Representation Theory in Higher Categories

From compact harmonic analysis
to coherent group actions

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November 2026

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Part I Compact groups and harmonic analysis

CHAPTER 1

Haar measure and compact group actions

A compact group acts on itself by translation. Its Haar integral assigns an average to every continuous function in a manner compatible with this action. The same integral averages the action on any compact space or continuous unitary representation of the group. Robert begins with these constructions and uses them to pass from general representations to unitary ones [22, Part I, §§1–2].

1.1 The normalized Haar integral

Throughout this part G is a compact Hausdorff group, e is its identity, and dg is Haar probability measure. The Haar measure theorem gives a unique left-translation-invariant regular Borel probability measure on G [22, Part I, §1]. In integral form its normalization and left invariance read

$$\int_G 1\,dg = 1, \qquad \int_G f(ag)\,dg = \int_G f(g)\,dg \quad (a \in G,\ f \in C(G)).$$

Right translation commutes with left translation. The right translate of the normalized measure is therefore another left-invariant probability and equals dg. Inversion carries a left-invariant probability to a right-invariant one, and right invariance has the same uniqueness property by applying inversion. It follows that inversion preserves dg as well. All three change-of-variable identities extend to integrable functions.

The measure has full support. If $U \subseteq G$ is open and nonempty, its left translates cover G, so finitely many of them cover G. Left invariance and normalization then give $1 \le N\, m_G(U)$ for some finite N. In particular, any continuous function vanishing almost everywhere vanishes everywhere. On a finite group the same invariance assigns mass $1/|G|$ to each point. For the circle $\mathbb{T} = \{z \in \mathbb{C} : |z| = 1\}$, the pushforward of $d\theta/(2\pi)$ from an interval of length 2π is invariant under every rotation. Uniqueness identifies it with Haar measure. These elementary cases use different coordinate descriptions of the same invariant probability.

1.2 Averaging along orbits

Let G act continuously on a compact Hausdorff space X, with gx denoting the action. Write $q : X \to G\backslash X$ for the orbit map. The orbit relation is the image of the compact space $G \times X$ under $(g, x) \mapsto (x, gx)$, and is closed in $X \times X$. The orbit space is consequently compact Hausdorff. For $f \in C(X)$ define

$$(P_X f)(x) = \int_G f(g^{-1}x)\, dg.$$

The integrand varies continuously in (g, x). Compactness of G makes this variation uniform in g for x near any fixed point, so $P_X f$ is continuous. Translating the integration variable shows that $P_X f(ax) = P_X f(x)$ for every $a \in G$. Conversely, if f is invariant, its integrand equals $f(x)$ and $P_X f = f$. Consequently P_X is an idempotent operator with image $q^*C(G\backslash X) = C(X)^G$. It preserves constants and nonnegative real functions, and $\|P_X f\|_\infty \le \|f\|_\infty$. If $b \in C(X)^G$, then

$$P_X(bf) = b\, P_X f.$$

This module identity makes P_X a conditional expectation of the algebra of continuous functions onto its invariant subalgebra.

An equivariant continuous map $u : X \to Y$ between compact G-spaces induces an orbit map $\bar{u} : G\backslash X \to G\backslash Y$. Pullback commutes with averaging because $u(g^{-1}x) = g^{-1}u(x)$. Writing $\bar{P}_X$ for the average regarded as a function on the orbit space gives the commutative square

$$\begin{array}{ccc} C(Y) & \xrightarrow{\ u^*\ } & C(X) \\ \downarrow{\scriptstyle \bar{P}_Y} & & \downarrow{\scriptstyle \bar{P}_X} \\ C(G\backslash Y) & \xrightarrow{\ \bar{u}^*\ } & C(G\backslash X). \end{array}$$

This identity explains how averaging is transported through maps of actions. At the level of an individual orbit, $(P_X f)(x)$ equals the integral of f against the pushforward of dg under $g \mapsto gx$.

Suppose ν is a G-invariant regular Borel probability on X. Integrating the definition of P_X and using Fubini gives $\int_X f\, d\nu = \int_X P_X f\, d\nu$. Equivalently this last integral is $\int_{G\backslash X} \bar{P}_X f\, d(q_*\nu)$. For a transitive action, $P_X f$ is constant. Hence the integral of every $f \in C(X)$ against any invariant probability agrees with the orbit average, proving uniqueness by the Riesz representation theorem.

For the same ν, the orbit construction gives a description of invariant vectors in $L^2(X, \nu)$. Cauchy–Schwarz for Haar probability gives $|P_X f(x)|^2 \le P_X(|f|^2)(x)$. Integrating this inequality shows that P_X extends to a contraction on $L^2(X, \nu)$. For $f, h \in C(X)$, changing variables $x = gy$ and then $g \mapsto g^{-1}$ shows $\langle P_X f, h\rangle = \langle f, P_X h\rangle$. The extension is consequently an orthogonal projection. Its range is the

closure of $q^*C(G\backslash X)$, which is $q^*L^2(G\backslash X, q_*\nu)$ because pullback is an isometry and continuous functions are dense for the regular measure $q_*\nu$. Translation by G is strongly continuous on $L^2(X,\nu)$: continuity on $C(X)$ in the supremum norm follows by joint continuity and compactness of X, and translations are isometries on $L^2(X,\nu)$. Hence the extension of P_X agrees with vector averaging by density. If f is invariant in $L^2(X,\nu)$, each translated vector equals f, so $P_X f = f$. In particular an L^2 invariant on a homogeneous space with its invariant probability is represented almost everywhere by a constant.

1.3 Homogeneous spaces and quotient integration

For a closed subgroup $H \le G$, the space G/H consists of cosets gH and carries the transitive left G-action. Its invariant probability $\mu_{G/H}$ is the pushforward under $q_H : G \to G/H$. Conversely, given $x_0 \in X$ in a transitive compact Hausdorff G-space, put $H = \{g : gx_0 = x_0\}$. It is closed, and $gH \mapsto gx_0$ is a continuous bijection from compact G/H onto Hausdorff X. This identifies the orbit average with integration on G/H.

For $f \in C(G)$, the function $g \mapsto \int_H f(gh)\,dh$ is constant on cosets, where dh denotes Haar probability on H. Its integral with respect to dg equals $\int_G f\,dg$ by Fubini and right invariance. Since $\mu_{G/H} = (q_H)_* dg$, we obtain

$$\int_G f(g)\,dg = \int_{G/H} \left(\int_H f(gh)\,dh \right) d\mu_{G/H}(gH). \tag{1.1}$$

The inner integral is well defined without a section of q_H. It is the conditional expectation for the right H-action on G, using compactness of H.

Example 1.1 (Rotations of the sphere)**.** The group $\mathrm{SO}(3)$ acts transitively on S^2. The subgroup fixing $e_3 = (0,0,1)$ rotates the e_1e_2-plane, so the orbit map identifies S^2 with $\mathrm{SO}(3)/\mathrm{SO}(2)$. Its invariant probability μ is normalized surface area by uniqueness, since rotations preserve surface area.

The vector $b = \int_{S^2} x\,d\mu(x)$ is fixed by every rotation. A nonzero vector is moved by a half-turn about a perpendicular axis, so $b = 0$. The real symmetric matrix $M = \int_{S^2} xx^{\mathsf{T}}\,d\mu(x)$ satisfies $aMa^{\mathsf{T}} = M$ for every $a \in \mathrm{SO}(3)$. Transitivity on unit vectors makes $v^{\mathsf{T}}Mv$ constant on S^2, hence M is a scalar matrix by polarization. Its trace is $\int_{S^2} |x|^2\,d\mu(x) = 1$. We obtain

$$\int_{S^2} x_i\,d\mu(x) = 0, \qquad \int_{S^2} x_i x_j\,d\mu(x) = \tfrac{1}{3}\delta_{ij}.$$

In particular the orbit average of $x \mapsto (u^{\mathsf{T}}x)^2$ is $|u|^2/3$ for any real vector u. The second moment follows from the invariant tensor and its trace.

1.4 Projection onto invariant vectors

Let U be a strongly continuous unitary action of G on a complex Hilbert space $\mathcal{H}$. For each $v \in \mathcal{H}$, continuity of the orbit map makes its compact image separable and bounded, so the vector integral

$$P_U v = \int_G U(g)v \, dg$$

exists as a Bochner integral. This defines a bounded linear operator of norm at most one. Left invariance yields $U(a)P_U = P_U$, and P_U acts identically on the closed fixed space $\mathcal{H}^G$. Invariance under inversion and unitarity give $P_U^* = P_U$. Thus the average is the orthogonal projection onto $\mathcal{H}^G$. For two unitary actions, every bounded intertwiner $T : \mathcal{H} \to \mathcal{K}$ satisfies $TP_U = P_V T$, since bounded linear maps pass through the vector integral. In particular, a bounded map to a representation with trivial action is equivariant exactly when it factors through P_U.

Writing ι_U for the inclusion of the trivial representation on $\mathcal{H}^G$, one has $P_U \iota_U = \mathrm{id}_{\mathcal{H}^G}$ when P_U is viewed as a map to its image. Every equivariant bounded map from a trivial representation takes values in $\mathcal{H}^G$, so it factors uniquely through ι_U. These two factorizations express the adjunctions between the fixed-vector functor and the functor giving a space its trivial action. They also recover the orthogonal decomposition into the trivial summands of U and their invariant orthogonal complement.

CHAPTER 2

Unitary representations, intertwiners, and tensors

A representation of a compact group G on a finite-dimensional complex vector space V is a continuous homomorphism $\pi : G \to \mathrm{GL}(V)$. For a Hilbert space $\mathcal{H}$ we consider unitary representations, using strong continuity, which means $g \mapsto \pi(g)v$ is norm continuous for every $v \in \mathcal{H}$. An intertwiner $A : V \to W$ satisfies $A\pi(g) = \sigma(g)A$ for the two actions, and in Hilbert spaces is assumed bounded. Inner products are linear in their first argument. We use the representation constructions in Robert [22, Part I, §2] to identify the spaces of intertwiners and then to prove complete reducibility.

2.1 Unitarization and tensor constructions

Choose a positive definite Hermitian inner product $\langle -, - \rangle_0$ on V and set

$$\langle v, w \rangle_G = \int_G \langle \pi(g)v, \pi(g)w \rangle_0 \, dg.$$

For $v \neq 0$ the integrand of $\langle v, v \rangle_G$ is positive and its integral is positive. Right invariance gives $\langle \pi(a)v, \pi(a)w \rangle_G = \langle v, w \rangle_G$. Every continuous finite-dimensional representation can therefore be made unitary. More generally the same procedure applies to a strongly continuous action by bounded invertible operators on a Hilbert space when $\sup_{g \in G} \|\pi(g)\|$ is finite. For each fixed v the orbit is bounded by compactness, so the uniform boundedness principle gives a uniform bound C for such a G-action. The inverse operators obey the same bound. Thus

$$C^{-2}\|v\|_0^2 \leq \langle v, v \rangle_G \leq C^2 \|v\|_0^2.$$

The average equips the same vector space with an equivalent complete Hilbert norm.

If V, W are representations, $V \otimes W$ carries $g(v \otimes w) = \pi(g)v \otimes \sigma(g)w$. The linear dual $V^\vee$ carries $g\lambda = \lambda \circ \pi(g)^{-1}$. In finite dimension the evaluation and coevaluation maps

$$V^\vee \otimes V \longrightarrow \mathbb{C}, \quad \lambda \otimes v \longmapsto \lambda(v), \qquad \mathbb{C} \longrightarrow V \otimes V^\vee, \quad 1 \longmapsto \sum_i e_i \otimes e^i$$

are equivariant, with the trivial action on $\mathbb{C}$ and with $(e_i), (e^i)$ dual bases. Evaluation is invariant because the two actions cancel. Under $V \otimes V^\vee \simeq \mathrm{End}(V)$, the coevaluation element corresponds to id_V and is invariant for conjugation. Composing evaluation and coevaluation in either of the two triangle identities amounts to expanding v or λ in its basis. This makes the category of finite-dimensional representations a symmetric monoidal category with duals. For Hilbert tensor products of unitary representations, the action on elementary tensors extends by density to a strongly continuous unitary action. The finite-dimensional construction of coevaluation uses the finite identity sum explicitly.

2.2 Averaging linear maps

On $\mathrm{Hom}(V, W)$ define the conjugation action $g \cdot A = \sigma(g) A \pi(g)^{-1}$. The fixed vectors are exactly the intertwiners. For finite-dimensional unitary representations this is a unitary action for the Hilbert–Schmidt inner product $\langle A, B \rangle = \mathrm{Tr}(AB^*)$, where the trace is taken on W. The projection onto invariant vectors from Chapter 1 gives

$$E_{V,W}(A) = \int_G \sigma(g) A \pi(g)^{-1} \, dg.$$

Linear maps $S : V' \to V$ and $T : W \to W'$ that are intertwiners induce the map $A \mapsto TAS$. The integrand commutes with precomposition by S and postcomposition by T, giving the square

$$\begin{array}{ccc} \mathrm{Hom}(V, W) & \xrightarrow{A \mapsto TAS} & \mathrm{Hom}(V', W') \\ \downarrow{\scriptstyle E_{V,W}} & & \downarrow{\scriptstyle E_{V',W'}} \\ \mathrm{Hom}_G(V, W) & \xrightarrow{A \mapsto TAS} & \mathrm{Hom}_G(V', W'). \end{array}$$

Equivariant maps can also be described as invariant tensors, since $V^\vee \otimes W \simeq \mathrm{Hom}(V, W)$ carries the tensor action to this conjugation action. Therefore

$$(V^\vee \otimes W)^G \simeq \mathrm{Hom}_G(V, W).$$

The identity explains how multiplicity questions become questions about invariant vectors in tensor products.

Proposition 2.1 (Schur's lemma)**.** *An intertwiner between nonzero finite-dimensional irreducible complex representations is either zero or an isomorphism. Every endomorphism of such a representation is scalar.*

Proof. The kernel and image of an intertwiner are invariant subspaces. Irreducibility therefore makes every nonzero intertwiner injective and surjective. If A is an intertwining endomorphism, choose a complex eigenvalue λ of A. The map $A - \lambda I$ is an intertwiner with a nontrivial kernel, so it is zero. □

As a consequence, if V is irreducible of dimension d, then $E_{V,V}(A) = (\operatorname{Tr} A/d)I$. The averaging operator has scalar image by Schur's lemma, and conjugation preserves trace. Thus $\operatorname{Tr} A$ specifies its scalar component.

2.3 Complete reducibility

For a finite-dimensional unitary representation, the orthogonal complement of an invariant subspace M is invariant. Indeed for $w \in M^{\perp}$ and $v \in M$, one has $\langle \pi(g)w, v\rangle = \langle w, \pi(g)^{-1}v\rangle = 0$. Induction on dimension decomposes V as an orthogonal direct sum of irreducibles. For an irreducible U, the evaluation map

$$U \otimes \operatorname{Hom}_G(U, V) \longrightarrow V, \qquad u \otimes A \longmapsto Au$$

is an isomorphism onto the U-isotypic subspace, the sum of the subrepresentations equivalent to U. To see this, decompose V into irreducibles and apply Schur's lemma to each block. Equivalent copies contribute one-dimensional Hom spaces, and the other blocks contribute zero. In particular the multiplicity $\dim \operatorname{Hom}_G(U, V)$ is independent of the chosen decomposition.

Theorem 2.2 (Complete reducibility in Hilbert space)**.** *Every strongly continuous unitary representation of a compact group is a Hilbert direct sum of finite-dimensional irreducible representations.*

Proof. Let $v \neq 0$ in the representation space $\mathcal{H}$, and let B_v be the positive rank-one operator $w \mapsto \langle w, v\rangle v$. Form

$$K_v = \int_G B_{\pi(g)v}\, dg.$$

The map $g \mapsto B_{\pi(g)v}$ is continuous in operator norm. Its integral is an operator-norm limit of finite linear combinations of rank-one operators, hence compact. The operator is positive, and $\langle K_v v, v\rangle = \int_G |\langle v, \pi(g)v\rangle|^2\, dg > 0$. The strict inequality uses continuity at e and full support of Haar measure. Since $\pi(a)B_{\pi(g)v}\pi(a)^{-1} = B_{\pi(ag)v}$, left invariance shows that K_v commutes with the action. The compact self-adjoint spectral theorem gives a nonzero eigenvalue of K_v with nonzero finite-dimensional eigenspace [19]. The eigenspace is invariant and contains an irreducible subrepresentation by the finite-dimensional argument.

Choose a maximal orthogonal family of irreducible subspaces by Zorn's lemma, using the union as upper bound for each chain of such families. The orthogonal complement of their closed span is invariant by unitarity. If it were nonzero, the argument with K_v inside that complement would produce a further irreducible summand. Thus the closed span is $\mathcal{H}$, and its decomposition is the Hilbert direct sum of the family. □

There is also a choice-independent Hilbert space of multiplicities. Fix an irreducible finite-dimensional unitary U and let $M_U = \mathrm{Hom}_G(U, \mathcal{H})$. If $A, B \in M_U$, then B^*A is an intertwiner because the two actions are unitary. Write $B^*A = \langle A, B\rangle_{M_U} I_U$ using Schur's lemma. The form is the Hilbert–Schmidt inner product divided by $\dim U$, as taking the trace shows. Since U is finite dimensional, the ambient Hilbert–Schmidt space is a finite Hilbert direct sum of copies of $\mathcal{H}$. The equations defining M_U are closed, so M_U is complete for the scaled norm.

For elementary tensors the evaluation map satisfies

$$\langle Au, Bv\rangle_{\mathcal{H}} = \langle A, B\rangle_{M_U} \langle u, v\rangle_U.$$

It extends to an isometry $U \otimes M_U \to \mathcal{H}$ whose range is the closed span of the images of the intertwiners A. If U and U' are inequivalent irreducibles and $A \in M_U$, $B \in M_{U'}$, then $B^*A = 0$ by Schur's lemma. The ranges for inequivalent irreducibles are therefore orthogonal. Complete reducibility makes their Hilbert direct sum all of $\mathcal{H}$. This identifies the canonical blocks without choosing irreducible subspaces inside them, a distinction that will be useful for the regular representation.

Example 2.3 (An invariant tensor from a rotation representation)**.** Let $V = \mathbb{C}^3$ with the complexified standard action of $\mathrm{SO}(3)$. The rotations by π about the three coordinate axes are diagonal and have distinct joint eigenspaces $\mathbb{C}e_i$. A nonzero subspace invariant under the group therefore contains a coordinate vector, by applying the joint spectral projections of these diagonal rotations. Rotations taking one coordinate axis to another imply that it contains all three. This proves irreducibility over $\mathbb{C}$. The invariant part of $V^\vee \otimes V$ is thus spanned by the identity tensor. For $A = B_{e_3}$, its averaged conjugate has trace one and equals $I/3$, recovering the second moment of the sphere in Chapter 1.

CHAPTER 3

The Peter–Weyl theorem

For a unitary representation π of dimension d_π, fix an orthonormal basis $e_1, \dots, e_{d_\pi}$ and write

$$u_{ij}^{\pi}(x) = \langle \pi(x) e_j, e_i \rangle.$$

Let $\widehat{G}$ contain one representative from each equivalence class of irreducible unitary representations. Write $\mathcal{A}(G)$ for the linear span of their matrix coefficients. The Peter–Weyl theorem gives uniform approximation by $\mathcal{A}(G)$ and an orthonormal basis of $L^2(G)$. Robert treats these conclusions in Part I, §§4–5 [22]. Density follows from finite-dimensional eigenspaces of convolution operators. Averaging intertwiners determines the normalization of the basis.

3.1 Convolution operators and finite spectra

Write $L_a f(x) = f(a^{-1}x)$ for left translation. The operators L_a define a strongly continuous unitary action on $L^2(G)$. For continuous f, compactness gives $L_a f \to f$ uniformly as $a \to e$. Density of continuous functions in L^2 for a regular Borel measure on a compact Hausdorff space extends this convergence to all $f \in L^2(G)$, using $\|L_a\| = 1$. The same argument gives strong continuity of $f(x) \mapsto f(xa)$.

For $\phi \in C(G)$, put

$$(T_\phi f)(x) = \int_G f(y)\phi(y^{-1}x)\, dy = \int_G \phi(z) f(xz^{-1})\, dz.$$

The second expression follows by Haar invariance under inversion and translation. For the first expression, Cauchy–Schwarz gives

$$\|T_\phi f\|_\infty \le \|\phi\|_2 \|f\|_2.$$

The kernel $k(x,y) = \phi(y^{-1}x)$ is continuous, and compactness makes it vary continuously in x in the supremum norm over y. This shows that $T_\phi f$ has a continuous representative for every $f \in L^2(G)$. Finite sums of product functions $a(x)b(y)$ are uniformly dense in $C(G \times G)$ by the complex Stone–Weierstrass theorem. The corresponding integral operators have finite rank. The operator norm of the difference on $L^2(G)$ is bounded by the uniform norm of the difference of kernels, since dg is a probability. Thus T_ϕ is compact.

Substitution $y = az$ in the first expression yields $T_\phi L_a = L_a T_\phi$. If $\phi^\sharp(z) = \overline{\phi(z^{-1})}$, the adjoint kernel of T_ϕ is that of $T_{\phi^\sharp}$, so $T_\phi^* = T_{\phi^\sharp}$. In particular a real

symmetric ϕ, with $\phi(z) = \phi(z^{-1})$, yields a self-adjoint operator. The compact self-adjoint spectral theorem gives finite-dimensional eigenspaces for its nonzero eigenvalues, and their closed span is the closure of its range [19].

Let M be such an eigenspace. It is invariant under L_a, which acts on it through a finite-dimensional unitary representation ρ. If $T_\phi f = \lambda f$ with $\lambda \neq 0$, the function $\lambda^{-1} T_\phi f$ is its continuous representative. It is unique by full support. Write $w \in M$ for the vector representing evaluation at e, so $f(e) = \langle f, w \rangle$ on M. For $f \in M$,

$$f(x) = \langle \rho(x^{-1}) f, w \rangle = \overline{\langle \rho(x) w, f \rangle}.$$

The expression on the right is a coefficient of the conjugate representation on $\overline{M}$. This representation is a finite direct sum of irreducibles by Chapter 2. Hence $M \subseteq \mathcal{A}(G)$.

3.2 Approximate identities and the two density statements

For each symmetric neighborhood U of e, choose a nonnegative continuous function supported in U and positive at e. Such a function exists by normality of the compact Hausdorff space G, after choosing a closed neighborhood of e contained in U. Replace it by the average of itself and its composition with inversion. Normalize its positive integral to one and denote the result by ϕ_U. The neighborhoods are directed by reverse inclusion. For $p = 2$ or $p = \infty$, with $f \in C(G)$ when $p = \infty$, the second expression for T_{ϕ_U} gives

$$\| T_{\phi_U} f - f \|_p \leq \sup_{z \in U} \| f(\cdot\, z^{-1}) - f \|_p.$$

The right side tends to zero by the continuity of translations in each of the stated norms.

Each $T_{\phi_U} f$ lies in the L^2-closure of the sum of its nonzero eigenspaces, hence in the L^2-closure of $\mathcal{A}(G)$. The approximate identity proves density of $\mathcal{A}(G)$ in $L^2(G)$. To obtain uniform density, let $f \in C(G)$ and $\varepsilon > 0$. Choose U so that $\| f - T_{\phi_U} f \|_\infty < \varepsilon/2$. Then select $q \in \mathcal{A}(G)$ with $\| f - q \|_2 < \varepsilon / (2 \| \phi_U \|_2)$. Since

$$u^\pi_{ij}(x z^{-1}) = \sum_k u^\pi_{ik}(x) u^\pi_{kj}(z^{-1}),$$

convolution by ϕ_U takes each coefficient to a finite linear combination of coefficients of the same representation. Thus $T_{\phi_U} q \in \mathcal{A}(G)$. The supremum norm estimate for the kernel gives $\| f - T_{\phi_U} q \|_\infty < \varepsilon$.

3.3 Schur orthogonality

For $\pi, \sigma \in \widehat{G}$ define $E_{jl} : V_\sigma \to V_\pi$ by taking the l-th basis vector to the j-th basis vector and the others to zero. The ik-entry of the averaged map $E_{V_\sigma, V_\pi}(E_{jl})$ is $\int_G u^\pi_{ij}(x)\overline{u^\sigma_{kl}(x)}\,dx$. For $\pi \neq \sigma$, this map is zero by Schur's lemma. For $\pi = \sigma$, its trace is δ_{jl}, and it equals $\delta_{jl} I/d_\pi$ by Chapter 2. Hence

$$\int_G u^\pi_{ij}(x)\overline{u^\sigma_{kl}(x)}\,dx = \begin{cases} d_\pi^{-1}\delta_{ik}\delta_{jl}, & \pi = \sigma, \\ 0, & \pi \neq \sigma. \end{cases}$$

Together with density, this proves that the functions $\sqrt{d_\pi}\,u^\pi_{ij}$ form an orthonormal basis of $L^2(G)$. In particular, writing $\chi_\pi = \operatorname{Tr}\pi = \sum_i u^\pi_{ii}$ gives $\langle \chi_\pi, \chi_\sigma \rangle = \delta_{\pi\sigma}$. For finite-dimensional V with character χ_V, its multiplicity of π is $\int_G \chi_V(x)\overline{\chi_\pi(x)}\,dx$ by additivity of the trace.

3.4 Class functions and characters

Conjugation is a continuous action of G on itself. Its invariant functions, called class functions, are constant on each conjugacy class. The measure dg is invariant under conjugation by bi-invariance. Chapter 1 therefore identifies the orthogonal projection onto the class functions in $L^2(G)$ as

$$(Qf)(x) = \int_G f(a^{-1}xa)\,da \quad (f \in C(G)).$$

For a matrix coefficient u^π_{ij}, average the matrix $\pi(a)^{-1}\pi(x)\pi(a)$ before taking its ij-entry. By Schur's lemma and invariance under inversion, the matrix average equals $(\chi_\pi(x)/d_\pi)I$. It follows that $Qu^\pi_{ij} = \delta_{ij}\chi_\pi/d_\pi$. Applying Q to the dense coefficient span in $L^2(G)$ proves that the characters form an orthonormal basis of the closed subspace of class functions. Applying the uniform contraction Q to uniform approximants proves that finite sums of characters are uniformly dense in the continuous class functions.

In particular, distinct conjugacy classes are distinguished by irreducible characters. Indeed the space of conjugacy classes is the compact Hausdorff orbit space for the conjugation action, so two distinct classes are separated by a continuous function on that orbit space. Its pullback is a continuous class function. If all characters had equal values on the two classes, their finite linear combinations and uniform limits would have equal values there as well.

3.5 Reconstruction by matrix images

The coefficient functions are closed under multiplication and complex conjugation. Products are coefficients of tensor products and conjugates are coefficients of

the conjugate actions. These finite-dimensional representations decompose into irreducibles, so the products and conjugates belong to $\mathcal{A}(G)$. The trivial representation supplies constants. Uniform density implies that these functions separate distinct points, because $C(G)$ does. In particular, two elements of G with the same image in every finite-dimensional irreducible representation coincide.

For each finite subset $F \subseteq \widehat{G}$, let G_F be the compact image of G in $\prod_{\pi \in F} \mathrm{U}(d_\pi)$. Projection of matrix coordinates supplies the maps $G_{F'} \to G_F$ for $F \subseteq F'$. The resulting homomorphism $G \to \varprojlim_F G_F$ is injective by separation of points. If $(x_F)_F$ is compatible, the closed preimages of the points x_F in G have the finite intersection property. For any finite collection of such preimages, a lift of the coordinate in their union belongs to the intersection. Compactness supplies a point of G lying in every preimage. Thus the homomorphism is surjective, and compactness and Hausdorffness make it a homeomorphism. The matrix images carry a system of finite-dimensional observations from which the compact group is recovered as a topological group.

CHAPTER 4

Fourier transform, Plancherel, and convolution

The orthogonal basis in Chapter 3 groups Fourier coefficients into one matrix per irreducible representation. This grouping turns convolution into multiplication in each matrix algebra. We take dg to be Haar probability and use the convolution product in Robert, Part I, §6 [22]. We record the multiplication order both for the integrated representation and for the transform with $\pi(x)^*$, since the two choices attach different conventions to the same convolution.

4.1 Convolution and the two transform conventions

For $f, h \in C(G)$, define

$$(f * h)(x) = \int_G f(y)h(y^{-1}x)\,dy, \qquad f^*(x) = \overline{f(x^{-1})}.$$

Haar invariance and Fubini give $\int_G \int_G |f(y)h(y^{-1}x)|\,dy\,dx = \|f\|_1\|h\|_1$, so convolution extends by continuity to a bounded bilinear product on $L^1(G)$. For continuous f, h, k, the substitution $y = uz$ shows that both $((f * h) * k)(x)$ and $(f * (h * k))(x)$ equal

$$\int_G \int_G f(u)h(z)k(z^{-1}u^{-1}x)\,dz\,du.$$

This proves associativity, which extends to $L^1(G)$ by density. Inversion invariance likewise gives $(f * h)^* = (h^*) * (f^*)$ and $(f^*)^* = f$.

Translations also control convolution on $L^2(G)$. If $f \in L^1(G)$ and $h \in L^2(G)$, the expression $f * h$ is the vector integral of $y \mapsto f(y)L_y h$. Since L_y is unitary, $\|f * h\|_2 \le \|f\|_1\|h\|_2$. The change of variables $z = y^{-1}x$ expresses $h * f$ as the average of $h(\,\cdot\, z^{-1})$ weighted by $f(z)$ and gives the same bound. These formulas agree with convolution on continuous functions and determine its extension to the indicated spaces.

For an irreducible unitary π, put

$$\Phi_\pi(f) = \int_G f(x)\pi(x)\,dx, \qquad \widehat{f}(\pi) = \int_G f(x)\pi(x)^*\,dx.$$

Each matrix integral is bounded in operator norm by $\|f\|_1$. For continuous inputs, substituting $x = yz$ in the integral of a convolution gives

$$\Phi_\pi(f * h) = \Phi_\pi(f)\Phi_\pi(h), \qquad \widehat{f * h}(\pi) = \widehat{h}(\pi)\widehat{f}(\pi).$$

The second formula uses $\pi(yz)^* = \pi(z)^*\pi(y)^*$. Both identities extend to $L^1(G)$. In the integrated convention, $\Phi_\pi(f^*) = \Phi_\pi(f)^*$ by $x \mapsto x^{-1}$. For $f^\vee(x) = f(x^{-1})$, one has $\widehat{f}(\pi) = \Phi_\pi(f^\vee)$ and $(f * h)^\vee = h^\vee * f^\vee$. In particular $\widehat{f^*}(\pi) = \widehat{f}(\pi)^*$ as well.

4.2 Plancherel and inversion

Because dg is a probability, $L^2(G) \subseteq L^1(G)$. In an orthonormal basis of V_π,

$$\widehat{f}(\pi)_{ij} = \int_G f(x)\overline{u^\pi_{ji}(x)}\,dx = \langle f, u^\pi_{ji}\rangle.$$

Applying Parseval to $\sqrt{d_\pi}\,u^\pi_{ij}$ yields the Plancherel identity

$$\|f\|^2_{L^2(G)} = \sum_{\pi \in \widehat{G}} d_\pi \operatorname{Tr}(\widehat{f}(\pi)\widehat{f}(\pi)^*).$$

Equipping $\operatorname{End}(V_\pi)$ with the inner product $d_\pi \operatorname{Tr}(AB^*)$, the transform is a unitary isomorphism from $L^2(G)$ onto the Hilbert direct sum of these matrix spaces. Every square-summable family of weighted matrices occurs, because its entries prescribe an arbitrary square-summable family of coordinates in the Peter–Weyl orthonormal basis.

For a finite subset $F \subseteq \widehat{G}$, the orthogonal projection retaining its blocks is

$$f_F(x) = \sum_{\pi \in F} d_\pi \operatorname{Tr}(\widehat{f}(\pi)\pi(x)).$$

The trace expands as $\sum_{i,j} \widehat{f}(\pi)_{ij} u^\pi_{ji}(x)$, which gives the coefficients of the orthogonal projection with the prescribed basis normalization. As F increases over finite subsets, $f_F \to f$ in $L^2(G)$. This form applies even if $\widehat{G}$ is uncountable. For each fixed f at most countably many blocks have positive squared norm, since for each $n \geq 1$ there are only finitely many whose squared norm exceeds $1/n$. A sufficient condition for uniform convergence is $\sum_\pi d_\pi \|\widehat{f}(\pi)\|_1 < \infty$, where $\|A\|_1$ for a matrix denotes its trace norm. Indeed $|\operatorname{Tr}(A\pi(x))| \leq \|A\|_1$ since $\pi(x)$ is unitary. Under the summability condition, the series gives the continuous representative of f.

4.3 Translation and isotypic projections

For $L_a f(x) = f(a^{-1}x)$ and $R_b f(x) = f(xb)$, Haar substitutions give

$$\widehat{L_a f}(\pi) = \widehat{f}(\pi)\pi(a)^*, \qquad \widehat{R_b f}(\pi) = \pi(b)\widehat{f}(\pi).$$

Thus the two commuting translation actions correspond to right and left multiplication on each Fourier block. As a representation for the pair of actions, the block $\mathrm{End}(V_\pi) \simeq V_\pi \otimes V_\pi^\vee$ has the first factor acted on by R_b through $\pi(b)$ and the dual factor acted on by L_a through $\pi(a)^\vee$. The coefficient functions themselves carry the same actions under the inverse unitary transform.

For $p_\pi = d_\pi \chi_\pi$, orthogonality yields $\widehat{p_\pi}(\sigma) = I$ if $\sigma = \pi$ and zero otherwise. Convolution by p_π therefore projects to the π-block. Since the transform is injective on $L^2(G)$, the convolution is the same from either side on $C(G)$ and the identity extends to $L^2(G)$ by continuity. These kernels are central in $L^1(G)$ by a further density argument. In particular

$$p_\pi * p_\sigma = \delta_{\pi\sigma} p_\pi, \qquad p_\pi^* = p_\pi.$$

The explicit kernel makes projection to a block an intrinsic operation on functions, independent of the basis chosen in V_π.

4.4 Fourier series on the circle

Example 4.1 (Fourier series on the circle)**.** For the circle $\mathbb{T}$, irreducibles are one-dimensional because each $\pi(z)$ commutes with the representation and is scalar by Schur's lemma. Its continuous characters are $z \mapsto z^n$ for $n \in \mathbb{Z}$. Indeed a continuous circle character composed with $t \mapsto e^{it}$ has a continuous real lift under $t \mapsto e^{it}$, chosen to vanish at 0. Additivity of the original homomorphism forces the lift to be additive, since its failure of additivity would be continuous with values in $2\pi\mathbb{Z}$ and zero at the origin. Continuity makes the lift ct, and 2π-periodicity of the original function forces $c \in \mathbb{Z}$. Thus

$$\widehat{f}(n) = \frac{1}{2\pi}\int_0^{2\pi} f(e^{it})e^{-int}\,dt, \qquad f(z) = \sum_{n\in\mathbb{Z}} \widehat{f}(n) z^n \quad \text{in } L^2.$$

The convolution identity is the usual scalar product of Fourier coefficients, since the matrices have dimension one.

For a continuous circle function, an average of the orthogonal partial sums always converges uniformly. To see the mechanism, use the kernels

$$K_N(z) = \frac{1}{N}\left|\sum_{j=0}^{N-1} z^j\right|^2 = \sum_{|k|<N}\left(1 - \frac{|k|}{N}\right) z^k \qquad (N \geq 1).$$

The second equality counts the pairs i, j in the product with $i-j = k$. In particular $K_N \geq 0$ and its integral is the constant coefficient 1. For $z \neq 1$, geometric summation gives $K_N(z) \leq 4/(N|1-z|^2)$. Outside any fixed neighborhood of 1 the denominator $|1-z|^2$ has a positive lower bound, so the mass of K_N there tends to zero.

Given $\varepsilon > 0$, choose a neighborhood U of 1 so that $|f(xz^{-1}) - f(x)| < \varepsilon$ for all x and $z \in U$. The mass and concentration estimates give

$$\|f * K_N - f\|_\infty \leq \varepsilon + 2\|f\|_\infty \int_{\mathbb{T}\setminus U} K_N(z)\, dz.$$

The right side has limit superior at most ε, proving uniform convergence. The convolution has coefficients $(1 - |k|/N)\widehat{f}(k)$ for $|k| < N$ and zero outside that range, so it equals the arithmetic mean of the N partial sums with cutoffs $0, \ldots, N-1$. The kernels supply explicit positive finite Fourier approximations, complementing the general convolution approximate identities of Chapter 3.

4.5 Matrix blocks in noncommutative convolution

Example 4.2 (Multiplication order for SU(2))**.** The defining representation π on $\mathbb{C}^2$ is irreducible. For each unit vector $v = (a, b)$, the matrix with columns v and $(-\bar{b}, \bar{a})$ lies in SU(2), so the orbit of a unit vector is the entire unit sphere and spans the space. Put $f = 2u^{\pi}_{12}$, $h = 2u^{\pi}_{21}$, and let E_{ij} be the elementary matrix. Their only nonzero Fourier blocks are $\widehat{f}(\pi) = E_{21}$ and $\widehat{h}(\pi) = E_{12}$. Thus

$$\widehat{f * h}(\pi) = E_{12}E_{21} = E_{11}, \qquad \widehat{h * f}(\pi) = E_{21}E_{12} = E_{22}.$$

Inversion gives $f * h = 2u^{\pi}_{11}$ and $h * f = 2u^{\pi}_{22}$. The two functions are distinct, since their matrix coefficients belong to distinct vectors of the orthogonal basis. More generally, each irreducible block carries the full operator algebra on its representation space, with opposite multiplication for the adjoint Fourier convention.

For any compact G, finite sums of coefficients form a convolution algebra. The transforms of such functions are exactly the finitely supported families of matrices by finite Fourier inversion. Thus $\mathcal{A}(G)$ under convolution is the algebraic direct sum of matrix algebras in the integrated convention. In the adjoint convention the same description uses the opposite algebras. A finite subset $F \subseteq \widehat{G}$ determines the subalgebra of coefficients supported in F under the adjoint transform. Its identity is $\sum_{\pi \in F} p_\pi$, whose transform is the identity on every retained block and vanishes elsewhere. As F grows, convolution by this identity converges strongly to the identity on $L^2(G)$, by the Plancherel formula.

CHAPTER 5

Characters, averaging, and projectors

A character is the trace of an action. Its integral against the conjugate of another character counts equivariant maps. Both facts come from taking the trace of a concrete projection, and inserting additional operators into that trace produces finer invariants. We use the compact group's normalized Haar measure throughout this chapter. The classical character theory is developed in [22, §7] and [11].

5.1 The trace of the invariant projection

For a finite-dimensional unitary representation V of a compact group G, its character is

$$\chi_V(g) = \operatorname{tr}(\rho_V(g)).$$

Conjugacy invariance follows from conjugacy invariance of trace. Additivity on direct sums, multiplicativity on tensor products, and the duality rule follow from the corresponding matrix operations, giving

$$\chi_{V\oplus W} = \chi_V + \chi_W, \qquad \chi_{V\otimes W} = \chi_V\chi_W, \qquad \chi_{V^*}(g) = \chi_V(g^{-1}) = \overline{\chi_V(g)}.$$

Thus the character respects the operations in the symmetric tensor category of representations, with product of characters computed pointwise.

The linear action on $\operatorname{Hom}_{\mathbb{C}}(V,W)$ is $\rho(g)A = \rho_W(g)A\rho_V(g)^{-1}$. Its fixed points are the intertwiners. The isomorphism $V^*\otimes W \to \operatorname{Hom}_{\mathbb{C}}(V,W)$, sending $\lambda\otimes w$ to $v \mapsto \lambda(v)w$, is equivariant. Let $P_X = \int_G \rho_X(g)\,dg$ on any such representation X. Invariance under left multiplication shows that the image of P_X consists of fixed vectors, and P_X is the identity on fixed vectors. The measure is invariant under inversion, so unitarity gives $P_X^* = P_X$. The resulting projector is compatible with the tensor–Hom isomorphism as follows.

$$\begin{array}{ccc} V^*\otimes W & \xrightarrow{\ \sim\ } & \operatorname{Hom}_{\mathbb{C}}(V,W) \\ {\scriptstyle P_{V^*\otimes W}}\downarrow & & \downarrow{\scriptstyle P_{\mathrm{Hom}}} \\ (V^*\otimes W)^G & \xrightarrow{\ \sim\ } & \operatorname{Hom}_G(V,W). \end{array}$$

The codomain of a projector in this diagram is its image, with the inclusion into the original space understood.

Theorem 5.1 (Character inner product)**.** *For finite-dimensional unitary V, W,*

$$\langle \chi_V, \chi_W \rangle_G = \int_G \overline{\chi_V(g)} \chi_W(g)\, dg = \dim \mathrm{Hom}_G(V, W).$$

In particular, distinct irreducible characters are orthogonal, and each irreducible character has norm one.

Proof. The trace of the orthogonal projection on X^G is its dimension. Taking $X = V^* \otimes W$, linearity of integration and the tensor and dual character formulas identify this trace with the displayed integral. Schur's lemma gives dimensions zero and one when V, W are irreducible. □

For each irreducible U, the integer $\langle \chi_U, \chi_V \rangle_G$ is the multiplicity of U in V. Complete reducibility now shows that the character determines V up to equivalence. The linear independence of irreducible characters also embeds the representation ring into the ring of continuous class functions. The algebra of finite linear combinations of characters uses pointwise multiplication. The central projectors in the next section instead use convolution.

5.2 Isotypic summands and tensor multiplicities

Let U be irreducible of dimension d_U. The part of V of type U is the image of the evaluation

$$U \otimes \mathrm{Hom}_G(U, V) \longrightarrow V, \qquad u \otimes a \longmapsto a(u).$$

Choosing a decomposition of V into irreducibles verifies that this evaluation is injective and its image is precisely the sum of the copies of U. The image is canonical because the formula does not involve that decomposition.

Put $p_U(g) = d_U \overline{\chi_U(g)}$. For any V the operator

$$\rho_V(p_U) = \int_G p_U(g) \rho_V(g)\, dg$$

commutes with G by conjugacy invariance, and on an irreducible W it is scalar by Schur's lemma. Its trace there is $d_U \langle \chi_U, \chi_W \rangle_G$. The scalar is therefore one for $W \cong U$ and zero otherwise. It follows that this formula realizes the orthogonal projection onto the isotypic part. With $f * k(x) = \int_G f(g) k(g^{-1}x)\, dg$ the integrated action satisfies $\rho_V(f * k) = \rho_V(f)\rho_V(k)$. Thus $p_U * p_W$ has the same integrated action as $\delta_{U,W} p_U$ on every irreducible. Matrix coefficients are uniformly dense by Peter–Weyl. A continuous function that integrates to zero

against every matrix coefficient integrates to zero against its own conjugate, hence vanishes. This gives

$$p_U * p_W = \delta_{U,W} p_U$$

as an identity of functions. Each p_U is self-adjoint for the involution $f^*(g) = \overline{f(g^{-1})}$.

The multiplicity of U in a tensor product $V \otimes W$ is the contraction

$$N_{V,W}^U = \int_G \overline{\chi_U(g)} \chi_V(g) \chi_W(g)\, dg.$$

Associativity of tensor product gives, for irreducibles A, B, C, D,

$$\sum_X N_{A,B}^X N_{X,C}^D = \sum_Y N_{B,C}^Y N_{A,Y}^D,$$

where the sums run over irreducible isomorphism classes and have finite support. The two sides count the dimensions of the multiplicity spaces obtained by resolving $A \otimes B$ or $B \otimes C$ first. The intermediate labels X, Y list the channels in which the tensor product is resolved, and the associator identifies the two decompositions of the final multiplicity space.

For a finite group, Haar integral is the average of the summands indexed by group elements. If X is a finite G-set, its permutation representation has character $\chi_{\mathbb{C}[X]}(g) = |X^g|$. A fixed vector has constant coefficients on each orbit, so the vectors $\sum_{x \in O} x$ indexed by the orbits O form a basis of the fixed subspace. Pairing the character with the trivial character derives Burnside's orbit-counting formula.

$$|X/G| = \frac{1}{|G|} \sum_{g \in G} |X^g|.$$

The same trace argument consequently counts multiplicities for general unitary representations and orbits for finite permutation actions.

Example 5.2 (Two-dimensional tensors for S_3)**.** The permutation action of S_3 on $\mathbb{C}^3$ splits off the invariant line of constant coordinates. Its complement V of coordinate sum zero has character $2, 0, -1$ on the identity, transpositions, and three-cycles. These classes have sizes $1, 3, 2$, so the character has squared norm $(4 + 2)/6 = 1$ and V is irreducible. The character of $V \otimes V$ has values $4, 0, 1$. Its inner products with 1, the sign character, and χ_V are respectively $(4 + 2)/6$, $(4 + 2)/6$, and $(8 - 2)/6$, all equal to one. The identified summands account for the full dimension. The alternating square is the sign line, since the determinant of the action on the permutation space is the sign and the constant line contributes determinant one. Thus the symmetric square is $\mathbb{C} \oplus V$. The invariant metric displays the trivial summand in the symmetric square.

5.3 The Frobenius–Schur indicator

Ordinary character orthogonality detects self-duality. The Frobenius–Schur indicator further determines the symmetry of the invariant bilinear form when the representation is self-dual. For an irreducible U put

$$\nu(U) = \int_G \chi_U(g^2)\, dg.$$

Let $S(u \otimes v) = v \otimes u$. On a finite-dimensional space the identity $\operatorname{tr}(S(A \otimes B)) = \operatorname{tr}(AB)$ follows by summing the diagonal entries in a tensor product basis. The flip commutes with the diagonal G-action, and integration gives

$$\nu(U) = \operatorname{tr}\bigl(SP_{U\otimes U}\bigr) = \operatorname{tr}\bigl(S|_{(U\otimes U)^G}\bigr).$$

The decomposition into the $+1$ and -1 flip eigenspaces yields

$$\nu(U) = \dim(\operatorname{Sym}^2 U)^G - \dim(\textstyle\bigwedge^2 U)^G.$$

The tensor–Hom isomorphism gives $(U \otimes U)^G \cong \operatorname{Hom}_G(U^*, U)$. Its dimension is zero if $U^* \not\cong U$ and one otherwise. An invariant bilinear form on U belongs to $(U^* \otimes U^*)^G$, the dual of the invariant space by Haar averaging. The flip on this dual has the same sign. Each nonzero form is nondegenerate since its radical is a proper invariant subspace. The three outcomes are therefore 0 for a non-self-dual U, $+1$ for a nondegenerate invariant symmetric complex bilinear form, and -1 for a nondegenerate invariant alternating one.

For $G = U(1)$ an irreducible of weight n has $\chi(z^2) = z^{2n}$. The indicator is 1 for $n = 0$ and 0 for $n \neq 0$. The defining representation of $SU(2)$ preserves the nondegenerate alternating form $(v, w) \mapsto \det(v, w)$, hence has indicator -1. Here the additional trace has detected how the self-duality intertwiner behaves under interchange of its arguments.

CHAPTER 6

Compact induction and reciprocity

An H-representation can be transported around a compact homogeneous space G/H by allowing its vectors to vary equivariantly over G. Restriction moves in the other direction and leaves the underlying vector space intact. The adjunction between the two is a precise sense in which induction packages the solutions of all restriction problems. We use finite-dimensional unitary test representations and normalized Haar measure, following the compact setup of [22, §8].

6.1 Covariant functions and their completion

Let H be a closed subgroup of a compact group G, with σ a finite-dimensional unitary representation of H on W. The space of continuous covariant functions is

$$\mathcal{C}(G,W)^H = \{f \in C(G,W) : f(gh) = \sigma(h)^{-1} f(g)\}.$$

These are continuous sections of the quotient $(G \times W)/H \to G/H$, for the right action $(g,w)h = (gh, \sigma(h)^{-1}w)$. Indeed a section takes gH to $[g, f(g)]$. Each $[g,w]$ determines a unique w over a chosen representative g, and the relation under changing representatives is the covariance rule. Continuity of the recovered function f follows by restricting to a closed ball in W containing its values. The norm on the quotient is continuous, so such a ball exists by compactness of the image of the section. On that ball, the map from G times the ball to the pullback of the quotient along $G \to G/H$ is a continuous bijection from compact to Hausdorff.

Equip this function space with the L^2 norm from Haar measure on G. The pushforward measure on G/H gives the same norm since $g \mapsto \|f(g)\|$ is constant on each right coset. Define the induced Hilbert representation $\operatorname{Ind}_H^G W$ to be its completion, with G acting by $L_x f(g) = f(x^{-1}g)$. Left translation preserves covariance, is unitary, and is strongly continuous by continuity on the dense continuous section space. A concrete realization of this completion is the range in $L^2(G,W)$ of the orthogonal projection

$$(P_H f)(g) = \int_H \sigma(h) f(gh)\, dh.$$

Indeed $h \mapsto [f(g) \mapsto \sigma(h) f(gh)]$ is a unitary representation of H, and its orthogonal averaging projection is P_H. The representation commutes with left

translation. The projection maps continuous functions to continuous covariant functions, and the continuous functions are dense in $L^2(G, W)$. Its image is therefore exactly the completion defining induction.

6.2 Reciprocity with a finite-dimensional test

For $H \leq G$, write Res_H^G for restriction. Hom spaces with an induced representation as source or target consist of bounded intertwiners. The finite-dimensional test in the next theorem specifies where evaluation is legitimate.

Theorem 6.1 (Compact reciprocity)**.** *Let V and W be finite-dimensional unitary representations of G and H, respectively. There is a natural complex-linear bijection*

$$\mathrm{Hom}_G(V, \mathrm{Ind}_H^G W) \cong \mathrm{Hom}_H(\mathrm{Res}_H^G V, W).$$

Its inverse sends A to T_A with $(T_A v)(g) = A(\rho_V(g^{-1})v)$.

Proof. The formula for T_A gives a continuous function. H-equivariance of A gives

$$(T_A v)(gh) = A(\rho_V(h^{-1})\rho_V(g^{-1})v) = \sigma(h)^{-1}(T_A v)(g).$$

The equality $T_A \rho_V(x) = L_x T_A$ follows by multiplying inside the formula. Boundedness also follows directly from $\|T_A v\|_{L^2} \leq \|A\| \, \|v\|$.

To construct the forward map, first consider an irreducible G-representation U and the isotypic projector on $L^2(G, W)$, given by the integrated central function p_U of Chapter 5. Integration against left translation, followed by change of variable $y = x^{-1}g$, writes it as

$$f \longmapsto \left[g \longmapsto \int_G p_U(gy^{-1}) f(y) \, dy \right].$$

This continuous representative exists for every L^2 input. In fact $f \in L^1(G, W)$, and uniform continuity of the kernel on $G \times G$ proves continuity in g. The operator commutes with the right covariance projection. If its input is covariant in L^2, its continuous representative satisfies the covariance pointwise, because for each fixed h equality almost everywhere between the two continuous functions implies equality everywhere by full support of Haar measure.

For any intertwiner $T : V \to \mathrm{Ind}_H^G W$, the image belongs to the sum of the finitely many isotypic parts with types in V. Applying the corresponding finite sum of projectors shows that Tv has a unique continuous covariant representative. Define $Av = (Tv)(1)$. For $h \in H$, we compute

$$A\rho_V(h)v = (L_h T v)(1) = (Tv)(h^{-1}) = \sigma(h)Av.$$

Evaluation of $T\rho_V(g^{-1})v = L_{g^{-1}}Tv$ at the identity proves $T = T_A$. This identification respects precomposition by G-maps on V and postcomposition by H-maps on W directly from the displayed formulas. □

The intertwining identity can be visualized on continuous representatives as follows.

$$\begin{array}{ccc} V & \xrightarrow{T_A} & \mathcal{C}(G,W)^H \\ {\scriptstyle \rho_V(g^{-1})}\downarrow & & \downarrow{\scriptstyle L_{g^{-1}}} \\ V & \xrightarrow{T_A} & \mathcal{C}(G,W)^H \\ & \searrow^{A} & \downarrow{\scriptstyle \mathrm{ev}_1} \\ & & W. \end{array}$$

On the top right the composite downward is evaluation at g. Taking Hilbert adjoints on both sides of the bijection gives a second complex-linear bijection, $\mathrm{Hom}_G(\mathrm{Ind}_H^G W, V) \cong \mathrm{Hom}_H(W, \mathrm{Res}_H^G V)$. There are two conjugate-linear operations, so the final operation is complex-linear.

For $H = \{1\}$ and $G = U(1)$ the function $N^{-1/2}\sum_{n=0}^{N-1} z^n$ has norm one and value $\sqrt{N}$ at the identity. This rules out extending evaluation to a bounded functional on the whole induced Hilbert space. The continuous representatives in the proof arise from the finite-dimensional test, which is what makes the Hom formula meaningful.

6.3 Restriction to finitely many double cosets

Let K be a closed subgroup of G and assume $K\backslash G/H$ is finite. Each K-orbit $O_g = KgH/H$ is compact and hence closed in G/H. Its complement is a finite union of closed orbits, so O_g is open as well and has positive measure. Define

$$J_g = K \cap gHg^{-1}, \qquad \sigma_g(j) = \sigma(g^{-1}jg) \quad (j \in J_g).$$

Then there is a unitary decomposition

$$\mathrm{Res}_K^G \mathrm{Ind}_H^G W \cong \bigoplus_{KgH \in K\backslash G/H} \mathrm{Ind}_{J_g}^K W_g,$$

where W_g is W with the J_g-action σ_g. To see it, restrict a continuous section on O_g along $k \mapsto kg$. Its restriction $a(k) = f(kg)$ satisfies $a(kj) = \sigma_g(j)^{-1}a(k)$. Conversely the right covariance extends such an a by $f(kgh) = \sigma(h)^{-1}a(k)$. If $kgh = k'gh'$, then $k' = kj$ with $j \in J_g$ and $h = (g^{-1}jg)h'$. The covariance of a makes the formulas agree. Continuity follows since the continuous surjection $K \times H \to KgH$ from compact to Hausdorff is a quotient map. One extends by zero off the clopen orbit.

The restriction of quotient measure to O_g, divided by its total mass, is the K-invariant probability on K/J_g. Uniqueness follows by averaging a continuous function over K, which makes it constant on the homogeneous space. Thus the map $f \mapsto \mu(O_g)^{1/2} a$ preserves the Hilbert norm on this summand, and summing gives the claimed isomorphism. In the finite-group case this is the full double-coset decomposition. The finiteness hypothesis here gave an orthogonal sum of clopen pieces with nonzero weights.

For $W = \mathbb{C}$ trivial, the multiplicity of V in $L^2(G/H)$ is $\dim V^H$, by reciprocity and the duality of invariant functionals and vectors. For finite G, the character of this transitive permutation representation at x is the number of cosets fixed by x.

For a particular example, let $G = S_3$ and let $H = K$ be generated by the transposition (12). There are two K-orbits on the three cosets. The base coset has stabilizer K, and the remaining two form an orbit with trivial stabilizer. For W trivial, the restriction of $L^2(G/H)$ to K is $\mathbb{C} \oplus L^2(K)$. The second summand splits into the constant and the sign lines. This agrees with the restriction of $\mathbb{C} \oplus V$ from the permutation action, since on the coordinate-sum-zero subspace V the transposition has eigenvalues $1, -1$.

One can also induce the sign line of H. Its induced representation contains the sign line of G once, contains the trivial line zero times, and contains V once. These values follow by inspecting precisely the -1 eigenspaces in the three restrictions to H and applying reciprocity. The induction formula can thus be used without selecting a basis for the induced representation.

CHAPTER 7

Rotation groups and their tensor rules

The representations of $SU(2)$ are indexed by a single nonnegative integer. Their tensor products already give a nontrivial family of decompositions with a closed multiplicity formula. Passing to $SO(3)$ keeps precisely the representations with even index, and considering $SU(2)$ relative to its diagonal torus selects the same family by a different criterion. Classical treatments appear in [22, §10] and [13].

7.1 Highest weights and symmetric powers

Write T for the diagonal torus of $SU(2)$ with elements $t(z) = \mathrm{diag}(z, z^{-1})$, where $|z| = 1$. A finite-dimensional unitary representation V restricts to a sum of torus weight spaces $V_j = \{v : \rho(t(z))v = z^j v\}$ with $j \in \mathbb{Z}$. The differential of the action, complexified, is an action of $\mathfrak{sl}_2(\mathbb{C})$. For the usual diagonal matrix $h = \mathrm{diag}(1, -1)$ and raising and lowering matrices $e = E_{12}$, $f = E_{21}$,

$$[h, e] = 2e, \qquad [h, f] = -2f, \qquad [e, f] = h.$$

The operator h has eigenvalue j on V_j. The first two relations give $eV_j \subset V_{j+2}$ and $fV_j \subset V_{j-2}$.

Let V be irreducible, n its greatest torus weight, and $v \in V_n$ nonzero. Then $ev = 0$. Induction using the commutators gives

$$hf^j v = (n - 2j)f^j v, qquad ef^j v = j(n - j + 1)f^{j-1}v \quad (j \geq 1).$$

For the second identity apply $ef = fe + h$ to $f^{j-1}v$. If $f^r v \neq 0$ and $f^{r+1}v = 0$, the identity forces $(r + 1)(n - r) = 0$, so $r = n$. The span of $v, fv, \ldots, f^n v$ is invariant under h, e, f, hence under the connected group. It equals V by irreducibility. These vectors are independent by their distinct weights, and the formulas completely specify the action. In $\mathrm{Sym}^n \mathbb{C}^2$, the highest weight vector e_1^n generates the full monomial basis by repeated lowering. This representation is irreducible since any nonzero submodule has a greatest weight vector, and the only monomial weight line killed by e is the one spanned by e_1^n. We write $V_n = \mathrm{Sym}^n \mathbb{C}^2$ for this irreducible, now using the subscript to index irreducibles.

The weight diagram for V_3 records the lowering chain and the reverse action of e in the basis of iterated lowers.

$$3 \underset{3e_0}{\overset{f}{\rightleftarrows}} 1 \underset{4e_0}{\overset{f}{\rightleftarrows}} -1 \underset{3e_0}{\overset{f}{\rightleftarrows}} -3.$$

Here e_0 denotes the map sending the selected basis vector in the source line to the selected basis vector in the destination line. Thus $3e_0$, $4e_0$, and $3e_0$ are the three restrictions of e.

7.2 Clebsch–Gordan and the representation ring

The character of V_n on the torus is the Laurent polynomial

$$\chi_n(z) = z^n + z^{n-2} + \cdots + z^{-n}, \qquad (z - z^{-1})\chi_n(z) = z^{n+1} - z^{-n-1}.$$

Every element of $SU(2)$ is conjugate into T by unitary diagonalization. A scalar change to the diagonalizing matrix can make its determinant one and leaves the conjugation intact. Thus it suffices to calculate characters on T.

Theorem 7.1 (Clebsch–Gordan)**.** *For $m, n \geq 0$,*

$$V_m \otimes V_n \cong \bigoplus_{r=0}^{\min(m,n)} V_{m+n-2r}.$$

Proof. Assume $m \leq n$. Multiplication by $z - z^{-1}$ gives

$$(z - z^{-1})\chi_m(z)\chi_n(z) = \sum_{r=0}^{m} \bigl(z^{m+n+1-2r} - z^{m-n-1-2r}\bigr).$$

The negative contributions are the list $-z^{-(m+n+1-2r)}$ in reverse order. Hence the sum is $(z - z^{-1}) \sum_{r=0}^{m} \chi_{m+n-2r}(z)$. Cancel in the Laurent polynomial ring and use the fact that the character of a compact representation determines the isomorphism class. □

The first product $V_1 \otimes V_1 \cong V_2 \oplus V_0$ has two canonical summands given by the symmetrizer and antisymmetrizer. The antisymmetric square is the determinant line and is trivial. If $x = [V_1]$ in the representation ring, the product formula gives $[V_{n+1}] = x[V_n] - [V_{n-1}]$ for $n \geq 1$. Each $[V_n]$ is therefore a monic integral polynomial in x of degree n. The classes $[V_n]$ form an additive basis, so the ring is $\mathbb{Z}[x]$.

An explicit invariant contraction realizes the lower summand in the product with V_1. Put $\varepsilon(u, v) = \det(u, v)$. The linear map $V_1 \otimes V_n \to V_{n-1}$ on decomposable inputs is

$$u \otimes (v_1 \cdots v_n) \longmapsto \sum_{i=1}^{n} \varepsilon(u, v_i) v_1 \cdots \widehat{v_i} \cdots v_n.$$

It intertwines because ε is invariant. It is nonzero since the input $e_2 \otimes e_1^n$ maps to $-ne_1^{n-1}$. Irreducibility of the target makes it surjective, and equivariance together with the tensor rule identifies its kernel with V_{n+1}. The two terms in the basic tensor rule thus have concrete descriptions by multiplication and contraction.

The first repeated multiplicity arises in $V_1^{\otimes 3} \cong V_3 \oplus V_1^{\oplus 2}$. For an explicit pair of maps to V_1, take $(u,v,w) \mapsto \varepsilon(u,v)w$ and $(u,v,w) \mapsto \varepsilon(v,w)u$. The maps are independent, as shown by first setting $u = v = e_1, w = e_2$ and then setting $u = e_2, v = w = e_1$. They thus give a basis of the two-dimensional intertwiner space. The third contraction $(u,v,w) \mapsto \varepsilon(w,u)v$ is the negative sum of the other two.

$$\varepsilon(u,v)w + \varepsilon(v,w)u + \varepsilon(w,u)v = 0.$$

The left side is alternating in the three vectors and therefore factors through $\bigwedge^3 \mathbb{C}^2 = 0$. This relation is a concrete change of linear coordinates between possible choices of intermediate pair in a triple tensor product. In the space of equivariant maps, the contraction identity displays the dependence that the tensor rule counts.

7.3 The central quotient and spherical representations

The conjugation action on the real three-dimensional space $\mathfrak{su}_2$ preserves the positive inner product $(X,Y) \mapsto -\operatorname{tr}(XY)$. The determinant is one because $SU(2)$ is connected. The derivative has kernel the center of $\mathfrak{su}_2$, which is zero, and source and target Lie algebras both have dimension three. The image in $SO(3)$ is therefore open and equals $SO(3)$ by connectedness. The kernel consists of matrices commuting with all of $\mathfrak{su}_2$. Such a matrix is scalar because it also commutes with the complex span $\mathfrak{sl}_2(\mathbb{C})$, and its determinant one condition makes it either I or $-I$. This realizes $SO(3) \cong SU(2)/\{\pm I\}$.

The central element $-I$ acts on V_n by $(-1)^n$. Pullback along the quotient identifies $SO(3)$-representations with direct sums of the V_{2k}. Their tensor rule follows by taking even indices in Clebsch–Gordan. In particular the complexified three-dimensional vector representation V_2 has tensor square $V_4 \oplus V_2 \oplus V_0$. Its three summands are respectively traceless symmetric tensors, alternating tensors, and the metric line. The even subring of $\mathbb{Z}[x]$ is $\mathbb{Z}[x^2] = \mathbb{Z}[y]$, where $y = [V_2] = x^2 - 1$.

The form $\varepsilon^{\otimes n}$ restricts to a nonzero invariant bilinear form on V_n, as shown by evaluation on e_1^n, e_2^n. Interchanging its arguments multiplies it by $(-1)^n$. The Frobenius–Schur indicator of V_n is therefore $(-1)^n$ by Chapter 5. The same parity appears in reciprocity for $T \subset SU(2)$. The space V_n^T is one-dimensional for even n and zero for odd n by the torus weight formula. The orbit of a unit diagonal vector in $\mathfrak{su}_2$ is the two-sphere with stabilizer T. Consequently $L^2(S^2)$ has exactly one copy of each V_{2k} in its Hilbert direct sum decomposition. The

torus fixed line chooses the type appearing in spherical functions, while the central quotient certifies which types can act as rotations.

More generally let L_m be the torus line on which $t(z)$ acts by z^m. By reciprocity, the multiplicity of V_n in $\mathrm{Ind}_T^{SU(2)} L_m$ is the multiplicity of weight m in V_n. The weight list yields

$$\dim \mathrm{Hom}_{SU(2)}(V_n, \mathrm{Ind}_T^{SU(2)} L_m) = \begin{cases} 1 & n \geq |m| \text{ and } n \equiv m \pmod 2, \\ 0 & \text{otherwise.} \end{cases}$$

This identifies all the irreducible summands of the induced Hilbert space. In particular the center acts on the whole induced space by the common scalar $(-1)^m$. Taking $m = 0$ recovers the scalar spherical functions above.

CHAPTER 8

Compact Tannaka reconstruction

The tensor category of finite-dimensional representations records the group element g as a coherent family of linear maps $\rho_V(g)$. The coherence includes tensor products, the unit, and every intertwiner. Tannaka reconstruction asks whether these conditions characterize the families arising from individual elements. For compact groups the answer uses unitarity and the ability of invariant tensors to separate a compact subgroup from other unitary transformations. See [22, §9] and [4] for compact duality.

8.1 The group of coherent transformations

Let $\mathrm{Rep}_{\mathrm{fd}}(G)$ contain finite-dimensional continuous unitary representations of a compact Hausdorff group G and all linear intertwiners. It is symmetric monoidal by tensor product with the usual flip and rigid by the contragredient dual. The forgetful functor

$$F : \mathrm{Rep}_{\mathrm{fd}}(G) \longrightarrow \mathrm{Hilb}_{\mathrm{fd}}$$

preserves these structures and the adjoint of every morphism. Its target has finite-dimensional complex Hilbert spaces and all linear maps.

A unitary tensor automorphism of F is a family of unitary operators $a_V : F(V) \to F(V)$ satisfying $a_{\mathbf{1}} = 1$, $a_{V\otimes W} = a_V \otimes a_W$, and $F(f)a_V = a_W F(f)$ for every intertwiner $f : V \to W$. The equalities use the structural identifications of F. Naturality for the dual evaluation $V^* \otimes V \to \mathbf{1}$ implies $a_{V^*} = (a_V^{-1})^\vee$, where the superscript $\vee$ is the algebraic linear dual. The compatibility condition is the commutativity of

$$\begin{array}{ccc} F(V)^* \otimes F(V) & \xrightarrow{a_{V^*}\otimes a_V} & F(V)^* \otimes F(V) \\ & \searrow^{\mathrm{ev}} & \downarrow{\scriptstyle \mathrm{ev}} \\ & & \mathbb{C}. \end{array}$$

Indeed evaluating at $\lambda \otimes v$ yields $(a_{V^*}\lambda)(a_V v) = \lambda(v)$. Tensor coherence and the unit also imply that every invariant tensor w in a mixed tensor power of V is fixed by its corresponding operator. The vector w is the value at 1 of an intertwiner from the trivial representation.

8.2 Invariant tensors determine a compact subgroup

Lemma 8.1. *Let K be a compact subgroup of $U(E)$ for a finite-dimensional Hilbert space E. For each $p, q \geq 0$ let $R_{p,q}$ be the representation of $U(E)$ on $E^{\otimes p} \otimes (E^*)^{\otimes q}$. Then K is the set of all $u \in U(E)$ such that $R_{p,q}(u)w = w$ for all $w \in R_{p,q}^K$ and all p, q.*

Proof. Each $u \in K$ fixes those vectors by definition. Suppose $u \notin K$. Since the compact Hausdorff right-coset space $U(E)/K$ separates uK and K by a continuous function, there is a continuous right-K-invariant scalar function b on $U(E)$ with $b(u) = 1$ and $b(1) = 0$.

Polynomials in the matrix entries of $x \in U(E)$ and their complex conjugates form a unital self-adjoint subalgebra separating points. By Stone–Weierstrass it contains an element c with $\|c - b\|_\infty < 1/3$. Products of coefficients are coefficients of tensor products, and conjugate entries are entries of the contragredient representation in the dual basis. Hence c is a finite sum of coefficients $x \mapsto \ell(R_{p,q}(x)v)$. Right averaging over K preserves the uniform bound because b is right invariant. It changes each such coefficient to

$$x \longmapsto \ell(R_{p,q}(x)w), \qquad w = \int_K R_{p,q}(k)v \, dk \in R_{p,q}^K.$$

The averaged function still takes different values at u and at 1. For at least one of its summands $R_{p,q}(u)w$ differs from w. This proves the asserted description of K. □

In the defining representation of $SU(2)$ the invariant alternating tensor spans the determinant line. Its stabilizer inside $U(2)$ is $SU(2)$, so a single invariant in the lemma suffices there. In general the lemma permits arbitrarily large tensor powers and applies to every compact matrix quotient of G.

8.3 Reconstruction and functoriality

Theorem 8.2 (Compact Tannaka reconstruction)**.** *There is a natural isomorphism of topological groups*

$$G \longrightarrow \mathrm{Aut}^{\otimes}_{\mathrm{unitary}}(F), \qquad g \longmapsto (\rho_V(g))_V.$$

The automorphism group has the topology of pointwise convergence in the unitary groups of a set of representatives of the finite-dimensional representations.

Proof. An element of G defines a coherent unitary family by the tensor product and intertwiner identities. The map is continuous. Its kernel is trivial because finite-dimensional matrix coefficients separate points by Peter–Weyl. Conversely

let a be a coherent family and fix V. Take $K = \rho_V(G) \subset U(F(V))$. The mixed tensors fixed by K are the vectors invariant under G. Naturality and dual compatibility imply that a_V satisfies every stabilizer condition in Lemma 8.1. Thus $a_V \in \rho_V(G)$.

To obtain a common element of G, consider $S_V = \{g \in G : \rho_V(g) = a_V\}$. This is closed and nonempty. Given finitely many $V_1, \dots, V_n$, use $V = \bigoplus_i V_i$. Naturality for its inclusions shows that a_V restricts to the specified a_{V_i} on every summand. It follows that $S_V \subset \bigcap_i S_{V_i}$. Compactness gives an element in all S_V and point separation makes it unique. The topological assertion follows because the continuous bijection is from a compact space to a Hausdorff space. □

For a continuous homomorphism $\varphi : G \to H$, restriction along φ is a symmetric monoidal functor $\varphi^* : \mathrm{Rep}_{\mathrm{fd}}(H) \to \mathrm{Rep}_{\mathrm{fd}}(G)$, compatible with the two forgetful functors. A family for G restricts to one for H by taking its values on the representations $\varphi^* W$. The reconstructed map sends g to $\varphi(g)$ because $(\varphi^* \rho_W)(g) = \rho_W(\varphi(g))$. In particular reconstruction is compatible with the direction reversal from restricting representations.

8.4 The role of the structure on the category

For $U(1)$ the irreducible representations are the weight lines L_n, $n \in \mathbb{Z}$. Naturality on direct sums and their inclusions determines a family from its scalars a_n on the weight lines. Tensor compatibility becomes $a_{m+n} = a_m a_n$ and $a_0 = 1$. Allowing all complex-linear invertible components gives $a_n = z^n$ with $z \in \mathbb{C}^\times$. Requiring unitary components gives $|z| = 1$. With naturality and unitarity alone, the scalars on different weight lines can be assigned independently. This example isolates the roles of tensor compatibility and the Hilbert structure.

The proof reconstructs the group from a category equipped with its concrete symmetric tensor functor. It uses the actual morphisms out of the unit, the operation of tensoring transformations, duality, and the Hilbert structure. The ordinary representation category with only its underlying linear category structure records the simple objects and their multiplicity spaces. The fusion ring adds the dimensions of tensor multiplicity spaces. The fixed tensors used in the subgroup lemma are vectors in concrete Hilbert spaces, so the dimensions by themselves do not encode the stabilizer conditions.

For example, the dihedral group of order eight and the quaternion group of order eight both have abelianization of order four isomorphic to $C_2 \times C_2$. Each consequently has four linear characters. The remaining squared dimension is four, giving a unique two-dimensional irreducible S in each case. Twisting S by a character yields S, and S is self-dual by uniqueness. Therefore $S \otimes S$ contains each linear character once by tensor–Hom duality, exhausting its dimension. The two

representation rings are isomorphic as rings with distinguished irreducible bases, while the groups have different numbers of elements of order two. The symmetric structure supplies an additional distinction in the signs of the indicators. For the dihedral group, the real planar action complexifies to S with invariant symmetric form. For the quaternion group, the two-dimensional realization in $SU(2)$ has the invariant alternating form.

There is also an abstract recognition theorem for appropriate symmetric rigid C^*-tensor categories, which addresses existence of a compact group and a fiber functor from intrinsic category axioms [9]. Here the functor is already supplied and the theorem computes its unitary automorphism group.

CHAPTER 9

Haar measure and homogeneous spaces

9.1 Translations and the modular character

Let G be a locally compact Hausdorff group. Write $C_c(G)$ for the continuous complex functions of compact support. A left Haar measure is a nonzero positive Radon measure invariant under $x \mapsto gx$. Haar's theorem gives existence and uniqueness up to multiplication by a positive scalar. It also gives positive measure to each nonempty open set. We fix a left Haar measure dx and give compact groups total measure one. On a discrete group we use counting measure when working with a group algebra. On $\mathbb{R}^n$ we use Lebesgue measure. Thus the geometry determines invariance, and a choice of scale determines the numerical integral.

Although a right translation need not preserve dx, it commutes with left translation. Haar uniqueness therefore supplies a positive constant $\Delta_G(g)$ for each $g \in G$, with the convention

$$\int_G f(xg)\,dx = \Delta_G(g)^{-1} \int_G f(x)\,dx.$$

Composing right translations gives $\Delta_G(gh) = \Delta_G(g)\Delta_G(h)$. For fixed $f \in C_c(G)$ with nonzero integral, the left-hand side varies continuously with g, by uniform continuity on a common compact support locally in g. This proves that Δ_G is a continuous character. It is called the *modular function*. Equivalently $\mu(Eg) = \Delta_G(g)\mu(E)$ for Borel sets E, and inversion has the change of variables

$$\int_G f(x^{-1})\,dx = \int_G f(x)\Delta_G(x^{-1})\,dx. \tag{9.1}$$

For the latter, the pushforward under inversion and the measure with density $\Delta_G(x^{-1})$ are both right Haar measures. If one is c times the other, applying inversion twice and using the continuity and multiplicativity of Δ_G gives $c^2 = 1$. The positivity removes the sign.

A group is *unimodular* when $\Delta_G = 1$. Every abelian group is unimodular since left and right translation agree. Every compact group is unimodular because the

image of its modular function would be a compact subgroup of the positive real numbers under multiplication, whose only compact subgroup is trivial. Discrete groups are unimodular by counting measure. These facts put compact and discrete examples within a single measure convention.

Example 9.1 (Positive affine transformations of the line)**.** Write (a, b) for $t \mapsto at + b$, with $a > 0$. The product is

$$(a, b)(a', b') = (aa', b + ab').$$

Under left multiplication by (a_0, b_0), both coordinates have Jacobian factor a_0, and $a^{-2}\,da\,db$ is invariant. Under right multiplication by (a_0, b_0), the Jacobian for $(a, b) \mapsto (aa_0, b + ab_0)$ is a_0. In the integral of $f((a, b)(a_0, b_0))$ against $a^{-2}\,da\,db$, the replacement of a by the first output coordinate contributes a_0^2 and the Jacobian contributes a_0^{-1}. Therefore $\Delta_G(a_0, b_0) = a_0^{-1}$. The disagreement between the two Haar measures is already visible in this two-dimensional Lie group.

The affine calculation extends to semidirect products $K \ltimes \mathbb{R}^d$. Let $k \mapsto A_k$ be a continuous homomorphism to $\mathrm{GL}_d(\mathbb{R})$ and put $D(k) = |\det A_k|$. Use the product $(k, v)(h, w) = (kh, v + A_k w)$. Left multiplication by (k_0, v_0) preserves left Haar measure on the first coordinate and multiplies Lebesgue measure in the second by $D(k_0)$. The function $D(k)^{-1}$ compensates for this factor. For right multiplication by (h, w), the second coordinate changes by a translation once the first coordinate is fixed. Right translation in K and the identity $D(kh^{-1})^{-1} = D(k)^{-1}D(h)$ consequently give

$$d\mu_G(k, v) = D(k)^{-1}\,dk\,dv, \qquad \Delta_{K \ltimes \mathbb{R}^d}(h, w) = \frac{\Delta_K(h)}{D(h)}.$$

For unimodular K, this product is unimodular precisely when the action on $\mathbb{R}^d$ preserves volume. The action can therefore change the modular function even when the two groups used to form the product are unimodular.

9.2 Integration along cosets

Let H be a closed subgroup. The orbit space G/H is locally compact Hausdorff and the quotient map q is open. Choose left Haar measure dh on H. Averaging over the right H-orbit defines

$$(P_H f)(xH) = \int_H f(xh)\,dh, \qquad f \in C_c(G).$$

Changing x to xh_0 invokes left invariance on H. For x in a compact neighborhood of any fixed lift, the h contributing to the integral belong to a fixed compact subset of H. This gives continuity by dominated convergence. The support of $P_H f$ is contained in $q(\operatorname{supp} f)$, and P_H commutes with left translation.

Theorem 9.2 (Invariant measure on a quotient)**.** *The space G/H carries a nonzero G-invariant Radon measure if and only if $\Delta_G|_H = \Delta_H$. Under this condition there is exactly one such measure $d\nu$ satisfying*

$$\int_G f(x)\,dx = \int_{G/H} P_H f(y)\,d\nu(y) \qquad (f \in C_c(G)). \tag{9.2}$$

Proof. For each compact $K \subset G/H$ choose nonnegative compactly supported functions on G whose averages are positive on neighborhoods covering K. At a point of K, take any bump positive at a lift and use the full support of Haar measure on H. A finite sum k_0 has $P_H k_0 > 0$ on an open set containing K. Choose a continuous cutoff supported in that open set and equal to one on K. Multiplying k_0 by the pullback of this cutoff divided by $P_H k_0$ gives $k \geq 0$ in $C_c(G)$ with $P_H k = 1$ on K. In particular, $\varphi \in C_c(G/H)$ has a lift $k(\varphi \circ q)$, nonnegative if $\varphi \geq 0$.

If ν is invariant, the positive functional $f \mapsto \int P_H f\,d\nu$ is left invariant and nonzero, so it is a positive multiple of Haar integration. For $h \in H$, the group integral of $f(-h)$ scales by $\Delta_G(h)^{-1}$. Its quotient integral scales by $\Delta_H(h)^{-1}$, as is seen inside P_H. Comparing the two forces equality of the modular functions on H.

Conversely, suppose they agree on H. For $f, k \in C_c(G)$ one has

$$\int_G f(x) P_H k(xH)\,dx = \int_G k(y) P_H f(yH)\,dy.$$

Indeed, in the iterated integral on the left, the substitution $y = xh$ produces $\Delta_G(h)^{-1}$; then $h \mapsto h^{-1}$ produces the inverse factor by (9.1) in H. The integrals are absolutely convergent since the contributing elements h lie in $H \cap (\operatorname{supp} f)^{-1} \operatorname{supp} k$. For $P_H f = 0$, take k with $P_H k = 1$ on $q(\operatorname{supp} f)$ to see that the group integral of f vanishes. Consequently integration factors through P_H.

$$\begin{array}{ccc} C_c(G) & \xrightarrow{\;P_H\;} & C_c(G/H) \\ & \searrow^{\int_G} \quad \swarrow_{\int_{G/H}} & \\ & \mathbb{C} & \end{array}$$

Nonnegative lifts make the descended functional positive, and equivariance of P_H makes it invariant. The Riesz representation theorem supplies ν. Surjectivity of P_H gives uniqueness with the prescribed normalization. □

When H is compact the criterion always holds. When H is closed and normal it also holds. To see the latter, integrate $P_H f$ against left Haar measure of the quotient group. The resulting left invariant positive functional on G is nonzero by the lift construction. Rescaling Haar on the quotient establishes (9.2). This includes $G = \mathbb{R}^{n+k}$, $H = \mathbb{R}^k$ with its usual embedding, where the formula reduces to integration in separate variables.

9.3 A quotient with a density

For a general closed subgroup one uses a positive continuous ρ-function satisfying

$$\rho(xh) = \rho(x)\frac{\Delta_H(h)}{\Delta_G(h)}.$$

The quotient integration theorem gives such a function and a Radon measure ν_ρ with

$$\int_G f(x)\rho(x)\,dx = \int_{G/H} P_H f(y)\,d\nu_\rho(y)$$

[11, 22]. The ratio $\rho(g^{-1}x)/\rho(x)$ is constant on each right H-coset and is the Radon–Nikodym derivative of $g_*\nu_\rho$ with respect to ν_ρ at xH. To check the formula, integrate a test average $P_H f$ against the pushed measure. Equivariance of P_H replaces the integrand on G by $f(gx)\rho(x)$; changing variables by left translation replaces the density by $\rho(g^{-1}x)$. This is quasi-invariance and is enough to construct unitary actions on quotient L^2-spaces by multiplying the translated functions by the square root of the derivative.

In the affine group, take the dilation subgroup $H = \{(a, 0) : a > 0\}$. The quotient coordinate is b, with $(a_0, b_0) \cdot b = b_0 + a_0 b$. The modular criterion fails because H is abelian and $\Delta_G(a, 0) = a^{-1}$. With $\rho(a, b) = a$, the measures separate into da/a along the coset and Lebesgue measure db on the quotient. The derivative of the pushforward under (a_0, b_0) is a_0^{-1}. Thus the factor $a_0^{-1/2}$ makes the pullback along the inverse affine map a unitary operator on $L^2(\mathbb{R}, db)$.

CHAPTER 10

Integrated representations and group algebras

10.1 Convolution and integrated operators

For a locally compact group G, a unitary representation is a homomorphism $\pi : G \to U(\mathcal{H})$ with $g \mapsto \pi(g)u$ continuous in Hilbert space norm for every $u \in \mathcal{H}$. Its integrated form is

$$\pi(f)u = \int_G f(x)\pi(x)u\,dx, \qquad f \in L^1(G).$$

For compactly supported continuous f this is a norm integral on each vector, with $\|\pi(f)u\| \le \|f\|_1\|u\|$. Density gives the unique extension to $L^1(G)$. The terminology refers to this representation on $\mathcal{H}$ of the convolution algebra, whose structure is

$$(f * h)(x) = \int_G f(y)h(y^{-1}x)\,dy, qquadf^*(x) = \Delta_G(x^{-1})\overline{f(x^{-1})}.$$

Fubini's theorem and left translation yield $\|f*h\|_1 \le \|f\|_1\|h\|_1$, and the inversion formula (9.1) gives $\|f^*\|_1 = \|f\|_1$. By the same change of variables,

$$\pi(f * h) = \pi(f)\pi(h), \qquad \pi(f^*) = \pi(f)^*.$$

The two factors in the involution have separate origins. Complex conjugation and inversion describe the adjoint of the integrand, and $\Delta_G(x^{-1})$ is the density for inversion of the measure.

There are nonnegative $e_U \in C_c(G)$ of mass one supported in arbitrarily small neighborhoods U of the identity. For the first approximate identity relation,

$$\|e_U * f - f\|_1 \le \sup_{x \in U} \|f(x^{-1}\,-) - f\|_1 \longrightarrow 0.$$

Continuity of the L^1 translation action follows first on $C_c(G)$, using uniform control on a common compact support, and then on $L^1(G)$, using isometricity. To obtain a two-sided approximate identity, start with symmetric neighborhoods and replace e_U by $(e_U + e_U^*)/2$, which retains the stated properties. On every vector, strong continuity of π implies $\pi(e_U)u \to u$. Hence the integrated form is nondegenerate, meaning that the linear span of $\pi(L^1(G))\mathcal{H}$ is dense.

10.2 Recovering the group action

Theorem 10.1 (Integration and reconstruction)**.** *The operation of integration induces a bijection between strongly continuous unitary representations of G and nondegenerate bounded $*$-representations of $L^1(G)$ on a fixed Hilbert space. A bounded linear map intertwines two group representations exactly when it intertwines their integrated forms.*

Proof. Let σ be nondegenerate and write $L_g f(x) = f(g^{-1}x)$. On the span of vectors $\sigma(f)u$, set

$$U_g\Big(\sum_i \sigma(f_i)u_i\Big) = \sum_i \sigma(L_g f_i)u_i.$$

The identity

$$(L_g f)^* * (L_g h) = f^* * h$$

holds because $(f^* * h)(x) = \int_G \overline{f(y)}h(yx)\,dy$, in which translation of both inputs changes only the integration variable. To obtain this last formula from the definition of convolution, use inversion and the modular factor in f^*. Taking the norm squared of each finite sum now gives the same result before and after the proposed action. This makes U_g well defined and isometric; $U_{g^{-1}}$ is its inverse. Also $U_g U_h = U_{gh}$ on the dense subspace, so the extensions form a unitary representation. For a vector $\sigma(f)u$, continuity follows from

$$\|(U_g - U_h)\sigma(f)u\| \le \|\sigma\|\,\|L_g f - L_h f\|_1 \|u\|.$$

Density and the uniform bound $\|U_g\| = 1$ extend continuity to all vectors.

For $f, h \in L^1(G)$ and $u \in \mathcal{H}$, integrating $U_g\sigma(h)u$ against $f(g)\,dg$ gives $\sigma(f * h)u = \sigma(f)\sigma(h)u$. Thus the integrated form of U equals σ on a dense subspace. If one starts with a group representation π, the substitution $x = gy$ shows $\pi(g)\pi(f) = \pi(L_g f)$, so reconstruction recovers π. An intertwiner between group representations intertwines their integrals. In the opposite direction, the reconstruction formula on the dense span yields the intertwining identity for every g. □

In fact, any algebraic $*$-homomorphism from $L^1(G)$ into bounded operators is contractive. Extend it to a unital homomorphism on the unitization by adjoining the identity operator. An inverse in the unitization maps to an inverse, giving spectral inclusion. Applied to the positive operator $\sigma(f)^*\sigma(f)$, this bounds its norm by the spectral radius of $f^* * f$, which in turn is bounded by $\|f^* * f\|_1 \le \|f\|_1^2$. This supplies continuity even when the algebraic assignment is all that is initially known.

Conjugation of the group acts on its convolution algebra by the operators

$$(\alpha_g f)(y) = \Delta_G(g) f(g^{-1}yg).$$

Indeed, the pushforward of left Haar measure under $x \mapsto gxg^{-1}$ has constant density $\Delta_G(g)$, and the formula is the density for the pushed measure $f(x)\,dx$. Integration turns this conjugation action into a commutative square.

$$\begin{array}{ccc} \mathcal{H} & \xrightarrow{\sigma(f)} & \mathcal{H} \\ \downarrow{\scriptstyle U_g} & & \downarrow{\scriptstyle U_g} \\ \mathcal{H} & \xrightarrow{\sigma(\alpha_g f)} & \mathcal{H} \end{array}$$

Conjugation by U_g replaces each U_x in the integral by $U_{gxg^{-1}}$, yielding $U_g \sigma(f) U_g^{-1} = \sigma(\alpha_g f)$.

10.3 Full and reduced completions

The full norm on $L^1(G)$ is $\|f\|_{\mathrm{u}} = \sup_\pi \|\pi(f)\|$, where the supremum ranges over strongly continuous unitary representations. It is bounded by $\|f\|_1$. The left regular representation $(\lambda(g)u)(x) = u(g^{-1}x)$ on $L^2(G)$ is strongly continuous by density of $C_c(G)$. Its integrated form is convolution on the left, and it is faithful on $L^1(G)$. Indeed if $\lambda(f) = 0$, then $f * e_U = 0$ for every $e_U \in C_c(G) \subset L^2(G)$. The right approximate identity in L^1 then gives $f = 0$. The same argument proves that $\|f\|_{\mathrm{r}} = \|\lambda(f)\|$ is a norm.

Each of these norms satisfies the C^*-identity and hence defines a completion. In the reduced case the completion is directly realized as a closed operator algebra on $L^2(G)$. For the full norm, the equality follows from taking the supremum of the operator identities $\|\pi(f)^*\pi(f)\| = \|\pi(f)\|^2$. The inequality $\|f\|_{\mathrm{r}} \le \|f\|_{\mathrm{u}}$ and the closed-range property of $*$-homomorphisms give the surjection

$$\begin{array}{ccc} L^1(G) & \longrightarrow & C^*(G) \\ & \searrow^{\lambda} & \downarrow \\ & & C_r^*(G). \end{array}$$

The full algebra controls all nondegenerate unitary representations by reconstruction. The reduced algebra controls the ones continuous for the regular norm. In particular, this distinction concerns which representations are admitted by a completion of the same convolution operation.

For a discrete group with counting measure, point masses satisfy $\delta_g * \delta_h = \delta_{gh}$ and $\delta_g^* = \delta_{g^{-1}}$. Both completions are unital and generated by their corresponding unitary point masses. A family of unitaries obeying the multiplication law defines a representation of the full algebra by summing absolutely over the group, and every unital representation of the full algebra is obtained this way. For a nondiscrete group, integration applies to densities, while reconstruction still produces the individual unitaries on every nondegenerate representation. Thus

the reconstruction formula gives the role of a point mass even when the algebra is presented entirely through integrable functions.

A locally compact group is amenable when there exists a left invariant mean on $L^\infty(G)$. The amenability criterion for group C^*-algebras says that $C^*(G) \to C_r^*(G)$ is an isomorphism exactly when G is amenable [11]. The trivial representation defines the scalar functional $f \mapsto \int_G f(x)\,dx$ on the full algebra. Whether it factors through the reduced algebra already detects amenability.

Example 10.2 (A strict norm inequality for a free group)**.** Let $G = F_2$ have free generators s, t, and let f put mass $1/4$ at each of s, t, s^{-1}, t^{-1}. The trivial representation and the L^1 bound give $\|f\|_{\mathrm{u}} = 1$. On $\ell^2(F_2)$, $4\lambda(f)$ is the adjacency operator of the four-regular tree. Root the tree at the identity. Each unoriented edge from a parent to a child contributes to the absolute value of its quadratic form at most $2|u(\text{parent})||u(\text{child})|$. Bound this by $c|u(\text{parent})|^2 + c^{-1}|u(\text{child})|^2$, with $c = 1/\sqrt{3}$. A nonroot vertex occurs three times as a parent and once as a child, with total weight $2\sqrt{3}$. The root has weight $4/\sqrt{3}$. The adjacency operator is selfadjoint, so the bound on its quadratic form gives $\|\lambda(f)\| \le \sqrt{3}/2$. Consequently the trivial representation of F_2 lies outside the reduced completion.

CHAPTER 11

Intertwiners, weak containment, and discrete series

11.1 Schur's lemma with domains

In a unitary representation, the orthogonal complement of a closed invariant subspace is invariant. Thus irreducibility is equivalent to the absence of nontrivial commuting orthogonal projections. If a bounded operator commutes with the representation, its adjoint and selfadjoint parts also commute. The spectral projections of the selfadjoint parts commute by the spectral theorem, so each part is scalar. This proves the endomorphism assertion of Schur's lemma. More generally, if a bounded T intertwines irreducible π_1, π_2, then T^*T and TT^* are scalar. When $T \neq 0$, it is a positive multiple of a unitary intertwiner.

A densely defined operator A on $\mathcal{H}$ commutes with π in the domain sense if

$$\pi(g)\mathcal{D}(A) = \mathcal{D}(A), \qquad A\pi(g)u = \pi(g)Au \quad (u \in \mathcal{D}(A)).$$

For a closed operator this says that its graph is invariant in $\mathcal{H} \oplus \mathcal{H}$ for the diagonal unitary action. Equality of the translated domains is essential to adjointing the relation. If A is closed and densely defined, then A^* also commutes in this sense. In particular $|A| = (A^*A)^{1/2}$ has spectral projections commuting with π. For irreducible π these projections are zero or the identity. Since the projections of $|A|$ for $[0, n]$ increase strongly to the identity, one of them is already the identity on a nonzero representation. Hence $|A|$ is bounded and its domain, which equals $\mathcal{D}(A)$, is all of $\mathcal{H}$. Bounded Schur's lemma then makes A scalar. This argument also applies to the positive operator A^*A when A intertwines π with a unitary representation on a different Hilbert space.

11.2 Weak containment and matrix coefficients

Write $\pi \prec \sigma$ if $\|\pi(f)\| \leq \|\sigma(f)\|$ for every $f \in L^1(G)$. The same inequality then holds on $C^*(G)$. It is equivalent to

$$\ker \sigma \subseteq \ker \pi$$

for the integrated forms on $C^*(G)$. Indeed the norm inequality gives the kernel inclusion. Conversely the inclusion makes π factor through the quotient

$C^*(G)/\ker\sigma$. This quotient is isometric to the range of σ, and the resulting $*$-homomorphism is contractive. A closed invariant copy of π in σ entails weak containment by taking norms on that subspace.

With the inner product linear in its first variable, set $c_u(g) = \langle \pi(g)u, u\rangle$. The matrix coefficient characterization says that $\pi \prec \sigma$ if and only if every such c_u is a limit, uniformly on compact subsets, of finite sums of diagonal coefficients of σ [11]. The implication from approximation to the norm inequality can be seen directly. For $f \in C_c(G)$, integrate the approximating sums against $f^* * f$. The resulting numbers are bounded by $\|\sigma(f)\|^2$ times the sums of squared norms of the coefficient vectors. These sums converge to $\|u\|^2$ by approximation at the identity. It follows that $\|\pi(f)u\| \le \|\sigma(f)\|\|u\|$, and density treats $f \in L^1(G)$.

For $G = \mathbb{Z}$, let $u_N \in \ell^2(\mathbb{Z})$ be the indicator of $[-N, N]$ normalized to have norm one. Then

$$\langle \lambda(k)u_N, u_N\rangle = \frac{\max(2N + 1 - |k|, 0)}{2N + 1}.$$

These coefficients tend to one uniformly on finite subsets. The trivial representation is weakly contained in the regular one, while an invariant vector in the regular representation would be a constant sequence of finite squared sum and hence zero.

11.3 Square-integrable irreducibles

An irreducible unitary representation is in the *discrete series* if it embeds as a closed invariant subspace of the left regular representation. In a unimodular group its coefficients admit a criterion and a uniform orthogonality law [22, 11].

Theorem 11.1 (Formal degree)**.** *Suppose G is unimodular and π is irreducible. Then π belongs to the discrete series if and only if a coefficient $x \mapsto \langle u, \pi(x)v\rangle$ with $u, v \neq 0$ belongs to $L^2(G)$. In this case there is a unique $d_\pi > 0$, depending on the scale of Haar measure, with*

$$\int_G \langle u, \pi(x)v\rangle\, \overline{\langle u', \pi(x)v'\rangle}\, dx = d_\pi^{-1}\langle u, u'\rangle\langle v', v\rangle.$$

Proof. Assume first that the coefficient for $u_0, v_0 \neq 0$ is square integrable. Fix v_0 and set $W_{v_0}u(x) = \langle u, \pi(x)v_0\rangle$ on its maximal domain in $\mathcal{H}$ of vectors for which the coefficient is in L^2. It intertwines π with left translation on its domain, since

$$W_{v_0}(\pi(g)u)(x) = W_{v_0}u(g^{-1}x).$$

The domain contains the orbit span of u_0, which is dense by irreducibility. Convergence of u_n in Hilbert space implies pointwise convergence of its coefficients,

and norm convergence in L^2 admits an almost everywhere convergent subsequence. Hence W_{v_0} is closed. Its positive operator $W_{v_0}^* W_{v_0}$ has spectral projections commuting with π. The domain version of Schur's argument makes this operator a scalar multiple of the identity. The scalar is positive because the span of the orbit of v_0 is dense and a continuous coefficient that vanishes almost everywhere vanishes everywhere. Scaling W_{v_0} now gives an isometry realizing a closed subrepresentation of λ.

$$\begin{array}{ccc} \mathcal{H}_\pi & \xrightarrow{\pi(g)} & \mathcal{H}_\pi \\ \uparrow{\scriptstyle W_{v_0}} & & \uparrow{\scriptstyle W_{v_0}} \\ L^2(G) & \xrightarrow{\lambda(g)} & L^2(G). \end{array}$$

For each fixed u, the operator $v \mapsto \overline{\langle u, \pi(x)v\rangle}$, with its maximal square-integrability domain, is linear and closed for the same reason. Replacing v with $\pi(h)v$ right-translates the output by h. These right translations are unitary by unimodularity. The domain contains the orbit span of v_0, so the spectral projection argument gives a norm squared proportional to $\|v\|^2$, with constant depending on u. Evaluation at v_0 shows that the constant is $c\|u\|^2$ for a common $c > 0$. Polarizing in u and v gives the stated law with $d_\pi = c^{-1}$.

Conversely, let $J : \mathcal{H} \to L^2(G)$ embed π into λ isometrically. Take $w \in C_c(G)$ with $J^* w \neq 0$, which exists by density. For $u \neq 0$,

$$\langle u, \pi(x) J^* w\rangle = \langle Ju, \lambda(x) w\rangle = (Ju * w^*)(x).$$

The last identity follows by substituting the definition of w^* for a unimodular group. Convolution on the right with w^* is bounded on $L^2(G)$ by $\|w\|_1$, using Minkowski's inequality and invariance under right translations. This supplies the square-integrable coefficient. □

The coefficient embeddings also recover the multiplicity of a discrete-series representation in the closed subspace generated by its coefficients. Choose an orthonormal basis $(v_i)_{i \in I}$ of $\mathcal{H}_\pi$. By the formal degree identity, the ranges of $d_\pi^{1/2} W_{v_i}$ in $L^2(G)$ are mutually orthogonal isometric copies of π. Their closed sum is the closure of all coefficients of π, since expansion of an arbitrary v in the basis converges in the norm controlling $W_v u$. If π' is another discrete-series representation, the operator $W_v^* W_{v'}$ between the two Hilbert spaces intertwines π' and π. Schur's lemma forces it to vanish when the representations are inequivalent. The orthogonality of their coefficient spaces thus follows from the group action on $L^2(G)$.

For compact G with mass one, summing the coefficient norms over the first index of an orthonormal basis shows that $d_\pi = \dim \pi$. Rescaling Haar measure

by $a > 0$ rescales the formal degree by a^{-1}. A noncompact locally compact group has infinite Haar measure, since one can choose infinitely many disjoint left translates of a relatively compact identity neighborhood recursively. An irreducible of a locally compact abelian group is one-dimensional by Schur's lemma, and its coefficient of two unit vectors has modulus one. Thus a noncompact locally compact abelian group has empty discrete series even though its regular representation has a full spectral decomposition.

CHAPTER 12

Spectral decomposition and abelian Plancherel theory

12.1 Pontryagin duality and the convolution spectrum

Let A be a second-countable locally compact abelian group. Its Pontryagin dual $\widehat{A}$ is the group of continuous characters $\chi : A \to \mathbb{T}$, with pointwise multiplication and the compact-open topology. Pontryagin duality asserts that $\widehat{A}$ is locally compact and that the evaluation map

$$A \longrightarrow \widehat{\widehat{A}}, \qquad x \longmapsto [\chi \mapsto \chi(x)]$$

is an isomorphism of topological groups. A compact abelian group has discrete dual and a discrete abelian group has compact dual [11]. For $\mathbb{R}$ the characters are $x \mapsto e^{2\pi i x\xi}$, with $\xi \in \mathbb{R}$. For $\mathbb{Z}$ the characters are $n \mapsto z^n$, with $z \in \mathbb{T}$. For $\mathbb{T}$ they are $z \mapsto z^n$, indexed by $n \in \mathbb{Z}$. These dualities connect the topology of a group to the frequencies available to its representations.

Choose the Fourier transform with conjugation in the integral,

$$\widehat{f}(\chi) = \int_A f(x)\overline{\chi(x)}\,dx.$$

For L^1 inputs the equalities

$$\widehat{f * h} = \widehat{f}\,\widehat{h}, qquad\widehat{f^*} = \overline{\widehat{f}}$$

follow by translation and inversion. The full group C^*-algebra is commutative. Each of its characters corresponds by Chapter 10 to a one-dimensional unitary representation of A, which we label by its inverse χ to agree with the transform. These evaluations identify the spectrum with $\widehat{A}$, including topology. Compact-open convergence implies convergence on $C_c(A)$ by uniform convergence on supports and on $L^1(A)$ by its norm bound. Conversely let evaluations converge on the algebra, and choose f with $\widehat{f}(\chi) \neq 0$ at the limiting character. On a compact set of x the translates $L_x f$ form a compact set in $L^1(A)$. Evaluation functionals have L^1-norm bounded by one, so a finite-net argument makes convergence uniform on that set of translates. Dividing by the convergent nonzero denominator in

$$\overline{\chi(x)} = \frac{\widehat{L_x f}(\chi)}{\widehat{f}(\chi)}$$

gives compact-open convergence. The commutative Gelfand theorem now extends the Fourier transform to $C^*(A) \cong C_0(\widehat{A})$, with the full norm equal to the supremum of $|\widehat{f}|$.

If $B \subseteq A$ is a closed subgroup, duality sends the quotient to the annihilator

$$B^\perp = \{\chi \in \widehat{A} : \chi|_B = 1\}.$$

Every character trivial on B descends continuously by the quotient topology. The character extension theorem ensures that $\widehat{A} \to \widehat{B}$ is surjective and realizes $\widehat{B}$ as the topological quotient $\widehat{A}/B^\perp$ [11]. Inclusion and quotient exchange roles in the dual pair of short exact sequences.

$$0 \longrightarrow B \hookrightarrow A \twoheadrightarrow A/B \longrightarrow 0$$

$$0 \longrightarrow \widehat{A/B} \hookrightarrow \widehat{A} \twoheadrightarrow \widehat{B} \longrightarrow 0$$

12.2 From spectral measures to direct integrals

Let π be a strongly continuous unitary representation of A on a separable Hilbert space $\mathcal{H}$. Its integrated form on $C_0(\widehat{A})$ is nondegenerate. The spectral theorem gives a unique projection-valued Borel measure E on $\widehat{A}$, normalized by $E(\widehat{A}) = I$, with

$$\pi(f) = \int_{\widehat{A}} \widehat{f}(\chi)\, dE(\chi), \qquad \pi(x) = \int_{\widehat{A}} \overline{\chi(x)}\, dE(\chi).$$

For the second identity, multiply both sides by $\pi(f)$ and use reconstruction from Chapter 10. The product of the integrands is $\widehat{L_x f}$, and nondegeneracy allows one to pass from $\pi(f)\mathcal{H}$ to $\mathcal{H}$. Restriction to B pushes E forward along $r : \widehat{A} \to \widehat{B}$, since $\chi(b) = r(\chi)(b)$ for $b \in B$.

Choose a finite scalar measure ν with the same null sets as E, for instance a weighted sum of the measures $S \mapsto \langle E(S)u_j, u_j\rangle$ for a dense sequence of unit vectors and positive summable weights. The multiplicity form of the spectral theorem decomposes $\mathcal{H}$ into a measurable field of Hilbert spaces $\mathcal{H}_\chi$:

$$\mathcal{H} \cong \int_{\widehat{A}}^{\oplus} \mathcal{H}_\chi \, d\nu(\chi), \qquad (\pi(x)u)(\chi) = \overline{\chi(x)}u(\chi).$$

The squared norm of a section is the integral of its squared fiber norms. Under this realization $E(S)$ multiplies by the indicator of S. A point mass contributes an eigenspace whose dimension is the fiber multiplicity at that point. An atomless measure has zero projection on every singleton, so it supplies continuous spectrum with no joint eigenvectors. These assertions remain valid when an atomic and an atomless summand occur together.

Example 12.1 (An eigencharacter in the support of a continuous spectrum)**.** Consider the representation of $\mathbb{R}$ on $\mathbb{C}^2 \oplus L^2([0,1])$ where x acts by $e^{-\pi i x}$ on $\mathbb{C}^2$ and by multiplication by $e^{-2\pi i x t}$ on the second summand. Dominated convergence establishes strong continuity. Its spectral measure has the two-dimensional projection at frequency $1/2$ and multiplies the function summand by indicators of frequency sets. The spectral measure has scalar measure class given by Lebesgue measure on $[0,1]$ plus a point mass at $1/2$. The projection onto the eigencharacter therefore singles out two dimensions even though every neighborhood of that frequency supports a nonzero projection in the continuous summand. The distinction is visible at the singleton itself, whose indicator vanishes in the function summand.

For a general second-countable locally compact group, the useful extension to direct integrals of irreducibles requires an additional hypothesis. If $C^*(G)$ is Type I, the direct-integral theory organizes separable representations by a measure class on the unitary dual and measurable multiplicity data [22, 11]. For general groups the central decomposition of the generated von Neumann algebra is into factors, whose representations can fail to have the structure of irreducibles with multiplicity. In the abelian case every irreducible has dimension one and the projection-valued measure gives the whole decomposition directly.

12.3 Haar measures in Fourier duality

For a fixed Haar measure on A there is a unique Haar measure on $\widehat{A}$ making the Fourier transform a unitary map

$$L^2(A, dx) \xrightarrow{\mathcal{F}} L^2(\widehat{A}, d\chi).$$

This is the abelian Plancherel theorem, with the transform agreeing with the L^1 formula on $L^1 \cap L^2$. When $f, \widehat{f}$ are integrable, inversion gives a continuous representative by

$$f(x) = \int_{\widehat{A}} \widehat{f}(\chi)\chi(x)\, d\chi.$$

Changing dx by a factor a changes $d\chi$ by a^{-1}, as follows by substituting into Parseval's identity. On $\mathbb{T}$ the pair is probability Haar and counting measure on $\mathbb{Z}$. On $\mathbb{R}$, with the displayed character normalization, both measures are Lebesgue measure. The Plancherel transform takes left translation to scalar multiplication by $\overline{\chi(x)}$, so the regular representation corresponds to the field of one-dimensional fibers with dual Haar measure.

Example 12.2 (Spectrum under restriction to a lattice)**.** Apply the Fourier transform to translations of $\mathbb{R}$ and restrict to $\mathbb{Z}$. The restriction map on frequency

is $\xi \mapsto e^{2\pi i \xi}$, which identifies all frequencies separated by an integer. Writing $\xi = t + k$ with $0 \le t < 1$ and $k \in \mathbb{Z}$ gives a unitary identification

$$L^2(\mathbb{R}, d\xi) \cong \int_{[0,1)}^{\oplus} \ell^2(\mathbb{Z})\, dt, \qquad F \longmapsto \big(t \longmapsto (F(t+k))_{k\in\mathbb{Z}}\big).$$

Its norm identity follows by summing the integrals on the translated unit intervals. Translation by $n \in \mathbb{Z}$ acts on the fiber at t as multiplication by $e^{-2\pi i n t}$. Thus multiplicity grows under the restriction even though the scalar spectrum of the restricted action still has no atoms.

CHAPTER 13

The principal series of $\mathrm{SL}_2(\mathbb{R})$

A line in the real plane has a stabilizer, and a unitary character of that stabilizer supplies a line bundle over the space of all lines. Normalized induction acts on its square-integrable sections after accounting for the change of measure on the base. For $\mathrm{SL}_2(\mathbb{R})$, the action can be computed on the circle, where angular Fourier coefficients describe the restriction to the rotation subgroup. The noncompact directions then connect these Fourier modes. We use the rank-one principal series discussed by Robert and Knapp to derive the action and its irreducibility criterion in fixed coordinates [22, §18][15].

13.1 The stabilizer and its character

Write $G = \mathrm{SL}_2(\mathbb{R})$, $K = \mathrm{SO}(2)$, and

$$a_r = \begin{pmatrix} r & 0 \\ 0 & r^{-1} \end{pmatrix}, \qquad n_b = \begin{pmatrix} 1 & b \\ 0 & 1 \end{pmatrix}, \qquad M = \{I, -I\}.$$

Here $r > 0$ and $b \in \mathbb{R}$. Given $g \in G$, set $r = \|ge_1\|$ and choose the unique $k \in K$ whose first column is ge_1/r. Then $k^{-1}g$ is upper triangular with positive top-left entry. Its determinant fixes the second diagonal entry, and its remaining entry fixes b. This gives a unique factorization $g = ka_rn_b$. The stabilizer P of the line $[e_1]$ is MAN, so inclusion identifies K/M with G/P.

Conjugation satisfies $a_rn_ba_r^{-1} = n_{r^2b}$, and the corresponding Jacobian on N is r^2. The square root r appears in the normalized induction of the character

$$(\sigma I)a_rn_b \longmapsto \sigma^\epsilon r^{i\lambda}, \qquad \sigma \in \{1, -1\}, \quad \epsilon \in \{0, 1\}, \quad \lambda \in \mathbb{R}.$$

In the equivariant function model the defining covariance is

$$F(g(\sigma I)a_rn_b) = \sigma^\epsilon r^{-1-i\lambda}F(g).$$

The exponent of r can also be obtained directly from the change of variables below. This specifies the sign of the spectral parameter and the square root independently of notation for the modular character.

Let $u_\theta = (\cos\theta, \sin\theta)$, and equip the circle with $d\theta/(2\pi)$. Restriction to K identifies the induced Hilbert space with the closed subspace of functions satisfying

$$f(u_{\theta+\pi}) = (-1)^\epsilon f(u_\theta)$$

almost everywhere. Geometrically, this is the space of sections of the even or odd line bundle on $\mathbb{RP}^1$. The orbit map and its compact restriction fit into the commutative diagram

$$\begin{array}{ccc} K & \hookrightarrow \text{ incl } \longrightarrow & G \\ \downarrow{\scriptstyle k\mapsto kM} & & \downarrow{\scriptstyle g\mapsto gP} \\ K/M & \xrightarrow{\sim} & G/P. \end{array}$$

The angular measure is invariant under K. Its transformation under G supplies a nonconstant density factor even though the induced action is unitary. In intrinsic terms, the smooth equivariant functions describe sections of the line bundle associated to the unitary character, tensored with half-densities on G/P. A half-density has an integrable squared absolute value. Choosing angular measure trivializes the half-density bundle and makes its transformation a square root of a Jacobian. The parity character still determines the transition law on the double cover, while the imaginary exponent supplies a phase along the A-coordinate. In this realization, unitarity follows from transport of densities, independently of a Fourier calculation.

13.2 The unitary action

For $u \in S^1$, define $T_g(u) = gu/\|gu\|$. Left translation on equivariant functions, followed by restriction to K, is given by

$$(\pi_{\lambda,\epsilon}(g)f)(u) = \|g^{-1}u\|^{-1-i\lambda} f(T_{g^{-1}}(u)). \tag{13.1}$$

Indeed the normalized first column and its length determine the K and A coordinates of $g^{-1}k$. The formula preserves parity because $T_g(-u) = -T_g(u)$. The length factor has the cocycle identity

$$\|ghu\| = \|hu\|\,\|gT_h(u)\|.$$

Together with $T_gT_h = T_{gh}$, this yields the representation law. The derivative of the transformed angle can be computed by taking the determinant of the two vectors $g^{-1}u_\theta$ and $g^{-1}u'_\theta$ and dividing by the squared norm of the first. The determinant is 1, so

$$d\theta' = \|g^{-1}u_\theta\|^{-2} d\theta, \qquad u_{\theta'} = T_{g^{-1}}(u_\theta).$$

For real λ, the factor in absolute square in (13.1) is exactly this Jacobian. Substitution in the norm integral proves unitarity. For smooth f, the transformed

function converges uniformly to f as $g \to I$. Density and the unitary norm bound extend convergence to all of the Hilbert space.

There is also a coordinate on the open part of the line space obtained from $(1, x)$. The vector $(1, x)/\sqrt{1+x^2}$ covers half of the circle, and parity supplies the other half. Thus the compact norm is

$$\int_{\mathbb{R}} \left| f\left(\frac{(1,x)}{\sqrt{1+x^2}}\right)\right|^2 \frac{dx}{\pi(1+x^2)}.$$

This expression comes from angular measure. Moving to Lebesgue measure on the same coordinate requires multiplying the representative by the square root of the displayed density. In either realization the group action transports the same section.

For a concrete angular distortion, the diagonal element a_r sends the line with slope x to one with slope $r^{-2}x$. The ratio of the resulting angular densities depends on x unless $r = 1$. In particular the pullback of the angular function by slope transport alone changes its squared norm. The modulus in (13.1) compensates for this change, while the phase $e^{-i\lambda \log \|g^{-1}u\|}$ leaves that norm calculation unchanged. Different values of λ enter the infinitesimal action through this phase.

13.3 Compact weights and the infinitesimal action

Let k_ϕ be rotation through ϕ, and $e_j(u_\theta) = e^{ij\theta}$. The characters of K appearing in $\pi_{\lambda,\epsilon}$ have weights $-j$ for $j \equiv \epsilon \pmod 2$, each with multiplicity one. The minus sign comes from the inverse in (13.1). For $X \in \mathfrak{sl}_2(\mathbb{R})$, set

$$q_X(\theta) = u_\theta \cdot X u_\theta, qquads_X(\theta) = u'_\theta \cdot X u_\theta.$$

Differentiating the norm and transformed angle of $\exp(-tX)u_\theta$ gives derivatives $-q_X$ and $-s_X$. On smooth sections this proves the operator formula

$$d\pi(X)f = (1 + i\lambda)q_X f - s_X \partial_\theta f.$$

In particular, for

$$H = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \qquad S = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \qquad J = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix},$$

the pairs (q_X, s_X) are respectively $(\cos 2\theta, -\sin 2\theta)$, $(\sin 2\theta, \cos 2\theta)$, and $(0, 1)$. Extending the derived action $\mathbb{C}$-linearly yields

$$\begin{aligned} d\pi(H + iS)e_j &= (1 + i\lambda + j)e_{j+2}, \\ d\pi(H - iS)e_j &= (1 + i\lambda - j)e_{j-2}, \qquad\qquad d\pi(J)e_j = -ije_j. \end{aligned}$$

Thus the group action depends on the maps between successive K-types as well as on the collection of types.

Proposition 13.1. *The unitary representation $\pi_{\lambda,\epsilon}$ is irreducible if $\lambda \neq 0$, and also if $\lambda = 0$ and $\epsilon = 0$.*

Proof. An invariant closed subspace of a unitary representation has an invariant orthogonal complement. Its projector Q commutes with rotations and therefore acts by a scalar $p_j \in \{0, 1\}$ on each Fourier mode. It also commutes with the derived action on smooth modes by differentiation. If $1+i\lambda+j \neq 0$, commutation with $d\pi(H + iS)$ gives $p_j = p_{j+2}$. These coefficients are all nonzero in the two stated regimes. All the p_j coincide, and the Fourier modes span the Hilbert space. □

13.4 The exceptional parameter

At $\lambda = 0$ in the odd-parity model, both coefficients that connect e_{-1} and e_1 vanish. Invariance under the group itself can be seen on the circle. Let $C = \left(\begin{smallmatrix} 1 & -i \\ 1 & i \end{smallmatrix}\right)$, as in the disk model of Chapter 14, and write $h = CgC^{-1}$. Set $z = e^{-2i\theta}$ and consider functions of the form $e^{-i\theta}P(z)$ for a polynomial P. Their closed span consists of all odd Fourier modes of negative index. If $g^{-1}u_\theta = \ell u_{\theta'}$, then

$$h^{-1}\begin{pmatrix} z \\ 1 \end{pmatrix} = \ell e^{i(\theta'-\theta)}\begin{pmatrix} e^{-2i\theta'} \\ 1 \end{pmatrix}.$$

The bottom coordinate is $j(h^{-1}, z)$ in the disk notation. Hence the transformed boundary function is

$$e^{-i\theta}j(h^{-1}, z)^{-1}P(h^{-1}z).$$

The factor following $e^{-i\theta}$ is holomorphic on a neighborhood of the closed disk. Its Taylor series has only nonnegative powers of z, so the boundary function has only negative odd Fourier modes. Polynomial density and boundedness give invariance of the closed span. At $\lambda = 0$ the circle multiplier is real, so complex conjugation establishes invariance of the positive odd modes. The projector proof applied within each string proves that each is irreducible. The weighted holomorphic construction of the next chapter gives a different family with one-sided weight strings and a square-integrability property.

CHAPTER 14

Discrete series on the disk

The unit disk carries an action of $\mathrm{SU}(1,1)$ by fractional linear maps. A power of the complex denominator acts on holomorphic functions, and a compatible weight makes this action unitary. We will compute the compact weights, prove irreducibility, and embed the resulting representation in the left regular representation. The last step gives a Haar-normalized coefficient norm and explains the lower bound on the integral parameter [22, §17][15].

14.1 The group, cocycle, and measure

Write

$$g = \begin{pmatrix} a & b \\ \overline{b} & \overline{a} \end{pmatrix}, \qquad |a|^2 - |b|^2 = 1.$$

Its action on the disk D is $gz = (az + b)/(\overline{b}z + \overline{a})$. With $j(g,z) = \overline{b}z + \overline{a}$, composition of the fractional linear maps gives

$$j(gh, z) = j(g, hz)j(h, z).$$

Two consequences of the determinant condition are

$$1 - |gz|^2 = (1 - |z|^2)|j(g,z)|^{-2}, \qquad \frac{d(gz)}{dz} = j(g,z)^{-2}.$$

The first follows by subtracting the squared modulus of the numerator from that of the denominator. The second follows by quotient differentiation. Both will enter the representation norm.

For an integer $n \geq 2$, define the weighted Bergman space $\mathcal{A}_n$ by the norm

$$\|f\|_n^2 = \frac{n-1}{\pi} \int_D |f(z)|^2 (1 - |z|^2)^{n-2}\, dx\, dy, \qquad f \in \mathcal{O}(D).$$

The monomials are orthogonal by angular integration, with

$$\|z^m\|_n^2 = (n-1) \int_0^1 t^m (1-t)^{n-2}\, dt = \frac{m!(n-1)!}{(m+n-1)!}.$$

For a holomorphic function, applying angular orthogonality to its Taylor series on each circle and then using monotone convergence identifies its squared norm with

the weighted sum of squared Taylor coefficients. Conversely, by Cauchy–Schwarz every sequence in that weighted ℓ^2 space gives a Taylor series converging on compact subsets of D, since the reciprocals of the squared monomial norms grow polynomially. This proves completeness and density of polynomials. In particular, the inner product, linear in its first variable, satisfies $\langle f, 1\rangle = f(0)$ and $\|1\|_n = 1$.

14.2 The holomorphic representation

Define

$$(D_n(g)f)(z) = j(g^{-1}, z)^{-n} f(g^{-1}z). \tag{14.1}$$

Integer n gives a single-valued multiplier on the group. The cocycle law gives the group law for D_n. With $z = gw$, the square of the multiplier contributes $|j(g, w)|^{2n}$, the weight contributes $|j(g, w)|^{-2(n-2)}$, and the area Jacobian contributes $|j(g, w)|^{-4}$. Their product is one. For polynomials, as g tends to the identity, the functions in (14.1) converge uniformly on the disk because the denominators stay uniformly bounded away from zero. Unitarity and polynomial density prove strong continuity on the entire space.

The stabilizer of 0 is $K_D = \{\mathrm{diag}(e^{i\theta}, e^{-i\theta})\}$. A diagonal element k_θ sends z to $e^{2i\theta}z$. The multiplier in (14.1) contributes $e^{-in\theta}$, so z^m has weight $-n - 2m$ in this torus parameter. Complex conjugation of the representation reverses these weights. Changing the orientation of the torus parameter also reverses their sign.

Consider the one-parameter subgroup

$$g_t = \begin{pmatrix} \cosh t & \sinh t \\ \sinh t & \cosh t \end{pmatrix}.$$

Differentiating at zero, the argument $g_{-t}z$ contributes $z^2 - 1$, and the derivative of the multiplier is nz. The expansion is uniform on the disk for polynomials because the denominator stays away from zero for small t, so differentiation is valid in the Hilbert norm. With $X = g_0'$ this gives

$$dD_n(X)z^m = (n + m)z^{m+1} - mz^{m-1},$$

where the second term is zero for $m = 0$. A projector onto an invariant closed subspace commutes with the torus and hence is diagonal on the monomials. Commutation with $dD_n(X)$ makes all consecutive diagonal entries equal because $n + m$ is positive. Completeness of the monomials proves irreducibility.

14.3 Evaluation and kernel vectors

The monomial norms also identify the linear functionals of point evaluation. For $w \in D$, a kernel vector K_w with $f(w) = \langle f, K_w \rangle$ is given by

$$K_w(z) = \sum_{m \geq 0} \frac{z^m \overline{w}^m}{\|z^m\|_n^2} = (1 - z\overline{w})^{-n}.$$

The first equality follows by computing the inner product against each monomial, and the second is the binomial series. Its Hilbert norm is finite by the same series evaluated at w. In particular $K_0 = 1$. Unitarity and covariance of evaluation give the transformation

$$D_n(g) K_w = \overline{j(g, w)^{-n}} K_{gw}.$$

Indeed, pairing a test function with the left side is evaluation of $D_n(g^{-1})f$ at w. This yields the right side because the inner product is conjugate linear in its second argument. The orbit of the constant vector gives kernels at every point up to a nonzero scalar. Their span is dense because a function perpendicular to them vanishes on the disk.

The kernel also determines the geometry of the norm on each evaluation line. Since $\|K_w\|_n^2 = K_w(w) = (1 - |w|^2)^{-n}$, evaluation grows toward the boundary. Normalizing the kernels by $\|K_w\|_n$ removes this growth and leaves only the phase supplied by the cocycle under transport. The regular embedding below can therefore be read as a weighted integral of point evaluations over the orbit of the constant vector.

14.4 A regular-representation embedding

The orbit map $g \mapsto g0$ identifies G/K_D with D. Normalize Haar measure on $G = \mathrm{SU}(1,1)$ so that integrating a right K_D-invariant function is integration against

$$d\mu_D(z) = \frac{dx\, dy}{\pi(1 - |z|^2)^2}.$$

The two change-of-variable formulas above make this measure invariant. Explicitly, integration of a general function can be done first over the right cosets with the probability Haar measure of K_D, then with μ_D on the quotient. This is left invariant because the inner average defines a function on the quotient and translation carries cosets to cosets.

For the unit vector 1, the coefficient is

$$\langle D_n(g) 1, 1 \rangle = j(g^{-1}, 0)^{-n} = a^{-n}.$$

Since $1 - |g0|^2 = |a|^{-2}$, the squared modulus is a function of the quotient. Its integral is

$$\int_G |\langle D_n(g)1, 1\rangle|^2 dg = \int_D (1 - |z|^2)^n d\mu_D(z) = \frac{1}{n-1}. \tag{14.2}$$

In particular the same integral at $n = 1$ diverges. The entire embedding follows from the same substitution. For $f \in \mathcal{A}_n$, let

$$(W_n f)(g) = \sqrt{n-1}\,\langle D_n(g^{-1})f, 1\rangle.$$

The expression inside the inner product becomes $j(g, 0)^{-n} f(g0)$ after evaluation at zero. Its absolute square, integrated over G, is $\|f\|_n^2/(n-1)$. Thus W_n is an isometry of $\mathcal{A}_n$ into $L^2(G)$. With $L_h F(g) = F(h^{-1}g)$, the representation law gives the commutative diagram

$$\begin{array}{ccc} \mathcal{A}_n & \xrightarrow{W_n} & L^2(G) \\ \downarrow{\scriptstyle D_n(h)} & & \downarrow{\scriptstyle L_h} \\ \mathcal{A}_n & \xrightarrow{W_n} & L^2(G). \end{array}$$

The image is closed. It exhibits a discrete-series summand of the regular representation, with formal degree $n - 1$ in this Haar normalization by (14.2) and Schur orthogonality.

The evaluation formula also describes the phase along the compact fibers of the embedding. If $k_\theta \in K_D$, then $j(gk_\theta, 0) = e^{-i\theta} j(g, 0)$, and $gk_\theta 0 = g0$. Therefore $W_n f(gk_\theta) = e^{in\theta} W_n f(g)$. The image transforms by a single character of the right compact action, although its restriction to the left compact action includes all the weights computed above.

14.5 Real forms and central descent

The Cayley transformation $z \mapsto (z - i)/(z + i)$ identifies the upper half-plane with D. Conjugation by its matrix $C = \left(\begin{smallmatrix} 1 & -i \\ 1 & i \end{smallmatrix}\right)$ identifies $\mathrm{SL}_2(\mathbb{R})$ with $\mathrm{SU}(1, 1)$ as matrix groups. With the rotation convention of Chapter 13, $Ck_\theta C^{-1} = \mathrm{diag}(e^{-i\theta}, e^{i\theta})$, so on this transported rotation subgroup the weights of D_n are $n + 2m$. Either real form has central element $-I$ acting by $(-1)^n$. Exactly the even parameters of the displayed family descend through the quotient by $\{I, -I\}$.

On the universal covering group the lifted torus has a real angle parameter. There the logarithm of the denominator can be chosen along a lift of a path from the identity, continuously in z, which explains why the automorphy factor can be considered for nonintegral weights on a cover. Square integrability over the covering group itself is a different condition. Matrix coefficients of a unitary irreducible have absolute value invariant under its center since the center acts

by scalars. A nonzero coefficient with positive quotient integral therefore has infinite squared integral over an infinite discrete central fiber with its counting Haar measure. This distinction matters when using square integrability modulo the center.

CHAPTER 15

Direct integrals and type I

An irreducible subrepresentation of the regular action has a closed Hilbert summand. The continuous part of the regular action behaves differently. Its individual spectral parameters can have zero measure, even though sets of such parameters account for a nonzero part of the Hilbert space. Direct integrals organize that part. The type I condition determines when the irreducible labels and their multiplicities admit a standard measurable description [22, §§19–20][8, 11].

15.1 Fields of spaces and operators

Let (X, μ) be a standard Borel measure space with a σ-finite measure. A measurable field of separable Hilbert spaces H_x has a countable fundamental sequence of sections whose values span dense subspaces and whose scalar products are measurable. A section is measurable when its scalar products with this sequence are measurable. The direct integral

$$H = \int_X^{\oplus} H_x \, d\mu(x)$$

is the Hilbert space of measurable sections modulo almost-everywhere equality, with squared norm $\int_X \|\xi(x)\|^2 d\mu(x)$. The fundamental sequence allows the field to be expressed locally by measurable orthonormal coordinates. Counting measure recovers an orthogonal sum. With a nonatomic measure, a section supported on a singleton represents zero.

An essentially bounded measurable field $x \mapsto T_x \in \mathcal{B}(H_x)$ defines a decomposable operator T by pointwise application. The operator norm is the essential supremum of the fiber norms. A direct integral of representations uses this construction for each group element, so

$$(U(g)\xi)(x) = U_x(g)\xi(x).$$

Suppose each U_x is strongly continuous and the field is measurable for each g. Then a sequence $g_j \to e$ gives pointwise convergence $U_x(g_j)\xi(x) \to \xi(x)$. The squared difference is bounded by $4\|\xi(x)\|^2$. Dominated convergence gives strong continuity of U, using first countability of the groups considered here.

Diagonal multiplication operators M_a for $a \in L^\infty(X, \mu)$ commute with every decomposable operator. The converse, the commutant theorem for direct integrals, is that any bounded operator commuting with the diagonal algebra is decomposable. In the spectral decomposition below, the diagonal algebra is the center of the represented von Neumann algebra. Consequently, the group action singles out the operators that preserve the fiber structure.

15.2 The type I decomposition theorem

A von Neumann algebra is a factor when its center consists of the scalars. A type I factor is isomorphic to $\mathcal{B}(V)$ for some Hilbert space V. A locally compact group is of type I when every factor representation generates a type I factor, or equivalently when the full group C^*-algebra $C^*(G)$ is type I. The full group algebra integrates all strongly continuous unitary representations. For a second countable locally compact Hausdorff group it is separable. This separability is the hypothesis used in the measurable classification and disintegration theorem [8, 11].

Theorem 15.1 (Type I decomposition)**.** *Let G be a second countable, locally compact Hausdorff group of type I, and U a strongly continuous unitary representation on a separable Hilbert space. Its irreducible dual $\widehat{G}$ has a standard Borel structure, and there is a decomposition*

$$H_U \simeq \int_{\widehat{G}}^{\oplus} H_\pi \otimes M_\pi \, d\mu(\pi), \qquad U(g) \simeq \int_{\widehat{G}}^{\oplus} \pi(g) \otimes I_{M_\pi} \, d\mu(\pi).$$

Here μ can be chosen finite, and its measure class and the measurable multiplicity $\dim M_\pi$ are unique up to null sets when the multiplicity spaces are taken nonzero almost everywhere. The multiplicity values lie in the positive integers together with ∞.

The theorem uses the central decomposition of the von Neumann algebra of the representation, which disintegrates it into factors. In the type I case each factor acts as $\mathcal{B}(H_\pi) \otimes I$, and the irreducible label can be used as the measurable coordinate. If μ is replaced by an equivalent measure ν, then multiplying a section by $(d\mu/d\nu)^{1/2}$ implements the equivalence of Hilbert realizations. The multiplicity dimension records the residual freedom after choosing the irreducible label.

For instance, $U(G)'$, the von Neumann algebra of bounded operators commuting with every $U(g)$, is given by

$$U(G)' \simeq \int_{\widehat{G}}^{\oplus} I_{H_\pi} \otimes \mathcal{B}(M_\pi) \, d\mu(\pi).$$

Every such operator commutes with the center of $U(G)''$, so is decomposable. For a fixed member of a countable dense subset of G, commutation holds off a null set. Taking the union of these null sets and using strong continuity gives the fiber commutation identities simultaneously for the group. On a fiber, Schur's lemma identifies the commutant with $I_{H_\pi} \otimes \mathcal{B}(M_\pi)$, by taking matrix coefficients in a basis of M_π. Conversely every such bounded measurable field commutes with the action.

For a Borel subset $E \subseteq \widehat{G}$, the diagonal projection given by multiplication by $\mathbf{1}_E$ has range the integral over E. These central projections record spectral subsets. Projections in the commutant can also select subspaces of individual multiplicity spaces, so each invariant closed summand may be described by a measurable field of projections on M_π. A rank-one choice on M_π retains one copy at each parameter, wherever that space is nonzero. The resulting subrepresentation may still have exclusively nonatomic spectrum. It becomes an actual Hilbert sum of irreducibles when its measure is supported on countably many atoms.

15.3 Two spectral tests

On $L^2(\mathbb{R}, \mu)$, let $U(t)\xi(s) = e^{its}\xi(s)$, with μ a probability measure. The representations with Gaussian μ and with $\mu = \sum_{j \geq 1} 2^{-j}\delta_{q_j}$ for an enumeration of the rationals have the same topological spectral support, all of $\widehat{\mathbb{R}} = \mathbb{R}$. They are inequivalent. A simultaneous eigenvector for the character $t \mapsto e^{its_0}$ must be supported at s_0. Indeed, first impose the eigenvector relation for the countable set of rational t and then use continuity in t. The Gaussian model has no such nonzero vectors. The atomic model has one for every rational s_0. Thus multiplicity and measure class capture information that the topological support alone loses.

For a contrasting factor, consider the free group F_2 on two generators as a countable discrete group. Every nonidentity conjugacy class is infinite. If w is a nonempty reduced word, choose a letter x different from the inverse of the first letter and from the last letter. The words $x^n w x^{-n}$ are reduced and have strictly increasing length. In the von Neumann algebra $L(F_2)$ generated by left translations on $\ell^2(F_2)$, a central operator T has $T\delta_e$ constant on each conjugacy class. This follows by commuting with left translations and with the right translations, the latter commuting with the entire algebra. Square summability forces all the coefficients on nonidentity classes to vanish. The identity coefficient determines T, because right translates of δ_e span a dense subspace. Thus $L(F_2)$ is a factor.

Its state $\tau(T) = \langle T\delta_e, \delta_e \rangle$ is a faithful normal trace. Faithfulness follows from the same determination by $T\delta_e$. For the trace identity, the column $T\delta_e$ and the identity row of T have the same squared sum since T commutes with right translations, and their coefficients are related by inversion in the index. Hence

$\tau(T^*T) = \tau(TT^*)$, which implies the trace identity by polarization. The factor is infinite-dimensional because its group translation operators are linearly independent. An infinite-dimensional type I factor has a proper isometry, incompatible with a faithful finite trace. Therefore this factor representation disproves the type I property for F_2. More generally, Thoma's theorem characterizes countable discrete type I groups as those containing an abelian subgroup of finite index [1].

CHAPTER 16

Nonabelian Plancherel and induction

The representation used here is the left regular action on $L^2(G)$. Its spectral decomposition requires a measure normalized to the chosen Haar measure, and the individual Fourier values are operators. On a Plancherel fiber, left and right translations act by multiplication on opposite sides. This recovers matrix Fourier analysis on a compact group and ordinary Fourier analysis on an abelian group as cases of a single construction [22, §21][8, 11].

16.1 The operator Fourier transform

Let G be a second countable unimodular locally compact Hausdorff group of type I and fix left Haar measure dg. Unimodularity makes this also right Haar measure. For $f \in L^1(G)$ and a unitary irreducible π, define

$$\mathcal{F}_G f(\pi) = \pi(f) = \int_G f(g)\pi(g)\, dg,$$

in the sense of integration on vectors. This orientation has $\pi(g)$ under the integral, so that the transform preserves the order of left convolution. With $(f * h)(g) = \int_G f(x)h(x^{-1}g)\, dx$, substitution $g = xy$ and Fubini yield

$$\pi(f * h) = \pi(f)\pi(h), \qquad \pi(f^*) = \pi(f)^*, \qquad f^*(g) = \overline{f(g^{-1})}.$$

The adjoint identity uses invariance of Haar measure under inversion, which follows from unimodularity.

Theorem 16.1 (Plancherel)**.** *There is a unique Plancherel measure μ_G on the standard Borel dual, normalized by dg, such that for $f \in L^1(G) \cap L^2(G)$ almost all $\pi(f)$ are Hilbert–Schmidt and*

$$\|f\|_2^2 = \int_{\widehat{G}} \|\pi(f)\|_{\mathrm{HS}}^2 \, d\mu_G(\pi).$$

The transform extends to a unitary map from $L^2(G)$ onto $\int_{\widehat{G}}^{\oplus} \mathrm{HS}(H_\pi)\, d\mu_G(\pi)$.

The structural theorem supplies both the measure and surjectivity. Its norm equality implies, by polarization of the Hilbert–Schmidt inner product, the weak pairing formula

$$\langle f, h\rangle = \int_{\widehat{G}} \operatorname{tr}(\pi(f)\pi(h)^*) d\mu_G(\pi), \qquad f, h \in L^1(G) \cap L^2(G).$$

The integral exists by Cauchy–Schwarz. Scaling Haar measure by $c > 0$ scales the squared L^2 norm by c and the squared transform norm in each fiber by c^2. The new Plancherel measure is consequently $c^{-1}\mu_G$.

For compact G with probability Haar measure, Peter–Weyl gives $\mu_G(\{\pi\}) = \dim H_\pi$. For $G = \mathbb{R}$ with Lebesgue measure, the chosen sign in the integrated representation gives the Fourier kernel $e^{it\xi}$ and $\mu_G = d\xi/(2\pi)$.

Set $L_x f(g) = f(x^{-1}g)$ and $R_y f(g) = f(gy)$. Substitution of the new variable gives the diagram

$$\begin{array}{ccc} L^2(G) & \xrightarrow{\mathcal{F}_G} & \displaystyle\int_{\widehat{G}}^{\oplus} \mathrm{HS}(H_\pi)\, d\mu_G(\pi) \\ \Big\downarrow{\scriptstyle L_x R_y} & & \Big\downarrow{\scriptstyle A_\pi \mapsto \pi(x) A_\pi \pi(y)^{-1}} \\ L^2(G) & \xrightarrow{\mathcal{F}_G} & \displaystyle\int_{\widehat{G}}^{\oplus} \mathrm{HS}(H_\pi)\, d\mu_G(\pi). \end{array}$$

The fiber identification $v \otimes \overline{w} \mapsto \langle \cdot, w\rangle v$ identifies the left multiplicity space with $\overline{H_\pi}$. This explains how Peter–Weyl's multiplicity by dimension generalizes to Hilbert–Schmidt fibers. For a nonunimodular group the simple bare transform here is generally replaced by one with fiber operators accounting for the modular function [11].

16.2 The reduced convolution norm

The same transform computes an operator norm. For $f \in L^1(G)$, let $\lambda(f)$ be convolution by f on $L^2(G)$. The left action on a fiber is multiplication of a Hilbert–Schmidt operator by $\pi(f)$ on the left. This follows first by integrating the translation relation on vectors and then by continuity, since the fiber operators are bounded by $\|f\|_1$. Left multiplication by B on $\mathrm{HS}(H_\pi)$ has norm $\|B\|$. The upper bound follows from the Hilbert–Schmidt norm inequality. For the reverse bound, take a rank-one operator $A = \langle \cdot, w\rangle v$ for unit vectors w, v. Its Hilbert–Schmidt norm is one and the norm of BA is $\|Bv\|$. Taking the supremum over v and using the norm formula for decomposable operators yields

$$\|\lambda(f)\| = \operatorname*{ess\,sup}_{[\pi]\in\widehat{G}}^{\mu_G} \|\pi(f)\|_{\mathrm{op}}.$$

This is the norm used to complete the convolution algebra to the reduced group C^*-algebra. It is determined by the Plancherel measure class, whereas the squared-norm formula for f uses the normalization of the measure. In the abelian example with $G = \mathbb{R}$, the essential supremum agrees with the ordinary supremum of the continuous Fourier transform. Nonetheless, the decomposition of the left regular action has no one-dimensional translation-eigenspace, because any such

vector would be supported at one frequency, a set of Lebesgue measure zero. This separates operator norm information from the existence of a summand at a frequency.

16.3 Atoms and formal degree

If π is an atom of μ_G, the π-fiber is a closed summand, and choosing a unit vector in its multiplicity space embeds π into $L^2(G)$. Conversely, if π embeds into $L^2(G)$, its range projection commutes with the left action. The commutant description in Chapter 15 gives a measurable family of multiplicity projections. Uniqueness of central decomposition applied to that range puts its measure on $\{\pi\}$. It follows that π is an atom.

For a square-integrable irreducible π, write d_π for its formal degree in Schur orthogonality. Take unit vectors $v, w \in H_\pi$ and $c(g) = \langle \pi(g^{-1})v, w\rangle$, so $\|c\|_2^2 = d_\pi^{-1}$. As v varies, these coefficients exhibit a left-invariant copy of π, and the Fourier transform of c is thus supported at the atom π. To evaluate it there, use truncations of c on a compact exhaustion. Those truncations lie in $L^1 \cap L^2$. Their transforms converge in Hilbert–Schmidt norm at the atom, since it has positive measure. For vectors $a, b \in H_\pi$ the corresponding scalar integrals converge by Cauchy–Schwarz, since both coefficients are square-integrable. Schur orthogonality then gives

$$\langle (\mathcal{F}_G c)(\pi)a, b\rangle = d_\pi^{-1}\langle a, w\rangle\langle v, b\rangle.$$

Taking norms in the Plancherel formula shows $d_\pi^{-1} = \mu_G(\{\pi\})d_\pi^{-2}$, or $\mu_G(\{\pi\}) = d_\pi$. The disk summands in Chapter 14 have weight $n-1$ for their stated Haar normalization. The Plancherel measure for $\mathrm{SL}_2(\mathbb{R})$ also has absolutely continuous parts on the two principal series of Chapter 13. The rank-one Plancherel formula specifies their densities after a Haar normalization and a choice of spectral parameter [15]. An open interval of principal parameters has positive spectral measure, while a single interior parameter of either family has measure zero.

16.4 Induction and a continuous model

For a second countable compact group K, consider $G = K \times \mathbb{R}$ with probability Haar measure on K and Lebesgue measure on $\mathbb{R}$. If σ is an irreducible of K on V_σ, a vector of $\mathrm{Ind}_K^G \sigma$ is represented by $F(k,t) = \sigma(k)^{-1}u(t)$, with $u \in L^2(\mathbb{R}, V_\sigma)$. Translation by (k_0, t_0) induces

$$u(t) \longmapsto \sigma(k_0)u(t - t_0).$$

Apply the scalar Fourier transform with kernel $e^{it\xi}$ to the variable t. The result is the direct integral of $\sigma \otimes \chi_\xi$, where $\chi_\xi(t) = e^{it\xi}$, with measure $d\xi/(2\pi)$ and

multiplicity one. In this example, induction and formation of the direct integral can be compared explicitly by a unitary intertwiner on the section model.

To obtain the regular representation of G, apply compact Plancherel in the K coordinate and scalar Plancherel in the real coordinate. The first contributes the squared Hilbert–Schmidt norms weighted by $\dim V_\sigma$, and the second integrates these norms over ξ with measure $d\xi/(2\pi)$. Thus the measure on the family $\{\sigma\}\times\mathbb{R}$ is $\dim V_\sigma \, d\xi/(2\pi)$. Every such family has nonatomic measure. Even with compact matrix blocks in one coordinate, an individual fiber of the regular representation need not be a Hilbert summand. The additional regular multiplicity is $\overline{V_\sigma}$, while the induced action computed above fixes a single copy in each fiber.

Equivalently, one can first induce the left regular representation of K to G. Its space is $L^2(\mathbb{R}, L^2(K))$, and evaluation of the inner variable identifies this with $L^2(G)$ by Fubini. Under this identification the induced action is the left regular action computed above. Peter–Weyl's decomposition of $L^2(K)$ then integrates each σ with its full compact multiplicity. The two descriptions match because they use the same Fourier intertwiners on the independent variables, and changing between the multiplicity description and the Hilbert–Schmidt norm description only rescales the measure on each family.

CHAPTER 17

Actions, deloopings, and representations

The averaging operator for a compact group projects onto the vectors that every element fixes. To formulate invariance when vectors come with chain homotopies, we first need a notion of an action that records how the operators, their composites, and those homotopies fit together. A group determines a classifying category that supplies precisely the diagram shape for this purpose.

17.1 Three meanings of a classifying object

For an abstract group G, let BG denote the nerve of the category with one object $*$ and automorphism group G. An n-simplex of BG is a string of n composable group elements. Inner face maps multiply adjacent elements, outer face maps discard an endpoint, and degeneracies insert the identity. In an ordinary category $\mathcal{C}$, a functor from this one-object category chooses $V \in \mathcal{C}$ and a homomorphism $G \to \mathrm{Aut}_{\mathcal{C}}(V)$. Naturality of a transformation is the familiar intertwining condition

$$\begin{array}{ccc} V & \xrightarrow{\rho_V(g)} & V \\ \downarrow{\scriptstyle f} & & \downarrow{\scriptstyle f} \\ W & \xrightarrow{\rho_W(g)} & W. \end{array}$$

For an ∞-category $\mathcal{C}$, an action is a functor $BG \to \mathcal{C}$. Thus it gives an object, an equivalence for each g, and a compatibility between $\rho(gh)$ and the composite $\rho(g)\rho(h)$. The two-simplex for (g, h) supplies that compatibility. The three-simplex for (g, h, ℓ) compares the two ways to use the two-simplex data. The full functor keeps the comparisons in all degrees. A strict action on a chain complex defines such a functor after inverting quasi-isomorphisms. The following result explains the extent to which strict chain complexes suffice as a model.

A topological group G provides several other inputs. Forgetting the topology gives the abstract group G^{δ}. Taking the topological classifying space $B_{\mathrm{top}}G$ and then its fundamental ∞-groupoid gives a second input. For a well-pointed topological group the universal bundle with contractible total space gives $\Omega B_{\mathrm{top}}G \simeq G$

in spaces. Functors from $B_{\mathrm{top}}G$ to an ordinary category factor through its fundamental groupoid. For $G = S^1$, the space $B_{\mathrm{top}}G$ is $K(\mathbb{Z}, 2)$, so such local systems are constant up to equivalence. The weight representations $z \mapsto z^m$ on $\mathbb{C}$ are instead continuous representations in the analytic sense. A classifying stack that retains continuous maps into G recovers finite-dimensional continuous representations in Chapter 20.

17.2 Linearization of a discrete group

Write $D(k)$ for the derived ∞-category of modules over a commutative ring k, obtained by inverting the quasi-isomorphisms of complexes. We use $k[G]$ for the group algebra and $D(k[G])$ for the corresponding derived category of left modules.

Theorem 17.1 (Derived linearization)**.** *For a discrete group G there is an equivalence*

$$\mathrm{Fun}(BG, D(k)) \simeq D(k[G])$$

compatible with evaluation at $*$ and the forgetful functor to $D(k)$. For spectra the analogous equivalence identifies $\mathrm{Fun}(BG, \mathrm{Sp})$ *with modules over the associative ring spectrum $\mathbb{S}[G] = \Sigma^\infty_+ G$.*

Proof. Let $i : \{*\} \to BG$ pick the object, with source containing only the identity. The restriction $U = i^*$ has a left adjoint $i_!$ by left Kan extension. The indexing category for this extension at $*$ is the discrete set G. Consequently, $Ui_!V = \bigoplus_{g \in G} V$. The monad unit selects the summand at 1, and its multiplication sends the summand indexed by (g, h) to the one indexed by gh. This follows by postcomposing arrows out of $i(*)$ in the comma category for the Kan extension.

Evaluation is conservative, since equivalences in a functor category are evaluated pointwise and BG has a single object. It preserves all colimits, in particular the geometric realizations required by the monadicity theorem [18, §4.7.3]. The functor category is therefore equivalent to the category of algebras for the monad $Ui_!$. Extension and restriction of scalars between $D(k)$ and $D(k[G])$ give another instance of this monadic adjunction. Restriction is conservative, and it preserves colimits, as can be seen by computing coproducts and cofibers on underlying complexes. Its monad is $k[G] \otimes^L_k (-)$ with multiplication from the group algebra. The free k-basis of $k[G]$ identifies this monad, including its unit and multiplication, with the Kan-extension monad.

In spectra the coproduct of copies of V is the smash product of $\Sigma^\infty_+ G$ with V. Restriction from spectrum modules preserves colimits and detects equivalences on underlying spectra. The same monadic comparison gives the spectral assertion. □

The tensor product on the functor side acts diagonally. For group algebra modules this is the action induced by $k[G] \to k[G] \otimes_k k[G]$, $g \mapsto g \otimes g$. The equivalence therefore describes both the underlying linear category and its symmetric monoidal structure. The unit is k with trivial action. If $G = C_2$ and 2 is invertible in k, the complementary idempotents $(1 + g)/2$ and $(1 - g)/2$ decompose $k[C_2]$ as $k \times k$. The derived action category consequently identifies with $D(k) \times D(k)$. Under tensor product the component of sign $-$ is the sum of the two mixed-sign products, while the component of sign $+$ is the sum of the two equal-sign products.

17.3 Spaces of intertwiners

For objects X, Y of $\mathrm{Fun}(BG, D(k))$, write $\mathrm{map}_k(UX, UY)$ for the mapping spectrum in $D(k)$. It carries the conjugation action, which in a strict model sends a cochain f to $\rho_Y(g) f \rho_X(g^{-1})$. For a coherent model this action is induced by the functorial internal Hom and inversion on the groupoid. The mapping spectrum of equivariant maps is

$$\mathrm{map}_{\mathrm{Fun}(BG, D(k))}(X, Y) \simeq \mathrm{map}_k(UX, UY)^{hG}.$$

Indeed, the natural-transformation end over BG has a cosimplicial description whose term in degree n is $\prod_{(g_1,\dots,g_n)\in G^n} \mathrm{map}_k(UX, UY)$. Transporting the source of each component along its string of arrows gives the cosimplicial replacement for the limit of the conjugation action. The first and last cofaces use action maps and the inner cofaces compose arrows. Totalization imposes each naturality condition together with its higher compatibility. The notation $(-)^{hG}$ means this homotopy limit.

For a finite group with order invertible in k, the derived invariants of a module in degree zero are concentrated in degree zero, as follows from the idempotent argument of Chapter 18. Over a field of characteristic dividing $|G|$, higher cohomology can already appear when $X = Y = k$. Thus ordinary intertwiners form the degree-zero part of a mapping object with additional groups. The spectral model similarly has one underlying spectrum and its coherent action. Genuine equivariant spectra require further structure associated to subgroups, including representation suspensions and transfers.

CHAPTER 18

Homotopy fixed points, orbits, and Tate theory

Restriction along $BG \to *$ takes an object to its trivial action. Its left and right adjoints are respectively the colimit and limit over BG. For an action X in $D(k)$ write them as X_{hG} and X^{hG}. The adjunctions organize two familiar functors on modules:

$$X_{hG} \simeq k \otimes^{L}_{k[G]} X,$$
$$X^{hG} \simeq \mathrm{RHom}_{k[G]}(k, X).$$

Here k carries the trivial right action in the tensor product and the trivial left action in the derived Hom. These formulas follow from Theorem 17.1 and the tensor–Hom adjunctions for the augmentation $k[G] \to k$. For a module M in cohomological degree zero, $H^{-n}(M_{hG}) = H_n(G; M)$ and $H^n(M^{hG}) = H^n(G; M)$ for $n \geq 0$.

18.1 A resolution from the group nerve

Let $P_n = k[G^{n+1}]$ with diagonal left action and differential on basis vectors

$$d(g_0, \ldots, g_n) = \sum_{i=0}^{n} (-1)^i (g_0, \ldots, \widehat{g_i}, \ldots, g_n).$$

The augmentation $P_0 \to k$ sends every basis vector to 1. The differential is G-equivariant, and the deletion identities pair opposite signs in d^2. Each P_n is a free $k[G]$-module because its basis is a free G-set, with unique representatives having first coordinate 1. The augmented underlying k-complex is contracted by prepending 1 to every tuple, with the empty tuple used in degree -1. In $ds + sd$ all terms cancel except the deletion of this prepended entry. Thus the augmented complex is exact, giving a free resolution of k. To obtain a projective resolution of the trivial right module, equip the same complex with the right action given by the inverse of the left action.

For comparison with the categorical definition, identify the free orbit of $(g_0, \ldots, g_n)$ by the increments $h_j = g_{j-1}^{-1} g_j$. Under this convention an equivariant function is given by its values on representatives with $g_0 = 1$. Applying $\mathrm{Hom}_{k[G]}(-, M)$ gives the inhomogeneous cochains $C^n(G; M) = \mathrm{Map}(G^n, M)$.

The degree-one differential is

$$(dc)(g,h) = g\,c(h) - c(gh) + c(g).$$

A one-cocycle satisfies $c(gh) = c(g) + g\,c(h)$. This is the group law for a homomorphic section of $M \rtimes G \to G$, $g \mapsto (c(g), g)$. Changing such a section by conjugation with an element of M changes c by a coboundary. Therefore $H^1(G; M)$ parametrizes conjugacy classes of these sections. The degree-zero group of the homotopy limit is M^G, while the first cohomology group classifies these sections up to conjugacy.

18.2 The finite-group norm

For finite G, the stable norm is a natural map from homotopy orbits to homotopy fixed points. Its degree-zero map on a module is the ordinary norm $[m] \mapsto \sum_{g\in G} gm$. The map respects the coinvariant relations because left and right multiplication by any group element permute the summation. Define the Tate object as the cofiber:

$$X_{hG} \xrightarrow{\ N\ } X^{hG} \longrightarrow X^{tG} \longrightarrow X_{hG}[1].$$

The diagonal $BG \to BG \times BG$ has finite discrete homotopy fibers. The sum-to-product equivalence along that diagonal defines the norm for $BG \to *$ by the relative norm construction [20, Construction I.1.7 and Example I.1.11]. The same finite-index norm applies to complexes, either by the stable construction in $D(k)$ or by viewing the complexes as module spectra over the Eilenberg–Mac Lane ring spectrum of k. Limits, colimits, and cofibers in this comparison have the indicated underlying spectra.

Proposition 18.1 (Averaging and Tate vanishing)**.** *If $|G|$ is invertible in k, then $X^{tG} = 0$ for every X in* $\mathrm{Fun}(BG, D(k))$.

Proof. The element $e = |G|^{-1} \sum_{g\in G} g$ is a central idempotent. The augmentation identifies the left ideal $k[G]e$, and the right ideal $ek[G]$, with their respective trivial modules. In particular those trivial modules are projective. On a complex X the invariants and coinvariants therefore compute their derived functors degreewise. The map $[x] \mapsto ex$ identifies the coinvariants with eX, since $x - ex = |G|^{-1} \sum_g (x - gx)$ is zero in the coinvariants. The invariants are eX as well. On a module in degree zero the norm is multiplication by $|G|$ under these identifications. In particular, on the free $k[G]$-module it sends the basis class to the sum over the basis orbit. Both functors are exact and preserve coproducts here, by the direct-summand description. The free module and its shifts generate $D(k[G])$ under colimits, so the norm equivalence on that generator gives the asserted equivalence on every object. Its cofiber vanishes. □

For compact groups of infinite order the normalized Haar integral also provides an averaging projector on continuous unitary representations. Its existence uses the analytic theory of the target category. The discrete-group homotopy limit instead uses the entire bar construction on strings of group elements.

18.3 A modular cyclic group

Let $G = C_p$ with generator g and let $k = \mathbb{F}_p$. Setting $u = g - 1$ gives $k[G] \simeq k[u]/(u^p)$. The group norm element is u^{p-1} because $1 + t + \cdots + t^{p-1} = (t-1)^{p-1}$ over k. There is a free resolution of the trivial module:

$$\cdots \xrightarrow{u^{p-1}} k[G] \xrightarrow{u} k[G] \xrightarrow{\varepsilon} k.$$

Further left the maps alternate multiplication by u and u^{p-1}. Exactness follows from $\ker(u) = (u^{p-1})$, $\ker(u^{p-1}) = (u)$ and $\ker(\varepsilon) = (u)$. Both Hom into the trivial module and tensor product with it turn every differential into zero. Consequently $H_i(C_p; k) \simeq k$ and $H^i(C_p; k) \simeq k$ for every $i \geq 0$.

For $X = k$ the norm map on H^0 is zero because $p = 0$ in k. In the cofiber sequence, $H^j(X^{tC_p}) \simeq H^j(X^{hC_p})$ for $j > 0$, and $H^j(X^{tC_p}) \simeq H^{j+1}(X_{hC_p})$ for $j < -1$. The connecting segment at $j = -1, 0$ gives the kernel and cokernel of the degree-zero norm, both k. Thus the Tate object has one copy of k in each integer cohomological degree. This calculation uses the periodic resolution and the norm triangle [5, Chs. I and VI].

CHAPTER 19

Induction as change of groups

In Chapter 6 induction constructs an action on a space of sections over a homogeneous space. For discrete groups both Kan extensions along a group inclusion are available. Their behavior explains reciprocity and the role of double cosets.

19.1 Two Kan extensions

Let $i : BH \to BG$ come from the inclusion of discrete groups $H \subseteq G$. The functor i^* is restriction, with adjunctions $i_! \dashv i^* \dashv i_*$. Under the linearization of Theorem 17.1 these are

$$\operatorname{Ind}_H^G W = k[G] \otimes^L_{k[H]} W,$$
$$\operatorname{Coind}_H^G W = \operatorname{RHom}_{k[H]}(k[G], W).$$

The first occurrence of $k[G]$ is a left $k[G]$, right $k[H]$ bimodule. In the second it is a left $k[H]$, right $k[G]$ bimodule, with left G acting on the resulting Hom by $(gf)(a) = f(ag)$. The tensor formulas identify the adjoints by the natural tensor–Hom bijections on modules. Since $k[G]$ is free over $k[H]$ on either side, the displayed functors can be computed degreewise on complexes. At the module level, a map out of the induced object is determined by its values on $1 \otimes W$. A map from a G-module M into the coinduced object is determined by evaluation at 1, and has the explicit form $m \mapsto (a \mapsto \phi(am))$ for an H-map $\phi : M \to W$.

Proposition 19.1 (Finite-index comparison)**.** *If $[G : H]$ is finite, induction and coinduction are naturally isomorphic. More generally there is a natural comparison*

$$g \otimes w \longmapsto (a \longmapsto \operatorname{pr}_{k[H]}(ag)w),$$

where $\operatorname{pr}_{k[H]}$ is coefficient projection.

Proof. Projection is H-linear on each side. Right linearity gives the tensor relation, and left linearity gives H-equivariance of the resulting map in a. The left G-action sends the pairing for g to the pairing for its left translate. For a set T of representatives of the cosets gH, the inverse elements t^{-1} represent the cosets Hg. Induction has one direct-sum copy of W for each t, and coinduction has one product copy, recovered by evaluating at t^{-1}. As $\operatorname{pr}_{k[H]}(t^{-1}s)$ is 1 for

$s = t$ and 0 for distinct representatives, the map is the inclusion of the direct sum into the product. It is an isomorphism for finite T, degreewise on complexes as well. □

For example, if $H = 1$ and $G = \mathbb{Z}$, induction from k yields finitely supported sequences on $\mathbb{Z}$ and coinduction yields all sequences, after indexing by inversion where appropriate. The natural comparison has proper image over every nonzero k. In the continuous compact setting the L^2 completion and boundedness of intertwining operators enter the story. Finite index permits a sum/product comparison in either setting.

19.2 Double cosets and base change

Let K be another subgroup of G. The homotopy pullback $BK \times_{BG} BH$ is a groupoid of pairs of subgroup objects together with an arrow between their images. Choosing that arrow to go from the H object to the K object identifies the objects with $g \in G$, and morphisms with the action $(k, h) \cdot g = kgh^{-1}$. Its connected components are indexed by $K\backslash G/H$. The stabilizer at g is identified with $L_g = K \cap gHg^{-1}$ through projection onto K, and the projection onto H sends $\ell \in L_g$ to $g^{-1}\ell g$. Its classifying diagram is

$$\begin{array}{ccc} \coprod_{[g]} BL_g & \longrightarrow & BH \\ \downarrow & & \downarrow \\ BK & \longrightarrow & BG. \end{array}$$

For an H-object W write ${}^gW$ for its restriction along $L_g \to H$ given by conjugation by g^{-1}.

Theorem 19.2 (Mackey decomposition)**.** *For $W \in D(k[H])$,*

$$\operatorname{Res}^G_K \operatorname{Ind}^G_H W \simeq \bigoplus_{[g] \in K\backslash G/H} \operatorname{Ind}^K_{L_g} {}^gW.$$

Proof. As a $(k[K], k[H])$-bimodule, $k[G]$ is the direct sum of $k[KgH]$. For each component the bimodule map

$$k[K] \otimes_{k[L_g]} k[H] \longrightarrow k[KgH], \qquad k_0 \otimes h_0 \longmapsto k_0 g h_0$$

uses $\ell \cdot h_0 = (g^{-1}\ell g)h_0$ on the second factor. It is well defined since $\ell g = g(g^{-1}\ell g)$. Choose representatives t of the cosets $k_0 L_g$. Every element of KgH has the form tgh_0 and this form is unique. Indeed, $tgh_0 = t'gh_0'$ would force $t'^{-1}t \in L_g$ and hence $t = t'$ and $h_0 = h_0'$. This gives the bimodule isomorphism. Tensoring over $k[H]$ gives the stated decomposition, with derived tensor products computed using the free coset modules. □

The diagram explains the proof's shape. Restricting the induced object is left Kan extension along the left projection from the pullback after restriction along the top one. Its disconnected components contribute a coproduct. The stabilizer specifies the group along which each component is induced.

19.3 Shapiro's lemma and a permutation module

The right adjunction gives

$$\mathrm{RHom}_{k[G]}(k, \mathrm{Coind}_H^G W) \simeq \mathrm{RHom}_{k[H]}(k, W).$$

This is a derived version of Shapiro's lemma [5, Ch. III]. The left adjunction similarly identifies the homotopy orbits of an induced object with the homotopy orbits over H. That statement also follows by associativity of derived tensor product.

Take $G = S_3$, let H be the order-two subgroup generated by a transposition, and let $k = \mathbb{F}_2$ with trivial H-action. The three-letter permutation module is $k[G/H]$. Its invariants are the line of constant functions on the three cosets. Since the index is finite it is also coinduced, so that

$$H^q(S_3; k[G/H]) \simeq H^q(C_2; k) \simeq k \qquad (q \geq 0).$$

The last identification uses the cyclic resolution of Chapter 18. The fixed line is therefore one degree of a derived invariant with nonvanishing cohomology in all higher degrees. Restricting to H gives another description of the permutation action. There is one fixed coset and a pair forming a free orbit. The two stabilizers in the Mackey decomposition are H and 1, producing the direct sum of the trivial and regular H-modules.

CHAPTER 20

Continuous actions and the classifying stack

A topological group carries information about families of group elements. Retaining such families changes what an action means. A convenient categorical object records maps into the group from all test spaces and permits local gluing. A complementary algebraic model for profinite groups uses coefficients with the discrete topology and isolates exactly which finite quotients govern cochains.

20.1 The group as a sheaf

Use the site of topological spaces with covers by open subspaces, working in a fixed universe. A topological group G determines the sheaf of groups $\mathbf{G}(U) = C(U, G)$, with pointwise operations. Its classifying stack $B\mathbf{G}$ assigns to each U the groupoid of locally trivial principal right G-bundles on U and their isomorphisms. To see the connection with the group nerve, choose local sections s_i on a trivializing cover $\{U_i\}$. On intersections there are continuous transition functions g_{ij} with $s_j = s_i g_{ij}$ and $g_{ij} g_{jk} = g_{ik}$ on triple intersections. Conversely these functions glue the products $U_i \times G$ by $(x, a)_j \sim (x, g_{ij}(x)a)_i$. Changes of sections change the functions by the corresponding change of trivialization, and the glued bundles represent isomorphic objects. This realizes the stack as the stackification of the groupoid with a single object and automorphism sheaf $\mathbf{G}$.

The trivial torsor defines an atlas $* \to B\mathbf{G}$ that covers in the sense that every torsor is locally trivial. The fiber products in its nerve in degrees $0, 1, 2$ are respectively $*, G, G^2$. Over a test space U the degree-one fiber product gives the isomorphisms between trivial G-bundles, which are precisely the continuous U-families of left translations. The degree-two fiber product records two composable translations.

Proposition 20.1 (Continuous representation by descent)**.** *Finite-rank complex vector bundles on $B\mathbf{G}$ form a symmetric monoidal category equivalent to that of finite-dimensional continuous complex representations of G.*

Proof. Pull back a vector bundle along the trivial torsor. This gives a vector space V. The descent isomorphism on G is a bundle automorphism of $G \times V$ over G, so is represented by a continuous map $\rho : G \to \mathrm{GL}(V)$. Compatibility on G^2

is the condition $\rho(gh) = \rho(g)\rho(h)$, with normalization at the identity following from the degeneracy. Equivalently, for a bundle given by g_{ij} the representation glues the trivial vector bundles by $\rho(g_{ij})$. This construction is independent of local sections and commutes with pullback. A linear map between the fibers at the trivial torsor preserves descent exactly when it intertwines the action, as shown by the square

$$\begin{array}{ccc} G \times V & \xrightarrow{\rho_V} & G \times V \\ \downarrow{\scriptstyle 1\times f} & & \downarrow{\scriptstyle 1\times f} \\ G \times W & \xrightarrow{\rho_W} & G \times W. \end{array}$$

Here the horizontal maps act on the second coordinate. The local gluing construction and pullback are inverse by descent for vector bundles. Tensor product is computed fiberwise and takes tensor products of the transition operators. □

If $G = S^1$ the weight-m representation defines descent via the function $z \mapsto z^m$. The underlying vector bundle on the atlas is the same one-dimensional space for every m. The descent automorphism over G distinguishes the weights. The classifying space $B_{\mathrm{top}}G$ of Chapter 17 has its own local systems, while the stack uses continuous local trivializations of torsors.

20.2 Discrete coefficients for a profinite group

Let P be profinite and k a discrete field. An action on a discrete k-vector space M is continuous exactly when each vector has an open stabilizer. Such representations are often called *smooth*. Every vector is fixed by some open normal subgroup N, since the open stabilizer has finite index and one can intersect its finitely many conjugates. Thus M is the filtered union of M^N as N decreases. Each M^N is acted on by the finite quotient P/N.

Proposition 20.2. *If k has characteristic zero, invariants are exact on smooth k-representations of P.*

Proof. Invariants are left exact. Given a surjective P-equivariant map $M \to Q$ and an invariant vector $q \in Q$, lift it to $m \in M$. Choose an open normal N fixing m. The vector $|P/N|^{-1} \sum_{g \in P/N} gm$ is invariant and maps to q. This proves surjectivity on invariants. □

For trivial discrete coefficients k, the complex of continuous inhomogeneous cochains $C^n_{\mathrm{cts}}(P; k) = C(P^n, k)$ satisfies

$$H^n_{\mathrm{cts}}(P; k) \simeq \varinjlim_{N \trianglelefteq_o P} H^n(P/N; k).$$

Indeed, a continuous cochain is locally constant. For $n \geq 1$ it is constant on a neighborhood of each point that is a product of cosets of open normal subgroups. Compactness gives a finite such cover of P^n. Intersect all the finitely many subgroups occurring in the cover. The cochain is then constant on each product of cosets of that intersection, so factors through a single finite quotient. Inflation along quotients is compatible with the differential, and filtered colimits of vector spaces are exact. These cochains compute continuous group cohomology with discrete coefficients [21, Ch. 6].

Example 20.3 (Inflation for the p-adic integers)**.** For $P = \mathbb{Z}_p$ and $k = \mathbb{F}_p$, the finite quotients form the sequence C_{p^n} for $n \geq 1$. The group $H^1(C_{p^n}; k)$ is the group of additive homomorphisms to k, and each inflation is an isomorphism. Thus $H^1_{\mathrm{cts}}(P; k) \simeq k$. The degree-two group of each quotient is also one-dimensional, using the periodic resolution with differentials $g - 1$ and $\sum_{j=0}^{p^n-1} g^j$. Its class of the nonsplit extension $C_{p^{n+1}} \twoheadrightarrow C_{p^n}$ is nonzero. The extension is nonsplit because elements of its domain of order dividing p^n have image in the proper subgroup generated by the pth power. Pulling this extension back along that very surjection splits it via the diagonal of the fiber product. Consequently inflation in degree two is zero at every step and $H^2_{\mathrm{cts}}(P; k) = 0$.

There are abstract actions with no smooth structure on the given discrete coefficient space. The product $P = \prod_{n \geq 1} C_2$ is a vector space over $\mathbb{F}_2$ as an abstract group. The quotient by the subgroup of finite-support elements has a nonzero linear functional to $\mathbb{F}_2$, by extending a nonzero vector of the quotient to a basis. Composing with the quotient and then the sign character gives a nontrivial abstract one-dimensional complex representation. It is trivial on the dense finite-support subgroup. A continuous map with this property into $\{1, -1\}$ would be trivial everywhere. This exhibits an action of P^δ that cannot arise from continuous descent.

CHAPTER 21

Monoidal categories and fiber functors

A representation supplies two kinds of data. It supplies a vector space on which an unknown symmetry acts, and it supplies maps between vector spaces that the symmetry must respect. Tensor products allow such maps to relate different representations. Chapter 8 used continuous unitary representations to reconstruct a compact group. We now work over an arbitrary field k, retain the symmetric tensor product, and allow the reconstructed group to be evaluated on any commutative k-algebra. The coordinate algebra can then record infinitesimal symmetries, even when every k-valued point acts trivially.

21.1 Matrix coefficients from a coend

Let $G = \operatorname{Spec} H$ be an affine group scheme over k, so H is a commutative Hopf algebra. Write $\mathrm{Rep}_k(G)$ for the category of finite-dimensional right H-comodules and ω for the forgetful functor to finite-dimensional vector spaces. The tensor coaction multiplies the coefficients of the two factors. The unit object is k with coaction $1 \mapsto 1 \otimes 1$, and the antipode supplies the dual coaction. Thus ω is a faithful exact symmetric monoidal functor from a rigid abelian category. These properties are the starting data of neutral Tannaka theory [7].

A matrix coefficient associated to $v \in V$ and $\phi \in V^*$ is

$$c_{\phi,v} = (\phi \otimes \mathrm{id}_H)\rho_V(v) \in H.$$

When $f\colon V \to W$ is equivariant, $c_{\psi,f(v)} = c_{\psi f,v}$. This relation suggests a vector space constructed entirely from $(\mathrm{Rep}_k(G), \omega)$. Take the direct sum of all $\omega(V) \otimes \omega(V)^*$ over a small skeleton and quotient by the linear relations

$$[f(v), \psi]_W = [v, \psi f]_V.$$

Denote the result by C_ω. This is the coend $\int^V \omega(V) \otimes \omega(V)^*$. The quotient makes every compatible family of bilinear pairings out of the summands factor uniquely through C_ω.

Theorem 21.1 (Affine reconstruction)**.** *The map $[v,\phi]_V \mapsto c_{\phi,v}$ identifies C_ω with H as a commutative Hopf algebra. For every commutative k-algebra R there is a natural group isomorphism*

$$G(R) \;\cong\; \mathrm{Aut}^{\otimes}_R(\omega_R), \qquad \omega_R(V) = \omega(V)\otimes_k R.$$

Here naturality on the right is with respect to equivariant maps in $\mathrm{Rep}_k(G)$*.*

Proof. First consider H as a right comodule under Δ. Each $z \in H$ belongs to a finite-dimensional subcomodule of H. To see this, write $\Delta(z) = \sum_i z_i \otimes a_i$ with the a_i linearly independent. Coassociativity gives

$$\sum_i \Delta(z_i)\otimes a_i = \sum_i z_i \otimes \Delta(a_i).$$

Applying coordinate functionals on the last tensor factor shows that the span W of the z_i is a right subcomodule. Applying $\mathrm{id}\otimes\epsilon$ to $\Delta(z)$ gives $z \in W$. Also $z = (\epsilon|_W \otimes \mathrm{id})\Delta(z)$, so z is the coefficient of the pair $(z, \epsilon|_W)$. This proves surjectivity.

For injectivity, $v \mapsto c_{\phi,v}$ defines a comodule map $f_\phi\colon V \to H$ by coassociativity. Its composite with ϵ is ϕ. Given a finite sum of generators of C_ω, take $W \subset H$ to be the sum of the images of the corresponding maps f_ϕ. Each map now factors through the same finite-dimensional subcomodule W. The defining relations rewrite the entire sum as $[\sum f_\phi(v), \epsilon|_W]_W$. The sum vanishes in C_ω whenever its image vanishes in H.

The Hopf operations are visible in the fiber functor. For a basis e_i of V with dual basis e_i^*, the formulas are

$$\begin{aligned} [v,\phi]_V[w,\psi]_W &= [v\otimes w, \phi\otimes\psi]_{V\otimes W},\\ \Delta[v,\phi]_V &= \sum_i [e_i,\phi]_V \otimes [v,e_i^*]_V, \qquad \epsilon[v,\phi]_V = \phi(v),\\ S[v,\phi]_V &= [\phi, \mathrm{ev}_v]_{V^*}. \end{aligned}$$

The coproduct formula inserts the canonical element in $V\otimes V^*$, so it is independent of the basis. The tensor unit gives the multiplicative unit. The corresponding coefficient identities in H follow from the tensor and dual coactions. The linear isomorphism therefore identifies all the indicated operations.

An R-linear natural transformation u of ω_R determines a k-linear map on C_ω by $[v,\phi] \mapsto (\phi\otimes \mathrm{id}_R)(u_V(v\otimes 1))$. Conversely, any such map gives the matrix entries of unique R-linear endomorphisms, with the coend relations exactly enforcing naturality. These endomorphisms satisfy the tensor and unit identities precisely when the map $H \to R$ is a unital algebra homomorphism. In Sweedler notation the resulting transformation is

$$u_V(v\otimes 1) = \sum v_{(0)} \otimes a(v_{(1)}), \qquad a\colon H \to R.$$

The coproduct formula identifies composition of transformations with convolution. The inverse is supplied by aS, so every monoidal natural transformation in question is invertible. The formulas commute with base change in R. □

21.2 Three coordinate computations

For $G = \mathbb{G}_m$, write $H = k[t, t^{-1}]$ with $\Delta(t) = t \otimes t$. Writing a coaction as $\rho(v) = \sum_n v_n \otimes t^n$, coassociativity implies $\rho(v_n) = v_n \otimes t^n$, and the counit gives $v = \sum_n v_n$. Thus the representation category consists of finite-dimensional $\mathbb{Z}$-graded vector spaces with tensor product adding degrees. A monoidal automorphism of ω_R acts on the weight-n line by the n-th power of its value on the weight-one line. Naturality with respect to maps out of weight lines then determines it on all objects. Its parameter is any element of $R^\times$.

For $G = \mathbb{G}_a$ in characteristic zero, a representation is a polynomial matrix $T(t)$ satisfying $T(s + t) = T(s)T(t)$ and $T(0) = I$. Comparing the coefficient of s gives $T'(t) = NT(t)$, where $N = T'(0)$. Over k, this recurrence forces $T(t) = \sum_{i \geq 0} t^i N^i / i!$. Since $T(t)$ is a polynomial, N is nilpotent. Conversely every nilpotent N defines such a representation. A parameter $a \in R$ acts by the finite expression $\exp(aN)$. The tensor product sends N, M to $N \otimes I + I \otimes M$, which accounts for the addition law on the reconstructed parameter.

In characteristic $p > 0$, let μ_p have coordinate algebra $k[t, t^{-1}]/(t^p - 1)$. Its k-points consist only of 1. Over $R = k[\varepsilon]/(\varepsilon^2)$ the element $1 + \varepsilon$ is a nontrivial p-th root of unity. Its action on the weight-one line detects it. The coend and the R-valued automorphisms retain precisely the testing information that is invisible on k-points alone.

The standard representation of SL_n, for $n \geq 2$, gives a different way to recognize the coordinates from morphisms. Write $U = k^n$ and choose a basis $e_1, \ldots, e_n$. The volume functional $U^{\otimes n} \to k$ sends $u_1 \otimes \cdots \otimes u_n$ to the determinant of the matrix with columns u_i. It is an equivariant map. Naturality and the tensor identity therefore make the matrix u_U of a fiber automorphism preserve the volume functional over R. Evaluation on $e_1 \otimes \cdots \otimes e_n$ gives $\det(u_U) = 1$.

The coefficients $c_{e_i^*, e_j}$ of U generate the coordinate algebra $k[x_{ij}]/(\det(x_{ij}) - 1)$, since they are the coordinate functions x_{ij}. A tensor automorphism is determined by its entries on U by affine reconstruction. Every determinant-one matrix in $M_n(R)$ then supplies an evaluation homomorphism and hence an automorphism of the fiber functor on all representations. The factorization through the determinant relation holds over every R, including rings with nilpotents. This gives both the reason for the constraint on u_U and the sufficiency of that constraint.

21.3 The target of the fiber functor

Over $\mathbb{C}$, finite-dimensional continuous unitary representations of the circle also split into integral weights. The forgetful functor to Hilbert spaces retains norms. A unitary tensor natural automorphism acts on the weight-one line by a scalar of norm 1, giving S^1. The underlying functor to complex vector spaces permits an arbitrary nonzero scalar and gives $\mathbb{C}^\times$, the complex points of $\mathbb{G}_m$. The difference comes from the structure preserved by an automorphism of the chosen fiber functor.

A derived fiber functor takes values in a symmetric monoidal category of modules over a base ring spectrum. Its automorphisms and their homotopies form a space with composition. The ordinary calculation above uses the endomorphisms of finite vector spaces and a set of natural transformations. The corresponding derived questions require control of the entire mapping spaces and of the geometric conditions on the objects that a fiber functor preserves.

CHAPTER 22

Derived Tannaka duality and affine group stacks

Two fiber functors can be equivalent in several ways, and equivalences can themselves be homotopic along several paths. This data becomes relevant when the coefficients of a representation have higher homotopy groups. In this chapter A and B are connective E_∞-rings. Connectivity means $\pi_i A = 0$ for $i < 0$, and the same convention applies to modules. The notation Mod_A denotes the stable ∞-category of all A-module spectra, with tensor product over A. For a discrete ring a, taking $A = Ha$ recovers the derived category of a-modules with its mapping spaces.

22.1 The affine module category

The tensor unit in Mod_A is A. Its endomorphism spectrum is A itself. The multiplication and all commutativity coherences in this identification come from the symmetric monoidal structure on endomorphisms of the unit. Consequently, a colimit-preserving symmetric monoidal functor from Mod_A to Mod_B induces a map of E_∞-rings $A \to B$.

Theorem 22.1 (Affine pullback)**.** *Extension of scalars induces an equivalence of spaces*

$$\mathrm{Map}_{\mathrm{CAlg}}(A, B) \;\simeq\; \left(\mathrm{Fun}^{L,\otimes}(\mathrm{Mod}_A, \mathrm{Mod}_B)\right)^{\simeq}.$$

Here CAlg *is the ∞-category of E_∞-rings, and the superscript $^{\simeq}$ denotes the maximal ∞-groupoid. The functor attached to $f\colon A \to B$ sends M to $B \otimes_A M$, using f for the action of A on B.*

Proof. The ∞-category of spectra is initial among stable presentable symmetric monoidal ∞-categories with colimit-preserving tensor products [18, Corollary 4.8.2.19]. Thus each of the two module categories has its canonical functor from spectra, and passing to symmetric monoidal functors under spectra imposes a contractible choice of compatibility. Lurie's fully faithful module construction [18, Corollary 4.8.5.21] identifies the space of these functors under spectra with $\mathrm{Map}_{\mathrm{CAlg}}(A, B)$. The construction sends a map to extension of scalars. Its action on the endomorphisms of the tensor unit recovers the given map. This identifies

both constructions on mapping spaces, including equivalences and their higher homotopies. □

The underlying functor can be understood with free modules. An exact colimit-preserving functor of stable presentable categories commutes with tensoring by spectra. Thus the image of a free module $A \otimes X$, for a spectrum X, is $B \otimes X$. The free-forgetful adjunction resolves each A-module M by a simplicial diagram of free A-modules with realization M. The resolution becomes split after forgetting the action. Its image computes the bar construction for $B \otimes_A M$, with the right A-action on B supplied by f. The coherent action is needed on the face maps. The assignment of values on free objects alone would leave those maps undetermined.

Since tensoring over a connective ring is connective on two connective modules, these functors preserve connective objects. This also follows directly from the bar description, since spectra of connective modules, their smash products, and their geometric realizations are connective. They preserve perfect modules because a perfect A-module is a retract of a finite cell module and $B \otimes_A -$ carries its cell construction to cells over B. On affine spectra the variance is

$$\operatorname{Spec} B \longrightarrow \operatorname{Spec} A, \qquad \operatorname{Mod}_A \longrightarrow \operatorname{Mod}_B .$$

The operation on modules is geometric pullback. Knowing only the ordinary category of its homotopy classes would discard information about paths between maps of the spectral affine schemes.

For a first calculation of a mapping space, take A to be the free commutative Hk-algebra on one generator in degree zero. Its mapping space into a connective Hk-algebra B, over Hk, is $\Omega^\infty B$. This follows from the adjunction between the symmetric algebra and its underlying module. If k has characteristic zero, the free algebra is equivalent to $H(k[t])$. Indeed, its summand of symmetric degree n is the homotopy coinvariant object of $Hk^{\otimes_{Hk} n} \simeq Hk$ under the symmetric group. The higher group homology vanishes by averaging, and π_0 gives the usual symmetric algebra on a line.

For $B = Hk \oplus \Sigma Hk$ with square-zero multiplication on the second summand, points of this affine line have components indexed by k. At every component the fundamental group is the additive group of k. The affine theorem interprets these groups as loops in the space of Hk-linear pullback functors. In particular, the value $f(t) \in \pi_0 B$ determines a component of this space and the endomorphism construction recovers its paths as well.

22.2 A reconstruction theorem for stacks

For a spectral algebraic stack X, the category $\mathrm{QCoh}(X)$ is formed by gluing the module categories on its spectral affine charts by pullback, with their higher compatibility data. Its tensor unit is $\mathcal{O}_X$. Classifying stacks require this descent information. If G is a suitable affine group over a base and BG classifies its torsors, pulling back a quasi-coherent object along a point specified by a torsor gives an associated module on the test affine. Thus a geometric fiber carries a choice of torsor as part of the data.

Theorem 22.2 (Bhatt–Halpern-Leistner)**.** *Let X be a spectral algebraic stack with quasi-affine diagonal such that $\mathrm{QCoh}(X)$ is compactly generated. For a spectral affine scheme S, pullback identifies $\mathrm{Map}(S, X)$ with the space of colimit-preserving symmetric monoidal functors*

$$\mathrm{QCoh}(X) \longrightarrow \mathrm{QCoh}(S)$$

that preserve connective objects [2, Theorem 1.3].

The diagonal hypothesis controls the geometry of the overlaps used to compare charts. Compact generation means that the stable category is generated under colimits by a set of compact objects, whose mapping functors commute with filtered colimits. The theorem reconstructs the whole space of geometric fibers, including their automorphisms. Bhatt and Halpern-Leistner give a further version for noetherian spectral algebraic stacks with quasi-affine diagonal, requiring preservation of both connective and almost perfect objects [2, Theorem 1.4]. The stated hypotheses specify the scope of each reconstruction.

22.3 A line and its higher symmetries

Let k be a field of characteristic zero and work over Hk. Inverting the generator of the free commutative Hk-algebra from the affine calculation gives $H(k[t, t^{-1}])$. On a test algebra R, this localization selects those components of $\Omega^\infty R$ indexed by units, so it represents $\mathbb{G}_m$. A point of $B\mathbb{G}_m$ over $\operatorname{Spec} R$ is a line module, locally equivalent to R in degree zero, with equivalences as its arrows. Its associated fiber pulls the universal weight-one line back to that module L, and pulls the weight-n line back to $L^{\otimes n}$, using the dual for negative n. A tensor-invertible module such as ΣR is invertible with inverse $\Sigma^{-1}R$. For nonzero discrete R it is never locally isomorphic to the degree-zero module on a nonempty open of $\operatorname{Spec} R$. This illustrates the role of geometric conditions when extracting a stack of fibers from monoidal data.

At the trivial line, the automorphism space is $\mathrm{GL}_1(R)$, the union of components of $\Omega^\infty R$ indexed by units of $\pi_0 R$. Indeed, an R-linear endomorphism of the free

module R lies in $\Omega^\infty R$ and is an equivalence exactly when its π_0-class is a unit. The component of the trivial line in the stack of lines is therefore $B\mathrm{GL}_1(R)$.

For the split square-zero ring $R = Hk \oplus \Sigma Hk$, products of elements of the suspended summand vanish. Its zeroth homotopy ring is k, and its only nonzero positive homotopy group is $\pi_1 R = k$. Passing from $\Omega^\infty R$ to the union of its invertible components leaves its positive homotopy groups at 1 unchanged. Taking the classifying space shifts them by one degree and gives

$$\pi_1\big(B\mathrm{GL}_1(R)\big) \cong k^\times, \qquad \pi_2\big(B\mathrm{GL}_1(R)\big) \cong (k, +).$$

Over Hk the second group vanishes. The ordinary scalar is still present in the first group. The second group describes homotopies of automorphisms of the trivial fiber, and can be read from the full space of points.

CHAPTER 23

Categorical characters and traces

For an ordinary representation of a finite group, the trace of g is already a scalar. For an action on a linear category, the trace of g is a vector space on which the centralizer of g acts. Taking its ordinary character gives a function of two commuting group elements. This construction can distinguish finite actions that become indistinguishable when replaced by their ordinary permutation representations.

23.1 The trace space and conjugation

Let G be finite and let k be algebraically closed of characteristic zero. A finite semisimple k-linear category here has finitely many simple isomorphism classes, finite-dimensional morphism spaces, and every object is a finite direct sum of simples. A weak action on $\mathcal{C}$ consists of k-linear equivalences F_g with isomorphisms $\mu_{g,h}\colon F_gF_h \xrightarrow{\sim} F_{gh}$. They satisfy

$$\mu_{gh,j}\,(\mu_{g,h}F_j) = \mu_{g,hj}\,(F_g\mu_{h,j}).$$

We use $F_e = \mathrm{Id}_{\mathcal{C}}$ and identity unit constraints. This is the finite semisimple setting of categorical character theory [12].

Write $T_g = \mathrm{Nat}(\mathrm{Id}_{\mathcal{C}}, F_g)$. This is a finite-dimensional vector space. A transformation is determined by its components on the finite set of simple representatives because naturality against the injections and projections determines its component on each direct sum. The endomorphisms of a simple are the scalars, so any choice of those components extends by additivity and naturality. Thus there is a vector space decomposition

$$T_g \cong \bigoplus_{s \in \mathrm{Irr}(\mathcal{C})} \mathrm{Hom}_{\mathcal{C}}(s, F_g s).$$

In this formula $\mathrm{Irr}(\mathcal{C})$ means a chosen set of simple representatives.

Conjugation gives a map $T_g \to T_{hgh^{-1}}$. To define it, set $g' = hgh^{-1}$ and form the composite

$$F_h \xrightarrow{F_h\eta} F_hF_g \xrightarrow{\mu_{h,g}} F_{hg} = F_{g'h} \xrightarrow{\mu^{-1}_{g',h}} F_{g'}F_h, \qquad \eta \in T_g.$$

Precomposition by F_h is an equivalence of functor categories, so this has the form $\Theta_h(\eta)F_h$ for a unique $\Theta_h(\eta) \in T_{g'}$.

Proposition 23.1. *The maps Θ_h compose according to multiplication in G. If g, h commute, then*

$$\chi_{\mathcal{C}}(g,h) = \operatorname{tr}(\Theta_h \colon T_g \to T_g)$$

is invariant under simultaneous conjugation of (g,h).

Proof. To compare $\Theta_r\Theta_h$ with Θ_{rh}, precompose with F_rF_h and put $g'' = rg'r^{-1}$. After the common first arrow $F_rF_h\eta$, the two pastings are from $F_rF_hF_g$ to $F_{g''}F_rF_h$, using $\mu_{r,h}$ to compare F_rF_h with F_{rh}. Compose both pastings with the isomorphism from $F_{g''}F_rF_h$ to $F_{g''rh}$ obtained from the compositors. For the iterated pasting the coherence for (r, g', h) reduces this to $\mu_{r,hg}(F_r\mu_{h,g})$. For the single transport it is $\mu_{rh,g}(\mu_{r,h}F_g)$. They agree by coherence for (r,h,g). Faithfulness of precomposition with F_rF_h now gives the composition law. In particular, Θ_r identifies the action of h on T_g with the action of rhr^{-1} on $T_{rgr^{-1}}$. The two linear maps have the same trace. □

The conjugation groupoid has objects $g \in G$ and arrows labeled h from g to hgh^{-1}. It can be obtained by taking loops in the one-object groupoid BG, with morphisms given by conjugation. The trace spaces and their transport form a functor from this groupoid to vector spaces. Its loop based at g and labeled h exists exactly when $gh = hg$.

23.2 Permuted simples and fixed subgroups

For a finite G-set S, let $\mathcal{C}_S$ be the category of S-graded finite-dimensional vector spaces. The strict action F_g permutes the simple objects according to $s \mapsto gs$. A simple contributes k to T_g precisely when $gs = s$, with a basis given by the identity at its coordinate. Therefore $T_g \cong k[S^g]$. If g, h commute, Θ_h permutes the basis by h. The trace of this permutation matrix is

$$\chi_{\mathcal{C}_S}(g,h) = |S^{\langle g,h\rangle}|.$$

For $S = G/H$, a coset xH belongs to this fixed set precisely when $x^{-1}\langle g,h\rangle x \subseteq H$. This criterion places the categorical character among the fixed-point counts of subgroups.

For example, take $G = C_2 \times C_2$, let S be the disjoint union of a free orbit and two fixed points, and let T be the disjoint union of G/H over the three subgroups H of order two. Both have six elements, and each nonidentity element fixes two points in either set. On T these two points lie in the unique summand whose stabilizer contains that element. Consequently the ordinary permutation characters coincide. Two independent elements generate G, and S has two points fixed by G whereas T has zero. The corresponding categorical characters take different values on the pair.

23.3 Conjugation data in the Drinfeld center

The same groupoid governs the Drinfeld center of Vec_G, the tensor category of finite-dimensional G-graded vector spaces. The tensor product has $(V \otimes W)_z = \bigoplus_{xy=z} V_x \otimes W_y$. An object of the center consists of V and natural isomorphisms $c_X \colon V \otimes X \to X \otimes V$ satisfying the half-braiding identity under tensor products.

For the line k_h in degree h, put $c_{k_h}(v \otimes 1_h) = 1_h \otimes a_h(v)$. Preservation of degrees makes a_h a map $V_g \to V_{h^{-1}gh}$. The half-braiding identity on $k_h \otimes k_j$ gives $a_{hj} = a_j a_h$, and the unit identity gives $a_e = \mathrm{id}$. Thus $h \mapsto a_{h^{-1}}$ defines conjugation transport on the graded fibers. Conversely, these maps define $c_X(v \otimes x_h) = x_h \otimes a_h(v)$ on homogeneous tensors. This formula is natural in graded maps and satisfies the required identity on each tensor of homogeneous vectors. Maps in the center correspond exactly to graded linear maps intertwining all the transports. The tensor product in the center is convolution of the resulting bundles over the group multiplication [10].

Restricting such a bundle to one representative g in each conjugacy class yields a representation of $C_G(g)$. Conversely, given a representation W of $C_G(g)$, the induced vector space $k[G] \otimes_{k[C_G(g)]} W$ carries the grading $\deg(x \otimes w) = xgx^{-1}$ and the conjugation transport given by left multiplication. The balanced tensor product makes the definition independent of the coset representative. These two constructions are inverse by transport from the representative fiber, and taking their sum over conjugacy classes describes the whole center as a linear category. Under this description the family T_g of a categorical action gives an object of the center. The values of its categorical character are the characters on its centralizer fibers.

For $G = S_3$, the identity class has centralizer S_3, the transposition class has centralizer C_2, and the class of three-cycles has centralizer C_3. The corresponding irreducible dimensions in the center, measured on the underlying graded vector space, are

representative	class size	dimensions
e	1	$1, 1, 2$
(12)	3	$3, 3$
(123)	2	$2, 2, 2$

Indeed, induction as above multiplies a centralizer representation's dimension by the class size. The sum of the squares is 36. More generally, summing by conjugacy class gives $\sum_{[g]} [G : C_G(g)]^2 |C_G(g)| = |G|^2$, using the ordinary sum-of-squares identity in each centralizer.

The additive and multiplicative rules for these traces can be seen on simple fibers. A direct sum of two finite semisimple categories with componentwise actions has the disjoint union of their simple objects, giving the direct sum of the two trace bundles. For the tensor product of the categories over k with

diagonal action, the simple objects are pairs, and their morphism spaces are tensor products of the corresponding morphism spaces. The trace space at g is consequently the tensor product of the two trace spaces at g, with transport given by the two transport maps tensored together. This tensor product is fiberwise. The convolution on the center uses the group multiplication and therefore has a different formula. On pairs of commuting elements the direct sum adds the numerical characters, and the fiberwise tensor product multiplies them.

CHAPTER 24

Projective actions, cocycles, and twists

The equation expressing compatibility of two group elements can fail by a scalar even when their actions on projective space are well defined. Associativity relates those scalars for triples. In a tensor category, the associator itself can be a scalar on homogeneous simple objects. Its pentagon compares products of scalars for quadruples. In both cases, changing chosen comparison maps leads to a cohomology class. The cochain degree records how many group elements enter an individual comparison map.

24.1 Lifting projective representations

Let G be a group, k a field, and V a nonzero finite-dimensional k-vector space. A projective representation is a homomorphism $G \to \mathrm{GL}(V)/k^\times$. Choose representatives ρ_g, with $\rho_e = I$. Then for a unique function $\alpha\colon G^2 \to k^\times$,

$$\rho_g \rho_h = \alpha(g,h)\rho_{gh}.$$

The function is normalized, meaning its value is 1 if either argument is e. Associativity gives the 2-cocycle identity

$$\alpha(g,h)\alpha(gh,j) = \alpha(h,j)\alpha(g,hj).$$

Replacing representatives by $b(g)\rho_g$, where $b(e) = 1$, multiplies $\alpha(g,h)$ by $(\delta b)(g,h) = b(g)b(h)/b(gh)$.

Proposition 24.1. *The projective representation lifts to* $\mathrm{GL}(V)$ *precisely when the class of* α *in* $H^2(G,k^\times)$ *is trivial, with the trivial action on* $k^\times$.

Proof. The quotient by normalized coboundaries makes the class independent of the chosen representatives. A lift supplies representatives with multiplier 1. Conversely, if the class is trivial, choose b with $\delta b = \alpha^{-1}$. The new representatives $b(g)\rho_g$ multiply exactly according to the group law and retain the same image in the quotient. □

An arbitrary normalized cocycle defines a central extension on $k^\times \times G$ by

$$(u,g)(v,h) = (uv\alpha(g,h), gh).$$

The cocycle supplies associativity, normalization gives the identity, and the formulas yield an inverse because $\alpha(g, g^{-1}) = \alpha(g^{-1}, g)$. An α-projective representation becomes a representation of this extension by $(u, g) \mapsto u\rho_g$. Conversely, restricting such a representation to the chosen section $g \mapsto (1, g)$ recovers the projective operators whenever the central subgroup acts by its given scalars.

The same equations define the twisted group algebra $k^\alpha[G]$, with basis E_g and product $E_g E_h = \alpha(g, h)E_{gh}$. Each E_g is invertible, so unital modules are equivalent to linear actions with multiplier α. Rescaling the basis realizes the change by a coboundary as an isomorphism of the corresponding algebras.

The multiplier behaves functorially under basic operations. If ρ and σ have multipliers α and γ, then $g \mapsto \rho_g \otimes \sigma_g$ has multiplier $\alpha\gamma$, by taking the two product equations in the two tensor factors. The contragredient operators $g \mapsto (\rho_g^{-1})^*$ have multiplier α^{-1}. Thus $V \otimes V^*$ carries an honest action, identified with conjugation on $\mathrm{End}_k(V)$. This associated representation is independent of representative choices. Its independence can also be seen in the conjugation formula, where the scalar multiplying the operator cancels with the scalar multiplying its inverse.

24.2 Two commuting elements with anticommuting lifts

Assume $\operatorname{char} k \neq 2$ and take $G = C_2 \times C_2$, written additively. Choose matrices on k^2

$$X = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \qquad Z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

They satisfy $X^2 = Z^2 = I$ and $ZX = -XZ$. Thus the projective images of X and Z commute. The representatives $\rho_{(a,b)} = X^a Z^b$ have multiplier $\alpha((a, b), (c, d)) = (-1)^{bc}$. The scalar commutator of the representatives of $(1, 0)$ and $(0, 1)$ is -1 and cannot change when either is rescaled. Since a linear representation of an abelian group has commuting images, the multiplier class is nontrivial.

The images of the four basis elements are I, X, Z, XZ. The two diagonal matrices I, Z span the diagonal subspace, and X, XZ span the off-diagonal subspace. This uses $2 \neq 0$. The four images form a basis of $M_2(k)$, so $k^\alpha[G] \cong M_2(k)$. This gives a projective simple module of dimension two over k. Ordinary simple modules for G are all one-dimensional because the commuting involutions diagonalize with eigenvalues 1 and -1 in k.

24.3 Associator twists

Let $\omega \colon G^3 \to k^\times$ be normalized. On finite-dimensional G-graded vector spaces use the usual graded tensor product and change its associator on degrees g, h, j

by the scalar $\omega(g,h,j)$, in the direction $(U \otimes V) \otimes W \to U \otimes (V \otimes W)$. Its pentagon holds precisely when

$$\omega(h,j,\ell)\omega(g,hj,\ell)\omega(g,h,j) = \omega(gh,j,\ell)\omega(g,h,j\ell).$$

The equality follows by evaluating the two paths in the pentagon on four homogeneous factors. The triangle uses normalization. This is the tensor category Vec_G^{ω} for a normalized 3-cocycle ω [10].

Consider an identity functor from Vec_G^{ω} to $\mathrm{Vec}_G^{\omega'}$ with tensorator $F(U) \otimes F(V) \to F(U \otimes V)$ given by a normalized scalar $\beta(g,h)$ on degrees g,h. The tensorator identity is

$$\omega'(g,h,j)\beta(h,j)\beta(g,hj) = \omega(g,h,j)\beta(g,h)\beta(gh,j).$$

It asserts $\omega'/\omega = (\delta\beta)^{-1}$. Conversely, the equality makes those tensorators coherent, so each normalized coboundary change gives such a monoidal equivalence. The choice of direction for the tensorator accounts for the inverse.

For $G = C_2$ written with entries $a,b,c \in \{0,1\}$, let $\omega(a,b,c) = (-1)^{abc}$ and again assume $\operatorname{char} k \neq 2$. Expansion of the pentagon exponents in $\mathbb{F}_2$ gives the same sum on both sides, so this is a 3-cocycle. For every normalized 2-cochain β, the value $\delta\beta(1,1,1)$ equals 1: the two occurrences of $\beta(1,1)$ cancel and the other terms involve the identity. Since $\omega(1,1,1) = -1$, this associator cannot be made trivial by such tensorators.

The graded lines remain dualizable after twisting. For $X = k_g$, choose $X^* = k_{g^{-1}}$, coevaluation $k_e \to X \otimes X^*$ with scalar 1, and evaluation $X^* \otimes X \to k_e$ with scalar v. The two triangle composites have scalars

$$v\,\omega(g,g^{-1},g), \qquad v\,\omega(g^{-1},g,g^{-1})^{-1}.$$

The pentagon for (g,g^{-1},g,g^{-1}), together with normalization, makes these factors equal. Taking $v = \omega(g,g^{-1},g)^{-1}$ gives both triangle identities. Applying the same construction to $k_{g^{-1}}$ provides duality in the other direction. Finite direct sums have duals obtained by taking the corresponding direct sums of dual lines, with evaluation and coevaluation pairing the summands. For finite G, this gives a semisimple rigid category with finitely many simple objects, each tensor-invertible.

24.4 The multiplier in a categorical character

A normalized 2-cocycle α gives a weak G-action on finite-dimensional vector spaces, regarded as a linear category. Take each F_g to be the identity functor and let $\mu_{g,h} = \alpha(g,h)\mathrm{id}$. Associativity of a weak action is the 2-cocycle equation in this case. In Chapter 23, the trace space T_g was defined by $\mathrm{Nat}(\mathrm{Id}, F_g)$. Here

it is a single copy of k. For $gh = hg$, the conjugation transport first applies $\mu_{h,g}$ and then $\mu_{g,h}^{-1}$. Thus the scalar on T_g is

$$\chi(g,h) = \frac{\alpha(h,g)}{\alpha(g,h)}.$$

A coboundary is symmetric on commuting pairs, so this expression depends only on the multiplier class. For the pair from the two generators in the matrix example, the value is -1.

The same ratio appears when conjugating ρ_g by ρ_h in a projective representation with multiplier α, since the two orders of the product have the same group index. If the ratio differs from 1, ordinary matrix trace gives $\operatorname{tr} \rho_g = 0$. In the categorical action the ratio itself survives as an operation on a one-dimensional trace space. The scalar twists are detected by the categorical character even for actions whose individual functors are all identities.

For fixed g, the ratio is a one-dimensional character of $C_G(g)$. One can prove this either by the composition law for conjugation transport or directly in the central extension. The commutator of a lift of $h \in C_G(g)$ with a lift of g lies in the central subgroup and equals the ratio. When two commutators lie in the center, the commutator for the product of their first entries is the product of the two commutators. This gives multiplicativity in h and explains how the degree-two cocycle produces an ordinary character on each centralizer.

CHAPTER 25

Hilbert modules, correspondences, and Morita equivalence

A covariant representation combines an algebra action with a group action on a Hilbert space. A correspondence replaces the scalar field of that Hilbert space by a second algebra. The replacement makes it possible to compose two generalized actions by tensoring over their common algebra. In particular, an algebra of kernels may carry the same module theory as the scalars even when its elements, multiplication, and ideals arise from a nontrivial geometric action.

25.1 Hilbert modules and their adjoints

A right Hilbert B-module E, for a C^*-algebra B, has a B-valued inner product. We use the convention linear in the second variable for module inner products. The identities

$$\langle e, fb\rangle_B = \langle e, f\rangle_B b, \qquad \langle e, f\rangle_B^* = \langle f, e\rangle_B$$

relate multiplication to the involution. Positive definiteness and completeness are taken with respect to $\|e\| = \|\langle e, e\rangle_B\|^{1/2}$. In B itself the inner product is b^*c. In a Hilbert space with scalar right action it is its ordinary Hilbert inner product. The bundle example is the space of continuous sections of a Hermitian vector bundle over a compact base X, with pointwise $C(X)$-action and pointwise inner product.

An adjointable endomorphism T has a bounded B-linear adjoint T^* with $\langle Te, f\rangle = \langle e, T^*f\rangle$. These operators form $\mathcal{L}_B(E)$. For $e, f \in E$ put

$$\theta_{e,f}(u) = e\langle f, u\rangle_B.$$

Its adjoint is $\theta_{f,e}$, and $\theta_{e,f}\theta_{u,v} = \theta_{e\langle f,u\rangle_B,v}$. Their closed linear span is the C^*-algebra $\mathcal{K}_B(E)$. For the module B the operators $\theta_{b,c}$ are left multiplications by bc^*, whose linear span is dense in B. Thus $\mathcal{K}_B(B)$ is B, including for nonunital B. On a Hilbert space this construction gives the ordinary compact operators. For other coefficient algebras the term compact refers to the module construction. See [17, Chs. 1, 4, and 7] for the general theory.

25.2 The balanced tensor product

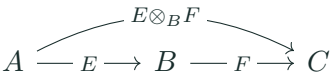
An A-to-B correspondence consists of E_B and a nondegenerate $*$-homomorphism $A \to \mathcal{L}_B(E)$. A representation of A on a Hilbert space is an A-to-$\mathbf{C}$ correspondence. Let F be a B-to-C correspondence, with left action ϕ. On the algebraic balanced product $E \odot_B F$, use

$$\langle e \otimes f, e' \otimes f' \rangle_C = \langle f, \phi(\langle e, e' \rangle_B) f' \rangle_C.$$

For the first balanced relation, replacing e by eb changes the operator inside the inner product to $\phi(b)^* \phi(\langle e, e' \rangle_B)$. This gives the same value as replacing f by $\phi(b) f$. The second relation follows by conjugate symmetry.

To see positivity, let $S : B^n \to E$ be $S(b_1, \dots, b_n) = \sum_i e_i b_i$. Its adjoint has i-th coordinate $\langle e_i, - \rangle_B$, so the matrix $[\langle e_i, e_j \rangle_B]$ is positive because it represents $S^* S$. Applying ϕ entrywise gives a positive operator on F^n. Its quadratic value on $(f_1, \dots, f_n)$ is the inner product of $\sum_i e_i \otimes f_i$ with itself. Quotient by the elements of squared norm zero and complete to obtain $E \otimes_B F$. The A-action descends as a bounded action. For $a \in A$, the Gram matrix of $(a e_i)$ is bounded above by $\|a\|^2$ times the Gram matrix of (e_i), by applying $S^*(-)S$ to $a^* a \le \|a\|^2 1$ in $\mathcal{L}_B(E)$.

If P is a C-to-D correspondence, the two bracketings give the same inner products on triples of pure tensors. The assignment $(e \otimes f) \otimes p \mapsto e \otimes (f \otimes p)$ therefore extends to a surjective unitary. The pentagon for four correspondences holds because both composites agree on pure tensors. The module A_A is an identity correspondence by multiplication and nondegeneracy. With unitary intertwiners as 2-morphisms, the algebras and correspondences form a bicategory.



25.3 A compact free action gives compact operators

Let G be compact with Haar probability and act on itself by left translation. On $\mathcal{H} = L^2(G)$, the covariant representation is multiplication by $C(G)$ together with $U_g \xi(x) = \xi(g^{-1} x)$. If F is a continuous function $(g, x) \mapsto F(g, x)$, the integrated form has kernel K_F:

$$\int_G F(g, x) \xi(g^{-1} x)\, dg = \int_G K_F(x, y) \xi(y)\, dy, \qquad K_F(x, y) = F(x y^{-1}, x).$$

The change $y = g^{-1} x$ preserves Haar measure because compact groups are unimodular and inversion preserves their Haar probability. The correspondence $F \mapsto K_F$ is a linear bijection of the continuous function spaces, with inverse

$F(g,x) = K(x, g^{-1}x)$. Substituting the variables in the crossed-product convolution and applying Fubini gives

$$K_{F*Q}(x,z) = \int_G K_F(x,y) K_Q(y,z)\, dy, \qquad K_{F^*}(x,y) = \overline{K_F(y,x)}.$$

For example, multiplication before the change of variable gives the integrand $F(s,x)Q(s^{-1}g, s^{-1}x)$ at (g,x). At $g = xz^{-1}$ this is precisely the kernel integrand after taking $y = s^{-1}x$.

The full crossed-product norm on F is bounded above by $\int_G \sup_x |F(g,x)|\, dg$, since this bounds every covariant integrated form. Finite sums of $e(x)\overline{f(y)}$ with $e, f \in C(G)$ are uniformly dense in continuous kernels by Stone–Weierstrass. Each finite sum belongs to the matrix algebra of kernels on some finite-dimensional subspace of $C(G)$, obtained by orthonormalizing the span of its e's and f's in $L^2(G)$. All covariant integrated forms restrict to contractive $*$-representations of this matrix algebra. The integrated form on $L^2(G)$ realizes its matrix norm. Thus the full norm and the kernel operator norm agree on finite sums and, by the displayed bound, on all continuous kernels. Finite-rank operators with continuous vectors are norm dense in $\mathcal{K}(\mathcal{H})$, since $C(G)$ is dense in $\mathcal{H}$. It follows that

$$C(G) \rtimes G \simeq \mathcal{K}(L^2(G)). \tag{25.1}$$

This is the translation case of the crossed-product imprimitivity machinery developed in [24].

25.4 The inverse correspondence

A right Hilbert B-module is full if its inner products span a dense linear subspace of B. An A-to-B correspondence that is full and whose left action identifies A with $\mathcal{K}_B(E)$ is an imprimitivity bimodule. It implements strong Morita equivalence. For (25.1), let $E = \mathcal{H}$ as a $\mathcal{K}(\mathcal{H})$-to-$\mathbf{C}$ correspondence. Its conjugate vector space $\overline{E}$ has right action $\bar{e} \cdot a = \overline{a^* e}$ and $\mathcal{K}(\mathcal{H})$-valued inner product $\langle \bar{e}, \bar{f} \rangle = \theta_{e,f}$. The left scalar action is ordinary multiplication on the conjugate space. The maps

$$\begin{aligned} E \otimes_{\mathbf{C}} \overline{E} &\longrightarrow \mathcal{K}(\mathcal{H}), \quad e \otimes \bar{f} \longmapsto \theta_{e,f}, \\ \overline{E} \otimes_{\mathcal{K}(\mathcal{H})} E &\longrightarrow \mathbf{C}, \qquad \bar{e} \otimes f \longmapsto \langle e, f \rangle \end{aligned}$$

preserve balances, inner products, and the left actions. For instance, in the first map the product of the adjoint of $\theta_{e,f}$ with $\theta_{e',f'}$ is $\theta_{f\langle e,e'\rangle,f'}$, exactly the tensor inner product. Their images are dense by the rank-one formulas and the existence of a unit vector. The maps are therefore unitary. Composing twice with these correspondences recovers the starting module by the tensor associator. This yields the inverse equivalences on nondegenerate Hilbert-space representations and explains why a free transitive compact action and a point give the same stabilized invariants in Chapter 26.

25.5 Recovering a Hilbert-space representation

The Morita equivalence can also be seen inside an arbitrary nondegenerate representation of the kernel algebra. Let $\rho : \mathcal{K}(\mathcal{H}) \to \mathcal{L}(\mathcal{V})$ be such a representation and choose a unit vector $e_0 \in \mathcal{H}$. With $p = \theta_{e_0,e_0}$ let $M = \rho(p)\mathcal{V}$. Define

$$W : \mathcal{H} \odot M \longrightarrow \mathcal{V}, \qquad W(e \otimes m) = \rho(\theta_{e,e_0})m.$$

Using $\theta^*_{e,e_0}\theta_{f,e_0} = \langle e, f\rangle p$ gives $\langle W(e \otimes m), W(f \otimes n)\rangle = \langle e, f\rangle\langle m, n\rangle$. The range contains $\rho(\theta_{e,f})v$ for every $v \in \mathcal{V}$, since $\theta_{e,f} = \theta_{e,e_0}\theta_{e_0,f}$ and $\rho(\theta_{e_0,f})v \in M$. Nondegeneracy and density of rank-one operators therefore make W a unitary onto $\mathcal{V}$. The rank-one identities also give $W(ae \otimes m) = \rho(a)W(e \otimes m)$. Thus the multiplicity space M classifies ρ up to unitary equivalence. Choosing another unit vector changes the corner space by the unitary supplied by the corresponding rank-one operator. For the original action $G \curvearrowright G$, the integrated operator includes both translation and multiplication. This result classifies that pair up to multiplicity, whereas a representation of translation alone has the many independent Peter–Weyl types.

CHAPTER 26

Equivariant K-theory and crossed products

A crossed product retains the compatibility between functions on a space and motion by a group. Its K-theory expresses part of that compatibility in terms of stable classes. A projection p in a matrix algebra over $C(X)$ determines a vector bundle by taking the range $p(x)$ at every x. The analogous definition for a general C^*-algebra uses the module pA^n. For unital A, $K_0(A)$ is the Grothendieck group of stable equivalence classes of projections with addition by block sum. The group $K_1(A)$ uses path components of unitaries after adjoining identity blocks. For a nonunital A one uses its unitization and takes the kernel of the map to the scalar summand. These constructions are stable under passage to compact operators on a full Hilbert module, by Morita invariance [3].

26.1 The compact comparison

Suppose G is a second-countable compact group and A a separable G-C^*-algebra with a continuous action. In the compact Green–Julg theorem the trivial algebra $\mathbf{C}$ occurs in the first Kasparov variable. The natural isomorphisms are

$$KK_i^G(\mathbf{C}, A) \simeq K_i(A \rtimes G), \qquad i = 0, 1.$$

One description descends a class to a correspondence from $C^*(G)$ to $A \rtimes G$ and takes its Kasparov product with the class of the projection for the trivial representation. In $C^*(G)$ this is the continuous function $g \mapsto 1$: convolution with itself remains 1 because Haar has mass one. The same compact theorem applies to the reduced product since compact groups are amenable. For $A = C(X)$ with X compact metrizable and G compact Lie, $KK_0^G(\mathbf{C}, C(X))$ identifies with the Grothendieck group $K_G^0(X)$ of equivariant complex vector bundles, by the equivariant projection description. Odd topological K-theory is defined by suspension. The crossed product identification is an isomorphism of groups. Tensor multiplication of bundles is extra structure on the geometric side [3, Sec. 11.7][23].

The hypotheses in this statement do work. For example, a general locally compact group has no Haar probability and the constant function need not furnish a projection in its group algebra. To see how the compact theorem

behaves, it helps to compare the two transitive extremes, a point with stabilizer G and G itself with trivial stabilizer.

26.2 Point, free orbit, and homogeneous orbit

At a point $K_G^0(*) = R(G)$, the Grothendieck group of finite-dimensional continuous complex representations, with multiplication from tensor product. The full group C^*-algebra satisfies

$$C^*(G) \simeq \bigoplus_{\pi \in \widehat{G}}^{c_0} \operatorname{End}(V_\pi).$$

Indeed, the integrated map uses $\pi(F) = \int_G F(g)\pi(g)\,dg$. The full norm is the supremum of these matrix norms because every unitary representation decomposes into irreducibles. Peter–Weyl orthogonality supplies a continuous function for any single matrix block, with all other blocks zero. Finite sums of matrix coefficients are uniformly dense in continuous functions, hence dense in $L^1(G)$. The estimate $\|\pi(F)\| \le \|F\|_1$ then gives precisely the c_0-completion. For finite sums of full matrix algebras the K_0-coordinate is matrix rank and the unitary groups are path connected. Passing to the limit over finite sets of blocks yields

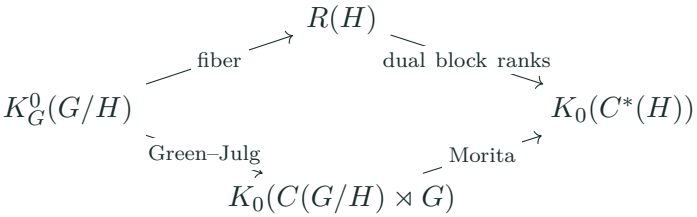
$$K_0(C^*(G)) = \bigoplus_{\pi \in \widehat{G}} \mathbf{Z}, \qquad K_1(C^*(G)) = 0.$$

The limit can also be seen from the norm topology. A projection in matrices over the c_0-sum has finitely many nonzero components since a nonzero projection has norm one. A unitary with scalar part 1 has tail uniformly near 1. On that tail one contracts it by the logarithm, while separately contracting each of the finitely many remaining finite-dimensional unitaries.

For the free orbit G, an equivariant bundle E has canonical trivialization over the group given by $(g, v) \mapsto gv$ for $v \in E_1$. Its inverse sends $w \in E_g$ to $(g, g^{-1}w)$. The vector space E_1 therefore determines its class and $K_G^0(G) = \mathbf{Z}$. On the algebraic side, Chapter 25 identifies $C(G) \rtimes G$ with the compacts, so this agrees with the compact-operator rank calculation.

For a closed subgroup H of a compact Lie group G, evaluate a G-equivariant vector bundle on G/H at $1H$. The fiber is an H-representation and an inverse constructs $G \times_H V$ from V. The defining relation is $(gh, v) \sim (g, hv)$; the repair of local coordinates on two sections of $G \to G/H$ is consequently the action of the transition function on V. For an input bundle E, the formula $[g, v] \mapsto gv$ gives an equivariant linear bijection on fibers and a bundle isomorphism by local triviality. Since both operations preserve direct sum and tensor product, this proves $K_G^0(G/H) \simeq R(H)$ as rings. On the algebra side, Green's imprimitivity

theorem gives a Morita equivalence from $C(G/H) \rtimes G$ to $C^*(H)$. Thus either path computes the even group and $K_1(C(G/H) \rtimes G) = 0$ [24, Thm. 4.22].



The block convention in this diagram can be checked using projections. Present an equivariant bundle E as the range of $p(x)$ in a trivial bundle with finite-dimensional unitary fiber representation ρ. The covariance equation is $p(gx) = \rho(g)p(x)\rho(g)^*$. In matrices over the crossed product, the descended class comes from the kernel $F(g, x) = p(x)\rho(g)$: it is the product of the bundle projection with the averaging projection for the diagonal action. Covariance and normalization of Haar give $F * F = F = F^*$.

Under the homogeneous imprimitivity equivalence, the block indexed by an irreducible H-representation η acts on sections of $E_\eta = G \times_H V_\eta$; it uses multiplication and the induced G-action U^η_g. There F acts as p times the averaging operator for $\rho(g) \otimes U^\eta_g$. These orthogonal projections commute by covariance. The range is the space of invariant sections of $E \otimes E_\eta$, and averaging continuous sections has image in the finite-dimensional space determined by an H-invariant value at o, since transitivity forces $s(go) = gs(o)$. Continuous sections are dense, and the finite-dimensional image is closed. Thus the range in the Hilbert space has dimension

$$\dim(E_o \otimes V_\eta)^H = \dim \mathrm{Hom}_H(V^*_\eta, E_o).$$

Thus with integrated block convention $\eta(F) = \int_H F(h)\eta(h)\, dh$, the rank of a descended bundle records the multiplicity of the dual type in its fiber. This specifies the diagram on actual bundles, and differences follow by additivity. Labelling the blocks by dual representations absorbs this indexing convention.

26.3 The rotation action on the sphere

Take $G = \mathrm{SO}(3)$ and $H = \mathrm{SO}(2)$, the rotations about the north–south axis. The orbit is S^2, and

$$K^0_{\mathrm{SO}(3)}(S^2) = \mathbf{Z}[z, z^{-1}], \qquad K_1(C(S^2) \rtimes \mathrm{SO}(3)) = 0,$$

where z is the weight-one character of H. Tensoring by a representation of G acts on the Laurent polynomial ring through its restriction to H. This follows by taking the fiber of $E \otimes (S^2 \times W)$ at the north pole: the H-action there is the tensor product of the action on E_{1H} and the restricted action on W.

Write w for the class of the complexified defining representation of G. Its restriction is $1+z+z^{-1}$. The degree-ℓ irreducible V_ℓ has weights $-\ell, -\ell+1, \ldots, \ell$ on H, each once, as in the rotation calculation of [22, Sec. 10]. Restricted characters distinguish representations of G: every rotation is conjugate to an element of H, and characters distinguish compact representations. Multiplying the weight sum by $1+z+z^{-1}$ gives the recurrence $w[V_\ell] = [V_{\ell+1}]+[V_\ell]+[V_{\ell-1}]$ for $\ell \geq 1$. Thus w generates $R(G)$, and its powers are independent since the Laurent polynomial w^n has highest exponent n with coefficient one. Over $R(G) = \mathbf{Z}[w]$, the equivariant K-group of the sphere is free with basis $1, z$. In fact $z^{-1} = w-1-z$ and $z^2 = (w-1)z - 1$ prove generation. If $a(w) + b(w)z = 0$, apply $z \mapsto z^{-1}$ and subtract. Then $b(w)(z - z^{-1}) = 0$ in an integral domain, so $b(w) = a(w) = 0$. This determines the action of every virtual rotation representation on the stable bundle classes.

CHAPTER 27

Equivariant convolution and learning

When a group acts on inputs, an equivariant layer takes the same action on its outputs. For rotation data, this might mean rotating a field of scalar intensities, or rotating both the base point and the components of a tangent vector field. The feature type specifies the representation on the output. With the input and output types fixed, Schur's lemma gives the full space of bounded linear layers. The descriptions by kernels in [16, 6] are especially useful when a layer has finite spectral support or is defined on fields with an appropriate kernel class.

27.1 The bounded linear classification

Let G be compact and let P, Q be unitary Hilbert representations. Write their Peter–Weyl decompositions as

$$P = \widehat{\bigoplus}_{\pi \in \widehat{G}} V_\pi \otimes M_\pi, \qquad Q = \widehat{\bigoplus}_{\pi \in \widehat{G}} V_\pi \otimes N_\pi.$$

Each multiplicity space M_π or N_π carries the trivial action. The bounded equivariant linear maps $T : P \to Q$ are exactly

$$T = \widehat{\bigoplus}_{\pi \in \widehat{G}} (1_{V_\pi} \otimes A_\pi), \quad A_\pi \in \mathcal{L}(M_\pi, N_\pi), \quad \|T\| = \sup_\pi \|A_\pi\| < \infty. \tag{27.1}$$

Here $\mathcal{L}(M_\pi, N_\pi)$ denotes bounded linear maps of Hilbert spaces.

For the proof, the operator $d_\pi \int_G \overline{\chi_\pi(g)} U_g \, dg$, with $d_\pi = \dim V_\pi$, is the orthogonal projection onto the π-isotypic part: Schur's lemma makes the integral scalar on each irreducible, and character orthogonality determines its trace. A bounded intertwiner commutes with these projections on the two spaces. To compute the π-block, for $m \in M_\pi$ and $n \in N_\pi$ apply the block to $v \otimes m$ and pair its N_π-coordinate with n. The resulting endomorphism of V_π commutes with π, so it is scalar. Equivalently, in an orthonormal basis of V_π, the off-diagonal operator entries $M_\pi \to N_\pi$ have every scalar matrix element zero and all diagonal entries agree. Their common value is A_π. The block norm is $\|A_\pi\|$. Conversely, a bounded family of those norms gives a bounded operator on the dense algebraic direct sum by summing the squared norm estimates, and its completion intertwines the two actions. Testing on a single isotypic part gives equality of the norms.

For finitely many types and finite multiplicities, the number of complex parameters is

$$\sum_{\pi \in \widehat{G}} (\dim M_\pi)(\dim N_\pi).$$

For scalar $L^2(S^2)$, M_ℓ is one-dimensional for every integer $\ell \geq 0$, because the $H = \mathrm{SO}(2)$-fixed subspace of each V_ℓ has dimension one. With c input channels and d output channels each spectral block is an arbitrary d-by-c matrix, applied in the same way to all $2\ell + 1$ coordinates of V_ℓ.

27.2 Regular convolution and its scope

On the left regular representation $L^2(G)$, a right convolution operator $T_k f = f * k$ for $k \in L^1(G)$ commutes with left translation by change of variables. With Haar probability, use

$$(f * k)(x) = \int_G f(y) k(y^{-1}x)\, dy, \qquad \widehat{f}(\pi) = \int_G f(g)\pi(g)^*\, dg.$$

The Fourier multiplier is $\widehat{f * k}(\pi) = \widehat{k}(\pi)\widehat{f}(\pi)$: substitute $x = yz$ in the double integral and multiply $\pi(z)^*\pi(y)^*$ in that order. Left translation by a acts on $\widehat{f}(\pi)$ by right multiplication by $\pi(a)^*$, so the position of $\widehat{k}(\pi)$ agrees with (27.1). Peter–Weyl orthogonality gives an actual continuous kernel for every finite-support family of matrix multipliers.

For $k \in L^1(G)$ the convolution norm is bounded by $\|k\|_1$. Indeed, rewriting the convolution as $\int_G k(z) f(xz^{-1})\, dz$ expresses it as an integral of right translations, all L^2-isometries. Approximation of k in L^1 by finite Peter–Weyl sums shows that every such T_k is compact on $L^2(G)$. For an infinite compact group the identity is bounded and equivariant and has infinite-dimensional range, so it cannot have an L^1 convolution kernel. It is convolution by the Dirac measure at the identity. Formula (27.1) applies to all bounded intertwiners independently of such kernel realizations.

27.3 Pointwise layers and feature types

Let $X = G/H$, and let V, W be finite-dimensional H-representations with actions ρ_V, ρ_W. A pointwise activation sends the fiber coordinate v of $[g, v]$ to $[g, \sigma(v)]$. The relation $[gh, v] = [g, \rho_V(h)v]$ makes this rule independent of the representative exactly when

$$\sigma(\rho_V(h)v) = \rho_W(h)\sigma(v) \qquad (h \in H).$$

The same formula then implies G-equivariance under left translation of g. An arbitrary scalar activation works for scalar fields, since the stabilizer acts trivially

on the fiber. A coordinatewise activation on a vector fiber need not satisfy the relation. For a unitary fiber a map $v \mapsto a(\|v\|)v$ satisfies it. If the activation is Lipschitz, it induces a Lipschitz map on L^2-sections over the finite invariant base measure: the pointwise estimate integrates, and $\|\sigma(v)\| \le \|\sigma(0)\| + L\|v\|$ ensures square integrability. A nonlinear layer need not preserve the spectral truncation of its input.

27.4 A rotation-invariant quadratic feature

Give S^2 its rotation-invariant probability measure. For real scalar inputs define

$$(Tf)(x) = 5 \int_{S^2} P_2(x \cdot y) f(y)\, d\mu(y), \qquad P_2(t) = \tfrac{1}{2}(3t^2 - 1).$$

The kernel uses only the dot product, so it is unchanged by simultaneous rotation of x and y. For fixed y, it is a traceless quadratic in x on the sphere. To determine its action, take $f_A(y) = y^{\mathsf{T}} A y$ for a real symmetric traceless matrix A. The moments are

$$\int y_i y_j \, d\mu = \tfrac{1}{3}\delta_{ij}, \qquad \int y_i y_j y_k y_l \, d\mu = \tfrac{1}{15}(\delta_{ij}\delta_{kl} + \delta_{ik}\delta_{jl} + \delta_{il}\delta_{jk}).$$

For the fourth moment, the polynomial in v obtained by integrating $(v \cdot y)^4$ is rotation invariant and homogeneous. Its value at a unit vector is $1/5$, from the sphere's area measure in the height coordinate $t \in [-1, 1]$. Comparing coefficients, or polarizing $\|v\|^4/5$, gives the tensor written above. The second moment is obtained from $\|y\|^2 = 1$ and rotation symmetry. Consequently

$$\int f_A \, d\mu = 0, \qquad \int (x \cdot y)^2 f_A(y)\, d\mu(y) = \tfrac{2}{15} x^{\mathsf{T}} A x.$$

Thus $Tf_A = f_A$. The kernel is symmetric and the range lies in the traceless quadratics, so T is their orthogonal projection.

Apply a scalar square after the projection and then pool by integration. The projection is finite rank with continuous outputs, so the square is defined and integrable for every L^2 input. It is an equivariant pointwise layer followed by an invariant pooling layer.

$$L^2(S^2; \mathbf{R}) \xrightarrow{\ T\ } V_2^{\mathbf{R}} \xrightarrow{\ u \mapsto u^2\ } C(S^2; \mathbf{R}) \xrightarrow{\ \int_{S^2}(-)\, d\mu\ } \mathbf{R}$$

Its value on f_A is $2\,\mathrm{tr}(A^2)/15$, from the fourth moment. For example, $A = \mathrm{diag}(1, -1, 0)$ gives $4/15$, and a constant input gives zero. A channel version first projects each channel and then applies any matrix of linear coefficients to those projected channels, as allowed by (27.1); each squared pooled output is invariant for every choice of those coefficients.

27.5 Kernels for associated bundles

Take the spaces of sections of $E = G \times_H V$ and $F = G \times_H W$ with invariant measures and Hermitian metrics. Suppose an intertwiner T has finite spectral support. For each retained G-type there are only finitely many copies by compact Frobenius reciprocity, since V and W are finite dimensional. The retained subspaces consist of continuous sections, given by matrix coefficients as in Chapter 28. For an orthonormal basis s_j of the retained input, define $K(x, y) : E_y \to F_x$ by

$$K(x, y)v = \sum_j (Ts_j)(x)\langle s_j(y), v\rangle.$$

Use the Hermitian product linear in the second argument. The orthonormal expansion gives $(Ts)(x) = \int K(x, y)s(y)\, d\mu(y)$ and makes the formula independent of the chosen basis. Replacing s_j by the translated basis and using $TU_a = U_aT$ shows $K(ax, ay) = aK(x, y)a^{-1}$, with the two actions interpreted on the appropriate fibers. Conversely, a continuous kernel satisfying this covariance defines a bounded equivariant operator by change of variables and the uniform bound on the kernel over the compact base. For each output point one can transport the kernel at o to that point, and the result is independent of the transporting element precisely because the kernel at o intertwines the H-actions on the fibers over y and hy. This describes how global rotation equivariance and local fiber coordinates constrain the same layer.

CHAPTER 28

An orbit, its stabilizer, and its invariants

The sphere carries an ordinary topology, an action of SO(3), and several complexes of sections. Each operation applied to this data has a different output. Taking the quotient as a set gives one point. Taking the quotient as a groupoid retains the rotations about an axis. Passing to a Hilbert space of sections takes a representation of those rotations and makes a representation of the whole group. The computation below compares these operations for the same line bundles.

28.1 A transitive action as a groupoid

Let G be a compact Lie group, H a closed subgroup, and $X = G/H$. The action groupoid $G \ltimes X$ has objects $x \in X$ and arrows $(g, x) : x \to gx$. The map from the one-object groupoid of H sends its object to $o = 1H$ and its arrows to the arrows $o \to o$. This map is fully faithful because $\{g : go = o\} = H$. It is smoothly essentially surjective because $g \mapsto go$ is a surjective submersion with local sections. At the level of equivariant vector bundles, the resulting comparison is the fiber construction of Chapter 26. The bundle action records how to transport a fiber along every arrow of the action groupoid, including the automorphisms of o.

The quotient stack has a related description that also keeps track of families. Over a smooth test manifold S, an object of $[X/G]$ is a principal right G-bundle $P \to S$ with a map $u : P \to X$ such that $u(pg) = g^{-1}u(p)$. The subset $Q = u^{-1}(o)$ is a principal right H-bundle. To see its local triviality, choose a local section $p : S_0 \to P$, lift $u(p(s))$ to $a(s) \in G$ on smaller neighborhoods using sections of $G \to G/H$, and use $p(s)a(s)$ as the local section of Q. Right multiplication on this section by $h \in H$ gives all of Q over the neighborhood.

Conversely, an H-bundle $Q \to S$ gives $P = Q \times_H G$ with $u([q, g]) = g^{-1}H$. This is well defined because replacing (qh, g) by (q, hg) leaves $g^{-1}H$ unchanged. Its fiber over o recovers Q by $q \mapsto [q, 1]$. In the other direction $[q, g] \mapsto qg$ is a G-bundle isomorphism giving the starting u. Both constructions are functorial in bundle isomorphisms and commute with pullback to another test manifold. This proves the equivalence of quotient stacks $[G/H \,/\, G] \simeq BH$, where BH in the stack description assigns principal H-bundles to tests.

The Borel construction $EG \times_G X$ has a corresponding statement for classifying spaces. The map $[e, gH] \mapsto egH$ is a homeomorphism onto EG/H for a contractible free universal G-space EG, with inverse $eH \mapsto [e, H]$. The latter quotient is a model of the classifying space of H. For $H = \mathrm{SO}(2) \simeq S^1$, one can use $\mathbf{CP}^\infty$ as this classifying space. Its second integral cohomology is infinite cyclic, and the universal line associated to the weight-one character has first Chern class a generator [14, Ch. 3]. These are cohomological invariants of the homotopy quotient. The ordinary set orbit captures only transitivity.

28.2 Weights, bundles, and section representations

Return to $G = \mathrm{SO}(3)$, $H = \mathrm{SO}(2)$, with o the north pole of S^2. Write $\chi_m(e^{it}) = e^{imt}$ for the weight $m \in \mathbf{Z}$ character and $L_m = G \times_H \mathbf{C}_m$. With $[gh, v] = [g, \chi_m(h)v]$, a section is represented by a function $f : G \to \mathbf{C}$ satisfying $f(gh) = \chi_m(h)^{-1} f(g)$. Completing under the invariant probability measure gives the compact induced representation

$$L^2(S^2, L_m) = \mathrm{Ind}_H^G \chi_m.$$

Compact Frobenius reciprocity calculates the multiplicity of any continuous finite-dimensional irreducible (V, π) as the dimension of $\mathrm{Hom}_H(V|_H, \mathbf{C}_m)$. Concretely, an H-map b gives the intertwiner with v-section $f_v(g) = b(\pi(g^{-1})v)$. The covariance equation follows by moving $\pi(h^{-1})$ across b. On the continuous matrix coefficients, evaluation at 1 gives the inverse formula. Haar projection onto an irreducible isotypic type gives the same comparison in the Hilbert completion, as in compact Frobenius reciprocity [22, Sec. 8].

The weights of V_ℓ on H are the integers from $-\ell$ through ℓ, each with multiplicity one. Hence

$$L^2(S^2, L_m) \simeq \widehat{\bigoplus}_{\ell \geq |m|} V_\ell. \tag{28.1}$$

For $m = 0$, the bundle is the scalar line and this yields the multiplicities of the rotation layers in Chapter 27. For any two weights m, n, an equivariant bounded linear operator between their section spaces consists of one scalar for every $\ell \geq \max(|m|, |n|)$, with uniformly bounded moduli, by (27.1). The scalars depend on choices of unitary identifications of the common irreducibles. All absent types give zero blocks. This describes an operator between different field types even when they live over the same base.

28.3 Opposite weights with the same spectral types

Orient S^2 so that the action on its tangent plane at o is the standard complex rotation. The equivariant real tangent bundle, viewed as a complex line by the

oriented metric, is L_1, since its fiber representation at o is χ_1. Associated bundles commute with tensor product and duals, so $L_m = L_1^{\otimes m}$ with dual powers for negative m. The first Chern class of the tangent line is the Euler class of the tangent plane [14, Ch. 3]. Evaluation on the fundamental class gives $\chi(S^2) = 2$, hence

$$c_1(L_m) = 2m\,u, \qquad u \in H^2(S^2;\mathbf{Z}), \quad \langle u, [S^2]\rangle = 1.$$

The equivariant first Chern class is mc, where c is the weight-one generator on EG/H. The map from a fiber S^2 of $EG \times_G S^2 \to BG$ pulls back the associated equivariant line to the ordinary line L_m, so it sends c to $2u$.

$$S^2 \ \text{fiber inclusion}\ EG \times_G S^2 \longrightarrow\!\!\!\!\!\!\simeq\ BH$$

$$H^2(S^2;\mathbf{Z}) \ \text{restriction}\ H^2_G(S^2;\mathbf{Z}) \longleftarrow\!\!\!\!\!\!\simeq\ H^2(BH;\mathbf{Z})$$

In particular, L_1 and L_{-1} are inequivalent even as ordinary complex bundles, since their Chern classes are $2u$ and $-2u$. Their equivariant K-classes are z and z^{-1} in the Laurent polynomial ring. Nevertheless their section representations are unitarily equivalent as G-representations by (28.1). Such an equivalence is obtained by choosing a unitary intertwiner on each V_ℓ, $\ell \geq 1$. It cannot be induced by a complex bundle isomorphism. In this example the stabilizer character determines the bundle, while passing to the bare section representation identifies two different characters. The multiplication action of functions on the base is additional data in the induced covariant representation, as the transitive imprimitivity theorem makes explicit [24, Sec. 4.5].

28.4 Pointwise changes of weight

The distinction between bundles and spaces of sections gives a direct constraint on nonlinear features. Consider a rule $L_m \to L_n$ of the form $[g, v] \mapsto [g, \sigma(v)]$, and suppose $m \neq 0$. By the associated-bundle relation, it exists precisely when

$$\sigma(e^{imt}v) = e^{int}\sigma(v) \qquad (t \in \mathbf{R},\ v \in \mathbf{C}).$$

If σ takes a nonzero value, any element of the kernel of χ_m must lie in the kernel of χ_n, so m divides n. Conversely, if $n = km$ with $k > 0$, then $\sigma(v) = v^k$ works, and if $k < 0$, then $\sigma(v) = \overline{v}^{-k}$ works. For $k = 0$, $\sigma(v) = |v|^2$ works. When $m = 0$ and $n \neq 0$, covariance forces $\sigma = 0$ because the input action is trivial and the output has no fixed vectors. These polynomial activations always take continuous sections to continuous sections on the compact sphere. They also act on finite spectral subspaces, where all input sections are continuous. For

arbitrary L^2 inputs a higher power may require stronger integrability, as the pointwise covariance condition concerns the fibers alone.

For instance, from weight two to weight three no nonzero pointwise rule is possible. There is a nonzero bounded linear equivariant operator between the two section spaces: restrict to the copy of V_3 in the domain, identify it unitarily with the copy of V_3 in the target, and annihilate every other type. The two multiplicity calculations impose different conditions because the global operator uses transport across the sphere, while the pointwise rule performs its entire construction in one fiber.

APPENDIX A

Notation and conventions

G, H, K
: Groups, with H and K also used for subgroups of G. In the analytic chapters closed subgroups carry their own left Haar measure. The quotient G/H consists of the cosets gH.

dg, Δ_G
: Left Haar measure and the modular function. Compact Haar measure has mass one. We use $\int f(xg)\,dx = \Delta_G(g)^{-1}\int f(x)\,dx$. See Chapters 1 and 9.

$\widehat{G}$
: The set of unitary equivalence classes of irreducible continuous unitary representations, equipped with the topology or measurable structure specified in context. For abelian G it is the Pontryagin dual. See Chapters 3, 12, and 15.

$\pi(f)$, $\widehat{f}(\pi)$
: The integrated operator $\int f(g)\pi(g)\,dg$ and the compact Fourier coefficient $\int f(g)\pi(g)^*\,dg$, respectively. The latter reverses convolution order. See Chapters 4 and 10.

χ_V, Ind_H^G
: The character $g \mapsto \mathrm{tr}(\rho_V(g))$ of a finite-dimensional representation and induction from H to G. The continuous model uses sections with $f(gh) = \rho_H(h)^{-1}f(g)$. The discrete derived model uses $k[G] \otimes^L_{k[H]} (-)$. See Chapters 5, 6, and 19.

BG, $B_{\mathrm{top}}G$, $B\mathbf{G}$
: The classifying groupoid of a discrete group, the classifying space of a topological group, and the stack of locally trivial torsors, respectively. See Chapters 17 and 20.

$D(k)$, $mathrmSp$, $mathbbS[G]$
: The derived category in its stable ∞-categorical form, the category of spectra, and the group ring spectrum $\Sigma^\infty_+ G$. We use cohomological grading on complexes, so $H^n(X[1]) = H^{n+1}(X)$. See Chapter 17.

X^{hG}, X_{hG}, X^{tG}
: Homotopy fixed points, homotopy orbits, and the cofiber of the finite-group norm, respectively. See Chapter 18.

$\mathrm{QCoh}(X)$, $\mathrm{Perf}(X)$
: The stable ∞-category of quasicoherent complexes and the full subcategory of perfect complexes on X. See Chapter 22.

$C^*(G)$, $C^*_r(G)$, $A \rtimes G$
: The full and reduced group C^*-algebras, and the full crossed product for an action on A. See Chapters 10 and 26.

APPENDIX B

Subject index

The references link to the chapters where each term is developed.

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Characters Ordinary, 5. Categorical, 23. Tensor twist, 24.

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Convolution Compact Fourier, 4. Group algebras, 10. Equivariant layers, 27.

Correspondence Hilbert C^*-modules, 25.

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Discrete series Formal degree, 11. $\mathrm{SL}_2(\mathbb{R})$, 14.

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