$. This is the Chern cocycle in the chosen exponential normalization. The character has order m, so its first Chern class generates a cyclic group of order m. The curvature and the coarse space are both unable to show this class. The stack retains the discrete descent arrows, and the differential class retains their phases.

The same mechanism occurs on a quotient with a nontrivial atlas. Let $\Gamma = \mathbb{Z}/m$ rotate $\mathbb{C}P^1$ by $j \cdot [a : b] = [a : e^{2\pi ij/m} b]$. The corresponding unitary diagonal matrices act on the ambient $\mathbb{C}^2$ and carry each tautological line to the tautological line at its image. The projection connection of Chapter 28 is equivariant because these matrices preserve both the trivial differential and orthogonal projection. Since the group is finite, this is a descent connection on $[\mathbb{C}P^1/\Gamma]$. At $[1 : 0]$ the action on the fiber is trivial, and at $[0 : 1]$ the action is the inverse of χ above. Each fixed point defines a map $B\Gamma \to [\mathbb{C}P^1/\Gamma]$. The restrictions of the equivariant first Chern class along those two maps are respectively zero and a generator of $H^2(B\Gamma; \mathbb{Z})$. Its restriction to the atlas is the first Chern class of the tautological line on the sphere, which has Chern number -1. These are three different tests of the same characteristic class. If the tautological equivariant bundle were pulled back from the orbit space, each stabilizer would act trivially on its fiber, because the pullback from a space acts on fibers by the identity along

arrows mapping to identities. The action at the second fixed point disproves that descent to the coarse orbit space.

CHAPTER 27

Derived intersections and Atiyah classes

The first infinitesimal neighborhood of a diagonal gives a local description of connections. A derived intersection of that diagonal with itself retains the relations among its equations. This chapter works in smooth algebraic geometry over a characteristic-zero field k, using the algebraic differential d. Smooth real connections and algebraic connections have different existence properties. The comparison rests on the common Leibniz identity.

27.1 Intersecting schemes through a derived tensor product

For affine schemes mapping to $\operatorname{Spec} R$, let their coordinate algebras be B and C. The ordinary pullback has coordinate algebra $B \otimes_R C$. In derived algebraic geometry one uses $B \otimes^{\mathbf{L}}_R C$, the homotopy pushout of the corresponding commutative algebras. We use cohomological degrees, so

$$H^{-j}(B \otimes^{\mathbf{L}}_R C) = \operatorname{Tor}^R_j(B, C), \qquad j \geq 0.$$

The homotopy pushout can be modeled by commutative differential graded algebras in nonpositive degrees in characteristic zero [24, Section 2]. The formula is understood locally on schemes when the rings vary from chart to chart.

Consider two embeddings of the line $\{y = 0\}$ in $\mathbb{A}^2_k$, so $R = k[x, y]$ and $B = C = R/(y)$. The algebra obtained by adjoining one odd generator e of degree -1 to R with $de = y$ resolves B. Indeed its underlying complex is $R \xrightarrow{y} R$, with zero kernel and cokernel B. As a free extension in graded commutative algebras this is a Koszul model for the derived base change. Tensoring with C gives a differential of zero, and the resulting algebra is $k[x] \oplus k[x]e$, with $e^2 = 0$. The ordinary intersection is just the line, while H^{-1} records a normal equation that occurs on both sides.

For a different second embedding take the parabola $y = x^2$. On $R/(y - x^2) \cong k[x]$, the same differential is multiplication by x^2, hence it is injective. The degree-zero algebra is $k[x]/(x^2)$, with no higher Tor. Here failure of transversality appears as a nonreduced ordinary intersection. The two examples distinguish nilpotents in degree zero from negative-degree homology.

27.2 The diagonal and its self-intersection

Let X be smooth, separated and of finite type over k. Its diagonal $\Delta : X \to X \times_k X$ is a regular closed immersion. Writing $\mathcal{I}$ for its ideal, the conormal $\mathcal{I}/\mathcal{I}^2$ is $\Omega^1_{X/k}$. The identification takes the class of $p_2^* f - p_1^* f$ to df. Indeed modulo $\mathcal{I}^2$ the difference obeys Leibniz, and on an étale coordinate neighborhood the coordinate differences generate the conormal freely. The same statement also follows from the universal derivation of a commutative algebra by applying its defining property to the multiplication map.

Proposition 27.1 (Tor sheaves along the diagonal)**.** *The homology sheaves on the derived self-intersection $X \times^h_{X \times_k X} X$ satisfy*

$$\mathcal{H}^{-j}(\mathcal{O}_{X \times^h_{X \times_k X} X}) \cong \bigwedge^j \Omega^1_{X/k}.$$

These identifications respect the graded homology algebra.

Proof. Take local generators $f_1, \dots, f_d$ of the diagonal ideal forming a regular sequence. The Koszul algebra adjoins odd generators e_i with $de_i = f_i$. It resolves the diagonal, as shown by induction on the length of the sequence. Passing from $d-1$ to d takes the cone of multiplication by f_d on the preceding Koszul complex, with the shift that puts the new exterior generator in degree -1. The preceding homology is the quotient by the first $d-1$ generators. The regular-sequence hypothesis makes the multiplication injective on that quotient, so the cone has only the requisite degree-zero homology. This completes the induction. Pulling back along the diagonal kills the f_i, leaving the exterior algebra with zero differential. Under a change of generators, the induced map in degree -1 is the change-of-basis map of $\mathcal{I}/\mathcal{I}^2$. Multiplication extends the identification to the exterior powers and makes it independent of the local resolution. □

The loop-space expression here is categorical. Cotensoring by $S^1 \simeq * \coprod^h_{S^0} *$ turns the homotopy pushout into the pullback of the two diagonal maps. A degree-zero scheme of points would miss the derived directions of this pullback. The global Hochschild–Kostant–Rosenberg comparison for the smooth diagonal gives the corresponding splitting as a derived object. For other embeddings a local Koszul model by itself only determines the homology sheaves. The extension condition for a global splitting is treated in [26, Theorem 1].

27.3 First jets and the obstruction to a connection

For a vector bundle E on X, let Z be the first infinitesimal neighborhood of the diagonal, with $\mathcal{O}_Z = \mathcal{O}_{X \times X}/\mathcal{I}^2$. The two projections of Z to X are still distinct.

Define $J^1E = p_{1*}p_2^*E$, regarded as an $\mathcal{O}_X$-module by the first projection. There is a sequence

$$0 \longrightarrow \Omega^1_{X/k} \otimes E \longrightarrow J^1E \longrightarrow E \longrightarrow 0.$$

After trivializing E, this is the exact sequence coming from reduction $\mathcal{O}_Z \to \mathcal{O}_\Delta$; the conormal module multiplies by coefficients restricted to the diagonal. Thus the sequence is exact independently of the chosen frame. Its extension class

$$\mathrm{At}(E) \in \mathrm{Ext}^1_{\mathcal{O}_X}(E, \Omega^1_{X/k} \otimes E)$$

uses the first jet convention. Atiyah's obstruction construction treats the holomorphic counterpart [25, Sections 2–4].

The jet operator j pulls a section along p_2. On $\mathcal{O}_Z$, multiplication by p_2^*f differs from multiplication by p_1^*f by df in the conormal ideal. Thus $j(fs) = fj(s)+df\otimes s$. If $\sigma : E \to J^1E$ is an $\mathcal{O}_X$-linear splitting, $j - \sigma$ takes values in the kernel and satisfies the connection Leibniz rule. Conversely a connection ∇ gives the splitting $j - \nabla$. The extension class vanishes exactly when a connection of this kind exists. Its vanishing alone makes no integrability assertion. Flatness imposes the separate curvature equation.

Here is a coordinate account of the class. Take local frames with $e_j = e_i g_{ij}$ and in frame i take the connection $\nabla^{(i)}$ for which e_i is parallel. Write $\sigma_i = j - \nabla^{(i)}$. On a double overlap

$$\sigma_j - \sigma_i = \nabla^{(i)} - \nabla^{(j)}.$$

In the j-frame this difference is $g_{ij}^{-1}dg_{ij}$. The overlap differences form a Čech cocycle of $\Omega^1 \otimes \mathrm{End}\, E$, and changing the splittings adds a coboundary. This is the same inhomogeneous term that enters the connection descent law.

Proposition 27.2 (Tensor products and determinant)**.** *With the jet convention above, the class of $E \otimes G$ is the sum of $\mathrm{At}(E) \otimes 1_G$ and $1_E \otimes \mathrm{At}(G)$, with the tensor factors reordered so that both take values in $\Omega^1 \otimes E \otimes G$. The class of $\det E$ is the trace of $\mathrm{At}(E)$.*

Proof. On a common trivializing cover, the tensor connection built from the chosen local connections satisfies the Leibniz rule on $s \otimes t$ by differentiating each factor in turn. Taking the difference between the two local tensor connections on an overlap leaves the sum of the differences in each input, since all terms are $\mathcal{O}_X$-linear after subtraction. This proves the tensor formula on the extension cocycles. For the determinant use the local connection induced on the top exterior power. In a wedge of a frame, replacing its i-th vector by a multiple of any other frame element gives zero, so only the diagonal entries of the difference matrix survive. Their sum gives the trace formula. □

On $\mathbb{P}^1_k$, the tautological bundle has frames $(1, z)$ and $(w, 1)$ with $w = 1/z$, giving transition $g = z^{-1}$. Its jet cocycle is $-dz/z$. In the two standard charts,

$$H^1(\mathbb{P}^1, \Omega^1) = \frac{k[z, z^{-1}]\, dz}{k[z]\, dz + z^{-2} k[z^{-1}]\, dz}.$$

This quotient follows by rewriting regular forms in the w-chart using $dw = -z^{-2} dz$. The class of dz/z survives, so the tautological bundle has no algebraic connection. Over $\mathbb{C}$ the same underlying smooth line has a Hermitian connection, computed explicitly in Chapter 28. The change of coefficient sheaf accounts for the different answers.

CHAPTER 28

From a Hopf connection to surface holonomy

The Hopf fibration gives a sequence of explicit geometric models. Its line bundle over the two-sphere has a curvature form with nonzero period. On the total space this form admits a global primitive. The primitive and the curvature together give an integral 3-form, leading to surface holonomy on the total space and ordinary line holonomy on its loop space.

28.1 The tautological connection and its periods

Let $L \to \mathbb{C}P^1$ be the tautological Hermitian line. It is a subbundle of $\mathbb{C}P^1 \times \mathbb{C}^2$. Projecting the trivial differential orthogonally to L gives a metric connection. For a section s, define $\nabla s = \pi_L ds$. The Leibniz identity follows from $\pi_L s = s$, and metric compatibility follows because the perpendicular component of ds pairs to zero with any section of L.

In the affine z-chart use the unit frame

$$s_0(z) = \frac{(1, z)}{\sqrt{1 + |z|^2}}.$$

The connection matrix is $A_0 = s_0^* ds_0$, namely

$$A_0 = \frac{\bar{z}\, dz - z\, d\bar{z}}{2(1 + |z|^2)} = i\, \frac{r^2}{1 + r^2}\, d\theta, \qquad z = re^{i\theta}.$$

The polar expression holds away from the origin. The Cartesian formula is smooth there. On the chart $w = 1/z$ take $s_1 = (w, 1)/\sqrt{1 + |w|^2}$. On the overlap $s_1 = s_0 e^{-i\theta}$, so the two matrices obey $A_1 = A_0 - i\, d\theta$. This is the transformation law from Chapter 7.

Set $\omega = iF/(2\pi)$. On the z-chart,

$$\omega = -\frac{1}{\pi} \frac{r}{(1 + r^2)^2}\, dr \wedge d\theta, \qquad \int_{\mathbb{C}P^1} \omega = -1.$$

For the integral one first integrates over $r < R$ and sends R to infinity. The missing point has measure zero for the smooth form in the other chart. With the complex orientation this gives the first Chern number of L [1, 5]. Its sign

also follows by using Stokes on the two chart disks, since $A_0 - A_1 = i\,d\theta$ on their common boundary oriented from the z-disk.

On a positively oriented z-circle of radius R, the parallel transport equation for the coefficient in the frame is $\dot{v} = -A_0(\dot{\gamma})v$, so

$$\mathrm{Hol}_L(\gamma) = \exp\left(-2\pi i \frac{R^2}{1+R^2}\right).$$

The integral of ω over the disk it bounds is $-R^2/(1+R^2)$. Choosing the complementary filling with the same boundary changes that integral by an integer. The exponent therefore comes from a differential character with curvature ω, not just from a choice of potential on the chart.

28.2 An integral 3-form on the total space

The unit circle bundle $p : S^3 \to \mathbb{C}P^1$ of L has global real 1-form $\alpha = q^* dq/(2\pi i)$ for a point $q \in S^3$ viewed as a unit complex column vector. Its value on a positively oriented fiber integrates to one. In the frame $q = s_0(z)e^{i\phi}$ the form is $d\phi/(2\pi) + A_0/(2\pi i)$. Consequently

$$d\alpha = -p^*\omega, \qquad \eta = \alpha \wedge p^*(-\omega), \qquad \int_{S^3} \eta = 1.$$

The orientation here uses the positive circle and complex base orientations. The integral follows by fiber integration: on each trivialization only the $d\phi$ term contributes, and the local base integrals glue by a partition of unity. Also $d\eta = p^*((-\omega)\wedge(-\omega)) = 0$. Thus η represents the integral generator of $H^3(S^3;\mathbb{Z})$ in real cohomology.

One can construct the associated degree-three differential character directly. A smooth singular 2-cycle z in S^3 bounds a smooth singular 3-chain D. Put $h(z) = \int_D \eta \bmod \mathbb{Z}$. The difference between two choices of D is a 3-cycle, and integrality of η gives independence. Linearity follows by adding fillings. Taking $z = \partial D$ gives exactly the character boundary identity with curvature η. The integral class of h maps to the de Rham class of η; here the degree-three cohomology is free, so this class is the generator. This character can also be represented by a bundle gerbe with connection.

To see the overlap geometry, cover S^3 by the complements of the two poles, with local 2-forms B_N, B_S satisfying $dB_N = dB_S = \eta$. Orient the equator as the boundary of the northern hemisphere. Stokes gives $\int_{S^2}(B_N - B_S) = 1$. On the overlap this difference is a closed 2-form of integral periods because the equator generates its H_2. It can be realized as a line curvature there by the differential cohomology curvature sequence. For the convention in which the overlap curvature is $B_S - B_N$, take the dual line. Use the reverse overlap line as

its dual and the identity lines on repeated charts. Evaluation of the line against its dual supplies the product on alternating overlaps. Associativity reduces to the tensor identities for a line and its dual, which can be checked in a unit frame. This is the connective descent presentation whose curvature is η. The difference of local 2-forms carries the same integer that the original Hopf curvature detected.

28.3 Evaluating families of loops

Let LS^3 consist of smooth loops, with a smooth family defined by its adjoint smooth map to S^3. For a smooth singular 1-cycle c of families of loops, write $\mathrm{ev} : LS^3 \times S^1 \to S^3$ for evaluation. Then the transgressed holonomy of c is

$$(\tau h)(c) = h\big(\,\mathrm{ev}_*(c \times [S^1])\big).$$

A smooth singular fundamental cycle of S^1 is used in this expression. Replacing it by a homologous cycle changes the argument by a boundary evaluated through the one-dimensional circle factor. The lifted 3-chain has rank at most two under evaluation, so its curvature integral vanishes. This proves independence of that choice.

For a smooth 2-chain C of loop families, the boundary of $C \times [S^1]$ is $\partial C \times [S^1]$. The character property gives

$$(\tau h)(\partial C) = \int_C \tau\eta \bmod \mathbb{Z}, \qquad (\tau\eta)_\gamma(V, W) = \int_0^1 \eta_{\gamma(t)}(\dot\gamma(t), V(t), W(t))\, dt.$$

In the orientation of $C \times S^1$, moving the loop tangent from the final slot to the first makes two interchanges, explaining the sign. The gerbe-to-loop-space transgression theorem realizes this degree-two character as a line bundle with connection. Waldorf's convention uses inverse surface holonomy [27, Proposition 4.3.2]. The dual of his transgressed bundle realizes the positive evaluation convention above. Its fusion structure gives compatible products along triples of paths with common endpoints. The conditions on this structure for the inverse regression theorem are given in [27, Theorem A].

More explicitly, for paths γ_i from x to y that are constant near their endpoints, let ℓ_{ij} traverse γ_i and then γ_j backwards. The line fibers $T_{\ell_{ij}}$ of the transgression admit fusion maps

$$T_{\ell_{12}} \otimes T_{\ell_{23}} \longrightarrow T_{\ell_{13}}.$$

For four paths the two composites of fusion maps agree. The maps express the cancellation of the common path segment in the surface description. A bare line bundle over the loop space does not supply the fusion maps or their compatibility with transport. The cited theorem uses fusion bundles with connections satisfying the stated superficiality and compatibility conditions, and takes the homotopy category on the gerbe side.

The transgressed curvature has a nonzero integral period. For this, choose an embedded oriented 3-ball in $S^2 \times S^1$ avoiding both $S^2 \times \{t_0\}$ and $\{s_0\} \times S^1$. Collapse its complement to a point in S^3, using a smooth representative $f : S^2 \times S^1 \to S^3$ constant near the boundary of the ball. Such a representative is obtained in polar coordinates by mapping the radius to a polar angle from zero to π, with a smooth constant collar at π and a linear radial behavior near zero. Choosing the orientation of the chart gives degree $+1$. The adjoint family $S^2 \to LS^3$ takes values in based loops since $f(s, t_0)$ is constant. Its curvature period is

$$\int_{S^2} (\operatorname{adj} f)^*(\tau\eta) = \int_{S^2 \times S^1} f^*\eta = 1.$$

Thus this higher holonomy produces a line bundle with nontrivial first Chern class on loop space.

The examples identify the roles of the data at each step. A local matrix measures parallel transport, an integral class controls gluing, and a differential character compares fillings of cycles. Changes of presentation preserve these comparisons. Increasing the dimension of the cycle changes the descent data that carry the holonomy, and transgression converts one such comparison into another by evaluation on a family of loops.

Appendices

APPENDIX A

Notation and comparison conventions

A.1 Bundles, frames, and curvature

For a principal bundle $P \to X$ the group acts on the right. Local sections are written s_i, with $s_j = s_i g_{ij}$ on overlaps. Matrix group connection forms transform as

$$A_j = g_{ij}^{-1} A_i g_{ij} + g_{ij}^{-1} dg_{ij}, \qquad F_j = g_{ij}^{-1} F_i g_{ij}.$$

A column representing a section of the associated vector bundle changes by $v_j = g_{ij}^{-1} v_i$ for the defining representation. The local covariant derivative on the column is $dv_i + A_i v_i$. When the Lie algebra-valued form is expressed in matrices, $F = dA + A \wedge A$. In bracket notation the last term is $\frac{1}{2}[A, A]$. On an ordinary tangent bundle we take

$$R(X, Y)Z = \nabla_X \nabla_Y Z - \nabla_Y \nabla_X Z - \nabla_{[X,Y]} Z.$$

Sectional curvature uses $\langle R(X, Y)Y, X\rangle$ divided by $\langle X, X\rangle\langle Y, Y\rangle - \langle X, Y\rangle^2$.

For a unitary line $A = -2\pi i\, a$, the real first Chern form is $iF/(2\pi) = da$, and in a local frame parallel transport along a path is multiplication by $\exp(2\pi i \int a)$. The curvature on a general principal bundle is an adjoint-valued form on the base. Invariant polynomials give scalar differential forms. For a unitary vector bundle the total Chern form is $\det(I + iF/(2\pi))$. Changing the orientation of a real even-rank bundle changes the sign of its Euler class.

A.2 Differentials, local cochains, and transport

The Čech differential δ is the alternating sum of the restrictions obtained by omitting an index. In a Čech complex with p the Čech degree and q the form degree, the total differential is $D = \delta + (-1)^p d$ on bidegree (p, q). In degree two, if $g_{ij} = \exp(-2\pi i \lambda_{ij})$, then $a_j - a_i = d\lambda_{ij}$, and the integral first Chern cocycle for the positive exponential sequence is $-\delta\lambda$. The comparison $D(a + \lambda) = da + \delta\lambda$ verifies the sign in real cohomology. The gerbe chapter specifies its multiplication and cocycle conventions separately.

Manifolds used as smooth tests are Hausdorff, second countable, and finite dimensional. Smooth singular simplices, paths with sitting instants, and fiber integration are considered with the boundary, orientation, and support conventions

in the relevant chapters. In the path transport formulas, composing two paths means traversal of the first path followed by the second, with the corresponding linear maps composed in the usual function order.

A.3 Moduli and differential coefficients

$\mathbf{B}G$ is the smooth stack of right G-torsors, and $\mathbf{B}G_{\nabla}$ includes their connections. A smooth test X is identified with its representable sheaf when this causes no ambiguity. A superscript h on a pullback or a limit emphasizes the use of paths and their coherences. The pullback in Chapter 16 uses locally constant coefficient sheaves $K(\mathbb{Z}, n)$ and $K(\mathbb{R}, n)$. In differential cohomology I denotes the underlying integral class and R denotes curvature. Where R acts on a vector via a tangent connection it denotes the Riemannian curvature tensor. The section and page links in Appendix B point to the main treatments.

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APPENDIX B

Index of concepts and constructions

Select a page number to open the corresponding treatment.

Concept	Construction	Page
Adjoint bundle	Lie algebra bundle carrying curvature and covariant differentiation.	26, 58
Associated bundle	Turns a torsor and a representation into a vector bundle.	26, 55
Atiyah class	The extension class of the first jet sequence.	106
Basic form	Horizontal invariant form that descends to the base.	35, 47
Bianchi identity	Covariant differential of curvature and its higher analogue.	22, 86
Cartan model	Equivariant forms and the contraction term in their differential.	47
Characteristic form	Closed form obtained by evaluating an invariant polynomial on curvature.	35, 62
Čech descent	Local objects, transition maps, and higher comparisons.	18, 51
Chern classes	Classes represented over the reals by the normalized determinant form.	39, 70
Chern–Simons form	Transgresses the difference of characteristic forms.	43
Christoffel symbols	Local coefficients of the affine connection.	5
Classifying stack	$\mathbf{B}G$ for torsors, $\mathbf{B}G_{\nabla}$ for torsors with connection.	55, 58
Crossed module	Two groups, an action, and Peiffer identities presenting a strict 2-group.	82
Curvature	Obstruction detected by the square of the covariant differential and small loops.	22, 30
Curving	Local two-form involved in a gerbe connection.	78
Derived intersection	Homotopy pullback computed locally by a derived tensor product.	106
Differential character	Evaluates smooth cycles with a Stokes law for its curvature.	66

Concept	Construction	Page
Differential refinement	An integral class, a form, and a comparison of their real images.	62, 70
Dixmier–Douady class	Underlying degree-three characteristic class of a circle gerbe.	78
Euler class	Top class of an oriented real bundle, with a Pfaffian form in even rank.	39
Fake curvature	Difference between ordinary curvature and the boundary of the two-form in a strict two-connection.	86
Fiber integration	Pushforward reducing cohomological degree by fiber dimension.	74
Gauss and Codazzi equations	Compare ambient and intrinsic derivatives of a submanifold.	10
Gauss–Bonnet	Identifies the total Gaussian curvature with 2π times the Euler characteristic.	14
Gauge transformation	Changes a frame or acts as an isomorphism in the connection stack.	22, 58
Gerbe holonomy	Surface evaluation encoded by local two-forms and overlap data.	78, 90
Holonomy	Transport around a closed path or higher cycle.	30, 74
Homotopy pullback	A pair of maps with their specified compatibility path.	51, 62
Jacobi field	First variation of a geodesic through geodesics.	14
Levi–Civita connection	The metric-compatible tangent connection with zero torsion.	5
Pontryagin classes	Even Chern classes of the complexification with the conventional signs.	39, 98
Principal connection	An equivariant horizontal splitting, equivalently a normalized Lie algebra-valued one-form.	26
Quotient stack	Remembers inertia and descent along an action groupoid.	102
Reduction	An H-subbundle of a G-bundle, or a section of its G/H-bundle.	55
Sectional curvature	Scalar curvature associated to an unoriented tangent two-plane.	10
Smooth site	Manifolds as tests, with gluing over open covers.	51
String structure	A lift across the homotopy fiber of $\frac{1}{2}p_1$.	98

Concept	Construction	Page
Surface transport	Two-dimensional transport on bigons with their thin relations.	90
Thin homotopy	A rank-constrained homotopy respected by transport.	30, 90
Transgression	Integrates along a parameter family or along the evaluation circle.	43, 74
Weil algebra	Universal algebra for connection and curvature generators.	47, 94

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