RESEARCH MANUSCRIPT NOVEMBER 2026

Maxwell Electromagnetism in a Smooth ∞-Topos

Connections, constitutive laws, and the descent of sourced classical fields

Anonymous manuscript

Prepared for *Higher Categorical Semantics*

Abstract

We formulate classical Maxwell electromagnetism as a homotopy pullback in an ∞-topos of smooth families on Lorentzian spacetimes. Magnetic closure is built into the curvature of a circle connection. The remaining Maxwell equation is the fiber over a closed electric current of the composite $\mathrm{d} \star_g \mathrm{curv}$. Comparison with real field strengths identifies the integral-period image, its flat-character ambiguity, and its gauge automorphisms. A premetric formulation keeps the constitutive law visible and makes its base-change properties explicit. We prove descent in spacetime and parameter directions, analyze the obstruction posed by nonexact currents, and compare the field stack with Cauchy data, holonomy, duality, and the derived linearized equations. Historical and technical notes trace the passage from Maxwell and Heaviside to exterior forms, differential characters, locally covariant gauge theory, and cohesive higher toposes. A Rust companion renders four geometric configurations.

Keywords. Maxwell equations, smooth stacks, differential cohomology, circle connections, Lorentzian geometry, descent.

Preparation. AI assistance was used in research synthesis, mathematical exposition, and the accompanying software. Anonymous review copy.

Contents

1 The modern equations and their provenance **3**
1.1 Which equations are being categorified? . . . 3
1.2 From Maxwell to spacetime . . . 3
1.3 What the categorical account retains . . . 4
1.4 Which category carries the equations? . . . 5

2 Exterior equations and constitutive data **6**
2.1 Forms, orientations, and the current . . . 6
2.2 Recovering the four equations . . . 6
2.3 Naturality, conformal invariance, and characteristics . . . 7
2.4 Integration and charge balance . . . 8

3 Two smooth higher toposes and the family convention **9**
3.1 The smooth classifying environment . . . 9
3.2 The product site of parameters and spacetimes . . . 9
3.3 Relative forms and the mapping-stack distinction . . . 10
3.4 Descent before solving a PDE . . . 11

4 The connection stack and differential cohomology **12**
4.1 A Cech–Deligne presentation . . . 12
4.2 The curvature sequence and its stack refinement . . . 12
4.3 Why global integrality is different from local image . . . 13

5 The Maxwell solution object **15**
5.1 An equation map over closed currents . . . 15
5.2 The real theory and the integral lift . . . 16
5.3 Constitutive relations as spans . . . 16
5.4 Matching across a collar . . . 17

6 Global sectors, current obstructions, and holonomy **18**
6.1 An affine space of curvatures and a torsor of connections . . . 18
6.2 An exact obstruction for the real field problem . . . 18
6.3 Wilson data and thin transport . . . 19
6.4 Electric sources and magnetic defects . . . 20

7 Relative lifting and smooth parameter families **21**
7.1 The vertical coefficient sheaves . . . 21
7.2 The relative obstruction class . . . 21
7.3 A compact-fiber criterion . . . 22
7.4 Truncating before and after sections . . . 23

8 Cauchy data and the role of hyperbolicity **24**
8.1 Restriction to a spacelike hypersurface . . . 24
8.2 A complete ultrastatic comparison . . . 24
8.3 Exactness, time slices, and what is propagated . . . 25
8.4 Local solvability and the internal source map . . . 26

9 Electric–magnetic duality and higher gauge degree **28**
9.1 Real vacuum rotations . . . 28
9.2 Why rotations generally fail on the connection stack . . . 28
9.3 Relation to duality in quantum theory . . . 29
9.4 Higher-form Maxwell equations . . . 29
9.5 Normalization and changes of coefficient lattice . . . 30

10 Variation, observables, and derived refinements **31**
10.1 The Euler equation on a fixed topological sector . . . 31
10.2 A conserved presymplectic current . . . 31
10.3 The linear deformation complex . . . 32
10.4 Gauge arrows, observables, and local covariance . . . 33

11 Four geometric configurations **34**
11.1 A transverse plane wave . . . 34
11.2 A monopole on a punctured spatial slice . . . 34
11.3 A flat Aharonov–Bohm connection . . . 36
11.4 Quantized flux on a periodic three-torus . . . 37
11.5 Reproducing the scenes . . . 39

12 From differential forms to cells **40**
12.1 The exact chain-level structure . . . 40
12.2 Mesh incidence and metric approximation . . . 40
12.3 What would have to commute under discretization . . . 41
12.4 A cochain model of charge conservation . . . 42

13 Notes on the literature and scope of the formulation **43**
13.1 Primary texts and modern translation . . . 43
13.2 Differential cohomology and stacks . . . 44
13.3 Direct predecessors for the equation and the phase space . . . 45
13.4 Causal analysis, quantization, and computation . . . 46
13.5 Summary of hypotheses and claims . . . 47

References **48**

1 The modern equations and their provenance

1.1 Which equations are being categorified?

In a Cartesian frame classical electromagnetism is stated as four vector-calculus equations. On spacetime it is stated as two exterior differential equations together with a constitutive relation. In a smooth higher topos the equations can be imposed on a moduli object of connections, whose values carry both fields and their isomorphisms. The inverse comparison is part of the problem. Evaluating the moduli object on an ordinary spacetime should recover the exterior equations. A chosen Cartesian frame should then recover the four vector equations.

We fix the electric current and start with an ordinary circle gauge field on a background. The current is prescribed independently of any charged matter equation. There are two choices of global field content. The field strength can be any real two-form satisfying the equations, or it can be required to be the curvature of a circle connection. The theories agree locally at the curvature level. The first admits real magnetic periods. In the second the periods lie in a lattice, and a connection carries holonomy information. We retain both and construct the comparison map.

Fix a spacetime M with orientation and Lorentzian metric g of signature $(-+++)$, and put $\mathbb{T} = \mathbb{R}/\mathbb{Z}$. The normalization is such that the Wilson phase of a unit charge is $\exp(2\pi i \int A)$ in a local trivialization. Curvatures then have integral periods. With the positive vacuum coupling absorbed into the current and excitation,

$$\mathrm{d}F = 0, \qquad \mathrm{d}G = J, \qquad G = \star_g F. \tag{1.1}$$

Here F, G have degree two and J degree three. In unnormalized units the constitutive map includes a positive constant. On an unoriented spacetime the excitation and current take values in the orientation local system.

1.2 From Maxwell to spacetime

Maxwell's 1865 paper and 1873 *Treatise* are primary sources for the field-theoretic account and the use of displacement current [1, 2]. Their organization and symbols differ from current vector notation. The dynamical argument connects electromagnetic induction, the circuital law, displacement, and propagation of disturbances. Reading today's divergence and curl equations back into the primary texts amounts to translation between frameworks. Hunt describes changes in Maxwell's use of measurement and dynamical reasoning [3]. Jackson and Okun trace the emergence of the modern gauge concept from classical potential theory [4].

Heaviside's vectorial recasting is central to the compact modern form. His *Electromagnetic Theory* is a primary source for the operational style. Historical accounts

distinguish his early journal series from the later collected volumes [5, 6]. The compact form makes electric and magnetic fields available directly for evolution. Potentials remain useful for other calculations. A connection makes the difference between a potential in a gauge and a global geometric field explicit.

Minkowski's 1908 work places the electrodynamics of moving bodies in four-dimensional spacetime [7]. Different observers split the same antisymmetric field tensor differently. The Hodge operator introduces dependence on metric data. Exterior calculus separates that operation from the exterior derivatives. The premetric program developed by Hehl and Obukhov makes the distinction an organizing principle [8, 9, 10]. The ordinary and twisted forms below follow that separation.

1.3 What the categorical account retains

On each member of a sufficiently fine cover one chooses A_i with $F|_{U_i} = \mathrm{d}A_i$. On intersections there are transition functions g_{ij}, and their multiplication on triple intersections satisfies a cocycle law. Changing local trivializations changes the A_i and g_{ij} together. Compatible curvature forms can glue even when chosen potentials cannot be made into a single global one-form. A solution sheaf of field strengths handles the first operation. A solution stack of connections handles the gauge and gluing data too.

Let $\mathcal{C}$ be the connection stack and $\mathcal{J}$ the sheaf of closed currents on a site of Lorentzian backgrounds. The universal equation map is

$$E_g \colon \mathcal{C} \longrightarrow \mathcal{J}, \qquad E_g(P, \nabla) = \mathrm{d} \star_g \operatorname{curv}(P, \nabla).$$

The fiber over J is a homotopy pullback. The current sheaf is discrete, so a witness of equality of two currents is either absent or unique. Isomorphisms between connections in the fiber survive. The same construction makes base change explicit. Pullback commutes with d for all smooth maps and with $\star$ for suitable metric maps.

The differential-cohomological Maxwell kernel on isomorphism classes appears in Becker, Schenkel, and Szabo [26]. Benini, Schenkel, and Schreiber construct the Yang–Mills solution stack and its smooth families [27]. Cohesive higher toposes and differential refinements of classifying objects are developed by Schreiber [16]. The contribution here is a unified sourced presentation with the closed-current base, premetric relation, curvature image and full fibers, and parameter convention made explicit. It specializes and relates these established constructions.

There are three distinct locality conditions. A connection can be trivialized on sufficiently small spacetime opens. A smooth family can be restricted on a parameter manifold while leaving spacetime fixed. Local covariance describes the behavior under allowed spacetime embeddings, and for observables it has its own causal hypotheses. Sections 2–4 establish the exterior and stack input. Sections 5–6 compare solution

objects. Later sections address evolution, duality, variation, and geometry. Ordinary mathematical proofs and cited analytic results provide the evidence for the claims. The Rust scenes display selected exact fields and cocycles by sampling them.

1.4 Which category carries the equations?

One could put spacetimes in a category, send each to its set of classical solutions, and call functoriality a categorical treatment. That is a useful first level, but it does not specify the right relation between a local potential and a global gauge field. For that purpose we want objects over every background, arrows between objects over a background, and a gluing condition for these groupoids. The higher topos is the category in which the gluing condition is part of the ambient structure. Its limits impose the equations while retaining the arrows that are compatible with them.

A second choice concerns the role of the source. One can write a different solution functor for each current, or one can first construct the universal map to the sheaf of currents. The latter lets restriction of a background, fixing a source, and making a family of sources all use the same pullback rule. It also distinguishes algebraic exactness from categorical local surjectivity. A universal equation map can hit all germs even when its global sections miss sources with nonzero cohomology class. The local existence proof in Section 8 and the global obstruction in Section 6 exhibit both facts in the same model.

A third choice concerns what the Hodge operation acts on. It is a natural operation on the site with the chosen metric or compatible conformal data. It is not a map of sheaves of arbitrary smooth four-dimensional backgrounds with all smooth arrows. Separating the absolute smooth classifying stack from the metric site keeps that restriction visible. The resulting construction uses the ordinary connection classifier wherever connections are pulled back and uses the metric site whenever the Maxwell dynamics must also be pulled back.

2 Exterior equations and constitutive data

2.1 Forms, orientations, and the current

Let M be smooth, Hausdorff, second countable, and four dimensional. Write o_M for its real orientation local system. A twisted p-form is a section of $\Lambda^p T^*M \otimes o_M$. Locally an orientation identifies twisted and ordinary forms. Reversing that orientation changes the local representative of a twisted form by a sign. The exterior derivative on twisted forms uses the flat local system. Integration of a twisted top form requires no global orientation [8, 10].

The premetric equations with no magnetic source on the domain are

$$\mathrm{d}F = 0, \quad \mathrm{d}_o G = J, \qquad F \in \Omega^2(M), \quad G \in \Omega^2(M; o_M), \quad J \in \Omega^3(M; o_M). \tag{2.1}$$

The ordinary form F integrates over oriented two-chains. The excitation and current pair with chains carrying the appropriate local orientation coefficients. On an oriented spacetime we identify o_M with $\mathbb{R}$ and omit its subscript. Working with twisted G, J separates their transformation properties from the orientation used to write a Hodge operator.

Local conservation $\mathrm{d}_o J = 0$ follows by applying d_o again. For a global smooth excitation there is the further consequence $[J]_{\mathrm{dR}} = 0$ in twisted de Rham cohomology. Closure makes the integral vanish on boundaries of four-chains. Exactness makes the periods vanish on every closed three-cycle with the appropriate coefficients. The distinction matters on a spacetime where some closed three-cycles do not bound.

A linear local medium specifies a bundle map

$$\kappa \colon \Lambda^2 T^*M \longrightarrow \Lambda^2 T^*M \otimes o_M.$$

A nonlinear medium can use a fiber map or a relation on pairs. In vacuum κ is a positive scalar times the metric Hodge operator with its twist. This additional map fixes the vacuum propagation law. For a dispersive model the excitation can depend on earlier values of the field. A constitutive morphism on a site of opens then requires a suitable enlarged history space and a restriction rule. The simple bundle map describes local instantaneous constitutive laws.

2.2 Recovering the four equations

Use $M = \mathbb{R}_t \times \mathbb{R}^3$, metric $-\mathrm{d}t^2 + \mathrm{d}x^2 + \mathrm{d}y^2 + \mathrm{d}z^2$, and orientation $\mathrm{d}t \wedge \mathrm{d}x \wedge \mathrm{d}y \wedge \mathrm{d}z$. Put

$$F = B - \mathrm{d}t \wedge E, \qquad G = D + \mathrm{d}t \wedge H, \qquad J = \rho - \mathrm{d}t \wedge j. \tag{2.2}$$

Here E, H are spatial one-forms, B, D, j are two-forms, and ρ is a three-form. The vector current corresponds to $j = \iota_{\boldsymbol{j}} \operatorname{vol}_3$. For a spatial p-form a and a spatial $(p-1)$-form b the Lorentzian star is

$$\star_g a = (-1)^p \mathrm{d}t \wedge \star_3 a, \qquad \star_g(\mathrm{d}t \wedge b) = - \star_3 b. \tag{2.3}$$

Thus $D = \star_3 E$ and $H = \star_3 B$ in our vacuum normalization. Some conventions reverse the excitation and current simultaneously. With d_s the spatial differential, the three exterior derivatives are

$$\mathrm{d}F = \mathrm{d}_s B + \mathrm{d}t \wedge (\partial_t B + \mathrm{d}_s E), \tag{2.4}$$

$$\mathrm{d}G = \mathrm{d}_s D + \mathrm{d}t \wedge (\partial_t D - \mathrm{d}_s H), \tag{2.5}$$

$$\mathrm{d}J = \mathrm{d}t \wedge (\partial_t \rho + \mathrm{d}_s j). \tag{2.6}$$

The exterior equations are consequently equivalent to

$$\mathrm{d}_s B = 0, \quad \mathrm{d}_s E = -\partial_t B, \quad \mathrm{d}_s D = \rho, \quad \mathrm{d}_s H = j + \partial_t D. \tag{2.7}$$

Under Euclidean identification of one-forms with vectors and two-forms with flux densities, these are the two magnetic laws, electric Gauss law, and the Ampere–Maxwell law. In SI vacuum units one writes $\mathbf{D} = \varepsilon_0 \mathbf{E}$ and $\mathbf{H} = \mathbf{B}/\mu_0$. By $\mathrm{d}_s^2 = 0$ the evolution equations propagate both Gauss constraints when the source is conserved. This deduction uses only exterior algebra and holds before a Cauchy problem is solved.

2.3 Naturality, conformal invariance, and characteristics

For any smooth map f, pullback commutes with the exterior derivative. For the twisted equations one uses an identification $f^* o_N \simeq o_M$, available for a local diffeomorphism. The constitutive equation is preserved when

$$f^* \kappa_N = \kappa_M f^*. \tag{2.8}$$

Equal-dimensional orientation-preserving isometries satisfy this condition in vacuum. A map that lowers rank or changes metric data may fail it. The smooth topos containing connection data and the site on which the Maxwell operator is natural therefore serve different roles.

If $g' = e^{2\varphi} g$, the inner product on p-forms scales by $e^{-2p\varphi}$ and the volume form by $e^{4\varphi}$. Hence $\star_{g'}|_{\Omega^p} = e^{(4-2p)\varphi} \star_g$. The factor is one in degree two. If J is a fixed three-form, both the exterior equations and their connection moduli are invariant. If a source is specified as a fixed vector v and then converted to $J = \iota_v \operatorname{vol}_g$, the three-form generally changes. The source convention is part of the conformal statement.

The principal symbol shows how the constitutive law determines vacuum propagation. A real covector $\xi \neq 0$ and two-form polarization f satisfy $\xi \wedge f = \xi \wedge \star f = 0$. The first gives $f = \xi \wedge a$. Up to sign the second is $\iota_{\xi^\sharp} f = 0$, so

$$g^{-1}(\xi, \xi)a - a(\xi^\sharp)\xi = 0.$$

For nonnull ξ this forces $f = 0$. For null ξ, a lies in a three-dimensional annihilator and is defined modulo ξ. The quotient has two polarizations. Reconstructing the conformal metric from a general constitutive tensor requires additional hypotheses [9].

2.4 Integration and charge balance

Stokes's theorem gives $\int_{\partial C} F = 0$ and $\int_{\partial C} G = \int_C J$ on an oriented three-chain C. When C lies in a time slice these are Gauss laws. When C sweeps a spatial surface through time, its boundary includes the cylinder swept out by the boundary of the surface, which gives the appropriate electric circulation. On a closed spatial slice Σ one obtains $\int_\Sigma \rho = 0$. A spatial boundary permits an electric flux term. A puncture can produce new cycles. In particular a sphere around a removed point may carry magnetic flux while $\mathrm{d}F = 0$ everywhere on the domain.

3 Two smooth higher toposes and the family convention

3.1 The smooth classifying environment

Use a small representative of the site of smooth manifolds in a fixed universe, with open covers, or its equivalent Cartesian basis for the purposes of smooth stacks. Put $\mathcal{H}_{\mathrm{sm}} = \mathrm{Sh}_\infty(\mathrm{Mfd})$, the ∞-topos of sheaves of spaces on that site. A smooth manifold represents a sheaf. The functor of points of a stack evaluates it at every smooth test manifold T, and restriction pulls back along a smooth map of tests. The Yoneda principle identifies the evaluation at T with the mapping space from its representable. Limits are computed objectwise in sheaves of spaces. Sheafification is left exact, a fact that permits groupoid presentations followed by finite homotopy limits [11, 12, 13].

The basic classifying object is $\mathbf{B}\mathbb{T}_\nabla \in \mathcal{H}_{\mathrm{sm}}$. An ordinary manifold M maps into it by specifying a principal circle bundle with smooth connection. A map between two such maps is a bundle isomorphism preserving the connection. In the theory of principal ∞-bundles, the classifying object for the underlying bundle is the delooping of the smooth circle group object [14, 15]. The connection refines the bundle class. The resulting stack is 1-truncated. The higher topos is the ambient category and supports iterated deloopings and higher gluing. Ordinary electromagnetism itself has the gauge groupoid of a 1-form connection.

For interpretation in a cohesive model, one distinguishes smooth structure from homotopy type. A smooth circle group, a locally constant sheaf of circle coefficients, and the ordinary classifying space of the topological circle have different roles. Write $\underline{\mathbb{T}}^\delta$ for the locally constant circle-coefficient sheaf, with the circle used as an abstract abelian group. For example, local parallel sections of a flat circle bundle form a torsor under $\underline{\mathbb{T}}^\delta$. Allowing gauge functions that are arbitrary smooth maps gives the smooth group sheaf instead. We use this distinction concretely, so the main construction requires no choice of a cohesive modal axiomatization beyond the smooth site. For the broader cohesive framework see [16, 17].

3.2 The product site of parameters and spacetimes

Let $\mathcal{L}$ have oriented and time-oriented Lorentzian four-manifolds as objects. Its arrows are orientation- and time-orientation-preserving isometric open embeddings. We allow arbitrary such spacetimes for local descent, including open subsets of globally hyperbolic ones that are not themselves globally hyperbolic. Covering families are open inclusions with union the target. One can enlarge the arrows to conformal open embeddings if the constitutive law and the specification of current satisfy the conditions of Section 2. Write

$$\mathcal{D} = \mathrm{Mfd} \times \mathcal{L}, \qquad \mathcal{H}_{\mathcal{D}} = \mathrm{Sh}_\infty(\mathcal{D}).$$

The topology is generated by parameter open covers with spacetime unchanged and spacetime open covers with parameter unchanged. Arbitrary smooth arrows in the parameter factor allow reparametrization. We only require descent for the generated topology. Rectangles form a basis of the topology on an ordinary product, but a general cover of that product by rectangles need not have a global rectangular refinement of the form of one independent product of two fixed covers. We do not use that stronger assertion.

For a pair (T, M) let $\Omega^k_v(T, M)$ be the sections over $T \times M$ of $\mathrm{pr}_M^* \Lambda^k T^* M$. We use the full smoothness in (t, x) and differentiate only in x. Use d_v, $\star_v$, and $Z^k_v = \ker(d_v)$ for the vertical derivative, Hodge star, and closed forms. A connection along the fibers of $T \times M \to T$ is called a vertical connection. Its principal bundle is a smooth bundle over the total product, and its horizontal distribution is specified along vertical tangent directions. In a trivialization it is represented by a vertical one-form. A gauge transformation changes this form by the vertical logarithmic derivative of a circle-valued function. Denote the resulting groupoid-valued functor by $\mathcal{C}$.

Parameter restriction acts on this functor even for constant maps, and spacetime restriction acts even for inclusions that meet only part of a time slice. A nontrivial bundle pulled back from T admits a canonical vertical flat connection. It gives a legitimate family with trivial restriction to each individual spacetime fiber. It may remain nontrivial as a family. This is one reason to use groupoid-valued descent in the parameter direction. Smooth families of isomorphism classes alone can miss the bundle that makes the family possible.

3.3 Relative forms and the mapping-stack distinction

In $\mathcal{H}_{\mathrm{sm}}$, the internal exponential $\mathbf{B}\mathbb{T}_\nabla^M$ evaluated at T is

$$(\mathbf{B}\mathbb{T}_\nabla^M)(T) \simeq \mathrm{Map}(T \times M, \mathbf{B}\mathbb{T}_\nabla).$$

Its objects have a connection in every direction of the product. The curvature used in the Maxwell operator is vertical, and differentiation and Hodge star must ignore the parameter directions. Simply replacing M by $T \times M$ in the vacuum four-dimensional formula changes the dimension and the type of the metric. Restriction to the vertical tangent distribution defines a comparison from the full mapping stack to $T \mapsto \mathcal{C}(T, M)$.

On the product of ordinary manifolds every vertical connection extends to a full connection. Choose any full connection and restrict it vertically. The difference from the prescribed vertical connection is a vertical one-form on the base. Extend that form by zero along the product's parameter tangent distribution and add it to the chosen connection. For a fixed vertical object, the groupoid of full extensions with a specified identification of their vertical restriction is equivalent to a discrete affine space for

parameter-direction one-forms on the product. Indeed an arrow inducing the identity after the specified identifications must be the identity bundle map. Thus the forgetful comparison is essentially surjective at each pair and typically fails to be an equivalence. This is the concrete relative-field distinction behind the use of concretification for mapping stacks in [27]. We use the direct vertical description, without asserting that every construction called concretification uses the same ambient site.

3.4 Descent before solving a PDE

Forms glue by locality and smoothness. Closure is a local condition, so Z_v^k is a sheaf of abelian groups. Smooth principal bundles glue along open covers using transition isomorphisms. A compatible family of vertical connections glues since its local horizontal data agree under those isomorphisms. Bundle maps preserving them glue as smooth maps. These facts hold for the covering generators in either direction and their pullbacks. Therefore $\mathcal{C}$ is a stack and the vertical curvature map

$$\mathrm{curv}_v \colon \mathcal{C} \longrightarrow Z_v^2$$

is a map of stacks. Its definition follows from $d_v^2 = 0$ and the gauge-invariance of curvature.

For a single cover, the homotopy limit of its Cech diagram has objects given by local connections together with isomorphisms on double overlaps satisfying the triple-overlap identity. Morphisms are compatible families of connection-preserving maps. This reconstructs the descent groupoid. The difference from a set-valued equalizer is the explicit overlap isomorphisms. A 1-truncated sheaf of spaces is hypercomplete, so if hypercovers are used instead of Cech covers the same object satisfies their descent. No assumption of hypercompleteness of the entire ambient topos is required [11, 13].

Finally, at a fixed pair (T, M) the homotopy type of the value is the nerve of an ordinary groupoid. For example, a smooth path of gauge transformations is reflected by evaluation at a parameter interval. Such a path need not be a two-morphism between two distinct arrows in the groupoid at $T = *$. Keeping these levels distinct avoids confusing ordinary automorphisms with higher homotopy in a realization of a smooth moduli object.

4 The connection stack and differential cohomology

4.1 A Cech–Deligne presentation

For a smooth map $g : U \to \mathbb{T}$ write $d\log g$ for the real one-form obtained by differentiating a local real lift. It is independent of the lift and has integral periods. Use the transition convention

$$g_{ij} + g_{jk} = g_{ik}, \qquad A_j - A_i = d\log g_{ij}. \tag{4.1}$$

The group law for circle-valued functions is additive. A change of trivialization by $h_i : U_i \to \mathbb{T}$ sends

$$A_i \longmapsto A_i + d\log h_i, \qquad g_{ij} \longmapsto g_{ij} + h_j - h_i.$$

The data satisfy the usual normalization on equal indices and inverses by the cocycle law. On the product site use the same formulas with d_v on local functions. They are compatible with the usual geometric construction of a bundle from transition functions. One can define the connection stack by stackifying the action groupoid $\Omega^1 // C^\infty(-, \mathbb{T})$, where a gauge function acts by addition of its logarithmic derivative. The stackification adds the bundles that require more than one trivialization. This is the two-term smooth Deligne presentation, with circle functions in degree zero and one-forms in degree one for hypercohomology. Its first hypercohomology is the degree-two differential cohomology group $\widehat{H}^2(M;\mathbb{Z})$. For the alternate integral presentation, use the exponential sequence to replace circle functions by $\mathbb{Z} \to \Omega^0$ and shift degrees accordingly [18, 16, 19, 20].

In the geometric normalization we use $c(P)$ for the integral characteristic class with real image $[F]$. The corresponding Čech representative can be constructed by taking local primitives and their real logarithms and choosing the Čech/de Rham sign compatible with the stated positive Wilson convention. The test for a clutching example is the integral of F using the chosen orientation. This avoids identifying a transition function written in the inverse convention with one written in (4.1).

4.2 The curvature sequence and its stack refinement

Define $\Omega^2_{\mathbb{Z}}(M)$ to consist of closed real two-forms with integer integrals on smooth integral two-cycles. Then

$$0 \longrightarrow H^1(M;\mathbb{R}/\mathbb{Z}) \longrightarrow \widehat{H}^2(M;\mathbb{Z}) \xrightarrow{\text{curv}} \Omega^2_{\mathbb{Z}}(M) \longrightarrow 0. \tag{4.2}$$

Here H^1 is ordinary cohomology with constant coefficients. The zero-curvature objects are exactly flat circle bundles. Their isomorphism classes are characters of $H_1(M;\mathbb{Z})$: on a connected component, choose a base point and parallel transport. A flat connection has monodromy factoring through ordinary homotopy of loops, then through

abelianization. Conversely a character constructs the flat bundle by the universal cover and the monodromy action. Use componentwise constructions when M is disconnected.

For surjectivity, choose $c \in H^2(M;\mathbb{Z})$ mapping to the de Rham class of a given integral form F. Such a class exists by the integral-period characterization of the image under change of coefficients, using de Rham integration and the integral universal coefficient theorem. Choose a circle bundle with class c and any smooth connection with curvature F_0. Since $F - F_0$ is exact, choose a one-form a with $da = F - F_0$ and add it to the connection. Necessity of integrality follows from the de Rham image of the integral characteristic class, equivalently from the differential-character Stokes law discussed below.

There is also the characteristic-class exact sequence

$$0 \longrightarrow \Omega^1(M)/\Omega^1_{\mathbb{Z}}(M) \longrightarrow \widehat{H}^2(M;\mathbb{Z}) \xrightarrow{c} H^2(M;\mathbb{Z}) \longrightarrow 0. \tag{4.3}$$

The first map assigns a connection on the trivial bundle. Logarithmic derivatives of smooth circle functions are exactly the closed one-forms with integral periods, since integrating a closed integral form from a chosen base point defines a circle-valued function. Every bundle class admits a connection, so the last map is onto. The first sequence answers what is left after measuring curvature. The second answers what is left after fixing bundle type. Fixing both leaves the quotient of closed real one-forms by integral-period ones as the ambiguity of classes, provided the pair is compatible.

At the groupoid level, fix F in the image and choose a lift x. Tensoring with x gives an equivalence from the groupoid of flat connections to the curvature fiber over F. Its inverse is tensoring with the dual of x. Choosing another lift translates the equivalence by a flat object. The fiber is thus a torsor in the groupoid sense. The automorphism group of any point in it is $H^0(M;\mathbb{R}/\mathbb{Z})$, since a connection-preserving gauge transformation is a locally constant circle-valued function. The natural group structure is on the zero fiber. A nonzero curvature fiber has an action by it.

4.3 Why global integrality is different from local image

The map $\mathrm{curv} : \mathbf{B}\mathbb{T}_\nabla \to \Omega^2_{\mathrm{cl}}$ in the smooth topos is locally surjective by the Poincare lemma. It is an effective epimorphism of stacks, even though its map on global sections can miss nonintegral forms. The presheaf $U \mapsto \Omega^2_{\mathbb{Z}}(U)$ does not in general satisfy descent. Restrictions of a nonintegral closed two-form to sufficiently small balls are exact and integral. They glue as real forms, whose global integral periods need not be integers. The same local surjectivity holds in the product site because a spacetime contracting homotopy on a coordinate ball provides a primitive smooth in the passive parameters. The stack fiber, with overlap equivalences, retains the obstruction to selecting a global lift.

On ordinary manifolds the curvature fiber over zero is $\mathbf{B}\underline{\mathbb{T}}^{\delta}$ as a smooth stack. One can express the absolute curvature sequence before taking global sections as a fiber sequence

$$\mathbf{B}\underline{\mathbb{T}}^{\delta} \longrightarrow \mathbf{B}\mathbb{T}_{\nabla} \longrightarrow \Omega^{2}_{\mathrm{cl}}.$$

Delooping the flat kernel, the local lifting obstruction for a closed form has target $\mathbf{B}^{2}\underline{\mathbb{T}}^{\delta}$. On a manifold its class in $H^{2}(M;\mathbb{R}/\mathbb{Z})$ can be described by reducing the periods of F modulo integers, up to the sign convention for the obstruction map. This can be seen directly: local primitives differ by derivatives of real functions on pairwise overlaps. Exponentiating them gives would-be transition functions whose triple-overlap product is locally constant. Its cohomology class vanishes exactly when the functions can be corrected by locally constant factors to satisfy the circle cocycle condition. The same argument identifies the class as the image of the real de Rham class under the coefficient map to $\mathbb{R}/\mathbb{Z}$.

For vertical connections a flat object's automorphisms are smooth functions on $T \times M$ that are locally constant in the spacetime variable. They are generally not locally constant in T. Thus the absolute flat kernel and the vertical flat kernel have different parameter behavior. In particular the notation for the absolute constant-coefficient object is used only for the absolute curvature sequence or a fixed ordinary spacetime evaluation.

5 The Maxwell solution object

5.1 An equation map over closed currents

In $\mathcal{H}_{\mathcal{D}}$ write $\mathcal{J} = Z^3_v$ and let

$$P_g = d_v \star_v : Z^2_v \longrightarrow \mathcal{J}, \qquad E_g = P_g \operatorname{curv}_v : \mathcal{C} \longrightarrow \mathcal{J}. \tag{5.1}$$

The subscript g indicates the metric carried by the spacetime object. The formulas are natural under every arrow in $\mathcal{D}$. The target is closed because $d_v^2 = 0$. All maps are additive: on $\mathcal{C}$ the operation is tensor product of circle bundles with induced connection. Thus these are morphisms of abelian group objects in the sense appropriate to their truncation levels, and in particular morphisms of their underlying objects in the topos.

A prescribed source at (T, M) is a map $J : y(T, M) \to \mathcal{J}$. The solution family over the background representable is

$$\mathsf{Max}^{\mathbb{T}}_J = \mathcal{C} \times^h_{\mathcal{J}} y(T, M). \tag{5.2}$$

The real field-strength version replaces $\mathcal{C}$ by Z^2_v. Forming the homotopy pullback in the slice over $y(T, M)$ can also be expressed as taking the equalizer of E_g and the section J on the restriction of $\mathcal{C}$ to that slice. The analogous definition for a variable source uses any object B of the topos equipped with a source map $B \to \mathcal{J}$. This generality accommodates several prescribed source parameters or a subsheaf of admissible sources.

Theorem 5.1 (Recovery and descent)**.** *At an ordinary background $(*, M, J)$, the groupoid of sections of (5.2) over the background has objects (P, ∇) with $d \star_g F_\nabla = J$ and all connection-preserving bundle isomorphisms between them. The real version has the set of all F solving (1.1). Both constructions satisfy descent for the topology in Section 3. For a map $B' \to B$ over $\mathcal{J}$, pulling back the solution over B gives the solution over B'.*

Proof. Mapping out of a representable preserves limits. The target at a stage is a set, so the homotopy fiber condition is exactly equality of currents. Between objects in that fiber every isomorphism of connections is compatible with the required equality. The curvature of a connection is closed and a point of Z^2_v is closed by definition. This recovers both homogeneous and inhomogeneous exterior equations. The inputs are sheaves of spaces, whose limits are computed objectwise. Descent follows, also directly by gluing connections and using locality of their field equations. Base change is the associativity of pullbacks. □

In particular, for a cover $\{U_i \to M\}$ and a current on M, the groupoid at M is the homotopy limit of the groupoids on the Cech diagram with restricted currents. There

can be local Maxwell solutions on every member of a cover, without a compatible descent object. Descent identifies global solutions with compatible local data. It places no existence requirement on a compatible choice of local solutions.

5.2 The real theory and the integral lift

The two solution objects fit in a pullback square over Z_v^2:

$$\begin{array}{ccc} \mathsf{Max}_J^{\mathbb{T}} & \longrightarrow & \mathcal{C} \\ \downarrow & & \downarrow \mathrm{curv}_v \\ \mathsf{Max}_J^{\mathbb{R}} & \longrightarrow & Z_v^2. \end{array}$$

The square also works over a general source object. It shows that a connection solution is a real solution together with a differential lift. The theorem about integral periods applies to the square at a single spacetime. A vertical family lift over an arbitrary parameter manifold involves its own compatibility data. In particular the fiberwise criterion should be distinguished from an assertion about smooth descent for families with parameter gluing. The pullback itself gives the family criterion and retains all the gluing data.

Since curvature is an effective epimorphism, its base change is one as well. Thus the comparison is locally surjective in the site. At fixed ordinary spacetime, it may fail to be surjective on components. The apparent difference is accounted for by the order of operations. The image as an object of the topos is the sheaf image. The image of the map of global components is the pointwise set image, which remembers integral periods. Taking global sections need not preserve effective epimorphisms.

5.3 Constitutive relations as spans

Work with the premetric site of four-manifolds and orientation-local-system identifications, and use vertical ordinary and twisted forms as needed. An allowed local constitutive relation can be encoded by an object $\mathcal{K}$ over $\Omega_v^2 \times \Omega_{v,o}^2$. A subobject is sufficient for a relation of bare forms. A more general object can carry a material state witnessing the relation. The projections assign F and G. The premetric solution object is defined by the diagram

$$\mathcal{K} \xrightarrow{(d_v F, d_{v,o} G)} Z_v^3 \times Z_{v,o}^3 \xleftarrow{(0,J)} B.$$

The homotopy pullback implements the equations for all allowed constitutive data at once. To refine the magnetic variable to a connection, pull the result back along $\mathcal{C} \to \Omega_v^2$ using its curvature map. When $\mathcal{K}$ is the graph of κ and both projections are ordinary forms, the construction reduces to the previous one with $\star_g$ replaced by

κ. This follows by eliminating G using the graph condition, then using $d_v^2 = 0$ for connections.

For a prescribed constitutive map one can first restrict to morphisms preserving it. For a material model allowed to vary, one can keep its parameters as part of the source object and pull back simultaneously in source and material data. The needed naturality is commutation with restriction as in (2.8). When a relation allows multiple excitations for one F, projecting away G can change the solution problem even on a contractible spacetime. The use of a span specifies exactly which variables and material witnesses survive.

5.4 Matching across a collar

A gluing law also expresses one common use of boundary data. Suppose two open regions M_1, M_2 cover M and overlap on a collar C. For the restricted currents the solution groupoid is equivalent to

$$\mathsf{Max}^{\mathbb{T}}_{J}(M) \simeq \mathsf{Max}^{\mathbb{T}}_{J_1}(M_1) \times^{h}_{\mathsf{Max}^{\mathbb{T}}_{J_C}(C)} \mathsf{Max}^{\mathbb{T}}_{J_2}(M_2).$$

A point comprises solutions on both regions and an isomorphism of their restrictions. That isomorphism is part of the gluing information. Matching only curvature on the collar can leave a flat mismatch. If one instead prescribes a boundary connection on an embedded hypersurface, pullback of connections to that hypersurface is defined even though the Maxwell equation is not preserved by the dimension-changing map. One can take a homotopy fiber of this boundary restriction without equipping the hypersurface with a four-dimensional Maxwell operator. The prescribed boundary problem may be overdetermined or underdetermined as a PDE. The pullback records precisely the boundary condition and makes no analytic claim by itself.

6 Global sectors, current obstructions, and holonomy

6.1 An affine space of curvatures and a torsor of connections

At fixed M define

$$\mathcal{V}_g = \{F \in \Omega^2(M) : dF = 0,\ d \star_g F = 0\}, \qquad \mathcal{V}_{g,\mathbb{Z}} = \mathcal{V}_g \cap \Omega^2_{\mathbb{Z}}(M).$$

If a real solution exists at J, all real solutions are its translates by $\mathcal{V}_g$. If an integral one exists, all integral solutions are its translates by $\mathcal{V}_{g,\mathbb{Z}}$. These are consequences of linearity in the curvature. The full groupoid of solutions, when pointed by any connection solution, is equivalent by tensor product to the groupoid of vacuum connection solutions. There is no preferred translation unless a particular solution has been specified.

Theorem 6.1 (Global curvature comparison)**.** *On a fixed ordinary spacetime with prescribed current, the curvature map on components of* $\mathsf{Max}^{\mathbb{T}}_J$ *has image precisely the real solutions with integral periods. Over each such* F *the component set is a torsor for* $H^1(M;\mathbb{R}/\mathbb{Z})$ *and the automorphism group at each connection is* $H^0(M;\mathbb{R}/\mathbb{Z})$*. If a bundle class* c *is fixed as well, it must map to* $[F]$ *in real cohomology. The fiber of classes with both invariants fixed is a torsor for* $H^1(M;\mathbb{R})/\operatorname{im} H^1(M;\mathbb{Z})$*.*

Proof. A lift of an integral form exists by (4.2). Every lift of the form satisfies the inhomogeneous equation exactly when the form does. The flat fiber is the one in Section 4. Fixing c retains the flat classes with trivial characteristic class, which are precisely the stated quotient by the coefficient exact sequence. The existence for a compatible pair (F, c) follows by adjusting a connection on a bundle of class c by a one-form. □

The category remembers two distinct uses of topology. A torsion class in $H^2(M;\mathbb{Z})$ has zero real image, so curvature does not see a change of bundle class by a torsion class. A topologically trivial bundle can nevertheless have nontrivial flat holonomy, as on a circle. More generally, flat classes map onto the kernel of $H^2(M;\mathbb{Z}) \to H^2(M;\mathbb{R})$ by the Bockstein. For manifolds of finite CW type this kernel is the torsion subgroup. Without a finiteness hypothesis the coefficient kernel is the more general statement and is the one used in the proof.

6.2 An exact obstruction for the real field problem

The map $P_g : Z^2(M) \to Z^3(M)$ factors through $d\Omega^2(M)$. Therefore $[J]_{\mathrm{dR}} = 0$ is necessary. For the premetric theory with G independent of F, it is sufficient for the equation $dG = J$, since this is the definition of exactness. For vacuum Maxwell the

additional equation $dF = 0$ must be satisfied with $G = \star F$. There is a further exact linear-algebra description that does not need a Cauchy assumption.

Suppose J is exact and choose G_0 with $dG_0 = J$. All candidates are $G = G_0 + \eta$ for $\eta \in Z^2(M)$. Since $\star^{-1} = -\star$ on two-forms, the remaining equation is

$$P_g \eta = -d \star G_0. \tag{6.1}$$

The class of $d \star G_0$ in $d\Omega^2(M)/P_g Z^2(M)$ is independent of the choice of primitive. Vanishing of this class is necessary and sufficient for a real solution. This gives an obstruction quotient even on backgrounds where causal evolution is unavailable. It is an algebraic description of the range problem. It gives no elliptic or hyperbolic estimate. On the complete ultrastatic backgrounds of Section 8, a Cauchy construction shows that an exact current does admit a solution, including an integral-curvature one.

Even a closed current can fail the first obstruction. On $\mathbb{R} \times \mathbb{T}^3$ with unit spatial coordinates, $J = q\, dx \wedge dy \wedge dz$ is closed. If $q \neq 0$ its integral over any spatial slice is nonzero, which contradicts global Stokes for $J = dG$. Periodic identification of opposite faces makes the boundary terms cancel. A solution on every sufficiently small coordinate neighborhood is consistent with this failure of global solvability. The obstruction belongs to the current and arises before any magnetic integrality condition.

6.3 Wilson data and thin transport

For a differential character $h : Z_1^{\mathrm{sm}}(M) \to \mathbb{R}/\mathbb{Z}$ of degree two, the differential Stokes law is

$$h(\partial C) = \int_C F \pmod{\mathbb{Z}}.$$

For a circle connection the value on a closed path is the additive holonomy, with Wilson phase $\exp(2\pi i h)$. When $F = 0$ the formula factors through first homology. For nonzero curvature a homotopy of loops with swept surface C changes h by the integral of F over C. Thus ordinary homotopy is generally too coarse for transport. Thin homotopies have rank at most one, so the pullback of a two-form vanishes on them. Schreiber and Waldorf formulate parallel transport as a smooth functor on a thin path groupoid [22]. The differential-character picture records its abelian closed-loop invariants [19, 21].

Given an oriented surface with boundary γ, the phase on γ can be computed by its curvature integral, and two choices of spanning surface differ by an integral period. For a loop that does not bound in the domain the formula on boundaries does not determine its phase. This is the source of the flat ambiguity in Theorem 6.1. A measurement of F on every open set cannot by itself resolve a global flat character.

One can retain pointwise gauge by fixing a trivialization of the fiber over one point in each connected component. The residual automorphism of a framed connection

must be the identity, since a parallel gauge transformation with prescribed value at one point is determined throughout that component. This operation differs from passing to components of the unframed stack. It changes the allowed arrows and uses additional geometric data. Open-boundary framing has an analogous effect when the framed locus meets each component and the framing is included in the field object.

6.4 Electric sources and magnetic defects

An excision model uses a smooth complement of a defect. Magnetic periods there can be nonzero even though all local curvature forms are closed. A smooth extension across a deleted point would make a sufficiently small linking sphere a boundary, so Stokes would force its magnetic period to vanish. The same reasoning gives the corresponding electric period test for G when the extended current vanishes in the filled region. Distributional sources on a filled spacetime require distributional forms and additional conventions at the singular locus. The stack developed here describes the smooth complement. For the geometric example its topological content is the Chern class on the linking sphere, and the curvature supplies the local radial field.

7 Relative lifting and smooth parameter families

7.1 The vertical coefficient sheaves

The global-lift statement for one ordinary spacetime has a relative counterpart. For this purpose fix ordinary manifolds T and M and work on the ordinary topology of $X = T \times M$. Rectangular opens suffice for local calculations. Denote by $\mathcal{R}_v$ the sheaf of real smooth functions with $d_v f = 0$ and by $\mathcal{T}_v$ its circle-valued counterpart. These sheaves consist locally of functions of the parameter alone. In particular they differ from locally constant coefficient sheaves on X when T has positive dimension. There is an exact exponential sequence of sheaves

$$0 \longrightarrow \underline{\mathbb{Z}} \longrightarrow \mathcal{R}_v \longrightarrow \mathcal{T}_v \longrightarrow 0. \tag{7.1}$$

A locally vertical-constant circle function has a local vertical-constant real lift, which gives the surjectivity as a sheaf statement.

The vertical de Rham sequence resolves $\mathcal{R}_v$:

$$0 \longrightarrow \mathcal{R}_v \longrightarrow \Omega_v^0 \xrightarrow{d_v} \Omega_v^1 \xrightarrow{d_v} \Omega_v^2 \xrightarrow{d_v} \cdots .$$

On a product with a contractible coordinate ball in M, the usual radial homotopy formula differentiates only the spacetime coordinates and preserves smoothness in the parameters. It proves local exactness. The form sheaves are fine on the total product, since partitions of unity multiply sections of the vertical bundles. Fineness is a property of the terms and does not require d_v to commute with multiplication by arbitrary functions. Consequently the cohomology of the complex of global vertical forms computes $H^\bullet(X; \mathcal{R}_v)$.

A principal circle bundle with flat vertical connection has local vertical-parallel trivializations. Their transition functions belong to $\mathcal{T}_v$. Conversely gluing local zero connections along such functions gives a flat vertical connection. The groupoid of flat vertical objects is thus the groupoid of $\mathcal{T}_v$-torsors, with components $H^1(X; \mathcal{T}_v)$ and automorphisms $H^0(X; \mathcal{T}_v)$. For $T = *$ these are the ordinary flat invariants. The equality $H^0(X; \mathcal{T}_v) = C^\infty(T, \mathbb{T})$ when M is nonempty and connected is an immediate example of the changed coefficient convention.

7.2 The relative obstruction class

Theorem 7.1 (Vertical curvature lift)**.** *Let F be a closed vertical two-form on $T \times M$, and write $[F]_v$ for its class in $H^2(X; \mathcal{R}_v)$. A global vertical circle connection with curvature F exists precisely when $[F]_v$ is the image of a class in $H^2(X; \underline{\mathbb{Z}})$. When a lift exists, the groupoid of lifts is a torsor under the groupoid of $\mathcal{T}_v$-torsors on X.*

Proof. Use a good cover by products, obtained from good covers of both factors. On each product intersection the spacetime contracting homotopy gives vertical primitives and primitives of their closed differences. Choose A_i with $d_v A_i = F$, and real functions f_{ij} with $d_v f_{ij} = A_j - A_i$. The Cech coboundaries $(\delta f)_{ijk}$ are vertical-constant functions. With the compatible total-complex sign they represent $[F]_v$ using the vertical de Rham resolution. The would-be transitions f_{ij} mod $\mathbb{Z}$ have a Cech defect $(\delta f)_{ijk}$ mod $\mathbb{Z}$ in $\mathcal{T}_v$. Its class is the coefficient image of $[F]_v$ in $H^2(X; \mathcal{T}_v)$, up to the common choice of sign.

For the descent argument this class can equally be represented after refinement. The product intersections are acyclic for $\mathcal{R}_v$ by the vertical resolution and the explicit spacetime homotopy, and for $\underline{\mathbb{Z}}$ by ordinary contractibility. They are therefore acyclic for $\mathcal{T}_v$ by (7.1). When the defect class vanishes, one subtracts a Cech one-cochain of $\mathcal{T}_v$ from the would-be transitions. The corrected functions obey the cocycle law without changing the vertical logarithmic derivatives. The resulting data glue to a vertical connection. If a global connection exists, its own local data give vanishing of the same defect. Exactness of the cohomology sequence for (7.1) proves the criterion. The torsor statement follows by taking the difference of any two lifts using tensor product and duality, as in the absolute case. □

The construction shows how the obstruction can be found before solving for transition functions globally. In a parameterized Maxwell problem the input form is already a real solution and the relative obstruction concerns its connection lift. It is a separate operation from taking a primitive of the prescribed current. The two obstructions live in different coefficient systems and degrees.

7.3 A compact-fiber criterion

When M is a compact smooth manifold without boundary, smooth fiberwise integrality of F already suffices for a vertical lift on the product. To see this, treat each connected component of T independently and choose t_0. The periods on a fixed two-cycle vary continuously in t and are integers, so $[F_t] = [F_{t_0}]$ in real cohomology. Choose a circle connection on M with curvature F_{t_0} and pull it back to $T \times M$. The difference $\beta_t = F_t - F_{t_0}$ is a smooth family of exact forms. With any fixed auxiliary Riemannian metric on M, let G be its Hodge Green operator and put $a_t = \delta G \beta_t$. Hodge decomposition gives $da_t = \beta_t$. The operator is continuous on the Fréchet space of smooth forms and sends smooth families to smooth families, as follows by differentiating in the parameter in the smooth section topology on a compact manifold. Adding the vertical one-form a to the pulled-back connection gives the desired lift. This uses an auxiliary Riemannian operator only to pick a primitive. It leaves the Lorentzian equation and its constitutive law unchanged [24, 25].

Fiberwise integrality is necessary even without compactness, by restriction. The coefficient criterion of Theorem 7.1 is valid on arbitrary products of the stated manifolds. The compact argument gives one explicit way of passing from pointwise integrality to a smooth lift. In either case the choice of lift matters: tensoring by a bundle from T leaves every spacetime fiber curvature unchanged. Parameter topology can thus enter the moduli even when a lift has already been constructed.

7.4 Truncating before and after sections

The relative Poincare lemma gives two local facts. Every closed vertical two-form is locally the curvature of a connection. Two local lifts of the same form become isomorphic on a finer cover, since the difference is a flat vertical connection. It follows that the 0-truncation of $\mathcal{C}$ in the ∞-topos is Z_v^2. One can verify this using the sheafification of pointwise components. Since the Maxwell equation tests curvature in a sheaf of sets, its restriction to each source has the same property: the sheaf of local components of the connection solution object is the field-strength solution sheaf. This statement holds without asserting that any source has a local solution. If it has none on a given open, both sides are locally empty wherever that obstruction persists.

In contrast, for a fixed background the components of its global sections form the differential-cohomological solution set, and only its image is the subset of real global solutions with integral curvature. In symbols, the natural map

$$\pi_0 \operatorname{Map}(yM, \mathsf{Max}_J^{\mathbb{T}}) \longrightarrow \operatorname{Map}(yM, \tau_{\leq 0}\mathsf{Max}_J^{\mathbb{T}})$$

need not be injective or surjective, where the mapping spaces are taken in the slice over the fixed representable yM. The monopole, holonomy, and torus scenes show the two reasons: periods can obstruct a lift, and a lift can carry holonomy that is invisible to curvature. This comparison also explains the role of a higher topos even when the ordinary field equations have a sheaf-of-sets presentation.

8 Cauchy data and the role of hyperbolicity

8.1 Restriction to a spacelike hypersurface

Let $i : \Sigma \hookrightarrow M$ be a spacelike Cauchy hypersurface with the induced orientation. A connection solution determines

$$x_\Sigma = i^*(P, \nabla), \qquad D_\Sigma = i^* \star_g F, \qquad \rho_\Sigma = i^* J.$$

These obey $d_\Sigma D_\Sigma = \rho_\Sigma$. The magnetic datum is the curvature of x_Σ. In the real theory one specifies an independent closed magnetic two-form B_Σ. For a circle connection, the Cauchy datum x_Σ carries more than B_Σ: it retains a bundle class, Wilson data and its gauge automorphisms. In particular, a solution groupoid cannot generally be recovered from magnetic and electric flux densities alone.

At a chosen point of a hypersurface the metric normal and the tangential decomposition identify F with its tangential magnetic part and a normal electric part. The pullback of $\star F$ is equivalent to the normal electric datum. The two Gauss constraints involve only these pullbacks. The remainder of the equations determines evolution along the normal. Existence and uniqueness require a PDE theorem, in contrast to the restriction and constraint statements. The smooth splitting theorem of Bernal and Sanchez ensures that a globally hyperbolic spacetime has a smooth Cauchy temporal splitting, though its metric need not be ultrastatic [29]. For the general theory of normally hyperbolic operators and causal propagation see [30]. The abelian and nonabelian stacky Cauchy discussion in [27] takes the parameter dependence of the PDE as part of the problem.

8.2 A complete ultrastatic comparison

Take $M = \mathbb{R} \times \Sigma$ with $g = -dt^2 + h$ and a complete oriented Riemannian three-manifold (Σ, h) without boundary. Use the split of (2.2) and write $\rho(t) = r(t)\,\mathrm{vol}_h$, $j(t) = \star_h \jmath(t)$, where $\jmath$ is a one-form. The closure of J is

$$\partial_t r = \delta_h \jmath, \tag{8.1}$$

with δ_h the Riemannian formal adjoint of d_Σ. In this setting there is a direct construction of the connection from the data. We state it at the groupoid level for an ordinary background, without needing a parametrized smoothness theorem.

Theorem 8.1 (Ultrastatic Cauchy groupoid)**.** *For the complete ultrastatic spacetime just described and a smooth closed current J, restriction to $t = 0$ induces an equivalence from the groupoid of connection solutions to the groupoid of pairs (x_Σ, D_0) with $d_\Sigma D_0 = \rho(0)$. Its arrows are connection-preserving isomorphisms of x_Σ and preserve D_0. For real curvatures the restriction is a bijection onto (B_0, D_0) with $d_\Sigma B_0 = 0$ and $d_\Sigma D_0 = \rho(0)$.*

Proof. First take a circle bundle with initial connection x_Σ and curvature B_0. Extend the bundle by pullback to $\mathbb{R} \times \Sigma$ and work in temporal gauge, writing the spatial connection at time t as $x_\Sigma + a(t)$, with $a(0) = 0$. Then $B = B_0 + d_\Sigma a$, $E = -\dot{a}$, $D = -\star_h \dot{a}$, and $H = \star_h B$. The electric evolution law and the Gauss constraint become

$$\ddot{a} + \delta_h d_\Sigma a = \jmath - \delta_h B_0, \qquad \delta_h \dot{a} = r. \tag{8.2}$$

Set $u(t) = \int_0^t r(s)\, ds$. Solve the normally hyperbolic problem on one-forms

$$\ddot{a} + \Delta_h a = \jmath - \delta_h B_0 + d_\Sigma u, \qquad a(0) = 0, \qquad \dot{a}(0) = -\star_h D_0, \tag{8.3}$$

where $\Delta_h = d_\Sigma \delta_h + \delta_h d_\Sigma$. Completeness gives global hyperbolicity of this ultrastatic product. Standard smooth existence and uniqueness for the normally hyperbolic operator apply, including unrestricted smooth data by localization and finite propagation [30]. Since δ_h commutes with the Hodge Laplacian, $w = \delta_h a - u$ obeys $\ddot{w} + \Delta_h w = \delta_h \jmath - \dot{r} = 0$. Both initial data for w vanish, the second by the initial Gauss constraint. Thus $w = 0$, and (8.3) reduces to (8.2). The magnetic evolution is automatic from $B = B_0 + d_\Sigma a$ and $E = -\dot{a}$.

An arbitrary solution admits temporal gauge by parallel transport along each t-line from $t = 0$. This identifies its bundle with the pullback of the initial bundle and gives the same a by uniqueness of (8.3). An isomorphism of initial connections extends by pullback in these gauges. It is the unique connection-preserving extension because any extension must intertwine transport on the t-lines. This proves essential surjectivity and full faithfulness.

For the real problem use (8.3) for any closed B_0 without requiring a bundle. Conversely, set $a(t) = -\int_0^t E(s)\, ds$ for a given real solution. Faraday's law gives $B = B_0 + d_\Sigma a$. Then the electric equations give the same wave problem, which proves uniqueness. □

In these units $\delta_h D$ is not used as Gauss data. The condition is $d_\Sigma D = \rho$. The sign $\delta_h \dot{a} = r$ follows from $\delta_h = -\star_h d_\Sigma \star_h$ on one-forms and $D = -\star_h \dot{a}$. The other sign in (8.2) uses $\delta_h = \star_h d_\Sigma \star_h$ on two-forms. Writing the evolution in terms of the potential changes which constraints have to be built into initial data. The magnetic constraint is automatic for a connection, and the electric constraint is enforced by the scalar auxiliary equation.

8.3 Exactness, time slices, and what is propagated

The product retracts onto the initial slice. For a closed J its de Rham class vanishes precisely when $[\rho(0)] = 0$. If the class vanishes, choose D_0 with $d_\Sigma D_0 = \rho(0)$, and take the trivial initial bundle if desired. The theorem produces a solution with integral magnetic periods. Conversely global Stokes forces exactness. Hence on these

backgrounds exactness is the full existence obstruction for the sourced theory with no additional initial, boundary, or support requirements. The conclusion can fail after adding support or falloff conditions because those change the spaces of admissible primitives and solutions.

Any time slab containing the initial slice has the same Cauchy classification for the restricted source. The same wave argument works there by restriction, and its uniqueness gives equivalence of solution groupoids under restriction to the slab. This comparison uses the restriction of the same global prescribed source. Data specifying a different current outside the slab can lead to a different extension. A statement about all Cauchy embeddings or smooth families on nonultrastatic backgrounds needs appropriate parameter-dependent hyperbolic analysis.

The Cauchy groupoid also separates flux conservation from integrality. Under the deformation retraction, the magnetic class on spacetime is the one on the initial slice. The exterior equation propagates that class in the real theory, and a chosen circle connection supplies its integral lift. Flat holonomy data, when comparing two solutions of the same curvature, restrict and extend by the same homotopy type. Neither flux conservation nor the Lorentzian wave operator sets the initial integral class or its flat refinement.

8.4 Local solvability and the internal source map

The use of all closed currents as the target of the equation map is compatible with local analytic solvability. In fact, any smooth current germ can be solved by a connection on a neighborhood. The same statement holds for vertical families.

Proposition 8.2 (Local surjectivity of the equation map)**.** *The maps $E_g : \mathcal{C} \to Z^3_v$ and $P_g : Z^2_v \to Z^3_v$ are effective epimorphisms on the site of Section 3.*

Proof. Fix a spacetime point and choose a globally hyperbolic open neighborhood U of it in a sufficiently small coordinate neighborhood. Take a smaller contractible coordinate ball $V \subset U$ about the point, and a smooth cutoff χ compactly supported in V and equal to one on a smaller neighborhood W. The parameter-dependent Poincare lemma gives a vertical primitive G_0 of the prescribed closed current on $T \times V$. On $T \times U$ form $J' = d_v(\chi G_0)$ by extending the cutoff primitive by zero. For all parameters J' is supported in a single compact subset in the spacetime direction, and it agrees with J on W.

Use the formal Lorentzian codifferential δ. In the chosen signature and dimension it is $\star d \star$ on the degrees used here, so $j' = \star J'$ is coclosed. On one-forms the operator $\Box = d\delta + \delta d$ is normally hyperbolic, up to an overall sign convention. Let a be the retarded Green solution of $\Box a = j'$. The operators δ and $\Box$ commute in successive

degrees. Therefore δa solves the homogeneous scalar wave equation. Its retarded support forces it to vanish by uniqueness of the Cauchy problem. It follows that $\delta da = j'$, and hence $d \star da = J'$. The connection with potential $a|_W$ on the trivial bundle is the desired solution.

For families the same retarded operator acts on each parameter value. On forms supported in a fixed compact set it is continuous from the smooth section topology to the smooth section topology on compact sets of the output, by smooth Cauchy dependence and finite propagation [30]. A jointly smooth input with fixed spacetime support is smooth in the parameter in that topology. Applying the continuous linear operator commutes with its parameter derivatives and gives jointly smooth output. Thus the local lifts exist in the spacetime-cover direction for the whole family. This proves effective epimorphy. The claim for P_g follows by taking curvature of the local connection solution. □

Here the analytic argument is local even when the ambient spacetime has closed timelike curves or no global Cauchy surface. The globally hyperbolic neighborhood is part of the construction near the point. The cutoff is applied to a primitive, which preserves current conservation. Applying it directly to a current could introduce an unwanted derivative term. No retarded choice is required in the definition of the solution stack or in its descent property.

As an internal consequence, for any prescribed source the solution object is locally a torsor for the vacuum solution group object in the slice over the source background. The local nonemptiness follows from the proposition. The action law is tensor product, and the difference of two lifts of the same source is a vacuum object. A global section trivializes this torsor, whereas global exactness and the Cauchy analysis determine when such a section exists under additional hypotheses. This is the internal counterpart of the affine comparison of Section 6.

9 Electric–magnetic duality and higher gauge degree

9.1 Real vacuum rotations

On oriented Lorentzian four-manifolds, $\star_g^2 = -1$ on two-forms. In real vacuum both F and $\star F$ are closed, and

$$R_\theta(F) = \cos\theta\, F + \sin\theta\, \star F \tag{9.1}$$

defines an action of the circle of rotations on real vacuum fields. The identity $\star R_\theta(F) = \cos\theta\, \star F - \sin\theta\, F$ gives the composition law. The action is natural on the oriented metric site because it uses only the Hodge map and scalar multiplication. This describes rotations of classical curvature fields at fixed normalization.

With two prescribed closed three-form sources, put $dF = J_{\mathrm{m}}$, $d\star F = J_{\mathrm{e}}$. The same rotation acts on the source pair by

$$(J_{\mathrm{m}}, J_{\mathrm{e}}) \longmapsto (\cos\theta\, J_{\mathrm{m}} + \sin\theta\, J_{\mathrm{e}}, -\sin\theta\, J_{\mathrm{m}} + \cos\theta\, J_{\mathrm{e}}).$$

The usual electric-source theory keeps $J_{\mathrm{m}} = 0$, so a general rotation does not preserve that problem. A smooth circle connection always has closed curvature. A magnetic source requires changing the input geometry, for example using a complement of magnetic defects, or using a different singular formulation. The formula on source pairs is an identity for real forms, independently of the existence of a global ordinary circle lift.

9.2 Why rotations generally fail on the connection stack

The single-integrality condition only controls the periods of F. For a flat spatial three-torus, a constant vacuum field can have zero magnetic periods and an electric period of $\star F$ equal to any given real number. Its underlying circle bundle can be trivial. In particular a quarter-turn sends some such fields to curvatures that violate integrality. This prevents an action lifting (9.1) on all circle-connection solutions and compatible with the curvature map. Even when $\star F$ has integral periods, curvature alone does not determine a unique dual connection: its possible lifts form the flat fiber of Theorem 6.1.

There is an explicit duality-symmetric refinement. In vacuum define the groupoid of pairs (x, y) of connections with $\mathrm{curv}(y) = \star\,\mathrm{curv}(x)$. This is a pullback over the sheaf of two-forms, so it satisfies descent on the metric site. The functor

$$(x, y) \longmapsto (y, x^\vee)$$

preserves the condition because $\star^2 = -1$ and dualization reverses curvature. Its fourth iterate is canonically isomorphic to the identity. The fiber over an allowed real F is the product of the flat torsors for F and $\star F$. This construction keeps independent

flat choices for the two lifts. It is closely related to the dual differential-character formulation of Becker, Benini, Schenkel, and Szabo [28], which additionally develops the presymplectic and quantum aspects of duality in a locally covariant setting.

On a fixed manifold the subgroup of real rotations preserving all doubly integral sectors can depend on the realized period lattice. If both coordinates of a period pair can be chosen independently as integers, a rotation preserving every such pair must send the standard generators to integer pairs of unit norm. Its angle is a multiple of $\pi/2$. On a contractible background there are no such periods and the obstruction has a different form. Existence of a curvature rotation and existence of a canonical lift on connections are separate requirements even there.

9.3 Relation to duality in quantum theory

Witten studies the modular transformation of the abelian Euclidean partition function on a general four-manifold [31]. Its dependence on couplings and manifold topology differs from a classical Lorentzian fixed-background rotation. The moduli-stack construction only gives a classical input. Freed, Moore, and Segal show how torsion gives noncommuting flux observables in quantum Maxwell theory [32]. The classical groupoid's retention of flat classes is compatible with the need to track that torsion, but does not by itself supply commutation relations. The generalized-symmetry description treats closed F and $\star F$ as the forms behind magnetic and electric one-form symmetries of the appropriate vacuum theory [33]. The associated currents and extended operators should be distinguished from gauge transformations of a single potential.

9.4 Higher-form Maxwell equations

The same construction applies on oriented Lorentzian n-manifolds to an abelian p-form potential, $p \geq 1$, in a smooth differential cohomology model. The classifying object is $\mathbf{B}^p\mathbb{T}_\nabla$, with a closed curvature of degree $p+1$. The metric sends that curvature to a form of degree $n-p-1$. One takes the homotopy fiber of $d \star \mathrm{curv}$ over a closed current of degree $n-p$. These degrees follow directly from the form equation, with no appeal to a choice of gauge. In a vertical family one makes the same relative-form replacement in the higher Deligne complex.

At an ordinary manifold the flat fiber has homotopy groups

$$\pi_i \simeq H^{p-i}(M;\mathbb{R}/\mathbb{Z}), \qquad 0 \leq i \leq p,$$

after a base point is chosen when appropriate. Indeed, the absolute flat stack is the p-fold delooping of the locally constant coefficient sheaf. Mapping out of a manifold gives the cohomology groups in successive loop degrees. For $p = 1$ this reproduces

components and automorphisms of the flat connection groupoid. For $p > 1$ there are higher gauge isomorphisms, and descent genuinely uses homotopy groups above degree one. Degree two in spacetime dimension four is also the first self-matching case under the Hodge map. In other degrees the dual potential generally has a different form degree. The principal-bundle and cohesive formulations in [14, 16] supply the higher classifying objects, and [34, 35] relate such flux refinements to phase-space formulations.

9.5 Normalization and changes of coefficient lattice

Let $L = \lambda\mathbb{Z} \subset \mathbb{R}$ be a nonzero flux lattice. Replacing $\mathbb{T}$ by $\mathbb{R}/L$ and normalizing its logarithmic derivative with real values requires periods in L. With a vacuum constitutive relation $G = \gamma \star F$ for $\gamma > 0$, a dual lift of G may use a different lattice L_{e}. An identification of a two-lift system with its dual must therefore keep track of what is being interchanged: curvatures, coefficient lattices and coupling normalizations. In this paper the rotation angle applies to $(F, \star F)$ in common units and with the same lattice on each member when both are compact.

Consider the torus fields of (11.1) in these common units. They allow every real pair (n, e) if no coefficient lattice is imposed. With only the magnetic lift, the allowed pairs are $\mathbb{Z} \times \mathbb{R}$. A rotation preserving that subset must have zero sine, since the first component of the image of $(0, e)$ must be integral for all real e. Consequently the rotations by integer multiples of π are the only ones preserving all single-lift sectors on this background. They lift as the identity or dualization. With both lifts, the lattice on this cycle is $\mathbb{Z}^2$ and the quarter-turn acts. These examples locate the obstruction at the coefficient lattice, without requiring any claim about the quantum spectrum of charges.

10 Variation, observables, and derived refinements

10.1 The Euler equation on a fixed topological sector

Take the orientation and units of (1.1). In a trivialization, for compactly supported variations, the local Lagrangian is

$$-\tfrac{1}{2}F \wedge \star F + A \wedge J.$$

For a variation a of the connection the kinetic term is $-da \wedge \star F$. The product rule gives

$$\delta L = a \wedge (J - d \star F) - d(a \wedge \star F). \tag{10.1}$$

Thus stationarity with compactly supported one-form variations gives the sourced Maxwell equation. The other Maxwell equation comes from $F = dA$ locally and holds for every connection, including those on nontrivial bundles.

On a fixed bundle choose a reference connection ∇_0. Differences $\nabla - \nabla_0 = a_0$ are global one-forms, so the source contribution can be written relatively as $a_0 \wedge J$ when the relevant integrals are defined. The choice of reference changes this contribution by a field-independent term. Gauge transformations in the connected infinitesimal gauge algebra have $a = d\chi$ and the change of the source coupling is a boundary term when $dJ = 0$. A large gauge transformation may have logarithmic derivative an integral closed one-form with no global real potential. The change is then a pairing with J and needs a separate global check. For example it vanishes if $J = dG$ and the closed gauge shift has compact support. To define an exponentiated action for quantized or singular sources one may need differential cohomology and additional pairing data [23, 20]. The equation stack itself was defined for all smooth closed J, with possibly empty fibers.

For a constant theta-type contribution proportional to $F \wedge F$, the variation on a fixed bundle is an exterior derivative because $dF = 0$. It leaves the bulk equations invariant under compactly supported variations. It can still affect boundary terms or global sector weights. This is another case in which agreement of equation stacks does not identify all action-level or quantum data.

10.2 A conserved presymplectic current

Let a, b be tangent variations about a solution at fixed J, so $d \star da = d \star db = 0$. From the boundary term in (10.1) one obtains the antisymmetrized current

$$\omega(a, b) = a \wedge \star db - b \wedge \star da.$$

Its derivative vanishes: the equation removes the terms with $d \star da$ and $d \star db$, and $da \wedge \star db = db \wedge \star da$. Integrating over a Cauchy slice gives the corresponding antisymmetric

pairing when the restrictions have suitable support. If two slices bound a region and no flux escapes through other boundary components, Stokes gives the same value on both slices. For $a = d\chi$ one has $\omega(d\chi, b) = d(\chi \star db)$. With no contribution from the boundary these gauge directions are null. At a boundary the same exact term can survive. This connects boundary framings, phase-space reductions, and choices of boundary conditions [26, 37].

The current is computed inside a fixed topological sector, but the field stack includes all sectors. A continuous tangent calculation cannot detect the addition of a discrete torsion bundle class. It also does not equate field equations with an algebra of observables. One can construct holonomy observables, integrated curvature observables, or gauge-invariant functionals of matter. They use different structures and may obey different support conditions.

10.3 The linear deformation complex

For the infinitesimal abelian gauge symmetry the local linear complex is

$$\Omega^0 \xrightarrow{d} \Omega^1 \xrightarrow{d\star d} \Omega^3 \xrightarrow{d} \Omega^4. \tag{10.2}$$

Place the terms in degrees $-1, 0, 1, 2$. The two composites vanish by $d^2 = 0$. The first map represents infinitesimal gauge transformations. Degree zero cohomology is linearized solutions modulo these transformations. The last arrow records the linear Noether identity on equation residuals. Degree one cohomology measures the closed residuals not attained by the linearized operator in the chosen complex of sections. At this level a change of support conditions can change the cohomology.

The classical smooth stack is the homotopy fiber over an ordinary sheaf of currents. Its groupoid remembers gauge arrows, but the ordinary smooth site has no infinitesimal derived directions for residuals. In a derived treatment one can use a derived equation locus or a Batalin–Vilkovisky complex to retain higher relations among equations and infinitesimal directions. A two-term groupoid presentation and the four-term cochain complex answer different questions. Even using the same symbol for a homotopy fiber, the ambient category and the current object determine what the homotopy retains. For general derived and field-theoretic frameworks see [38, 39].

On an oriented manifold the pairing that integrates a k-form against a $(4-k)$-form, with compact support on a factor, pairs the outer two terms and the middle two terms of (10.2). Integration by parts connects d at the ends and makes $d \star d$ formally symmetric with the sign determined by wedge degree. This supplies the linear duality underlying the current formula. Constructing a shifted symplectic derived moduli object requires further geometric hypotheses on the ambient derived setting, the boundary, and the integration pairing. The formal integration identity is the claim used here.

10.4 Gauge arrows, observables, and local covariance

The solution functor is contravariant under the allowed open embeddings. An observable on a smaller background can be composed with restriction to define one on the larger background. This is the direction of a covariant observable assignment, once the class of allowed observables has been fixed. Covariance in that sense does not imply injectivity of observable maps. For example a cycle that does not bound in an open region can bound after inclusion into the larger one, changing the distinction between a Wilson datum and a curvature integral.

Brunetti, Fredenhagen, and Verch formulate local covariance for quantum field theory with specific causal embeddings [40]. Fewster and Lang distinguish universal and reduced free Maxwell theories, with different locality properties in the presence of topological sectors [41]. Becker, Schenkel, and Szabo identify topological contributions to the failure of injectivity in a differential-cohomological locally covariant model [26]. Benini, Schenkel, and Szabo analyze homotopy colimits and global observables for abelian gauge theory [42]. These works give a reason to specify the arrows and retained topological data before comparing any observable theory to the solution stack. Descent of configurations concerns gluing objects with isomorphisms on covers. It does not settle whether a quantized observable map is injective.

11 Four geometric configurations

The companion program uses Rust and Macroquad's three-dimensional camera and drawing primitives [52]. It generates every scene from the formulas described here. The wireframes, glyph lengths, and colors are display choices. A physical flux is given by the analytic integral, not by counting arrows. All scenes share an orbit camera, and the wave can be paused to compare its spatial electric and magnetic profiles. The period controls are discrete for integral flux and continuous up to the viewer's input step for real electric flux or a circle-valued phase.

11.1 A transverse plane wave

On Minkowski spacetime put $u = x - t$ and choose

$$A = \frac{a}{k}\sin(ku)\,dy, \quad F = a\cos(ku)(dx - dt) \wedge dy, \qquad k > 0.$$

Then $E = a\cos(ku)\,dy$ and the magnetic two-form is $B = a\cos(ku)\,dx \wedge dy$. In vector notation the magnetic direction is $+\partial_z$ where the common scalar is positive. The electric direction is $+\partial_y$. Their cross product points in the $+\partial_x$ direction even when both amplitudes are negative. Using Section 2,

$$\star F = a\cos(ku)(dt - dx) \wedge dz.$$

Both derivatives vanish because differentiating the scalar gives a factor of du. The star swaps transverse polarization data with a sign. The curvature is globally exact and there is no magnetic period on a closed cycle of Minkowski spacetime. The trivial bundle suffices. On this contractible background the curvature fiber has one isomorphism class but still has a circle of parallel automorphisms.

The viewer draws the x axis and places two arrows at each sampled x. The cyan electric arrows are in the y direction and the amber magnetic arrows are in the z direction. Two curves trace their tips. Advancing time translates the phase at speed one in the chosen units, rather than oscillating the propagation direction. The drawing is a line of samples from a plane wave, which is constant in the other two spatial coordinates. It is not a confined beam. The finite display window makes no assertion of finite integrated wave energy.

11.2 A monopole on a punctured spatial slice

Let $M = \mathbb{R}_t \times (\mathbb{R}^3 \setminus \{0\})$ with the flat metric restricted to the complement. Pull back a two-form from the unit sphere by the radial projection. In standard oriented angular

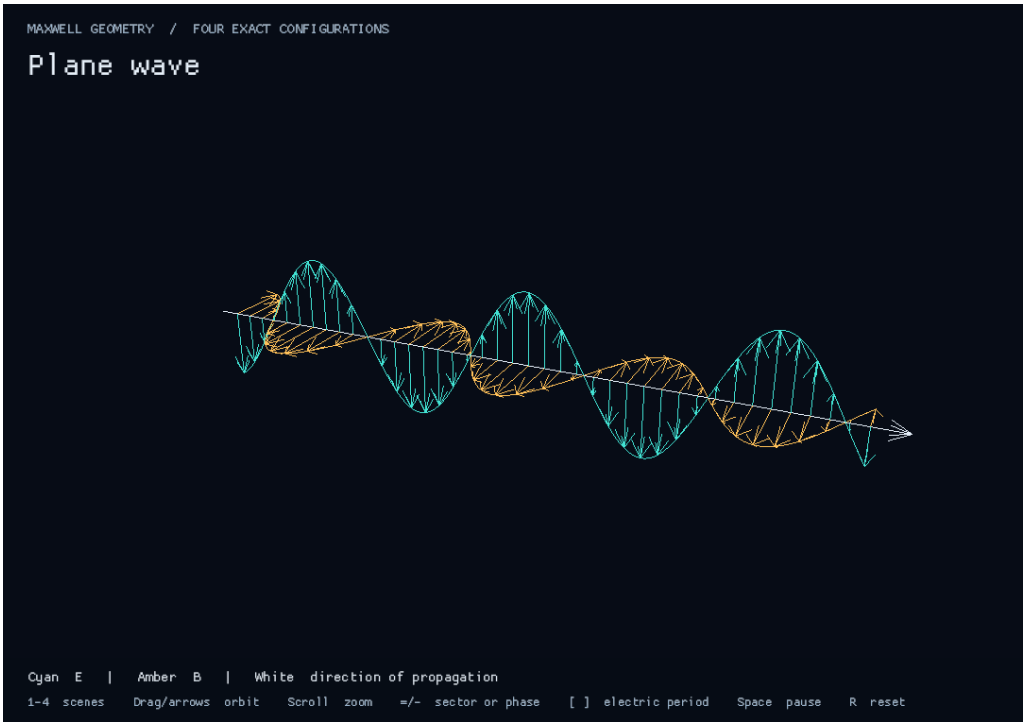


Figure 1: A paused plane wave. The profiles of the two transverse fields share a phase.

coordinates, for $n \in \mathbb{Z}$ let

$$F_n = \frac{n}{4\pi} \sin\theta \, d\theta \wedge d\phi, \qquad \int_{S^2} F_n = n.$$

Use charts omitting the south and north poles, respectively. Their potentials are

$$A_N = \frac{n}{4\pi}(1 - \cos\theta)d\phi, \qquad A_S = -\frac{n}{4\pi}(1 + \cos\theta)d\phi.$$

They are smooth at the included poles because the vanishing coefficient cancels the polar singularity of $d\phi$. Both have curvature F_n. On the overlap $A_S - A_N = -n\, d\phi/(2\pi)$, so in the convention of (4.1) use the transition function $g_{NS} = -n\phi/(2\pi)$ modulo integers. The transition phase in the viewer is consequently $\exp(-in\phi)$. The difference of the two oriented boundary integrals, $\int_{\mathrm{eq}}(A_N - A_S)$, gives the positively oriented period n. The winding read off from g_{NS} has the opposite sign with the chosen equator parameter. This makes the clutching and curvature sign conventions checkable together.

The local magnetic vector is radial with magnitude $|n|/(4\pi r^2)$ in units of the analytic field. The electric field vanishes. The corresponding spatial one-form H is $n\, dr/(4\pi r^2)$, so $dG = 0$ on the complement. A noninteger coefficient defines a real vacuum field on the same punctured spacetime but fails the curvature-lift test. A nonzero integral coefficient forbids smooth extension as a closed field over the origin. One can also

regard the integral example as the pullback of a degree-n circle bundle from the sphere. The underlying clutching geometry is the one that enters Dirac's monopole argument [43].

The viewer draws the two hemispheres in distinct colors. The physical chart overlap is an open band. The colored equator is a representative circle in that overlap, sufficient to display the transition winding. Radial glyphs are placed on a fixed sphere, so their common length encodes the sector at that radius. A negative sector reverses them. Setting the sector to zero gives the constant transition phase and removes radial glyphs. The omitted point is never part of the computational domain of the formulas.

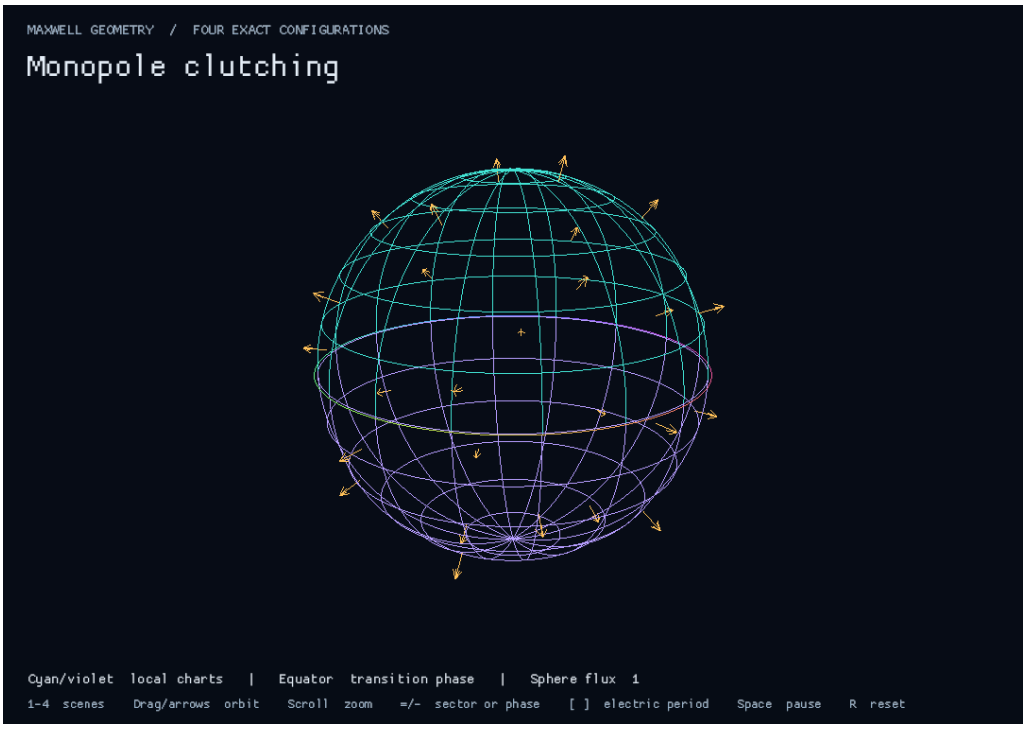


Figure 2: Clutching for unit monopole flux. The equatorial phase is sampled along one loop.

11.3 A flat Aharonov–Bohm connection

Use the spatial exterior of a solid straight cylinder, or equivalently the homotopy type of the complement of a line. In a polar coordinate on the exterior put

$$A_\alpha = \frac{\alpha}{2\pi} d\phi, \qquad F = 0, \qquad \alpha \in \mathbb{R}.$$

The angle is only locally real-valued, but $d\phi$ is a global closed one-form. A positive meridian has additive holonomy α mod $\mathbb{Z}$ and Wilson phase $\exp(2\pi i\alpha)$. Gauge transfor-

mations can shift α by an integer, using the circle map $m\phi/(2\pi)$ for $m \in \mathbb{Z}$. Conversely an equivalence preserves the meridian phase, so the flat isomorphism classes are exactly a circle. The automorphism group at every connection is another circle, describing constant gauge rather than a change of Wilson phase.

An exterior meridian does not bound in the domain. Differential Stokes determines holonomy on boundaries but leaves this phase free. In the physical Aharonov–Bohm setup a magnetic flux is confined to the removed interior, and the exterior phase has an interference interpretation [44]. The scene renders an idealized excluded cylinder and an exterior loop. A separate phase hand shows the image of that loop in the coefficient circle. It does not simulate a quantum wavefunction or interference pattern. Changing the phase changes the flat connection's class without drawing any local curvature.



Figure 3: An exterior meridian and its Wilson phase in the coefficient circle.

11.4 Quantized flux on a periodic three-torus

On $M = \mathbb{R}_t \times (\mathbb{R}/\mathbb{Z})^3$ let

$$F = n\,dy \wedge dz - e\,dt \wedge dx, \qquad \star F = e\,dy \wedge dz + n\,dt \wedge dx. \tag{11.1}$$

Both are closed. On an oriented yz-torus the magnetic period is n and the electric period of $\star F$ is e. The first is integral in the connection model, while e is an arbitrary

real number. The electric contribution to F is the exterior derivative of the global one-form $-et\,dx$. On the universal spatial cover, $A = ny\,dz - et\,dx$ gives the correct curvature. A shift $y \mapsto y+1$ changes this potential by $n\,dz$. The needed circle transition function $nz \bmod \mathbb{Z}$ is periodic in z when n is an integer. This is the quotient-cell version of the integral lift. One can also tensor with a flat connection to change one or more Wilson phases while preserving both periods.

The viewer uses a cube with identified opposite faces, a fundamental domain rather than an embedding of the whole three-torus in Euclidean three-space. The cyan face pair represents two copies of the same transverse yz-cycle. The amber and violet arrows represent magnetic and electric flux along the x direction, at offset positions so both can be seen. The controls change n in integer steps and e in real steps. For example, when $n = 0$ and $e = 0.43$, a quarter-turn sends the configuration to a real solution with magnetic period 0.43, which does not lift to a circle connection. The same periodic cell also explains why nonzero uniform total electric charge is forbidden: the outward face contributions cancel in pairs.

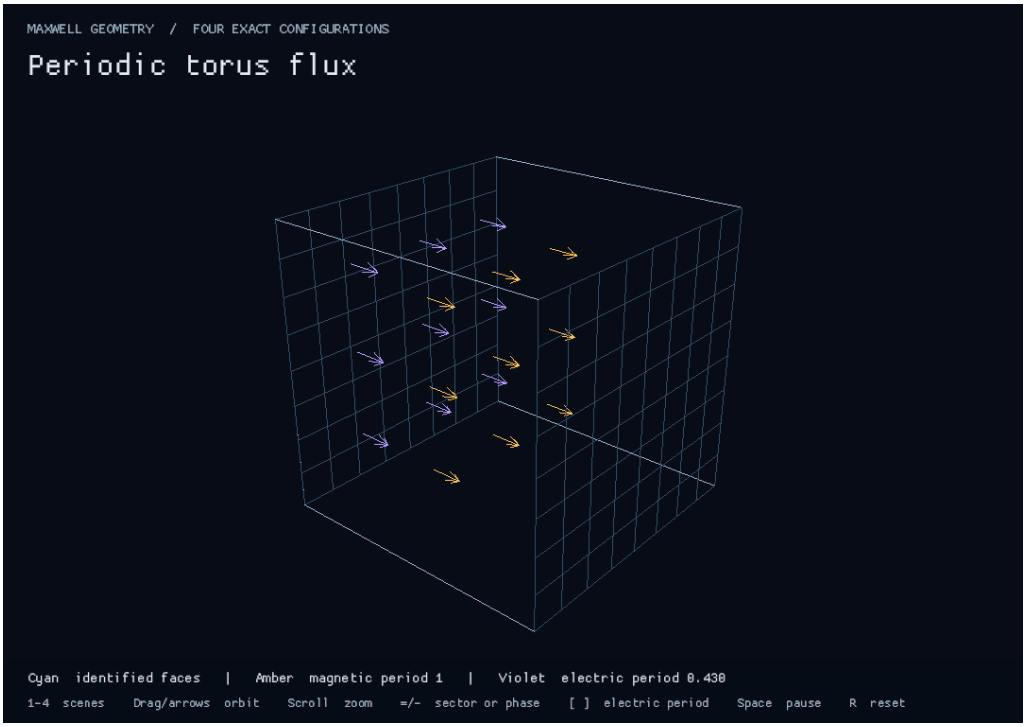


Figure 4: Two identified faces of a periodic cell and the independent flux controls.

11.5 Reproducing the scenes

The package contains the source and a locked dependency graph. From the viewer directory run `cargo run -locked`. Choose scenes with keys 1 to 4, orbit with mouse dragging or the arrow keys, zoom with the scroll wheel, and pause the wave with the space bar. The equals and minus keys adjust sector or phase. In the torus scene, the bracket keys adjust the electric period. Run `cargo run -locked - -capture DIR` to export one deterministic frame from each scene to a directory. This mode still uses a graphics context.

12 From differential forms to cells

12.1 The exact chain-level structure

A cell model can respect the exterior algebra that underlies the Maxwell equations. Let K be an oriented cellular or simplicial decomposition and let $C_k(K;\mathbb{Z})$ be its group of oriented chains. The coboundary on cochains is dual to the boundary operator. For a smooth triangulation the integration map I sends a form to its integrals on oriented simplices. Stokes gives

$$I(d\omega) = \delta I(\omega), \qquad \delta^2 = 0.$$

Thus the exact incidence identities behind conservation survive replacing forms by their integrals. A constitutive discretization requires further structure, such as a discrete Hodge operator or finite-element mass pairing. The incidence matrices alone contain no vacuum wave speed, impedance, or positivity of the discrete energy. That is the computational counterpart of the premetric decomposition.

For instance one may store a magnetic two-cochain on oriented faces and its homogeneous exterior equation on three-cells. If it is represented as the coboundary of a real one-cochain then its period on every cellular two-cycle is zero: $\langle \delta a, z\rangle = \langle a, \partial z\rangle$. Hence global potentials in this discrete model have the same topological limitation as a global real one-form. On the periodic torus cell, a representative of a nonzero magnetic flux class must be supplied separately or built into transition data. Periodic indexing alone does not generate that class from a single potential.

Likewise a compact circle-valued link variable captures the holonomy of an edge in a local lattice presentation. The product around a face determines a phase, hence the real face integral only modulo integers. Recovering real fluxes in a nontrivial integral sector can require choosing lifts with integer corrections and compatibility, or retaining chart and transition data. Such a discrete presentation is more than replacing each smooth potential value by an angle. The distinction follows from the same exact sequence that gives the continuum curvature fiber.

12.2 Mesh incidence and metric approximation

On a triangulation, Whitney forms supply a map from simplicial cochains to finite-element differential forms that intertwines coboundary and exterior derivative. The finite element exterior calculus develops subcomplexes and commuting projections in a framework that controls stability and approximation [45]. In computational electromagnetics Hiptmair describes the role of differential forms and the function spaces suited to Maxwell's equations [46]. Bossavit emphasizes how mesh geometry and Whitney forms enter the discrete constitutive law [47]. In lattice formulations

Chew and Teixeira describe related topological and metric factorizations [48, 49]. The incidence identity is common to these methods, while their inner products, placement of degrees of freedom, and analytic convergence arguments differ.

Yee's staggered update is a classical numerical construction for time-dependent Maxwell equations [50]. Its discrete field placement offers a useful comparison with the cell and dual-cell formulation. Stern, Tong, Desbrun, and Marsden develop discrete differential forms and variational integrators for electromagnetism [51]. In a compatible scheme, applying discrete divergence to a discrete curl gives zero by incidence. To conclude that an evolution preserves Gauss constraints one must additionally use a discrete current obeying the compatible continuity equation. A source added without that compatibility may defeat the constraint even if the underlying mesh identities hold.

A curvature glyph in the companion viewer is neither a cochain degree of freedom nor a numerical update. The viewer samples an exact field at selected points. A computational Maxwell solver would require discrete excitation data, a choice of temporal stepping or weak formulation, stability bounds and convergence analysis. The diagrams expose the topology and the field geometry independently of those numerical choices.

12.3 What would have to commute under discretization

For smooth forms and smooth oriented singular chains one has the commutative diagram

$$\begin{array}{ccc} \Omega^k(M) & \xrightarrow{d} & \Omega^{k+1}(M) \\ \downarrow I & & \downarrow I \\ C^k(K;\mathbb{R}) & \xrightarrow{\delta} & C^{k+1}(K;\mathbb{R}). \end{array}$$

There is no comparably automatic identity $I(\star F) = \star_K I(F)$ for arbitrary continuous forms and a fixed finite-dimensional cochain star. The integrated values of F can leave information needed to evaluate the integrated star on another collection of cells. A numerical star and projection therefore have to be justified within a chosen approximation class. A topology-preserving discrete model and a convergent constitutive scheme are distinct design requirements.

The analogous gluing question is whether chart restrictions, edge transport, and face phases reconstruct a well-defined global discrete connection. If a refinement subdivides an oriented loop, the product of its edge holonomies must agree with the original loop holonomy when representing the same smooth input. Subdividing a face must sum the integrals with their incidence signs. These identities follow from composition of transport and additivity of integration. Reconstructing a smooth connection from a finite grid would require additional interpolation choices. Gauge equivalence, metric approximation, and refinement are consequently different axes of comparison.

12.4 A cochain model of charge conservation

For a finite spacetime cell complex, let G_h be a two-cochain on the complex used for excitation and J_h a three-cochain. The identity $\delta G_h = J_h$ implies both $\delta J_h = 0$ and vanishing pairings of J_h with every three-cycle. It is the exact analogue of the two current conditions in the continuum. If the cells are periodic, the relevant chains are taken in the quotient complex. A sum of oriented cells representing a spatial slice has zero boundary because opposite-face incidences cancel. The global charge obstruction therefore survives an incidence-preserving discretization without a limiting argument. The choice of approximate Hodge operator can influence whether a prescribed exact discrete source is attained by a solution, just as constitutive data govern the continuum range problem.

A relative or boundary version changes the test cycles. On a domain cut into pieces, a three-chain can acquire a boundary on the cut. The excitation cochain evaluates on that boundary with the orientation of the piece. When two compatible pieces are glued, opposite orientations cancel the two contributions. Matching the cochain values yields this cancellation of flux. Matching the underlying connection can add conditions on Wilson data along the cut. This is a discrete analogue of the difference between the two types of collar gluing in Section 5. It follows at the level of chains without a particular discrete Hodge operator.

Finally, the four figures can be used to inspect which variables a discretization would need. A wave profile tests temporal evolution and the transverse metric relation. The monopole sphere tests a nontrivial flux class that cannot arise as a global real coboundary. The exterior loop tests a flat holonomy invariant even when all stored curvatures vanish. The periodic torus separates electric and magnetic periods. Together they distinguish four behaviors that can be conflated if a numerical representation stores only local field amplitudes.

13 Notes on the literature and scope of the formulation

13.1 Primary texts and modern translation

1. Maxwell's two primary sources. The 1865 paper and the 1873 *Treatise* record different presentations of the theory [1, 2]. The statements in this manuscript concern the present-day classical equations, so the source of their historical components and the source of our notation are different. Hunt is useful for distinguishing Maxwell's dynamical reasoning in the 1860s from the development of his measurement program [3]. The Smithsonian and Library of Congress copies give access to primary material without relying on a count of the equations taken from a modern retelling.

2. Heaviside and the role of the field variables. Heaviside's work brings the electric and magnetic variables forward and helps explain why the usual vector system has such a concise form. For a historical overview and the publication sequence see [6]; for the first collected volume see [5]. The modern split separates (E, B) from (D, H). Identifying them up to constants in vacuum is a constitutive specialization. This distinction remains important for the categorical generalization because the constitutive relation can be varied while the exterior equations stay fixed.

3. Spacetime and gauge history. Minkowski's 1908 article works in a spacetime setting and discusses electromagnetism in moving bodies [7]. It supplies historical context for treating electric and magnetic components as observer-dependent pieces of a unified object. The exterior form and Hodge conventions used here are modern ones. The history of gauge invariance has a separate development through the use and transformation of potentials. Jackson and Okun review that line in detail [4].

4. Premetric formulations. The treatment of excitation, twisted forms, and constitutive structure in [8] is much broader than the vacuum specialization here. Their constitutive program separates the differential equations from the possible physical response laws. The article [9] addresses metric reconstruction under restrictions on the constitutive tensor. Delphenich relates premetric equations to exterior differential systems [10]. The span in Section 5 is a convenient categorical representation of a permitted relation between fields, whether or not it comes from a metric reconstruction problem.

13.2 Differential cohomology and stacks

5. Differential characters. Cheeger and Simons's character description has two features used throughout this paper [19]. First, holonomy is a function on cycles with a differential Stokes law on boundaries. Second, the curvature of that function has integral periods. These facts give an intrinsic comparison of curvatures, fluxes, and Wilson phases. Bär and Becker supply a systematic treatment of operations such as product and fiber integration [21]. Hopkins and Singer give cochain and space-level constructions that are useful beyond ordinary degree two [20].

6. Quantization and compact gauge data. Freed explains charge quantization in terms of differential cohomology and generalized versions appropriate to other field theories [23]. In the classical Maxwell example the compact structure group fixes a magnetic integral lattice. This assumption is additional global input. The source equation does not impose it. Dirac's analysis of quantized singularities motivates the monopole geometry [43]. The Aharonov–Bohm example makes the complementary point: the curvature can be zero on the accessible domain while a Wilson phase remains [44].

7. Cohesion and characteristic cocycles. Schreiber develops differential cohomology internally to a cohesive higher topos and uses it to describe higher gauge fields and refinements of characteristic maps [16]. Schreiber and Shulman give a type-theoretic formulation of cohesion with gauge-theoretic applications [17]. Fiorenza, Schreiber, and Stasheff develop the Cech cocycle construction of differential characteristic classes from higher Lie data [18]. These works provide the context for treating a connection as a map into a differential classifying object. The present use of that context is deliberately concrete: the degree-two object has explicit transition functions, one-forms, and gauge arrows.

8. Generalized flux quantization. Sati and Schreiber place Maxwell-type phase spaces and higher examples in a framework in which flux quantization is additional global geometric structure [34, 35]. Their later higher-gauge exposition further develops the role of differential nonabelian cohomology [36]. The ordinary circle model used here keeps the standard degree-two coefficients fixed. Its relative lifting theorem does not select a generalized quantization law or extend to nonlinear self-sourcing equations by itself.

9. Principal bundles and descent. The classifying stack of the circle is the first instance of the principal ∞-bundle construction studied by Nikolaus, Schreiber, and

Stevenson [14, 15]. Hollander gives a model-categorical account of groupoid stacks [12]. Lurie's *Higher Topos Theory* places these in the category of sheaves of spaces [11]. Dugger, Hollander, and Isaksen describe hypercover localization in simplicial presheaves [13]. Because the field stack here is truncated, one can compare these descriptions without encountering a distinction between descent and hyperdescent for this particular object.

13.3 Direct predecessors for the equation and the phase space

10. The differential-cohomological Maxwell kernel. Becker, Schenkel, and Szabo compose a Lorentzian differential operator with curvature and take its kernel in a differential-cohomology setting [26]. Their analysis retains topological sectors and studies presymplectic structures, suitable duals and quantization. At the level of components, the vacuum degree-two fiber here has that same structure of a Maxwell kernel, with the codifferential equation equivalent to vanishing of $d \star F$. The explicitly sourced pullback in this paper replaces the zero point with an arbitrary closed current and retains the gauge groupoid.

11. The Lorentzian Yang–Mills stack. Benini, Schenkel, and Schreiber give a stack of Yang–Mills solutions with parametrized connections and a stacky Cauchy problem [27]. In the circle case the covariant derivative on adjoint-valued field strengths reduces to the ordinary exterior derivative. Their work is therefore the closest stack-theoretic predecessor, not merely a broad analogy with higher gauge theory. The product site and vertical connections here give an explicit presentation for fixed smooth prescribed sources. The ultrastatic proof in Section 8 is an ordinary-background comparison. It does not replace their analysis of parametrized Yang–Mills evolution.

12. Duality and the two-integral model. Becker, Benini, Schenkel, and Szabo implement abelian duality with two differential characters whose curvatures satisfy a Hodge relation [28]. The groupoid of dual lifts here is its classical stack-level analogue. Witten's partition-function calculation and the flux operators of Freed, Moore, and Segal belong to quantum theory [31, 32]. The classical quarter-turn describes a map of pairs before any action, Hilbert space or operator algebra is built. Gaiotto, Kapustin, Seiberg, and Willett put the magnetic and electric currents in the framework of higher-form symmetries [33].

13. Parallel transport. Schreiber and Waldorf formulate connections as smooth transport functors with local triviality and descent data [22]. Transport offers a complementary interpretation to the Deligne presentation. At fixed curvature the

possible changes in closed-loop transport are precisely flat characters. Conversely, constructing a smooth connection from abstract loop data requires smoothness and compatibility beyond assigning an arbitrary phase to every loop.

13.4 Causal analysis, quantization, and computation

14. Hyperbolic dynamics. Bernal and Sanchez provide smooth Cauchy hypersurfaces and temporal splittings [29]. Bär, Ginoux, and Pfäffle develop normal hyperbolicity, finite propagation, and Cauchy existence and uniqueness [30]. The ultrastatic case here uses these tools on a spatial one-form wave equation and on a scalar constraint propagator. Keeping the topology in the initial connection makes temporal gauge compatible with nontrivial bundles.

15. Observable locality. The locally covariant framework of Brunetti, Fredenhagen, and Verch specifies its background and observable categories and its causal conditions [40]. Fewster and Lang show how different treatments of topological contributions affect the Maxwell observable theory [41]. The homotopy-colimit construction of Benini, Schenkel, and Szabo is a route to global abelian observables that retains derived gluing data [42]. Such a construction is separate from the homotopy limit for configurations.

16. Boundary and derived directions. Boundary terms can change which gauge transformations are null directions of a phase-space pairing. Donnelly and Freidel develop boundary degrees of freedom in gauge theory and gravity [37]. The use of a derived deformation complex can be compared with the BV approach of Costello and Gwilliam and the derived-geometric framework of Pantev, Toën, Vaquié, and Vezzosi [38, 39]. Those references motivate retaining equation identities and tangent data in a derived enhancement. Our classical stack and its linear complex specify the input to such an enhancement.

17. Computational precedent. The compatible-discretization literature makes the exterior and metric split operational. Yee provides a classical staggered scheme, Chew and Teixeira discuss lattice equations and topology, and Bossavit and Hiptmair explain the geometry of finite elements [50, 48, 49, 47, 46]. Arnold, Falk, and Winther give a general Hilbert-complex approach [45]. Stern and coauthors combine discrete differential forms with a variational approach [51]. The examples in this manuscript concern the exact analytic fields underlying those possible approximations.

13.5 Summary of hypotheses and claims

The fixed-source solution object uses a metric or a compatible constitutive relation, a smooth closed prescribed source, and smooth connections on the stated domain. Its descent and base-change properties are formal consequences of the inputs and their naturality. The lift comparison requires integral periods for ordinary circle connections and uses relative cohomology of coefficient sheaves in families. The ultrastatic Cauchy theorem uses a complete spatial metric and no added support restrictions. The variational formulas use compactly supported changes or controlled boundary terms. Quantum duality, distributional fields on the filled defect, and convergence of a numerical solver require their own analytic and geometric input. Together these boundaries specify which results belong to the higher-topos model and which belong to neighboring formulations.

References

[1] J. C. Maxwell. A dynamical theory of the electromagnetic field. *Philosophical Transactions of the Royal Society of London* 155 (1865), 459–512. Source

[2] J. C. Maxwell. *A Treatise on Electricity and Magnetism*. 2 vols. Clarendon Press, 1873. Source

[3] B. J. Hunt. Maxwell, measurement, and the modes of electromagnetic theory. *Historical Studies in the Natural Sciences* 45 (2015), 303–339. Source

[4] J. D. Jackson and L. B. Okun. Historical roots of gauge invariance. *Reviews of Modern Physics* 73 (2001), 663–680. Source

[5] O. Heaviside. *Electromagnetic Theory*, vol. 1. The Electrician Printing and Publishing Company, 1893. Source

[6] C. Donaghy-Spargo and A. Yakovlev. Oliver Heaviside's electromagnetic theory. *Philosophical Transactions of the Royal Society A* 376 (2018), 20180229. Source

[7] H. Minkowski. Die Grundgleichungen für die elektromagnetischen Vorgänge in bewegten Körpern. *Nachrichten von der Gesellschaft der Wissenschaften zu Göttingen* (1908), 53–111. English translation by M. Saha, with Wikisource revisions. Source

[8] F. W. Hehl and Y. N. Obukhov. *Foundations of Classical Electrodynamics: Charge, Flux, and Metric*. Birkhäuser, 2003. Source

[9] Y. N. Obukhov and F. W. Hehl. Spacetime metric from linear electrodynamics. *Physics Letters B* 458 (1999), 466–470. Source

[10] D. H. Delphenich. Symmetries and pre-metric electromagnetism. Preprint, 2005. Source

[11] J. Lurie. *Higher Topos Theory*. Annals of Mathematics Studies 170. Princeton University Press, 2009. Source

[12] S. Hollander. A homotopy theory for stacks. *Israel Journal of Mathematics* 163 (2008), 93–124. Source

[13] D. Dugger, S. Hollander, and D. C. Isaksen. Hypercovers and simplicial presheaves. *Mathematical Proceedings of the Cambridge Philosophical Society* 136 (2004), 9–51. Source

[14] T. Nikolaus, U. Schreiber, and D. Stevenson. Principal infinity-bundles: General theory. *Journal of Homotopy and Related Structures* 10 (2015), 749–801. Source

[15] T. Nikolaus, U. Schreiber, and D. Stevenson. Principal infinity-bundles: Presentations. *Journal of Homotopy and Related Structures* 10 (2015), 565–622. Source

[16] U. Schreiber. Differential cohomology in a cohesive infinity-topos. Preprint, 2013, revised versions available. Source

[17] U. Schreiber and M. Shulman. Quantum gauge field theory in cohesive homotopy type theory. Preprint, 2014. Source

[18] D. Fiorenza, U. Schreiber, and J. Stasheff. Čech cocycles for differential characteristic classes: An infinity-Lie theoretic construction. *Advances in Theoretical and Mathematical Physics* 16 (2012), 149–250. Source

[19] J. Cheeger and J. Simons. Differential characters and geometric invariants. In *Geometry and Topology*, Lecture Notes in Mathematics 1167, Springer, 1985, pp. 50–80. Source

[20] M. J. Hopkins and I. M. Singer. Quadratic functions in geometry, topology, and M-theory. *Journal of Differential Geometry* 70 (2005), 329–452. Source

[21] C. Bär and C. Becker. Differential characters and geometric chains. Preprint, 2013. Source

[22] U. Schreiber and K. Waldorf. Parallel transport and functors. *Journal of Homotopy and Related Structures* 4 (2009), 187–244. Source

[23] D. S. Freed. Dirac charge quantization and generalized differential cohomology. *Surveys in Differential Geometry* 7 (2000), 129–194. Source

[24] R. Bott and L. W. Tu. *Differential Forms in Algebraic Topology*. Graduate Texts in Mathematics 82. Springer, 1982. Source

[25] F. W. Warner. *Foundations of Differentiable Manifolds and Lie Groups*. Graduate Texts in Mathematics 94. Springer, 1983. Source

[26] C. Becker, A. Schenkel, and R. J. Szabo. Differential cohomology and locally covariant quantum field theory. *Reviews in Mathematical Physics* 29 (2017), 1750003. Source

[27] M. Benini, A. Schenkel, and U. Schreiber. The stack of Yang–Mills fields on Lorentzian manifolds. *Communications in Mathematical Physics* 359 (2018), 765–820. Source

[28] C. Becker, M. Benini, A. Schenkel, and R. J. Szabo. Abelian duality on globally hyperbolic spacetimes. *Communications in Mathematical Physics* 349 (2017), 361–392. Source

[29] A. N. Bernal and M. Sánchez. Smoothness of time functions and the metric splitting of globally hyperbolic spacetimes. *Communications in Mathematical Physics* 257 (2005), 43–50. Source

[30] C. Bär, N. Ginoux, and F. Pfäffle. *Wave Equations on Lorentzian Manifolds and Quantization*. European Mathematical Society, 2007. Source

[31] E. Witten. On S-duality in abelian gauge theory. *Selecta Mathematica* 1 (1995), 383–410. Source

[32] D. S. Freed, G. W. Moore, and G. Segal. The uncertainty of fluxes. *Communications in Mathematical Physics* 271 (2007), 247–274. Source

[33] D. Gaiotto, A. Kapustin, N. Seiberg, and B. Willett. Generalized global symmetries. *Journal of High Energy Physics* 02 (2015), 172. Source

[34] H. Sati and U. Schreiber. Flux quantization on phase space. *Annales Henri Poincaré* 26 (2025), 895–919. Source

[35] H. Sati and U. Schreiber. Flux quantization. Preprint, 2024. Source

[36] H. Sati and U. Schreiber. Higher gauge theory via differential nonabelian cohomology. Preprint, 2026. Source

[37] W. Donnelly and L. Freidel. Local subsystems in gauge theory and gravity. *Journal of High Energy Physics* 09 (2016), 102. Source

[38] K. Costello and O. Gwilliam. *Factorization Algebras in Quantum Field Theory*, vol. 2. Cambridge University Press, 2021. Source

[39] T. Pantev, B. Toën, M. Vaquié, and G. Vezzosi. Shifted symplectic structures. *Publications Mathématiques de l'IHÉS* 117 (2013), 271–328. Source

[40] R. Brunetti, K. Fredenhagen, and R. Verch. The generally covariant locality principle: A new paradigm for local quantum field theory. *Communications in Mathematical Physics* 237 (2003), 31–68. Source

[41] C. J. Fewster and B. Lang. Dynamical locality of the free Maxwell field. *Annales Henri Poincaré* 17 (2016), 401–436. Source

[42] M. Benini, A. Schenkel, and R. J. Szabo. Homotopy colimits and global observables in Abelian gauge theory. *Letters in Mathematical Physics* 105 (2015), 1193–1222. Source

[43] P. A. M. Dirac. Quantised singularities in the electromagnetic field. *Proceedings of the Royal Society A* 133 (1931), 60–72. Source

[44] Y. Aharonov and D. Bohm. Significance of electromagnetic potentials in the quantum theory. *Physical Review* 115 (1959), 485–491. Source

[45] D. N. Arnold, R. S. Falk, and R. Winther. Finite element exterior calculus, homological techniques, and applications. *Acta Numerica* 15 (2006), 1–155. Source

[46] R. Hiptmair. Finite elements in computational electromagnetism. *Acta Numerica* 11 (2002), 237–339. Source

[47] A. Bossavit. Generalized finite differences in computational electromagnetics. *Progress in Electromagnetics Research* 32 (2001), 45–64. Source

[48] W. C. Chew. Electromagnetic theory on a lattice. *Journal of Applied Physics* 75 (1994), 4843–4850. Source

[49] F. L. Teixeira and W. C. Chew. Lattice electromagnetic theory from a topological viewpoint. *Journal of Mathematical Physics* 40 (1999), 169–187. Source

[50] K. S. Yee. Numerical solution of initial boundary value problems involving Maxwell's equations in isotropic media. *IEEE Transactions on Antennas and Propagation* 14 (1966), 302–307. Source

[51] A. Stern, Y. Tong, M. Desbrun, and J. E. Marsden. Geometric computational electrodynamics with variational integrators and discrete differential forms. Preprint, 2007, revised 2009. Source

[52] Macroquad contributors. Macroquad 0.4.16 API documentation, `Camera3D` and 3D drawing primitives. Source