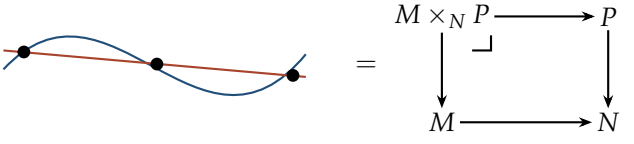
Higher Categorical Semantics

Differential Topology in Higher Categories

Mapping objects, derived intersections, spaces of choices, and cobordism



Anonymous

2026

About this text. This monograph is a higher-categorical treatment of the territory mapped by Morris W. Hirsch's *Differential Topology* (Graduate Texts in Mathematics 33, Springer, 1976): manifolds and maps, function spaces, transversality, vector bundles and tubular neighbourhoods, degrees and intersection numbers, Morse theory, cobordism, isotopy, and the classification of surfaces. It is an independent work. It reproduces none of that book's text or proofs verbatim; it is not affiliated with or endorsed by its publisher. Appendix B gives a concordance so that the two can be read side by side.

Status of results. Every numbered statement carries a tag: classical (standard differential topology, attributed), literature (established in the categorical, homotopical or derived-geometric literature, attributed), reformulation (a known result restated in a different language, with the restatement proved or sketched here), observation (an elementary remark we have not found stated in this form), or programme (a conjecture or a direction, not a result). No cited theorem is claimed as new.

Authorship and assistance. The text was drafted and typeset with a large language model (Anthropic's Claude) under the direction of the submitting author, who is responsible for it. Generative AI assisted with drafting, literature recall, mathematical consistency checks, computation, and typesetting. The numerical examples and a number of finite structural claims are recomputed by the accompanying script `verify-differential-topology.py` (Appendix C). Proofs and bibliographic details have not been independently refereed; corrections are welcome.

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Preface

Hirsch's *Differential Topology* [Hir76] is a book about what can be done to a smooth map by moving it a little. Its method is uniform across nine chapters. One topologizes the set $C^r(M, N)$ of maps; one shows that the maps with a desirable property (immersions, embeddings, maps transverse to a submanifold, Morse functions) form an open and dense, or at least residual, subset; one defines an invariant using a map in that subset; and one shows that the invariant does not depend on which map was used, because any two such maps are joined by a path that is itself generic one dimension up. Degree, intersection number, Euler characteristic, the cobordism class of a preimage, and the diffeomorphism type of a surface are all obtained in this way.

Each step of that method has a counterpart in the language of higher categories, and the counterparts are by now theorems in the literature and no longer analogies. The set of maps is the object of global points of a mapping object in a topos of smooth sets, whose homotopy type is its *shape*; the classification theorems of the subject (for vector bundles, for immersions, for manifolds up to cobordism) compute such shapes. Transversality is the condition under which a fibre product of manifolds exists and is correct; the category of manifolds has a universal completion in which every fibre product is correct, the ∞-category of derived manifolds, and the genericity theorems say that every object of the completion can be deformed back into the original category. Independence of choices is the statement that a space of choices (of tubular neighbourhoods, collars, metrics, embeddings in $\mathbb{R}^\infty$) is contractible, which is what makes a construction functorial. Invariants are traces. Decompositions by Morse functions are presentations by generators and relations of cobordism categories.

This book follows Hirsch's order and states each of his main theorems in that language, with attribution. It is not a replacement for his book, and it does not reprove his analytic theorems. The inverse function theorem, partitions of unity, the Morse–Sard theorem, and the existence and uniqueness of solutions of ordinary differential equations are used as stated. What changes is the form of the conclusions, and the last chapter says plainly how much that change is worth. In brief: of the three volumes in this series so far, this is the one in which the higher-categorical statements carry the most real content, because the subject is where a good part of higher category theory came from; and it is also the one in which the gap between a formal statement and its proof is

widest, because the proofs are analysis.

A companion volume, *Parametrized Neural Networks and Their Representations in Differential Topology*, applies the results of this book to smooth families of maps given by neural network architectures. It was written to be read after this one and cites it by chapter.

Introduction

What is assumed

The reader should know the definition of a smooth manifold and of a smooth map, the inverse function theorem, and the language of categories, functors, limits and adjunctions. Sheaves on a site, simplicial sets, and the idea of an ∞-category as a category with spaces of morphisms are used from Chapter 1 on; Appendix A collects the definitions and fixes the notation. Spectra appear in Chapters 6, 7 and 9 and are recalled where they enter. No result from stable homotopy theory beyond the statement of the Pontryagin–Thom theorem is needed to follow the text.

How the book is organized

The four parts follow the logical order of the source. Part I builds the ambient categories and treats the two genericity mechanisms, function spaces and transversality. Part II treats bundles and the invariants that come from them. Part III treats the three ways of taking a manifold apart or putting manifolds in families: Morse functions, cobordisms, isotopies; and ends with surfaces, where all three can be carried out completely. Part IV is an assessment. Figure 1 shows the dependencies.

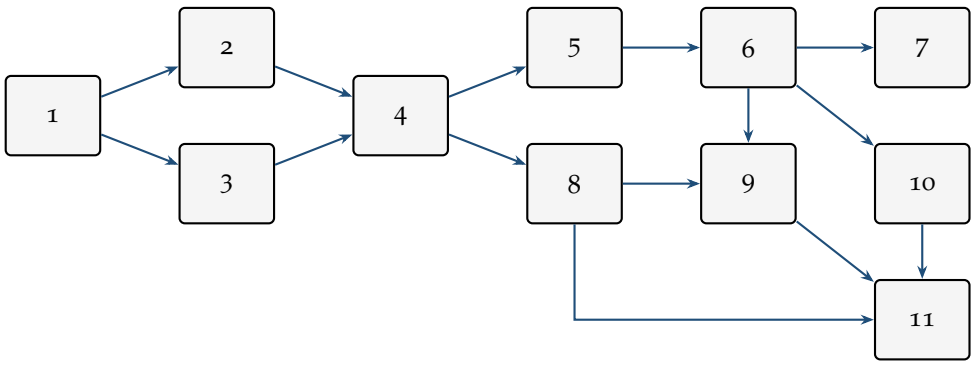


Figure 1: Dependencies among the chapters. Chapters 1–4 build the categories and the genericity theorems; 5–7 treat bundles, choices and numerical invariants; 8–11 treat Morse functions, cobordism, isotopy and surfaces.

A dictionary

The table lists, for each of the recurring notions of the source, the categorical object that this book uses in its place, and the chapter in which the translation is made.

In the classical text	Here	Ch.
the set $C^\infty(M,N)$ with a topology	the mapping object $[M,N]$ in smooth sets and its shape	3
the chain rule	functoriality of T and of Weil functors	2
a map transverse to a submanifold	a fibre product that the embedding into derived manifolds preserves	4
"by a small perturbation we may assume"	every derived manifold is cobordant to a manifold	4, 9
isomorphism classes of vector bundles	homotopy classes of maps to a classifying space; concordance	5
"unique up to isotopy"	a contractible space of choices	6
Euler characteristic, Lefschetz number	trace of an endomorphism of a dualizable object	7
passing a critical point	a homotopy pushout along a sphere	8
cobordism group	π_0 of a cobordism category; a homotopy group of a Thom spectrum	9
isotopy extension	a restriction map between embedding spaces is a fibration	10
classification of surfaces	generators and relations for the two-dimensional cobordism category	11

Conventions

Manifolds are smooth (C^∞), Hausdorff, second countable, and finite-dimensional; different components may have different dimensions unless a dimension is named. A *closed* manifold is compact without boundary. Hirsch works with C^r maps for $1 \le r \le \omega$ and proves that nothing is lost by passing to C^∞ (Theorem 1.2); we use the finite-differentiability classes only where a theorem needs them. "Space" means ∞-groupoid, that is, a homotopy type; the ∞-category of spaces is written $\mathcal{S}$. Citations of the form "Hirsch 3.2.7" mean Chapter 3, Section 2, item 7 of the source.

How to read the tags

A statement tagged classical or literature is someone else's theorem, with a reference; a proof is given only when it is short or when its form matters for what follows. A statement tagged reformulation is a restatement, and the text

says what it restates. Statements tagged observation are elementary; we have not found them written down in the form given and make no claim of depth. Statements tagged programme are not results.

Part I

Spaces of Smooth Things

Chapter 1

Manifolds and Smooth Maps

Hirsch's first chapter defines manifolds, maps, tangent bundles, embeddings and manifolds with boundary, and proves two theorems that the rest of his book uses constantly: the preimage of a regular value is a submanifold, and a compact manifold embeds in a Euclidean space. This chapter records the category those definitions produce, says exactly what is wrong with it as a category, and introduces the four enlargements that repair it. Each enlargement repairs a different defect, and it is worth seeing at the outset which is which.

1.1 The category of manifolds

Definition 1.1. Mfd is the category whose objects are smooth manifolds (Hausdorff, second countable, finite-dimensional, without boundary) and whose morphisms are smooth maps. Cart $\subset$ Mfd is the full subcategory on the objects $\mathbb{R}^n$, $n \geq 0$, and Open $\subset$ Mfd the full subcategory on open subsets of the $\mathbb{R}^n$.

The source works throughout with a differentiability class $1 \leq r \leq \omega$. The following theorem is the reason one may fix $r = \infty$.

Theorem 1.2 (Smoothing of differential structures)**.** [classical; Whitney; Hirsch 2.2.9, 2.2.10] *Let $1 \leq r < \infty$. Every C^r manifold is C^r diffeomorphic to a C^∞ manifold, and two C^∞ manifolds that are C^r diffeomorphic are C^∞ diffeomorphic.*

So the groupoids of C^r manifolds and C^r diffeomorphisms have the same sets of isomorphism classes for all $r \geq 1$. For $r = 0$ the statement fails in both directions: there are topological manifolds with no smooth structure and topological manifolds with several (Kervaire, Milnor). The theorem is proved by approximation in the strong topology (Chapter 3), and it is the first instance of the method described in the Preface.

1.2 What the category lacks

Mfd has finite products and countable coproducts, and little else. Three fibre products show the possibilities. In each, the cospan is $\mathbb{R}^k \xrightarrow{f} \mathbb{R} \xleftarrow{0} \mathrm{pt}$, so the fibre product, if it exists, is "the zero set of f".

Example 1.3 (Three zero sets)**.** (a) $f(x,y) = x^2 + y^2 - 1$. The value 0 is regular, the zero set is the circle, and the circle is the fibre product in Mfd.

(b) $f(x) = x^2$. A cone over the cospan is a manifold S with a smooth $u\colon S \to \mathbb{R}$ such that $u^2 = 0$; then $u = 0$. So the fibre product exists in Mfd and is a point. It is the wrong answer: the tangent functor does not preserve it (the fibre product of the tangent maps is $\{(x,v) : x^2 = 0,\ 2xv = 0\}$, a line), and it is unstable, since the zero set of $x^2 - \varepsilon$ is two points or none.

(c) $f(x,y) = xy$. The fibre product does not exist in Mfd (Proposition 1.4).

Proposition 1.4. [classical] *No manifold represents the functor* $S \mapsto \{u \in C^\infty(S, \mathbb{R}^2) : u_1 u_2 = 0\}$ *on* Mfd.

Proof. Suppose P does, with universal map $j\colon P \to \mathbb{R}^2$. Evaluating on $S = \mathrm{pt}$ shows that j is a bijection onto the cross $X = \{xy = 0\}$. Let $p_0 = j^{-1}(0)$ and let d be the dimension of P at p_0. The two axes are smooth curves in X through 0, so they lift to smooth injective curves $a, b\colon \mathbb{R} \to P$ through p_0 with $a(\mathbb{R}) \cap b(\mathbb{R}) = \{p_0\}$; hence $d \geq 1$. If $d = 1$, an injective continuous map between 1-manifolds is open, so $a(\mathbb{R})$ and $b(\mathbb{R})$ are both neighbourhoods of p_0 and meet in more than a point. If $d \geq 2$, then j has rank at most 1 everywhere, since X has empty interior, and by the constant rank theorem j fails to be injective on an open dense set where its rank is locally constant. □

There are also no mapping objects: outside degenerate cases there is no manifold N^M with $\mathsf{Mfd}(S \times M, N) \cong \mathsf{Mfd}(S, N^M)$. And quotients by even very good equivalence relations leave the category.

1.3 Smooth sets and diffeological spaces

The first repair is to keep the test objects and enlarge the class of things tested.

Definition 1.5. A family $\{U_i \to U\}$ in Cart is a covering if it is an open cover all of whose finite non-empty intersections are diffeomorphic to Euclidean spaces. A ***smooth set*** is a sheaf of sets on Cart for this coverage; $\mathsf{SmSet} = \mathsf{Sh}(\mathsf{Cart})$. For a smooth set X, the elements of $X(\mathbb{R}^n)$ are its ***plots*** of dimension n. A smooth set is ***concrete***, or a ***diffeological space***, if each plot is determined by the function $\mathbb{R}^n \to X(\mathrm{pt})$ it induces on points; $\mathsf{Dflg} \subset \mathsf{SmSet}$ is the full subcategory of these.

Theorem 1.6. [classical; Souriau, Iglesias-Zemmour [IZ13]; Baez–Hoffnung [BH11]]

(i) The functor $\mathsf{Mfd} \to \mathsf{SmSet}$, $M \mapsto C^\infty(-, M)$, *is fully faithful and takes values in* Dflg.

(ii) It preserves every limit that exists in Mfd, *and it sends an open cover of a manifold to a colimit diagram.*

(iii) SmSet *is a Grothendieck topos and* Dflg *a quasitopos. Both are complete, cocomplete and locally cartesian closed. The mapping object has plots* $[X,Y](\mathbb{R}^n) = \mathrm{Hom}(\mathbb{R}^n \times X, Y)$.

Proof of (i). A natural transformation $\alpha\colon C^\infty(-,M) \to C^\infty(-,N)$ gives a function $f = \alpha_{\mathrm{pt}}\colon M \to N$, and naturality with respect to the inclusions of points shows that α is composition with f. For every chart $\varphi\colon \mathbb{R}^m \to M$ the composite $f\varphi = \alpha(\varphi)$ is smooth, so f is smooth. □

By (ii), smooth sets inherit the fibre products of Example 1.3(a) and (b) unchanged, and in case (c) they supply one: the cross, with the plots being the smooth maps into $\mathbb{R}^2$ that land in it. By (iii) they supply the mapping object, and its plots are exactly what the source calls a smooth family of maps. Chapter 3 is about this object. What smooth sets do not repair is (b). A sheaf on Cart is tested only by reduced objects, and the equation $u^2 = 0$ has the same solutions as $u = 0$ in every $C^\infty(\mathbb{R}^n)$.

1.4 Function algebras

The second repair goes the other way and enlarges the class of test objects, by admitting algebras of functions that are not algebras of functions on a manifold.

Definition 1.7. A C^∞***-ring*** is a functor $A\colon \mathsf{Cart} \to \mathsf{Set}$ that preserves finite products. Its underlying set is $A(\mathbb{R})$; each smooth $h\colon \mathbb{R}^n \to \mathbb{R}$ gives an n-ary operation $A(h)\colon A(\mathbb{R})^n \to A(\mathbb{R})$. The category of these is written $C^\infty\mathsf{Ring}$. For a manifold M, the C^∞-ring $C^\infty(M)$ is $\mathbb{R}^n \mapsto C^\infty(M,\mathbb{R}^n)$.

Taking h to be addition and multiplication shows that a C^∞-ring has an underlying commutative $\mathbb{R}$-algebra. The free C^∞-ring on n generators is $C^\infty(\mathbb{R}^n)$, and quotients by ideals of the underlying algebra are again C^∞-rings.

Theorem 1.8. [literature; Moerdijk–Reyes [MR91], Ch. I, Thm. 2.8] *The functor* $\mathsf{Mfd} \to C^\infty\mathsf{Ring}^{\mathrm{op}}$, $M \mapsto C^\infty(M)$, *is fully faithful, and it sends transverse fibre products (Theorem 1.12) to pushouts: if* $f\colon M \to N$ *and* $g\colon P \to N$ *are transverse then* $C^\infty(M \times_N P) \cong C^\infty(M) \otimes^\infty_{C^\infty(N)} C^\infty(P)$, *the pushout in* $C^\infty\mathsf{Ring}$.

In Example 1.3(b) the pushout is $C^\infty(\mathbb{R})/(x^2)$. By Hadamard's lemma every smooth function is $g(x) = g(0) + g'(0)x + x^2h(x)$, so this ring is $\mathbb{R}[\varepsilon]/(\varepsilon^2)$, the dual numbers. Its "spectrum" has one point and a one-dimensional tangent space, which is what the tangent functor predicted. The ring has dimension 2, as does $C^\infty(\mathbb{R})/(x^2-\varepsilon) \cong \mathbb{R}\times\mathbb{R}$ for $\varepsilon > 0$. So C^∞-rings repair (b) on one side. For $\varepsilon < 0$ the function $x^2 - \varepsilon$ is invertible and the quotient is the zero ring; no

algebra of real functions will make the length constant. What is constant for all ε is the count with signs, $(+1)+(-1)=0$, and that is a statement about cobordism (Chapters 4 and 9).

The third and fourth repairs are homotopical. The third, the ∞-topos of smooth ∞-groupoids, is needed as soon as the objects being parametrized have automorphisms, as vector bundles and manifolds themselves do; it is introduced next. The fourth, derived manifolds, replaces the pushout of Theorem 1.8 by a homotopy pushout, and is the subject of Chapter 4. Table 1.1 shows which fibre product is seen where; the fourth row is the simplest case in which the derived answer differs from all the others.

zero set of	in Mfd	in SmSet	in $C^\infty\mathsf{Ring}^{\mathrm{op}}$	in DMfd
x^2+y^2-1	circle	circle	circle	circle
x^2	point	point	$\mathbb{R}[\varepsilon]/(\varepsilon^2)$	$\mathbb{R}[\varepsilon]/(\varepsilon^2)$
xy	none	cross	$C^\infty(\mathbb{R}^2)/(xy)$	$C^\infty(\mathbb{R}^2)/(xy)$
$0\colon \mathrm{pt}\to\mathbb{R}$	point	point	point	point of dimension -1

Table 1.1: Four fibre products $\mathbb{R}^k\times_{\mathbb{R}}\mathrm{pt}$ in four categories.

1.5 Smooth ∞-groupoids and shape

Definition 1.9. $\mathbf{H}=\mathsf{Sh}_\infty(\mathsf{Cart})$ is the ∞-topos of sheaves of spaces on Cart; its objects are ***smooth ∞-groupoids***, or smooth stacks. The constant-sheaf functor $\mathcal{S}\to\mathbf{H}$ has a left adjoint $\Pi\colon\mathbf{H}\to\mathcal{S}$, the ***shape***. Two points of $X(\mathrm{pt})$ are ***concordant*** if they are the restrictions to 0 and 1 of a point of $X(\mathbb{R})$.

Smooth sets are the 0-truncated objects of $\mathbf{H}$. The left adjoint exists because the objects of the site are contractible; there is a formula for it.

Theorem 1.10. [literature; see [BEBP24], [Sch13]] *For $X\in\mathbf{H}$, the shape is the geometric realization of the simplicial space obtained by evaluating on the extended simplices $\Delta^k_e=\{t\in\mathbb{R}^{k+1}:\sum t_i=1\}\cong\mathbb{R}^k$:*

$$\Pi X\ \simeq\ \operatorname*{colim}_{[k]\in\Delta^{\mathrm{op}}} X(\Delta^k_e).$$

For a manifold M, ΠM is the homotopy type of the underlying space of M, and $\pi_0\Pi X$ is the set of concordance classes of points of X whenever X is a smooth set.

For a manifold, the simplicial set $k\mapsto C^\infty(\Delta^k_e,M)$ is the smooth singular complex, and the statement that it has the homotopy type of M is the smooth approximation theorem (Hirsch 2.2.6) in its relative form. The functor Π is the device by which this book passes from smooth objects to homotopy types.

Nearly every classification theorem in later chapters has the form: for a certain smooth ∞-groupoid X of geometric structures, ΠX is such-and-such a space.

1.6 Regular values and the fibre products that exist

Definition 1.11. Smooth maps $f\colon M \to N$ and $g\colon P \to N$ are ***transverse***, written $f \pitchfork g$, if $df_x(T_xM) + dg_y(T_yP) = T_zN$ whenever $f(x) = g(y) = z$. If g is the inclusion of a submanifold A one writes $f \pitchfork A$. A point $z \in N$ is a ***regular value*** of f if $f \pitchfork \{z\}$.

Theorem 1.12 (Transverse fibre products). [classical; Hirsch 1.3.2, 1.3.3] *If $f \pitchfork g$ then $M \times_N P = \{(x,y) : f(x) = g(y)\}$ is a submanifold of $M \times P$ of dimension $\dim M + \dim P - \dim N$. It is the fibre product in* Mfd*, and its tangent bundle is $TM \times_{TN} TP$.*

Proof. By Exercise 1.5, $f \pitchfork g$ if and only if $f \times g \pitchfork \Delta_N$, so it suffices to treat the preimage $W = h^{-1}(A)$ of a submanifold $A \subset N$ of codimension k under a map $h \pitchfork A$. Locally A is the zero set of a submersion $\phi\colon V \to \mathbb{R}^k$, and transversality says that ϕh is a submersion at the points of W. The implicit function theorem then gives charts in which W is a coordinate subspace of codimension k and $T_xW = \ker d(\phi h)_x = dh_x^{-1}(T_{h(x)}A)$. A smooth map into $M \times P$ with image in an embedded submanifold is smooth into the submanifold, which gives the universal property. $\square$

The theorem identifies a class of cospans whose limits exist, are preserved by T, and by Theorem 1.8 are preserved by passage to function algebras. Chapter 4 shows that this class is large (every cospan is arbitrarily close to one in it) and that it determines a universal completion.

Example 1.13. Let $f\colon \mathbb{R}^{n\times n} \to \mathrm{Sym}_n$, $f(A) = A^{\mathsf{T}}A$. At $A \in \mathrm{O}(n)$ the differential $H \mapsto A^{\mathsf{T}}H + H^{\mathsf{T}}A$ is onto: $H = \tfrac{1}{2}AS$ maps to S. So $\mathrm{O}(n) = f^{-1}(I)$ is a manifold of dimension $n^2 - \tfrac{1}{2}n(n+1) = \tfrac{1}{2}n(n-1)$. The script checks the rank numerically for $n \le 4$.

1.7 Embeddings and idempotents

Theorem 1.14 (Whitney). [classical; Hirsch 1.3.5, 2.2.14] *Every manifold of dimension n admits an embedding into $\mathbb{R}^{2n+1}$ with closed image.*

Combined with the tubular neighbourhood theorem (Theorem 6.2), this says that every manifold is a smooth retract of an open subset of a Euclidean space. The converse also holds, and together they give a description of Mfd in which charts and atlases do not appear.

Theorem 1.15. [classical; attributed to Lawvere] Mfd *is the idempotent completion of* Open*: every idempotent in* Mfd *splits, and every manifold is a retract of an object of* Open.

Proof. The second statement was just explained. For the first, let $e\colon M \to M$ be smooth with $e^2 = e$; we show that $e(M) = \mathrm{Fix}(e)$ is a submanifold (Exercise 1.2 asks for the details). At a fixed point x the differential de_x is an idempotent matrix, so its rank equals its trace, which is continuous and integer-valued along $\mathrm{Fix}(e)$; let it be k near x. Rank is lower semicontinuous, so $\operatorname{rk} de_y \geq k$ for y near x, and $de_{e(y)}de_y = de_y$ gives $\operatorname{rk} de_y \leq \operatorname{rk} de_{e(y)} = k$. So e has constant rank k near x, and its image near x is a k-dimensional submanifold. □

The theorem explains two features of what follows. A product-preserving functor on Cart, such as a C^∞-ring, has a canonical value on every manifold, because the manifold is a retract of an open subset of some $\mathbb{R}^q$. And the universal completion of Theorem 4.12 has to be required to be idempotent complete.

1.8 Boundaries

Hirsch allows manifolds with boundary from the start, since cobordisms need them. A manifold with boundary is modelled on the half-space $\mathbb{R}^n_+ = \{x_n \geq 0\}$; its boundary ∂M is a manifold of dimension $n-1$. The category Mfd^∂ is less well behaved than Mfd: the product of two objects has corners, and ∂ is not a functor on all smooth maps. The notion that makes the regular value theorem work is the following.

Definition 1.16. A submanifold $A \subset M$ of a manifold with boundary is ***neat*** if $\partial A = A \cap \partial M$ and A is not tangent to ∂M at any point of ∂A.

Theorem 1.17. [classical; Hirsch 1.4.2] *Let $f\colon M \to N$ be smooth, with $\partial N = \emptyset$, and let z be a regular value of both f and $f|_{\partial M}$. Then $f^{-1}(z)$ is a neat submanifold of M with boundary $(f|_{\partial M})^{-1}(z)$.*

In smooth sets the half-line is the subobject $[0,\infty) \subset \mathbb{R}$, and a function on it is smooth in the sheaf sense exactly when it is the restriction of a smooth function on $\mathbb{R}$ [KM97, §24]. So manifolds with boundary embed fully faithfully in Dflg as well. The boundary is recovered from the object, as the set of points at which it is not locally Euclidean, although not functorially.

Notes

Diffeological spaces are due to Souriau; the standard reference is [IZ13], and the identification with concrete sheaves is in [BH11]. C^∞-rings come from

Lawvere's programme for synthetic differential geometry; see [MR91], [Koc06] and [Joy19]. The ∞-topos $\mathbf{H}$ and its shape are developed in [Sch13]; the formula of Theorem 1.10 is proved in [BEBP24] and in work of Pavlov and of Bunk. Manifolds with corners as a category are treated by Melrose and by Joyce.

Exercises

Exercise 1.1. Verify the second row of Table 1.1: the fibre product of $x \mapsto x^2$ and $0\colon \mathrm{pt} \to \mathbb{R}$ is a point in Mfd and in SmSet, and is corepresented by $\mathbb{R}[\varepsilon]/(\varepsilon^2)$ in $C^\infty\mathsf{Ring}$. Show that a map of C^∞-rings $C^\infty(M) \to \mathbb{R}[\varepsilon]/(\varepsilon^2)$ is a tangent vector to M.

Exercise 1.2. Complete the proof of Theorem 1.15: if $e\colon M \to M$ is a smooth idempotent then $\mathrm{Fix}(e)$ is a closed submanifold of M and $e\colon M \to \mathrm{Fix}(e)$ is smooth. Give an example in which $\mathrm{Fix}(e)$ has components of different dimensions.

Exercise 1.3. Carry out Example 1.13 in detail, and show that $T_I\mathrm{O}(n)$ is the space of skew-symmetric matrices.

Exercise 1.4. Let M, N be manifolds. Show that the smooth set $[M, N]$ is concrete, that its points are the smooth maps $M \to N$, and that $\mathrm{Hom}(S, [M, N]) \cong \mathrm{Hom}(S \times M, N)$ for every smooth set S.

Exercise 1.5. Show that $f\colon M \to N$ and $g\colon P \to N$ are transverse if and only if $f \times g\colon M \times P \to N \times N$ is transverse to the diagonal Δ_N.

Chapter 2

The Tangent Functor and Jets

The source introduces the tangent bundle in a few pages and the bundle of jets in a few more, and uses both as tools. Here they are treated as structure on the category. The chain rule is the statement that T is a functor; the symmetry of second derivatives is the statement that a certain algebra is commutative; the higher chain rule is the statement that jets compose. All product-preserving functors of this kind are classified by finite-dimensional algebras, and the axioms those functors satisfy can be isolated and imposed on other categories.

2.1 The tangent functor

On Cart put $T(\mathbb{R}^n) = \mathbb{R}^n \times \mathbb{R}^n$ and $Tf(x,v) = (f(x), Df_x v)$. The chain rule $D(gf)_x = Dg_{f(x)} Df_x$ is the statement $T(gf) = Tg\,Tf$.

Proposition 2.1. [classical] *T extends uniquely to a functor $T\colon \mathsf{Mfd} \to \mathsf{Mfd}$ with a natural transformation $p\colon T \Rightarrow \mathrm{id}$ such that T sends open embeddings to open embeddings and $T(U) = p_M^{-1}(U)$ for $U \subset M$ open. It preserves finite products and transverse fibre products, and each $p_M\colon TM \to M$ is a vector bundle.*

Proof. A manifold is the colimit in Mfd of the Čech diagram of an atlas, and the diagram consists of open embeddings between open subsets of $\mathbb{R}^n$, on which T is defined by the formula. The colimit of the image diagram exists and is TM. Products are preserved by the formula; transverse fibre products by Theorem 1.12. □

2.2 Weil algebras

Definition 2.2. A ***Weil algebra*** is a finite-dimensional commutative $\mathbb{R}$-algebra $A = \mathbb{R} \oplus \mathfrak{n}$ with $\mathfrak{n}$ a nilpotent ideal. It is a C^∞-ring in a unique way: for smooth h and $a = a_0 + n$ one puts $h(a) = \sum_\alpha D^\alpha h(a_0)\, n^\alpha / \alpha!$, a finite sum. The ***Weil functor*** of A is

$$T_A M = C^\infty\mathsf{Ring}\big(C^\infty(M), A\big),$$

the set of A-points of M, with the manifold structure for which $T_A(\mathbb{R}^n) = A^n$.

For $A = \mathbb{R}$ this is the identity functor (the $\mathbb{R}$-points of M are its points). For the dual numbers $W = \mathbb{R}[\varepsilon]/(\varepsilon^2)$ it is T, by Exercise 1.1. In coordinates

a map f acts on A-points by Taylor expansion in the nilpotent directions: $f(x + v\varepsilon) = f(x) + Df_x(v)\,\varepsilon$.

Theorem 2.3 (Classification of product-preserving bundle functors). [literature; Eck, Kainz–Michor, Luciano; [KMS93], §§35–36]

(i) For every Weil algebra A, T_A is a functor Mfd $\to$ Mfd *that preserves finite products, with a natural fibre bundle projection $T_A \Rightarrow \mathrm{id}$.*

(ii) Every product-preserving bundle functor on Mfd *is naturally isomorphic to T_A for a Weil algebra A, unique up to isomorphism.*

(iii) Natural transformations $T_A \Rightarrow T_B$ correspond bijectively to algebra homomorphisms $A \to B$.

(iv) $T_A \circ T_B \cong T_{A \otimes B}$, and $T_A \times_{\mathrm{id}} T_B \cong T_{A \times_{\mathbb{R}} B}$.

By (iii) and (iv), every natural operation on iterated tangent bundles is a homomorphism of small algebras. Table 2.1 lists the ones that matter.

natural transformation	algebra homomorphism	in coordinates
projection $p\colon T \Rightarrow \mathrm{id}$	$W \to \mathbb{R}$, $\varepsilon \mapsto 0$	$(x, v) \mapsto x$
zero section $0\colon \mathrm{id} \Rightarrow T$	$\mathbb{R} \to W$	$x \mapsto (x, 0)$
addition $+\colon T \times_{\mathrm{id}} T \Rightarrow T$	$W \times_{\mathbb{R}} W \to W$, $\varepsilon_1, \varepsilon_2 \mapsto \varepsilon$	$(x, v_1, v_2) \mapsto (x, v_1 + v_2)$
vertical lift $\ell\colon T \Rightarrow TT$	$W \to W \otimes W$, $\varepsilon \mapsto \varepsilon\eta$	$(x, v) \mapsto (x, 0, 0, v)$
flip $c\colon TT \Rightarrow TT$	$W \otimes W \to W \otimes W$, $\varepsilon \leftrightarrow \eta$	$(x, v, w, a) \mapsto (x, w, v, a)$

Table 2.1: The structure maps of the tangent functor as homomorphisms of Weil algebras. Here $W = \mathbb{R}[\varepsilon]/(\varepsilon^2)$, $W \otimes W = \mathbb{R}[\varepsilon, \eta]/(\varepsilon^2, \eta^2)$ and $W \times_{\mathbb{R}} W = \mathbb{R}[\varepsilon_1, \varepsilon_2]/(\varepsilon_i \varepsilon_j)$.

Proposition 2.4 (Symmetry of second derivatives). [reformulation; cf. [Koc06], [KMS93]] *The naturality of the flip c with respect to a map f is the equality $D^2 f_x(v, w) = D^2 f_x(w, v)$. It holds because $W \otimes W$ is commutative.*

Proof. Expanding in $W \otimes W$,

$$f(x + v\varepsilon + w\eta + a\varepsilon\eta) = f(x) + Df_x(v)\varepsilon + Df_x(w)\eta + \big(Df_x(a) + D^2 f_x(v, w)\big)\varepsilon\eta,$$

which is the formula for TTf. Exchanging ε and η exchanges v and w on the left; on the right it leaves the coefficient of $\varepsilon\eta = \eta\varepsilon$ in place. So $TTf \circ c = c \circ TTf$ if and only if $D^2 f_x(v, w) = D^2 f_x(w, v)$, and the exchange is an algebra automorphism, hence natural by Theorem 2.3(iii). □

The script evaluates polynomials in $\mathbb{R}[\varepsilon, \eta]/(\varepsilon^2, \eta^2)$ and confirms the expansion and its symmetry, and checks the chain rule in W against finite differences.

2.3 Tangent categories

The five maps of Table 2.1 and the equations among them can be abstracted.

Definition 2.5. A ***tangent category*** is a category $\mathcal{X}$ with an endofunctor T and natural transformations $p, 0, +, \ell, c$ as in Table 2.1, such that the fibre powers of each p_M exist and are preserved by T; $(p, 0, +)$ makes each TM a commutative monoid over M; ℓ and c are additive, $c^2 = \mathrm{id}$, $c\,\ell = \ell$, and the usual coherence equations hold; and the vertical lift is universal, in the sense that $TM \times_M TM \to TTM$, $(a, b) \mapsto T(+)\big(\ell(a), 0(b)\big)$, is the equalizer of $T(p)$ and $0\,p\,p$ [CC14, Def. 2.3].

The definition is Rosický's [Ros84], revived and developed by Cockett and Cruttwell. The last axiom says that a second-order tangent vector that is vertical is the same thing as a pair of first-order ones. It is what makes it possible to define the Lie bracket of vector fields, connections, and solutions of differential equations in any tangent category.

Theorem 2.6. [literature; Leung [Leu17]] *Let* Weil_1 *be the monoidal category of Weil algebras generated from* W *by* $\otimes$ *and* $\times_{\mathbb{R}}$. *A tangent structure on* $\mathcal{X}$ *is the same thing as a strong monoidal functor* $\mathsf{Weil}_1 \to \mathsf{End}(\mathcal{X})$ *that preserves the pullbacks expressing* $\times_{\mathbb{R}}$ *and the universality of* ℓ.

So Table 2.1 is the definition, read backwards. Examples of tangent categories include Mfd; $C^\infty\mathsf{Ring}^{\mathrm{op}}$ and the category of affine schemes, with T corepresented by $A \mapsto A[\varepsilon]$ and Kähler differentials; every model of synthetic differential geometry; and cartesian differential categories, which axiomatize Cart itself [CC14]. The definition has an ∞-categorical form, in which the Goodwillie calculus of functors is an example: the tangent bundle of an ∞-category is its fibrewise stabilization [BBC21].

2.4 Infinitesimals as objects

In $C^\infty\mathsf{Ring}^{\mathrm{op}}$ the algebra W is an object, $D = \operatorname{Spec} W$, and $T_A M$ is the set of maps $\operatorname{Spec} A \to M$. One would like TM to be the mapping object M^D.

Theorem 2.7. [literature; Dubuc [Dub79]; Kock [Koc06]; Moerdijk–Reyes [MR91]] *Let* $\mathcal{Z}$ *be the Cahiers topos, the topos of sheaves on the category of objects* $\mathbb{R}^n \times \operatorname{Spec} A$ *with* A *a Weil algebra. Then* $\mathsf{Mfd} \to \mathcal{Z}$ *is fully faithful and preserves transverse fibre products;* $\mathbb{R}^D \cong \mathbb{R} \times \mathbb{R}$ *(the Kock–Lawvere axiom); and* $(T_A M) \cong M^{\operatorname{Spec} A}$ *for every manifold* M *and Weil algebra* A.

In $\mathcal{Z}$ the tangent functor is an exponential, so it preserves all limits. The functor $(-)^D$ also has a right adjoint there, which is a statement about how

small D is. None of this is available in SmSet. The restriction of D to Cart is the terminal object, because a nilpotent element of $C^\infty(\mathbb{R}^n)$ is zero; a sheaf on Cart cannot tell D from a point. This is the same defect as in Example 1.3(b), and it is the reason that tangent functors on diffeological spaces are a subject with several inequivalent answers.

2.5 Jets

Definition 2.8. Two maps $f, g\colon M \to N$ have the same ***r-jet*** at x if $f(x) = g(x)$ and their partial derivatives up to order r agree at x in some, hence any, charts. $J^r(M,N)$ is the set of r-jets of maps $M \to N$ at all points of M; it is a manifold and a fibre bundle over $M \times N$ by source and target. A map f has an ***r-jet prolongation*** $j^r f\colon M \to J^r(M,N)$, a section of the source projection.

The fibre of $J^r(\mathbb{R}^m,\mathbb{R}^n)$ over $(0,0)$ is the space of polynomial maps $\mathbb{R}^m \to \mathbb{R}^n$ of degree at most r without constant term, of dimension $n\big(\binom{m+r}{r} - 1\big)$. For $r = 1$, $J^1(M,N) = \mathrm{Hom}(TM, TN)$ as a bundle over $M \times N$.

Proposition 2.9 (Jets compose). [classical] *For each r there is a category $\mathcal{J}^r$ whose objects are pairs (M,x) and whose morphisms $(M,x) \to (N,y)$ are the r-jets at x with target y. Taking the r-jet is a functor from germs of smooth maps to $\mathcal{J}^r$, and truncation gives functors $\mathcal{J}^r \to \mathcal{J}^{r-1}$. In coordinates, composition in $\mathcal{J}^r$ is composition of polynomials followed by truncation at degree r. The automorphism group G^r_m of $(\mathbb{R}^m,0)$ in $\mathcal{J}^r$ is a Lie group, $G^1_m = \mathrm{GL}_m(\mathbb{R})$, and the kernel of $G^r_m \to G^1_m$ is contractible.*

Proof. Functoriality is the higher chain rule: the derivatives of order at most r of gf at x are polynomials in those of f at x and of g at $f(x)$ (Faà di Bruno), so the r-jet of gf depends only on the r-jets, and associativity is inherited from composition of maps. The kernel consists of polynomial maps $x + (\text{terms of degree } 2,\dots,r)$ under truncated composition; scaling the higher terms to zero contracts it. □

For one variable and $r = 2$ the composition rule is $(a_1,a_2)\circ(b_1,b_2) = (a_1b_1,\ a_1b_2 + a_2b_1^2)$ for jets $a_1t + \frac{1}{2}a_2t^2$, which is $(gf)'' = g'f'' + g''(f')^2$. The script checks associativity of this and of the three-variable analogue. The last sentence of the proposition has a consequence that is used without comment throughout the subject: a structure defined by higher jets carries the same homotopical information as one defined by first jets, so the characteristic classes of a manifold are those of its tangent bundle.

Jets fit the framework of the previous sections: the functor $N \mapsto J^r_0(\mathbb{R}^m, N)$ of r-jets at the origin is the Weil functor of the algebra $\mathbb{R}[x_1,\dots,x_m]/\mathfrak{m}^{r+1}$.

2.6 Two derivatives of a composite

The tangent functor is covariant and pushes vectors forward. Its transpose pulls covectors back, and since T^* is not a functor on Mfd the transpose needs a different kind of category to live in.

Definition 2.10. Lens is the category whose objects are pairs (X, X') of Euclidean spaces and whose morphisms $(X, X') \to (Y, Y')$ are pairs $(f, f^\sharp)$ of smooth maps $f\colon X \to Y$ and $f^\sharp\colon X \times Y' \to X'$, composed by

$$(g, g^\sharp) \circ (f, f^\sharp) = \big(gf,\ (x, z') \mapsto f^\sharp(x, g^\sharp(f(x), z'))\big).$$

The ***reverse derivative*** of $f\colon \mathbb{R}^m \to \mathbb{R}^n$ is $R[f]\colon \mathbb{R}^m \times \mathbb{R}^n \to \mathbb{R}^m$, $R[f](x, w) = (Df_x)^{\mathsf{T}} w$.

Proposition 2.11 (Reverse chain rule). [literature; Fong–Spivak–Tuyéras [FST19]; Cockett et al. [CCG20]] *$f \mapsto (f, R[f])$ is a product-preserving functor* Cart $\to$ Lens.

Proof. $R[gf](x, w) = (Dg_{f(x)} Df_x)^{\mathsf{T}} w = (Df_x)^{\mathsf{T}} (Dg_{f(x)})^{\mathsf{T}} w = R[f](x, R[g](f(x), w))$, which is the composition law of Lens. □

On manifolds the same structure is the correspondence $T^*M \leftarrow f^*T^*N \to T^*N$. The forward functor computes a directional derivative of a composite at the cost of one evaluation; the reverse functor computes the gradient of a scalar-valued composite at the cost of one evaluation and one backward pass. In the companion volume the reverse functor is the backpropagation algorithm, and Proposition 2.11 is the reason it is compositional.

Notes

Weil introduced his algebras in 1953 as "points proches". The classification theorem is due independently to Eck, to Kainz and Michor, and to Luciano; [KMS93] is the standard account and treats natural operations systematically. Tangent categories are surveyed in [CC14]. For synthetic differential geometry see [Koc06] and [MR91]; the Cahiers topos is Dubuc's [Dub79]. Jet bundles as a comonad on bundles, with differential operators as its coKleisli maps, are a further piece of structure that this book does not use.

Exercises

Exercise 2.1. Show that $T(M \times N) \cong TM \times TN$ naturally, and that if z is a regular value of $f\colon M \to N$ then $T\big(f^{-1}(z)\big) = (Tf)^{-1}(0_z)$, where 0_z is the zero vector at z.

Exercise 2.2. Let $A = \mathbb{R}[x]/(x^3)$ and $B = W \otimes W$. Show that $T_A M$ is the bundle of 2-jets at 0 of curves in M, of fibre dimension $2 \dim M$, and that the homomorphism $A \to B$, $x \mapsto \varepsilon + \eta$, induces an embedding $T_A M \to TTM$ whose image is the fixed set of the flip intersected with $\{T(p) = p_T\}$.

Exercise 2.3. Derive the composition law for 2-jets of maps $(\mathbb{R}^m, 0) \to (\mathbb{R}^n, 0) \to (\mathbb{R}^k, 0)$ in terms of the linear and quadratic parts, and verify associativity directly.

Exercise 2.4. Verify the last three rows of Table 2.1: compute the natural transformations induced by the stated algebra homomorphisms in the coordinates (x, v, w, a) on $TT\mathbb{R}^n$.

Exercise 2.5. Show that composition in Lens is associative, and that the reverse derivative of a composite of k maps $\mathbb{R}^{n_0} \to \mathbb{R}^{n_1} \to \cdots \to \mathbb{R}^{n_k}$ with $n_k = 1$ can be evaluated with one forward pass storing the intermediate values and k transposed Jacobian–vector products.

Chapter 3

Function Spaces

The second chapter of the source puts two topologies on the set $C^r(M,N)$, proves that the important classes of maps are open in the stronger one, and proves approximation theorems by convolution and partitions of unity. This is the analytic engine of the book. The mapping object $[M,N]$ of Theorem 1.6 is a different kind of structure on the same set, and it is important to be clear about what each is for. The strong topology is for proving that properties are generic. The mapping object is for parametrizing, and its shape is the homotopy type of the space of maps. The two are compatible but not interchangeable.

3.1 Two topologies

Fix $0 \le r < \infty$. For $f \in C^r(M,N)$, charts (φ, U) on M and (ψ, V) on N, a compact $K \subset U$ with $f(K) \subset V$, and $\varepsilon > 0$, let $\mathcal{N}^r(f;\varphi,\psi,K,\varepsilon)$ be the set of $g \in C^r(M,N)$ with $g(K) \subset V$ and

$$\left\| D^k(\psi g \varphi^{-1})(x) - D^k(\psi f \varphi^{-1})(x) \right\| < \varepsilon \qquad \text{for } x \in \varphi(K),\ 0 \le k \le r.$$

Definition 3.1. The ***weak*** (or compact-open C^r) topology on $C^r(M,N)$ is generated by the sets $\mathcal{N}^r(f;\varphi,\psi,K,\varepsilon)$. The ***strong*** (or Whitney) topology has as a base the sets $\bigcap_i \mathcal{N}^r(f;\varphi_i,\psi_i,K_i,\varepsilon_i)$ over locally finite families of charts (φ_i, U_i) of M, with arbitrary $\varepsilon_i > 0$. On $C^\infty(M,N)$ each topology is the union of the ones induced from C^r, $r < \infty$. The spaces are written $C^r_W(M,N)$ and $C^r_S(M,N)$.

A weak neighbourhood controls a map and its derivatives on a compact set. A strong neighbourhood controls them everywhere, with a tolerance that may shrink to zero at infinity as fast as one likes. If M is compact the two agree. If M is not compact the strong topology is not metrizable and not even first countable, but it is the one in which the theorems below are true.

Theorem 3.2 (Open conditions). [classical; Hirsch 2.1.1–2.1.6] *For $r \ge 1$ the sets of immersions, submersions, embeddings, closed embeddings and diffeomorphisms are open in $C^r_S(M,N)$. For $r \ge 0$ the set of proper maps is open in $C^r_S(M,N)$.*

Example 3.3. Let $\beta \ge 0$ be a bump function on $\mathbb{R}$ supported in $[-1,1]$, with maximum at 0 and $\max \beta' = 1$, and put $g_n(x) = x - 2\beta(x-n)$. Then g_n

agrees with the identity on $[-n+1, n-1]$, so $g_n \to \mathrm{id}$ in $C^\infty_W(\mathbb{R},\mathbb{R})$; but $g_n' = 1 - 2\beta'(x-n)$ takes the value -1, so no g_n is a diffeomorphism or even an immersion. The set of diffeomorphisms of $\mathbb{R}$ is not weakly open. In the strong topology g_n does not converge to the identity: the neighbourhood $\{g : |g(x) - x| < \beta(0)e^{-x^2}\}$ contains none of them. The script confirms both statements numerically.

Theorem 3.4 (Approximation). [classical; Hirsch 2.2.6] *For $0 \le r < s \le \infty$, $C^s(M,N)$ is dense in $C^r_S(M,N)$. If f is already C^s on a neighbourhood of a closed set A, the approximating maps may be taken to agree with f on A.*

Since diffeomorphisms are open (Theorem 3.2) and smooth maps are dense, a C^r diffeomorphism can be approximated by a C^∞ one once source and target carry compatible smooth structures; this is the main step in Theorem 1.2. The same two theorems give the first comparison of homotopy types.

Corollary 3.5. [classical; Hirsch 5.1.5] *Every continuous map between manifolds is homotopic to a smooth map, and two smooth maps that are continuously homotopic are smoothly homotopic.*

3.2 The mapping object

By Theorem 1.6 the smooth set $[M,N]$ has as plots the smooth maps $F\colon \mathbb{R}^n \times M \to N$. A plot is a family $s \mapsto F_s$ of maps, smooth jointly in the parameter and the variable. Nothing is said about neighbourhoods. A topology can be recovered from the plots, as for any smooth set.

Definition 3.6. A set A of points of a smooth set X is ***D*-open** if $p^{-1}(A) \subset \mathbb{R}^n$ is open for every plot p of X. A subobject of X is D-open if its set of points is and it contains every plot whose points lie in it.

Theorem 3.7. [literature; Christensen–Sinnamon–Wu [CSW14]] *On $C^\infty(M,N)$ the D-topology of the mapping object $[M,N]$ is finer than the weak topology and coarser than the strong topology. If M is compact all three coincide.*

The first inequality says that a smooth family is continuous into the weak topology, which is easy. The second says that a D-open set is strongly open, which is not. For non-compact M the second is not an equality: a plot $\mathbb{R} \to [M,N]$ is in particular a path, and paths in the strong topology are very restricted.

Proposition 3.8. [classical] *Let M be non-compact. If $t \mapsto f_t$ is a continuous path in $C^0_S(M,N)$, there is a compact $K \subset M$ such that $f_t = f_0$ outside K for all $t \in [0,1]$.*

So the path components of $C^\infty_S(M,N)$ are far smaller than homotopy classes (Exercise 3.2), and the strong topology does not compute the homotopy type of anything one wants. It is a device for stating that a set of maps is large.

3.3 The homotopy type of a space of maps

The homotopy type attached to the mapping object is its shape (Definition 1.9). By Theorem 1.10 it is the realization of the simplicial set $k \mapsto C^\infty(\Delta^k_e \times M, N)$, whose vertices are smooth maps, whose edges are smooth homotopies, and so on.

Theorem 3.9 (Shape of mapping objects)**.** [literature; Berwick-Evans–Boavida de Brito–Pavlov [BEBP24], Thm. 1.1] *For a manifold M and any $F \in \mathbf{H}$ the natural map*

$$\Pi[M,F] \longrightarrow \mathrm{Map}(\Pi M, \Pi F)$$

is an equivalence of spaces. In particular $\Pi[M,N] \simeq \mathrm{Map}(M,N)$, the space of continuous maps, for manifolds M and N.

For $F = N$ a manifold and M compact this is classical, and it is a parametrized form of Corollary 3.5. In general it is a recent theorem, and it will be used in Chapter 5 with F a stack of bundles. On π_0 it says that concordance classes of sections of F over M are homotopy classes of maps from M to the space ΠF. The hypothesis that M is a manifold cannot be dropped; what is used is that M has partitions of unity and is locally contractible, which is where the analysis of this chapter enters.

Remark 3.10. Concordance is an equivalence relation on the points of any smooth set, but transitivity is not automatic: two plots $\mathbb{R} \to X$ that agree at a point need not glue to a plot. One reparametrizes so that each is constant near the ends (Exercise 3.4). The same device makes the smooth singular complex a Kan complex.

3.4 Subobjects cut out by open conditions

Definition 3.11. $\mathrm{Imm}(M,N)$, $\mathrm{Emb}(M,N)$ and $\mathrm{Diff}(M)$ are the subobjects of $[M,N]$ (of $[M,M]$) whose plots are the families $F\colon \mathbb{R}^n \times M \to N$ such that every F_s is an immersion, an embedding, a diffeomorphism.

For non-compact M this definition of Emb is the weakest reasonable one, and there are others (requiring $(s,x) \mapsto (s, F_s(x))$ to be an embedding, or requiring compact support). For compact M they agree.

Proposition 3.12. [reformulation; cf. Hirsch 2.1.1–2.1.6] *If M is compact, $\mathrm{Imm}(M,N)$, $\mathrm{Emb}(M,N)$ and $\mathrm{Diff}(M)$ are D-open subobjects of the mapping object. That is: in a smooth family of maps from a compact manifold, the set of parameters at which the map is an immersion (an embedding, a diffeomorphism) is open.*

Proof. A plot F induces a continuous map $\mathbb{R}^n \to C^\infty_W(M,N)$, since F and its derivatives in the M directions are uniformly continuous on compact subsets of $\mathbb{R}^n \times M$. For compact M the weak and strong topologies agree, and the three sets are open by Theorem 3.2. □

The reformulation is modest, but it is the form in which the companion volume uses Theorem 3.2: the parameters of a network at which it is an embedding of a compact data manifold form an open set. It is also the form that generalizes. A D-open subobject of a mapping object has the same local structure as the mapping object, and so inherits whatever infinite-dimensional manifold structure the latter has [KM97].

3.5 Genericity

Theorem 3.13 (Baire property). [classical; Hirsch 2.4.4] *Every weakly closed subset of $C^r_S(M,N)$ is a Baire space in the strong topology: a countable intersection of open dense subsets is dense.*

A subset is ***residual*** if it contains a countable intersection of open dense sets. The theorems of the next chapter assert that certain sets are residual, and Theorem 3.13 is what turns such an assertion into one about density. The proof realizes $C^r(M,N)$, by the prolongation $j^r\colon C^r(M,N) \to C^0(M, J^r(M,N))$, as a weakly closed set of sections of the jet bundle; the strong C^r topology is by definition the one induced from the strong C^0 topology on sections, and the Baire property is proved there by a diagonal argument.

Remark 3.14 (What the topos does not see). The D-topology is defined from the plots, so openness is a notion internal to smooth sets. Residuality is not. A residual set need not be D-open, and a set meeting every plot in a residual set of parameters need not be residual. We know of no formulation of the Baire property of $C^\infty_S(M,N)$ in terms of the sheaf $[M,N]$ alone. What the sheaf does see is the parametric form of genericity (Theorem 4.5): in a family rich enough to be a submersion onto its target, almost every member is generic. In applications that statement is usually the one wanted.

3.6 Jets and partial differential relations

The prolongation is a map of smooth sets $j^r\colon [M,N] \to \Gamma(J^r(M,N))$ into the smooth set of sections of the jet bundle over M. It is a monomorphism, and its image consists of the ***holonomic*** sections. A condition on maps that is defined by a subset $\mathcal{R} \subset J^r(M,N)$, such as being an immersion (the 1-jet is injective) or a Morse function (the 1-jet meets the zero locus only where the 2-jet is non-degenerate), gives a pullback square

$$\begin{array}{ccc} [M,N]_{\mathcal{R}} & \longrightarrow & \Gamma(\mathcal{R}) \\ \downarrow & & \downarrow \\ [M,N] & \xrightarrow{j^r} & \Gamma(J^r(M,N)) \end{array}$$

in which the right-hand side involves no differential equation: $\Gamma(\mathcal{R})$ is the object of ***formal solutions***. If $\mathcal{R}$ is open, the left-hand subobject is D-open when M is compact, by the argument of Proposition 3.12. Whether the top map induces an equivalence of shapes is the question answered, for immersions, by the Smale–Hirsch theorem (Theorem 10.7).

Notes

The weak and strong topologies and their properties are developed in Hirsch's Chapter 2 and, for C^∞ and jets, in Golubitsky–Guillemin and Mather. The convenient setting of [KM97] gives $C^\infty(M,N)$ the structure of an infinite-dimensional manifold whose smooth curves are the smooth families with locally compact support in the parameter; Proposition 3.8 is the reason for that restriction. The comparison of the D-topology with the classical ones is [CSW14]. Theorem 3.9 is called the smooth Oka principle by Sati and Schreiber; earlier special cases are due to Madsen and Weiss.

Exercises

Exercise 3.1. Verify the claims of Example 3.3. Show also that $\mathrm{Diff}(\mathbb{R})$ is not closed in $C^\infty_W(\mathbb{R},\mathbb{R})$.

Exercise 3.2. Prove Proposition 3.8 for $M = N = \mathbb{R}$. Deduce that the path component of f in $C^0_S(\mathbb{R},\mathbb{R})$ is the set of maps that agree with f outside a compact set.

Exercise 3.3. Show that a plot of $[M,N]$ is continuous as a map into $C^1_W(M,N)$, and give an example of a smooth family $F\colon \mathbb{R}\times\mathbb{R}\to\mathbb{R}^2$ of maps for which $\{s : F_s \text{ is an embedding}\}$ is not open.

Exercise 3.4. Let X be a smooth set. Show that concordance is an equivalence relation on $X(\mathrm{pt})$.

Exercise 3.5. A function $f\colon \mathbb{R} \to \mathbb{R}$ is Morse if $f''(x) \neq 0$ wherever $f'(x) = 0$. Show that the set of Morse functions is dense but not open in $C^\infty_W(\mathbb{R}, \mathbb{R})$.

Chapter 4

Sard's Theorem and Transversality

This is the chapter on which the source turns. Its results say that the transverse cospans of Theorem 1.12 are dense among all cospans, and that the same holds in families. The proofs reduce, by one piece of linear algebra, to a single analytic theorem. The higher-categorical counterpart is a universal property: there is a smallest enlargement of Mfd in which all fibre products exist and the transverse ones are unchanged, and the genericity theorems say that every object of the enlargement is cobordant to an object of Mfd.

4.1 The Morse–Sard theorem

Theorem 4.1 (Morse–Sard). [classical; Morse, Sard, Brown; Hirsch 3.1.3] *Let $f\colon M \to N$ be of class C^r with $r > \max\{0, \dim M - \dim N\}$. Then the set of critical values of f has measure zero in N, and the set of regular values is residual.*

The proof is an induction on $\dim M$ by slicing with Fubini's theorem, with the differentiability hypothesis used to control the image of the set where many derivatives vanish. Nothing in this book adds to it. The theorem is the source of every statement of the form "for almost all" below.

Corollary 4.2. [classical; Hirsch 3.1.4] *A compact manifold with non-empty boundary admits no smooth retraction onto its boundary.*

Proof. Let $\rho\colon M \to \partial M$ be one and y a regular value of ρ; it is automatically a regular value of $\rho|_{\partial M} = \mathrm{id}$. By Theorem 1.17, $\rho^{-1}(y)$ is a compact neat 1-dimensional submanifold with boundary $\rho^{-1}(y) \cap \partial M = \{y\}$. A compact 1-manifold has an even number of boundary points. □

The argument is the prototype for the whole subject: a count that would have to be odd is the boundary of something, hence even. Brouwer's fixed point theorem follows (Exercise 4.5).

4.2 The transversality theorems

Theorem 4.3 (Transversality). [classical; Thom; Hirsch 3.2.1] *Let $A \subset N$ be a submanifold and $1 \le r \le \infty$. The set of $f \in C^r(M, N)$ with $f \pitchfork A$ is residual, hence dense, in $C^r_S(M, N)$; it is open if A is closed.*

The proof uses a parametric statement, which is the form we shall use most, and the parametric statement rests on a lemma worth isolating.

Lemma 4.4 (Fibrewise transversality). [classical] *Let $F\colon V \times M \to N$ be smooth with $F \pitchfork A$, let $W = F^{-1}(A)$ and let $\pi\colon W \to V$ be the projection. Then $F_v = F(v,-)$ is transverse to A if and only if v is a regular value of π.*

Proof. Fix $(v,x) \in W$, $z = F(v,x)$, and let $L\colon T_vV \oplus T_xM \to Q = T_zN/T_zA$ be the differential of F followed by the quotient. By hypothesis L is onto, and $T_{(v,x)}W = \ker L$. Write L_M for the restriction of L to T_xM; then $F_v \pitchfork A$ at x means that L_M is onto, and $d\pi$ is the projection $\ker L \to T_vV$.

If L_M is onto and $a \in T_vV$, choose b with $L_M(b) = L(a,0)$; then $(a,-b) \in \ker L$ projects to a. Conversely, if the projection is onto and $q = L(a,b) \in Q$, choose $(a,b') \in \ker L$; then $q = L(a,b) - L(a,b') = L_M(b-b')$. □

Theorem 4.5 (Parametric transversality). [classical; Hirsch 3.2.7] *Let $F\colon V \times M \to N$ be of class C^r with $r > \max\{0, \dim M + \dim A - \dim N\}$ and $F \pitchfork A$. Then the set of $v \in V$ with $F_v \pitchfork A$ is residual in V, and its complement has measure zero.*

Proof. Apply Theorem 4.1 to $\pi\colon W \to V$, noting $\dim W - \dim V = \dim M + \dim A - \dim N$, and use Lemma 4.4. □

The lemma has a reading that will recur. The manifold W is the fibre product of the whole family with A, and it is always a manifold. The individual fibre products $F_v^{-1}(A)$ are its fibres over V. So a non-transverse member $F_{v_0}^{-1}(A)$ of the family is a singular fibre of a map between manifolds, and nearby regular fibres are its replacements. If F is merely a plot of $[M,N]$, with no transversality assumed, nothing can be said; the hypothesis $F \pitchfork A$ is what makes a family "rich enough" in the sense of Remark 3.14. It holds in particular when F is a submersion, for instance $F(v,x) = f(x) + v$ with $N = V = \mathbb{R}^n$ (Exercise 4.1).

Theorem 4.6 (Jet transversality). [classical; Thom; Hirsch 3.2.8, 3.2.9] *Let $A \subset J^r(M,N)$ be a submanifold and $r < s \le \infty$. The set of $f \in C^s(M,N)$ with $j^r f \pitchfork A$ is residual in $C^s_S(M,N)$, and open if A is closed.*

4.3 Dimension counts

A map transverse to a submanifold of codimension greater than $\dim M$ misses it. So genericity theorems become existence theorems as soon as one can compute a codimension.

Proposition 4.7. [classical] *The $m \times n$ real matrices of rank ρ form a submanifold of $\mathbb{R}^{m\times n}$ of dimension $\rho(m+n-\rho)$, that is, of codimension $(m-\rho)(n-\rho)$.*

Corollary 4.8. [classical; Whitney; Hirsch 2.2.12, 2.2.13, 6.1.2] *Let* $\dim M = m$, $\dim N = n$, *with* M *compact.*

(a) If $n \geq 2m$, *immersions are dense in* $C^\infty(M, N)$.

(b) If $n \geq 2m + 1$, *embeddings are dense in* $C^\infty(M, N)$.

(c) Morse functions are open and dense in $C^\infty(M, \mathbb{R})$.

Proof. (a) In $J^1(M, N)$ the jets of rank $\rho < m$ form submanifolds of codimension $(m - \rho)(n - \rho) \geq n - m + 1 > m$. A map whose 1-jet prolongation is transverse to all of them misses them. (b) For an immersion f, injectivity is the statement that $f \times f$ restricted to $M \times M \setminus \Delta$ misses Δ_N, of codimension $n > 2m$; one applies a multijet form of Theorem 4.6. (c) A function is Morse when df, as a section of T^*M, is transverse to the zero section, a closed submanifold of codimension m. □

The script checks Proposition 4.7 by computing the rank of the differential of a local defining map for several (m, n, ρ), and checks (b) in a small case by sampling linear projections.

4.4 Clean and transverse

Transversality is a sufficient condition for a fibre product in Mfd to be correct. It is useful to separate the two things it provides.

Definition 4.9. Maps $f\colon M \to N$ and $g\colon P \to N$ intersect ***cleanly*** if $Z = M \times_N P$ is a submanifold of $M \times P$ and $TZ = TM \times_{TN} TP$. The ***excess bundle*** is the vector bundle $E = \operatorname{coker}(TM \oplus TP \to TN)$ over Z, of rank $e = \dim N - \dim M - \dim P + \dim Z$.

Proposition 4.10. [classical] *If* f *and* g *intersect cleanly then* Z *is the fibre product in* Mfd *and* T *preserves it. They are transverse if and only if they intersect cleanly with* $e = 0$. *If* $M, P \subset N$ *are compact submanifolds meeting cleanly in* Z, *then for every sufficiently small generic perturbation* M' *of* M *the transverse intersection* $M' \cap P$ *is cobordant to the zero set of a generic section of* E *over* Z.

So a clean intersection has the right tangent spaces and the wrong dimension, by e, and the excess bundle measures exactly what a perturbation will cut away. A submanifold meets itself cleanly with E its normal bundle; the diagonal of $M \times M$ meets itself cleanly with $E = TM$. Example 1.3(b) is not clean. In each case there is an object that records the correct answer before any perturbation is made.

4.5 Derived manifolds

By Theorem 1.8 the function algebra of a transverse fibre product is a pushout of C^∞-rings. The derived fibre product is defined by the homotopy pushout.

Definition 4.11. A ***simplicial C^∞-ring*** is a product-preserving functor $\mathsf{Cart} \to \mathcal{S}$; these form an ∞-category $sC^\infty\mathsf{Ring}$, with homotopy pushouts written $\otimes^{\mathbb{L}}$. Each has homotopy groups π_k, with π_0 a C^∞-ring and π_k a π_0-module.

Theorem 4.12 (Universal property of derived manifolds). [literature; Carchedi–Steffens [CS19]; Spivak [Spi10]] *There is an ∞-category DMfd that has finite limits and is idempotent complete, with a functor $i\colon \mathsf{Mfd} \to \mathsf{DMfd}$ that preserves transverse fibre products and the terminal object, and that is universal with these properties: for every idempotent complete ∞-category $\mathcal{C}$ with finite limits, composition with i is an equivalence between finite-limit-preserving functors $\mathsf{DMfd} \to \mathcal{C}$ and functors $\mathsf{Mfd} \to \mathcal{C}$ preserving transverse fibre products and the terminal object. Moreover DMfd is equivalent to the opposite of the ∞-category of finitely presented simplicial C^∞-rings, with $i(M) = C^\infty(M)$, and i is fully faithful.*

The theorem says that nothing was chosen. Whoever wants all fibre products of manifolds, and wants the transverse ones to stay as they are, is led to this ∞-category and no other. The role of idempotent completeness was explained after Theorem 1.15. That the answer is an ∞-category and not a category is forced: the last row of Table 1.1 has no 1-categorical solution.

Locally every derived fibre product of manifolds is a derived zero locus, and those can be written down.

Proposition 4.13 (Koszul model). [literature; [Spi10], [CS19]] *Let $E \to M$ be a vector bundle of rank k with a section s, and let $Z(s) = M \times^h_{s,E,0} M$ be the derived zero locus. Its simplicial C^∞-ring corresponds to the commutative differential graded algebra*

$$\big(\Gamma(\Lambda^\bullet E^\vee),\ \iota_s\big)\colon \qquad \cdots \to \Gamma(\Lambda^2 E^\vee) \xrightarrow{\iota_s} \Gamma(E^\vee) \xrightarrow{\iota_s} C^\infty(M),$$

so $\pi_0 = C^\infty(M)/(s)$. If s is transverse to the zero section then $\pi_k = 0$ for $k > 0$ and $Z(s) = i\big(s^{-1}(0)\big)$. For maps $f\colon M \to N$ and $g\colon P \to N$, the derived fibre product $M \times^h_N P$ is the derived zero locus of a section of $(f \times g)^* TN$ over a neighbourhood of $M \times_N P$ in $M \times P$.

The number $\operatorname{vdim} Z(s) = \dim M - k$ is the ***virtual dimension***; for a derived fibre product it is $\dim M + \dim P - \dim N$ whether or not the maps are transverse.

Example 4.14. (a) $M = \mathbb{R}$, E trivial of rank 1, $s = x^2$. The complex is $C^\infty(\mathbb{R}) \xrightarrow{x^2} C^\infty(\mathbb{R})$, which is injective, so $\pi_1 = 0$ and $Z(s)$ is the classical object $\mathbb{R}[\varepsilon]/(\varepsilon^2)$ of virtual dimension 0: not a manifold, and not derived either.

(b) $M = \mathrm{pt}$, $E = \mathbb{R}$, $s = 0$. The complex is $\mathbb{R} \xrightarrow{0} \mathbb{R}$; $\pi_0 = \pi_1 = \mathbb{R}$. This is the point of virtual dimension -1 in Table 1.1.

(c) $s = 0$ on any $E \to M$: the algebra is $\Gamma(\Lambda^\bullet E^\vee)$ with zero differential, of virtual dimension $\dim M - k$. It is the derived self-intersection of the zero section, and by the tubular neighbourhood theorem the derived self-intersection of any submanifold with normal bundle E.

(d) The diagonal: $M \times^h_{M\times M} M$ has functions $\Gamma(\Lambda^\bullet T^*M)$, the differential forms placed in homological degrees, and virtual dimension 0. Its "number of points" will turn out to be the Euler characteristic (Chapter 7).

Every object of DMfd has a tangent complex at each point. For $X = M \times^h_N P$ at (x, y) over z it is

$$\mathbb{T}_X = \left[T_xM \oplus T_yP \xrightarrow{df-dg} T_zN\right]$$

in degrees 0 and 1, with H^0 the fibre product of tangent spaces and H^1 the fibre of the excess bundle. Its Euler characteristic is the virtual dimension.

Theorem 4.15 (Transversality, derived form). [reformulation; cf. [Spi10], [CS19]] *For smooth maps $f\colon M \to N$ and $g\colon P \to N$, the derived fibre product $M \times^h_N P$ lies in the essential image of $i\colon \mathsf{Mfd} \to \mathsf{DMfd}$ if and only if $f \pitchfork g$.*

Proof. If $f \pitchfork g$, then i preserves the fibre product by Theorem 4.12. Conversely, if $M \times^h_N P \simeq i(Z)$ for a manifold Z, its tangent complex is concentrated in degree 0 at every point, so $H^1 = T_zN/(df\,T_xM + dg\,T_yP)$ vanishes at every point of the fibre product, which is transversality. □

Compare the situation in Mfd. There a fibre product may fail to exist, or exist and be wrong, or exist and be right without one's knowing which. In DMfd it always exists and is always right, and it is a manifold in exactly the transverse case.

4.6 Genericity as a statement about derived manifolds

Theorem 4.16. [literature; Spivak [Spi10]] *Compact derived manifolds that are locally derived zero loci have a cobordism theory, and i induces an isomorphism from the cobordism ring of manifolds to it. In particular a compact $X = M \times^h_N P$ has a class $[X]$ in the cobordism group $\mathfrak{N}_d$ of dimension $d = \dim M + \dim P - \dim N$ (Chapter 9), represented by $M \times_N P'$ for any $g' \pitchfork f$ sufficiently close to g.*

Surjectivity is the transversality theorem: perturb. Injectivity, and the independence of the perturbation, is Lemma 4.4 applied to a path of perturbations: the fibre product of the path with A is a manifold with boundary

whose two ends are the two candidates. Putting Theorems 4.12, 4.15 and 4.16 together gives a statement of the method of the source that makes no mention of perturbation until the end.

Principle 4.17 (Generic position). [reformulation; cf. Hirsch, Ch. 3; [Spi10]] *Every finite diagram of manifolds has a limit in* DMfd*. If the limit is compact it has a well-defined cobordism class. The class is represented by the honest limit of any sufficiently generic perturbation of the diagram, and two such representatives are cobordant by the limit of a generic path between the perturbations.*

Example 4.18. The zero set of $x^2 - \varepsilon$ on $\mathbb{R}$, as an oriented 0-manifold, is $\{+\sqrt{\varepsilon}\}$ with sign $+$ and $\{-\sqrt{\varepsilon}\}$ with sign $-$ for $\varepsilon > 0$, and empty for $\varepsilon < 0$. Both have class $0 \in \Omega_0 = \mathbb{Z}$, and the cobordism between them is the parabola $\{x^2 = \varepsilon\}$ in the (ε, x)-plane, a manifold by Lemma 4.4. The derived zero set at $\varepsilon = 0$ is the dual numbers, with class 0. Its length as an algebra is 2, which is the count of complex roots; the cobordism class and the length are different invariants and only the first is stable under real perturbation.

An object of negative virtual dimension, such as Example 4.14(b), has class zero for trivial reasons and is nevertheless not empty. The cobordism class is the part of a derived manifold that perturbation can reach, and it is the only part the classical theory uses.

Notes

Transversality is Thom's; the parametric form and its use as the organizing principle are the substance of Hirsch's Chapter 3 and of Abraham's work. Derived manifolds were introduced by Spivak, after Lurie's sketch in his work on derived algebraic geometry; variants are due to Borisov and Noel and to Joyce, and the universal property is [CS19], building on [Mac17]. Clean intersections are Bott's. The excess bundle and the associated formula are Quillen's in cobordism and Fulton–MacPherson's in algebraic geometry.

Exercises

Exercise 4.1. Let $f\colon M \to \mathbb{R}^n$ be smooth and $A \subset \mathbb{R}^n$ a submanifold. Show that $f + v$ is transverse to A for almost every $v \in \mathbb{R}^n$. Deduce that almost every translate of a curve in $\mathbb{R}^3$ misses a given curve.

Exercise 4.2. Prove Proposition 4.7.

Exercise 4.3. Compute the homotopy groups of $\mathbb{R} \otimes^{\mathbb{L}}_{C^\infty(\mathbb{R})} \mathbb{R}$, where both maps are evaluation at 0, and of $C^\infty(\mathbb{R}) \otimes^{\mathbb{L}}_{C^\infty(\mathbb{R})} \mathbb{R}$, where the first map is $x \mapsto x^2$. Compare with Example 4.14.

Exercise 4.4. Let $U \subset \mathbb{R}^m$ be open and $f\colon U \to \mathbb{R}$ smooth. Show that $f_a(x) = f(x) + \langle a, x\rangle$ is a Morse function for almost every $a \in \mathbb{R}^m$.

Exercise 4.5. Deduce from Corollary 4.2 that every continuous map $D^n \to D^n$ has a fixed point.

Part II

Bundles and Invariants

Chapter 5

Vector Bundles and Classifying Maps

The first half of Hirsch's fourth chapter proves that a vector bundle over $B \times I$ is a product in the I direction, deduces that homotopic maps pull back isomorphic bundles, and classifies bundles over an n-manifold by homotopy classes of maps to a Grassmannian. The three statements are one statement about a stack: it is concordance-invariant, so its sections are classified by maps to its shape. The same theorem, applied to other stacks, gives de Rham's theorem and the classification of principal bundles, and in later chapters the classification of immersions and of manifolds up to cobordism.

5.1 The stack of vector bundles

Definition 5.1. For a manifold M, $\mathsf{Vect}_k(M)$ is the groupoid of real vector bundles of rank k over M and their isomorphisms. Pullback makes $M \mapsto \mathsf{Vect}_k(M)$ a functor $\mathsf{Mfd}^{\mathrm{op}} \to \mathsf{Gpd}$.

Proposition 5.2. [classical] Vect_k *is a stack: bundles and their isomorphisms glue along open covers. Its restriction to* Cart *is equivalent to* $U \mapsto B\,C^\infty(U, \mathrm{GL}_k)$*, the groupoid with one object and automorphisms the smooth maps* $U \to \mathrm{GL}_k(\mathbb{R})$*. Hence* $\mathsf{Vect}_k \simeq B\,\mathrm{GL}_k$*, the delooping in* $\mathbf{H}$ *of the sheaf of groups represented by* $\mathrm{GL}_k(\mathbb{R})$*, and*

$$\mathsf{Vect}_k(M) \;\simeq\; \mathbf{H}(M, B\,\mathrm{GL}_k).$$

Proof. Descent for bundles is the construction of a bundle from transition functions satisfying the cocycle condition. Over a Euclidean space every bundle is trivial (by Theorem 5.3 below, since $\mathbb{R}^n$ is contractible), which identifies the restriction. An object of $\mathbf{H}$ is determined by its restriction to Cart. □

So "a vector bundle is a map to $B\,\mathrm{GL}_k$" is true as stated, in $\mathbf{H}$, with no homotopy classes taken; a cocycle (g_{ij}) on a cover is the same map written out on the Čech nerve of the cover. The object $B\,\mathrm{GL}_k$ is not a smooth set. Its value on a point is the groupoid of k-dimensional vector spaces, and it is the first place in this book where the ∞-topos of Definition 1.9, and not just its 0-truncated part, is needed.

5.2 Homotopy invariance

Theorem 5.3. [classical; Hirsch 4.1.5, 4.2.4] *Every vector bundle over $M \times I$ is isomorphic to $(\xi|_{M\times 0}) \times I$. Consequently, if $f_0, f_1 \colon M \to N$ are homotopic and ξ is a bundle over N, then $f_0^*\xi \cong f_1^*\xi$.*

The proof chooses a connection, or in Hirsch's version covers $M \times I$ by product neighbourhoods on which the bundle is trivial and patches with a partition of unity; either way a bundle over $M \times I$ is transported along the I direction. In the language of Definition 1.9, two bundles over M are concordant if they are the ends of a bundle over $M \times \mathbb{R}$, and the theorem says that concordant bundles are isomorphic. The converse is trivial.

5.3 Classification

Let $\mathrm{Gr}_k(\mathbb{R}^s)$ be the Grassmannian of k-planes in $\mathbb{R}^s$, a compact manifold of dimension $k(s-k)$, with tautological bundle γ_k.

Theorem 5.4 (Classification of vector bundles)**.** [classical; Hirsch 4.3.4] *Let ξ be a vector bundle of rank k over a manifold M of dimension n. If $s \geq k+n$ there is a map $f \colon M \to \mathrm{Gr}_k(\mathbb{R}^s)$ with $f^*\gamma_k \cong \xi$, and if $s > k+n$ it is unique up to homotopy. So for $s > k+n$,*

$$\pi_0 \mathsf{Vect}_k(M) \;\cong\; [M, \mathrm{Gr}_k(\mathbb{R}^s)].$$

Hirsch proves existence by embedding the bundle in a trivial one, which is a transversality argument, and uniqueness by the same argument one dimension up. The dimension restrictions are the connectivity of the inclusion $\mathrm{Gr}_k(\mathbb{R}^s) \to \mathrm{Gr}_k(\mathbb{R}^\infty) = B\mathrm{O}(k)$.

Theorem 5.5 (Concordance classification)**.** [literature; [BEBP24]; cf. Theorem 3.9] *For every $F \in \mathbf{H}$ and every manifold M, the set of concordance classes of sections of F over M is in natural bijection with $[M, \Pi F]$, and more precisely $\Pi[M, F] \simeq \mathrm{Map}(M, \Pi F)$.*

For $F = B\,\mathrm{GL}_k$ the shape is the classifying space of the topological group $\mathrm{GL}_k(\mathbb{R})$, which is $B\mathrm{O}(k)$ by Gram–Schmidt (Exercise 6.1). Concordance classes are isomorphism classes by Theorem 5.3. So Theorem 5.5 gives $\pi_0\mathsf{Vect}_k(M) \cong [M, B\mathrm{O}(k)]$, which is Theorem 5.4 with $s = \infty$; and it gives more, namely the homotopy type $\mathrm{Map}(M, B\mathrm{O}(k))$ for the space of all rank-k bundles over M, whose fundamental group at ξ is the group of components of the gauge group of ξ.

Table 5.1 lists other instances. In each row the left-hand side is a geometric structure that can be pulled back, the right-hand side is a space, and the

F (sections over M)	shape ΠF	classical name
maps to a manifold N	N	smooth approximation (Corollary 3.5)
vector bundles of rank k	$BO(k)$	Theorem 5.4
principal G-bundles, G a Lie group	BG	classification of principal bundles
vector bundles with connection	$BO(k)$	connections form a contractible space
closed differential p-forms	$K(\mathbb{R}, p)$	de Rham's theorem
all differential p-forms	pt	forms are contractible
bundles of closed d-manifolds	$\coprod_{[W]} B\,\mathrm{Diff}(W)$	classification of manifold bundles

Table 5.1: Instances of Theorem 5.5. The union in the last row is over diffeomorphism classes of closed d-manifolds.

theorem says that deformation classes of the structure on M are homotopy classes of maps into the space.

The fifth row deserves a comment. Two closed p-forms on M are concordant exactly when they are cohomologous (Exercise 5.4 for $p = 1$), so the theorem says $H^p_{\mathrm{dR}}(M) \cong [M, K(\mathbb{R}, p)]$, which is de Rham's theorem. The sheaf of closed forms is not homotopy invariant; its concordance classes are.

5.4 Bundles as idempotents

Theorem 5.6. [classical; Swan; Hirsch 4.3.3] *Every vector bundle ξ over a manifold M has a complement: a bundle η with $\xi \oplus \eta$ trivial. The functor of sections is an equivalence from vector bundles over M to finitely generated projective $C^\infty(M)$-modules, taking $\oplus$ and $\otimes$ to $\oplus$ and $\otimes_{C^\infty(M)}$.*

So the category of vector bundles over M is the idempotent completion of the category of trivial bundles, just as Mfd is the idempotent completion of Open (Theorem 1.15), and for the same reason: finite dimension bounds the number of charts needed. A complement exhibits ξ as the image of an idempotent endomorphism e of a trivial bundle $M \times \mathbb{R}^s$, that is, of a smooth map from M to the manifold of idempotent $s \times s$ matrices of rank k. Projecting an idempotent to the orthogonal projection with the same image identifies that manifold, up to homotopy, with $\mathrm{Gr}_k(\mathbb{R}^s)$. A classifying map and a complement are the same datum.

Example 5.7. Over the circle $\mathbb{RP}^1 = \mathbb{R}/\pi\mathbb{Z}$ the matrix $e(\theta) = \left(\begin{smallmatrix} \cos^2\theta & \cos\theta\sin\theta \\ \cos\theta\sin\theta & \sin^2\theta \end{smallmatrix}\right)$ is an idempotent of rank 1; its image is the tautological line bundle. A unit vector in the image transported once around the

circle returns to its negative, so the bundle is the Möbius band. Over S^n the idempotent $e(x) = I - xx^{\mathsf{T}}$ on $\mathbb{R}^{n+1}$ has image TS^n and complement the trivial normal line. The script checks both.

5.5 Orientations and tangential structures

A bundle is ***orientable*** if its transition functions can be chosen in $\mathrm{GL}_k^+(\mathbb{R})$; an ***orientation*** is such a choice up to equivalence. In terms of classifying maps, an orientation is a lift of $M \to B\mathrm{O}(k)$ through the double cover $B\mathrm{SO}(k) \to B\mathrm{O}(k)$, and the obstruction to its existence is the class $w_1(\xi) \in H^1(M; \mathbb{Z}/2)$ pulled back from the generator of $H^1(B\mathrm{O}(k); \mathbb{Z}/2)$. A manifold is orientable if its tangent bundle is.

Definition 5.8. A ***tangential structure*** is a map of spaces $\theta\colon B \to B\mathrm{O}(d)$. A ***$\theta$-structure*** on a d-manifold M is a lift of the classifying map $\tau_M\colon M \to B\mathrm{O}(d)$ of TM through θ.

Orientations ($B = B\mathrm{SO}(d)$), framings ($B = \mathrm{pt}$), spin structures, almost complex structures, and maps to a fixed background space X ($B = B\mathrm{O}(d) \times X$) are all of this form. Because θ-structures pull back along open embeddings and are defined by a lifting problem in spaces, the stack of d-manifolds with θ-structure is again an object to which Theorem 5.5 applies; this is how the tangential structures of Chapter 9 enter.

Theorem 5.9. [classical; Hirsch 4.4.7] *A closed non-orientable n-manifold does not embed in a simply connected $(n+1)$-manifold. In particular $\mathbb{RP}^{2n}$ does not embed in $\mathbb{R}^{2n+1}$.*

Proof. Let $M \subset N$ be such an embedding with normal line bundle ν. A loop in N transverse to M meets it in an even number of points, since the count modulo 2 is a homotopy invariant (Theorem 7.4) and the loop is null-homotopic. A loop in M along which ν is non-trivial can be pushed off M to a loop meeting M once. So ν is trivial, and $TM \oplus \nu \cong TN|_M$ gives $w_1(TM) = w_1(TN|_M) = 0$, as N is simply connected. $\square$

5.6 The tangent bundle as a map

The tangent bundle of an m-manifold is a point τ_M of $\mathrm{Map}(M, B\mathrm{O}(m))$. For a submanifold $M \subset \mathbb{R}^{m+k}$ the classifying map can be taken to be the Gauss map $x \mapsto T_xM \in \mathrm{Gr}_m(\mathbb{R}^{m+k})$, and the normal bundle is classified by $x \mapsto (T_xM)^{\perp}$. The normal bundle depends on the embedding, but its class in $[M, B\mathrm{O}]$, where $B\mathrm{O} = \mathrm{colim}_k B\mathrm{O}(k)$, does not; this ***stable normal bundle*** ν_M is inverse to τ_M under direct sum, and Chapter 6 shows that it is well defined as a point of a contractible space of choices. Characteristic classes of M are the pullbacks

along τ_M or ν_M of classes in $H^*(B\mathrm{O}; \mathbb{Z}/2) = \mathbb{Z}/2[w_1, w_2, \dots]$. Every invariant of manifolds used in Chapters 7 and 9 is obtained from τ_M, as a consequence of the remark after Proposition 2.9.

Notes

Steenrod's book is the classical source for classification by maps to Grassmannians; Hirsch's treatment by transversality is his own. The stack-theoretic formulation is standard; see [Sch13]. The equivalence with projective modules is Swan's, after Serre. Theorem 5.5 has a long prehistory: Brown representability, Gromov's sheaf-theoretic h-principle, and Madsen and Weiss's use of concordance in the proof of the Mumford conjecture.

Exercises

Exercise 5.1. Show that the tautological line bundle over $\mathbb{RP}^1$ is not trivial, and that its total space is the open Möbius band.

Exercise 5.2. Let $e\colon M \to \mathbb{R}^{s\times s}$ be smooth with $e^2 = e$ and constant rank k. Show that the image of e is a vector bundle whose module of sections is $e\,C^\infty(M)^s$, a direct summand of a free module.

Exercise 5.3. Show that $TS^n \oplus \mathbb{R}$ is trivial, and conclude that the composite $S^n \to B\mathrm{O}(n) \to B\mathrm{O}$ classifying the stable tangent bundle is null-homotopic.

Exercise 5.4. Show that two closed 1-forms on M are concordant (are the restrictions to $M \times \{0\}$ and $M \times \{1\}$ of a closed 1-form on $M \times \mathbb{R}$) if and only if they differ by an exact form.

Exercise 5.5. Show that isomorphism classes of line bundles over M correspond to $H^1(M; \mathbb{Z}/2) = \mathrm{Hom}(\pi_1 M, \mathbb{Z}/2)$, and count them for the torus and for the Klein bottle. Which one is the determinant of the tangent bundle?

Chapter 6

Tubular Neighbourhoods, Collars, and Spaces of Choices

The second half of Hirsch's fourth chapter proves that a submanifold has a tubular neighbourhood, that the boundary of a manifold has a collar, and that each is unique up to isotopy. The uniqueness statements are what allow later constructions (connected sums, gluing of cobordisms, the Pontryagin–Thom collapse) to be called well defined. "Unique up to isotopy" says that a space of choices is connected. In every case in this chapter the space of choices is in fact contractible, and that stronger statement is what makes the constructions functorial.

6.1 Tubular neighbourhoods

Definition 6.1. Let $M \subset N$ be a submanifold, both without boundary. A ***tubular neighbourhood*** of M in N is a vector bundle $\xi = (p, E, M)$ with an embedding $\varphi\colon E \to N$ onto an open set that restricts to the identity on the zero section. The ***normal bundle*** is $\nu = \nu(M, N) = TN|_M / TM$.

The differential of φ along the zero section identifies ξ with ν, so one may take $\xi = \nu$.

Theorem 6.2 (Tubular neighbourhoods). [classical; Hirsch 4.5.2, 4.5.3] *Every submanifold $M \subset N$ has a tubular neighbourhood, and any two are isotopic relative to M.*

The existence proof embeds N in a Euclidean space, uses the normal exponential map $(x, v) \mapsto x + v$ there, which is a local diffeomorphism along the zero section by the inverse function theorem, and retracts back to N. For the unit circle in the plane the map is $(\theta, t) \mapsto (1 + t)(\cos\theta, \sin\theta)$, with Jacobian determinant $1 + t$; it is an embedding for $t > -1$. In general the radius is limited by the focal points of M (Exercise 6.3).

Theorem 6.3. [literature; folklore; see [God07]] *The space* $\mathrm{Tub}(M, N)$ *of tubular neighbourhoods $\varphi\colon \nu \to N$ whose differential along the zero section induces the identity of ν is contractible.*

Idea of proof. Given two, first shrink one fibrewise until its image lies in the image of the other; this is a canonical path. Then $\psi = \varphi_0^{-1}\varphi_1$ is an embedding of ν in itself, fixing the zero section, with fibre derivative the identity there, and $\psi_t(v) = \psi(tv)/t$ for $0 < t \leq 1$ extends smoothly to $t = 0$ by the identity. Both steps depend continuously on parameters. □

With boundary, the right notion is that of a neat submanifold (Definition 1.16), and the theorems hold with tubular neighbourhoods that restrict to tubular neighbourhoods in the boundary (Hirsch 4.6.3, 4.6.4).

6.2 Collars and gluing

Theorem 6.4 (Collars). [classical; Brown; Hirsch 4.6.1] *For a manifold M with boundary there is an embedding $\partial M \times [0,\infty) \to M$ extending the identity of $\partial M \times 0$. Any two are isotopic, and the space of collars is contractible.*

Theorem 6.5 (Gluing). [classical; Hirsch 8.2.1, 8.2.3] *Let P, Q be manifolds with boundary and $f\colon \partial Q \to \partial P$ a diffeomorphism. The space $P \cup_f Q$ has a smooth structure restricting to the given ones on P and Q; any two such structures are diffeomorphic by a diffeomorphism supported near the seam; and isotopic gluing maps give diffeomorphic results.*

The smooth structure is built by choosing collars on both sides and declaring the union of the two collars to be a product neighbourhood of the seam. So composition of cobordisms is defined only after a choice, and the choice is from a contractible space. A category whose composition depends on contractible choices is an ∞-category presented in the usual way, and this is the elementary reason that cobordism categories are ∞-categories (Chapter 9). The dependence on the isotopy class of f is real: gluing two discs along a diffeomorphism of S^{n-1} not isotopic to one that extends over the disc can give an exotic sphere (Exercise 6.5).

6.3 Contractible spaces of choices

Principle 6.6 (Choices). [reformulation] *Let $p\colon \mathcal{E} \to \mathcal{B}$ be a map of spaces whose fibres are contractible. Then p is an equivalence, and every functor on $\mathcal{E}$ factors essentially uniquely through $\mathcal{B}$. A construction on objects b that requires a choice c from a contractible space $\mathcal{E}_b$ is therefore as good as a construction on the objects alone.*

Table 6.1 lists the cases used in this book. The first four are convex or affine and need no argument. The fifth and sixth are Theorems 6.3 and 6.4. The seventh is Whitney's theorem with parameters.

choice	what it makes well defined
inner product on a vector bundle	reduction $B\,\mathrm{GL}_k \simeq B\mathrm{O}(k)$; sphere and disc bundles
Riemannian metric on M	exponential maps, gradients
connection on a bundle	parallel transport, Theorem 5.3
partition of unity subordinate to a cover	gluing of local constructions
tubular neighbourhood of $M \subset N$	collapse maps, normal data
collar of ∂M	gluing, composition of cobordisms
embedding $M \to \mathbb{R}^\infty$	stable normal bundle; a model of $B\,\mathrm{Diff}(M)$

Table 6.1: Contractible spaces of choices.

Theorem 6.7. [classical; Whitney] *For every manifold M the space* $\mathrm{Emb}(M, \mathbb{R}^\infty) = \mathrm{colim}_q \mathrm{Emb}(M, \mathbb{R}^q)$ *is weakly contractible.*

It is non-empty by Theorem 1.14. Contractibility is a two-step linear homotopy: move a family of embeddings into the odd coordinates, then slide it to a fixed embedding placed in the even coordinates (Exercise 6.2). The group $\mathrm{Diff}(M)$ acts freely on $\mathrm{Emb}(M, \mathbb{R}^\infty)$, and the quotient, the space of submanifolds of $\mathbb{R}^\infty$ diffeomorphic to M, is therefore a model for the classifying space $B\,\mathrm{Diff}(M)$. A map $B \to B\,\mathrm{Diff}(M)$ is a family of submanifolds of $\mathbb{R}^\infty$ parametrized by B, that is, a bundle of manifolds with fibre M; this is the last row of Table 5.1.

6.4 The collapse map and Thom spaces

Definition 6.8. The ***Thom space*** $\mathrm{Th}(\xi)$ of a vector bundle ξ over a compact base is the one-point compactification of its total space; equivalently the quotient $D(\xi)/S(\xi)$ of the unit disc bundle by the unit sphere bundle, for any inner product.

Proposition 6.9. [classical] $\mathrm{Th}(\xi \times \eta) \cong \mathrm{Th}(\xi) \wedge \mathrm{Th}(\eta)$; $\mathrm{Th}(\xi \oplus \mathbb{R}^j) \cong \Sigma^j \mathrm{Th}(\xi)$; *the Thom space of the trivial bundle $M \times \mathbb{R}^k$ is $\Sigma^k M_+$; and the Thom space of the tautological line bundle over $\mathbb{RP}^n$ is $\mathbb{RP}^{n+1}$.*

Let $M \subset \mathbb{R}^{n+k}$ be a closed submanifold of dimension n with a tubular neighbourhood $\varphi\colon \nu \to U$. The ***Pontryagin–Thom collapse*** is the map

$$c\colon S^{n+k} = \mathbb{R}^{n+k} \cup \{\infty\} \longrightarrow U^+ \cong \mathrm{Th}(\nu)$$

that is φ^{-1} on U and sends everything else to the point at infinity. More generally an embedding $M \subset N$ of closed manifolds gives $N_+ \to \mathrm{Th}(\nu(M, N))$. By

Theorem 6.3 and Principle 6.6 the collapse is a well-defined point of a mapping space, not merely a homotopy class. It is a map in the wrong direction, from the ambient manifold to (a twisted suspension of) the submanifold, and it is the geometric source of all wrong-way maps in cohomology.

The Thom space depends on the embedding only through a suspension. By Theorem 6.7 the desuspended suspension spectrum

$$M^{-TM} = \Sigma^{-(n+k)}\Sigma^\infty \operatorname{Th}(\nu)$$

is well defined; it is the Thom spectrum of the stable normal bundle $\nu_M = -TM$.

Theorem 6.10 (Atiyah duality). [classical; Atiyah [Ati61]; Milnor–Spanier] *For a closed manifold M the suspension spectrum $\Sigma^\infty_+ M$ is dualizable in the stable homotopy category, with dual M^{-TM}. The coevaluation $\mathbb{S} \to \Sigma^\infty_+ M \wedge M^{-TM}$ is the collapse $S^{n+k} \to \operatorname{Th}(\nu)$ followed by the diagonal $\operatorname{Th}(\nu) \to M_+ \wedge \operatorname{Th}(\nu)$. The evaluation is the collapse for the diagonal embedding $M \to M \times M$, whose normal bundle is TM.*

Dualizability and its use are recalled in Appendix A. The theorem is the homotopy-theoretic form of Poincaré duality, and Chapter 7 uses it to compute Euler characteristics.

6.5 The normal bundle without choices

There is a short exact sequence $0 \to TM \to TN|_M \to \nu \to 0$, so the normal bundle is the cofibre of the map of tangent bundles; in the notation of Chapter 4 it is the degree-1 part of the relative tangent complex of $M \to N$. The tubular neighbourhood theorem says that the germ of N along M is diffeomorphic to the germ of ν along its zero section. There is a canonical object behind the non-canonical diffeomorphism.

Proposition 6.11 (Deformation to the normal bundle). [classical] *There is a manifold $\mathcal{D}(M,N)$ with a submersion to $\mathbb{R}$ whose fibre over $t \neq 0$ is N and whose fibre over 0 is $\nu(M,N)$, containing $M \times \mathbb{R}$ as a closed submanifold. It is functorial for maps of pairs. A tubular neighbourhood is a trivialization of this family near $M \times \mathbb{R}$.*

For the diagonal $M \subset M \times M$ this is Connes's tangent groupoid, interpolating between the pair groupoid of M and the tangent bundle. The proposition illustrates a general pattern: where the classical text makes a choice and proves uniqueness up to isotopy, there is usually a canonical object in a larger category of which the choices are trivializations.

By Example 4.14(c) the derived self-intersection $M \times^h_N M$ depends only on the normal bundle: its functions are $\Gamma(\Lambda^\bullet \nu^\vee)$. Its virtual dimension is $2\dim M - \dim N$, and its cobordism class is that of the zero set of a generic section of ν (Proposition 4.10).

Notes

Hirsch's proofs of uniqueness of tubular neighbourhoods and collars give isotopies depending continuously on the data, which is most of the way to contractibility; the statements with parameters are spelled out in [Godo7] and in the literature on cobordism categories [GMTW09]. The collapse map is Pontryagin's for framed submanifolds and Thom's in general. For Thom spectra of virtual bundles and their multiplicative properties see [ABGHR14]. The deformation to the normal cone is due to Fulton and MacPherson in algebraic geometry; the smooth version is in Connes's and Higson's work on index theory.

Exercises

Exercise 6.1. Show that the space of inner products on a finite-dimensional real vector space is convex, and that $\mathrm{O}(k) \to \mathrm{GL}_k(\mathbb{R})$ is a homotopy equivalence. Deduce $B\,\mathrm{GL}_k(\mathbb{R}) \simeq B\mathrm{O}(k)$.

Exercise 6.2. Prove Theorem 6.7 for π_0: any two embeddings $M \to \mathbb{R}^\infty$ are isotopic through embeddings. Explain why the straight-line homotopy between two embeddings in $\mathbb{R}^q$ need not be an isotopy.

Exercise 6.3. For a closed plane curve with curvature κ and inward unit normal n, show that $(s,t) \mapsto \gamma(s) + t\,n(s)$ has Jacobian determinant $1 - t\kappa(s)$. Find the largest r such that the map is a local diffeomorphism for $|t| < r$ when the curve is the ellipse with semi-axes $a > b$.

Exercise 6.4. Prove Proposition 6.9.

Exercise 6.5. Let f be a diffeomorphism of S^{n-1} and $\Sigma_f = D^n \cup_f D^n$. Show that Σ_f is homeomorphic to S^n, and diffeomorphic to S^n if f extends to a diffeomorphism of D^n.

Chapter 7

Degree, Intersection Numbers, and the Euler Characteristic

Hirsch's fifth chapter is where the method of the Preface first produces numbers. Each invariant is a signed count of points in a transverse preimage, and each proof of invariance exhibits two such counts as the two ends of a compact oriented 1-manifold. In the terms of Chapter 4, each invariant is the class in $\Omega_0 = \mathbb{Z}$ (or $\mathfrak{N}_0 = \mathbb{Z}/2$) of a compact derived manifold of virtual dimension zero. This chapter records the invariants in that form and then gives the second, independent description of the Euler characteristic and the Lefschetz number as traces.

7.1 Degree

Definition 7.1. Let M, N be closed oriented n-manifolds with N connected, $f\colon M \to N$ smooth, and y a regular value. The ***degree*** is $\deg f = \sum_{x \in f^{-1}(y)} \operatorname{sign} \det df_x$. Without orientations the ***mod*** 2 ***degree*** is $\#f^{-1}(y)$ modulo 2.

Theorem 7.2. [classical; Brouwer; Hirsch 5.1.6, 5.1.8]

(a) $\deg f$ *does not depend on the regular value, and homotopic maps have the same degree.*

(b) If $M = \partial W$ *with* W *compact oriented and* f *extends to* W*, then* $\deg f = 0$*.*

(c) If W *is connected and* $N = S^n$*, the converse of (b) holds: a map* $\partial W \to S^n$ *of degree zero extends to* W*.*

Proof of (a) and (b). Let $F\colon W \to N$ extend f, and let y be a regular value of F and of f (Theorem 4.1). By Theorem 1.17, $F^{-1}(y)$ is a compact oriented 1-manifold with boundary $f^{-1}(y)$, and the boundary orientation is the sign in the definition. The signed number of boundary points of a compact oriented 1-manifold is zero. Homotopy invariance is the case $W = M \times I$. Independence of y follows because any two points of a connected manifold are exchanged by a diffeomorphism isotopic to the identity (Chapter 10). □

For example $z \mapsto z^k$ on the circle has degree k, and the antipodal map of S^n has degree $(-1)^{n+1}$ (Exercise 7.1). The script computes the first both by counting preimages and as a winding-number integral.

Theorem 7.3 (Hopf). [classical; Hopf; Hirsch 5.1.10] *Let M be a closed connected n-manifold. If M is oriented,* $\deg\colon [M, S^n] \to \mathbb{Z}$ *is a bijection. If M is not orientable, the mod* 2 *degree $[M, S^n] \to \mathbb{Z}/2$ is a bijection.*

7.2 Hopf's theorem as a classification theorem

There are two ways to see Theorem 7.3 as an instance of something general, and they are dual.

The first is geometric. For a closed manifold M let $\mathcal{F}_k(M)$ be the set of closed submanifolds $A \subset M$ of codimension k equipped with a framing of the normal bundle $\nu(A, M)$. For any k, taking the preimage of a regular value gives a bijection

$$[M, S^k] \;\cong\; \mathcal{F}_k(M) \,/\, \text{framed cobordism in } M \times I,$$

with inverse the collapse $M \to \mathrm{Th}(\nu(A, M)) \cong \Sigma^k A_+ \to S^k$ of Chapter 6. This is the Pontryagin–Thom construction inside M [Pon55]. For $k = n$ a framed submanifold is a finite set of points with signs, and a framed cobordism cancels pairs of opposite sign; if M is connected and oriented the classes are counted by the total sign, and if M is non-orientable a point can be carried around an orientation-reversing loop and return with the opposite sign, so only the parity remains.

The second is homotopical. The map $S^n \to K(\mathbb{Z}, n)$ inducing an isomorphism on π_n is $(n+1)$-connected, so $[M, S^n] \to [M, K(\mathbb{Z}, n)] = H^n(M; \mathbb{Z})$ is a bijection for $\dim M \le n$; and $H^n(M; \mathbb{Z})$ is $\mathbb{Z}$ or $\mathbb{Z}/2$ according to orientability. The two readings are identified by Poincaré duality, $H^n(M; \mathbb{Z}) \cong H_0(M; \mathbb{Z}^w)$ with twisted coefficients, a class in the latter being a finite set of points with local orientations.

7.3 Intersection numbers

Theorem 7.4. [classical; Hirsch 5.2.1] *Let $f\colon M \to N$ be a map of oriented manifolds and $A \subset N$ a closed oriented submanifold, with M compact and* $\dim M + \dim A = \dim N$. *If $f \pitchfork A$, put $\#(f, A) = \sum_{x \in f^{-1}(A)} \pm 1$, the sign comparing $df_x(T_xM) \oplus T_{f(x)}A$ with $T_{f(x)}N$. Then homotopic maps transverse to A have the same intersection number. Without orientations the same holds modulo* 2.

So $\#(f, A)$ is defined for every f, by perturbing. In the language of Chapter 4 no perturbation is needed to define it:

$$\#(f, A) = \left[M \times_N^h A\right] \in \Omega_0 = \mathbb{Z},$$

the cobordism class of the derived fibre product, which is compact of virtual dimension zero (Theorem 4.16). Degree is the case $A = \{y\}$.

Example 7.5. On the torus $\mathbb{R}^2/\mathbb{Z}^2$ the closed curve of slope class (a, b) and the curve of class (c, d), in general position, meet in $|ad - bc|$ points all of the same sign, and the intersection number is $ad - bc$. The script counts the intersection points for several pairs.

7.4 The Euler number

Definition 7.6. Let ξ be an oriented vector bundle of rank n over a closed oriented n-manifold M, with zero section Z. Its ***Euler number*** is $\chi(\xi) = \#(Z, Z)$, the signed number of zeros of a section transverse to Z. The Euler characteristic of M is $\chi(M) = \chi(TM)$.

Theorem 7.7. [classical; Hopf; Hirsch 5.2.2, 5.2.3, 5.2.5, 5.2.10]

(a) If ξ has a nowhere-zero section then $\chi(\xi) = 0$; if M is connected the converse holds.

(b) $\chi(\xi) = 0$ if n is odd.

(c) $\chi(S^n) = 1 + (-1)^n$, and $\chi(M \times N) = \chi(M)\chi(N)$.

(d) (Poincaré–Hopf) For a vector field on M with isolated zeros, the sum of the indices is $\chi(M)$.

The Euler number is the class of the derived self-intersection of the zero section, which by Example 4.14(c) is the derived manifold with functions $\Gamma(\Lambda^\bullet \xi^\vee)$. Part (b) is the observation that $s \mapsto -s$ is an automorphism of this object that multiplies its orientation by $(-1)^n$. The converse in (a) is an instance of Theorem 7.2(c): the zeros of a section can be gathered into a disc by an isotopy, and the section on the boundary sphere then has degree zero and extends without zeros.

7.5 Traces

The second description starts from a definition that makes no mention of manifolds (Appendix A). In a symmetric monoidal category an object X is ***dualizable*** if there are $X^\vee$, a coevaluation $\eta\colon \mathbf{1} \to X \otimes X^\vee$ and an evaluation

$\epsilon\colon X^\vee \otimes X \to \mathbf{1}$ satisfying the triangle identities. The ***trace*** of an endomorphism f of such an X is the composite

$$\operatorname{tr}(f)\colon\ \mathbf{1} \xrightarrow{\eta} X \otimes X^\vee \xrightarrow{f\otimes 1} X \otimes X^\vee \xrightarrow{\text{swap}} X^\vee \otimes X \xrightarrow{\epsilon} \mathbf{1},$$

an element of the commutative monoid $\operatorname{End}(\mathbf{1})$, and $\chi(X) = \operatorname{tr}(\mathrm{id}_X)$.

Proposition 7.8. [classical; Dold–Puppe [DP80]] $\operatorname{tr}(fg) = \operatorname{tr}(gf)$; $\operatorname{tr}(f \otimes g) = \operatorname{tr}(f)\operatorname{tr}(g)$; *and symmetric monoidal functors preserve dualizable objects and traces. In finite-dimensional vector spaces* tr *is the trace of a matrix. In graded vector spaces with the Koszul sign in the symmetry it is the alternating sum of the traces on the graded pieces.*

Theorem 7.9 (Lefschetz–Hopf, after Dold and Puppe)**.** [classical; [DP80]] *Let M be a closed manifold and $f\colon M \to M$ a map. The trace of $\Sigma^\infty_+ f$ in the stable homotopy category is an integer, an element of $\pi_0 \mathbb{S} = \mathbb{Z}$, and:*

(i) it equals the Lefschetz number $L(f) = \sum_i (-1)^i \operatorname{tr}\big(f_* \mid H_i(M;\mathbb{Q})\big)$;

(ii) it equals the intersection number of the graph of f with the diagonal in $M \times M$; if the fixed points of f are non-degenerate, this is $\sum_{f(x)=x} \operatorname{sign}\det(1 - df_x)$.

In particular $\chi(M) = \sum_i (-1)^i \dim H_i(M;\mathbb{Q}) = \#(\Delta,\Delta) = \chi(TM)$.

Sketch. (i) Rational homology is a symmetric monoidal functor from finite spectra to graded vector spaces, by the Künneth theorem; apply Proposition 7.8. (ii) By Theorem 6.10 the coevaluation is the collapse onto a tubular neighbourhood of $M \subset \mathbb{R}^{n+k}$ and the evaluation is the collapse onto a tubular neighbourhood of the diagonal. The composite defining the trace is a map $S^{n+k} \to S^{n+k}$ whose preimage of the base point of the last collapse is the set of fixed points of f, and the local degree at a non-degenerate fixed point is the sign of $\det(1 - df_x)$. □

So the equality of the two definitions of the Euler characteristic, which the source reaches at the end of its treatment of Morse theory (Hirsch 6.3.8), is the statement that a trace can be computed either after applying a monoidal functor or before. The multiplicativity $\chi(M \times N) = \chi(M)\chi(N)$ is $\operatorname{tr}(f \otimes g) = \operatorname{tr}(f)\operatorname{tr}(g)$. Additivity over pushouts, $\chi(A \cup B) = \chi(A) + \chi(B) - \chi(A \cap B)$, is a property of traces in stable settings [Mayo1], and gives the formula $\chi(M) = \sum (-1)^k \nu_k$ in terms of the numbers of critical points of a Morse function (Chapter 8).

Example 7.10. A matrix $A \in \mathrm{GL}_2(\mathbb{Z})$ acts on the torus. On H_0, H_1, H_2 its traces are 1, $\operatorname{tr} A$, $\det A$, so $L(A) = 1 - \operatorname{tr} A + \det A = \det(1 - A)$. The fixed points are the solutions of $(A-1)x \in \mathbb{Z}^2$ modulo $\mathbb{Z}^2$; there are $|\det(A-1)|$ of them and each has index $\operatorname{sign}\det(1-A)$. The script enumerates them for several A.

7.6 The diagonal, three ways

The self-intersection of the diagonal has now appeared in three forms.

Proposition 7.11. [reformulation; cf. Hirsch 5.2, 6.3.8; [DP80]] *For a closed manifold M and $f\colon M \to M$, the following integers are equal:*

(i) the signed number of fixed points of a generic perturbation of f;

(ii) the class in Ω_0 of the derived fixed-point object $M \times^h_{M\times M} M$, the fibre product of the graph of f and the diagonal;

(iii) the trace of $\Sigma^\infty_+ f$.

For $f = \mathrm{id}$ the object in (ii) has functions the differential forms $\Gamma(\Lambda^\bullet T^*M)$ in homological degrees, and the integer is $\chi(M)$.

Form (i) is the definition in the source; it requires a choice and a proof of independence. Form (ii) requires no choice but lives in DMfd. Form (iii) requires no manifold: it makes sense for any dualizable object, and it is the one that generalizes, to parametrized families, to equivariant settings, and to bicategories, where the Lefschetz number acquires a refinement due to Reidemeister [PS14].

The derived fixed-point object of the identity is the smooth counterpart of the derived loop space of algebraic geometry. That its functions are differential forms is the Hochschild–Kostant–Rosenberg theorem in this setting, and the Euler characteristic is then the "dimension" of M in the sense in which the dimension of a vector space is the trace of its identity map.

Example 7.12 (Linking numbers)**.** For disjoint oriented closed curves $K, L \subset \mathbb{R}^3$ the linking number is the degree of the Gauss map $K \times L \to S^2$, $(x, y) \mapsto (y - x)/|y - x|$. It equals the intersection number of L with any oriented surface bounded by K, and it is given by Gauss's integral. For the Hopf link it is ± 1, and the script evaluates the integral numerically.

Notes

The treatment of degree by regular values and one-dimensional cobordisms is Pontryagin's and is the subject of Milnor's *Topology from the Differentiable Viewpoint*. The trace in a symmetric monoidal category and its application to fixed point theory are due to Dold and Puppe; see also [May01] for additivity and [PS14] for an exposition and the bicategorical refinement. Derived loop spaces and their relation to differential forms are due to Ben-Zvi and Nadler and to Toën and Vezzosi in the algebraic setting.

Exercises

Exercise 7.1. Show that $z \mapsto z^k$ has degree k on S^1 and that the antipodal map of S^n has degree $(-1)^{n+1}$. Deduce that every vector field on S^{2m} has a zero.

Exercise 7.2. Verify Example 7.10: for $A \in \mathrm{GL}_2(\mathbb{Z})$ with $\det(A-1) \neq 0$, the induced map of the torus has exactly $|\det(A-1)|$ fixed points, each of index $\operatorname{sign}\det(1-A)$.

Exercise 7.3. Prove $\operatorname{tr}(fg) = \operatorname{tr}(gf)$ and $\operatorname{tr}(f \otimes g) = \operatorname{tr}(f)\operatorname{tr}(g)$ from the definition, and show that in graded vector spaces with the Koszul sign rule the trace of a degree-preserving endomorphism is the alternating sum of the traces of its components.

Exercise 7.4. Compute $\chi(S^n)$ from a vector field with two zeros, and again as $\sum (-1)^i \dim H_i$.

Exercise 7.5. Let K be the unit circle in the xy-plane and L the circle of radius 1 centred at $(1,0,0)$ in the xz-plane. Compute the linking number as the intersection number of L with the disc bounded by K.

Part III

Decompositions and Moduli

Chapter 8

Morse Theory

A Morse function on a manifold is a generic point of the mapping object $[M, \mathbb{R}]$, and Hirsch's sixth chapter shows how much of the manifold such a point determines: a decomposition into handles, a cell structure, inequalities for the Betti numbers, the Euler characteristic. In categorical terms a Morse function presents the homotopy type of M as an iterated pushout, and presents M itself as a composite of elementary cobordisms. This chapter states the classical theorems in that form, and then turns to what the source does not treat: the structure on the set of all Morse functions, which is where the presentations become canonical.

8.1 Morse functions

Definition 8.1. A critical point p of $f\colon M \to \mathbb{R}$ is ***non-degenerate*** if the Hessian of f at p is a non-degenerate quadratic form; its ***index*** is the number of negative eigenvalues. f is a ***Morse function*** if all its critical points are non-degenerate. Its ***type*** is $(\nu_0, \dots, \nu_n)$, with ν_k the number of critical points of index k.

Theorem 8.2. [classical; Morse; Hirsch 6.1.1, 6.1.2] *Near a non-degenerate critical point of index k there are coordinates in which $f = f(p) - x_1^2 - \cdots - x_k^2 + x_{k+1}^2 + \cdots + x_n^2$. Morse functions form an open dense subset of $C^\infty_S(M, \mathbb{R})$.*

Density was derived in Corollary 4.8(c) from jet transversality: f is Morse when df is transverse to the zero section of T^*M.

Example 8.3. On the torus with coordinates (θ, φ) let $f = (R + r\cos\varphi)\cos\theta$, $R > r > 0$, the height function of a torus standing on its edge. The critical points are the four points with $\theta, \varphi \in \{0, \pi\}$, and the Hessian there is diagonal with entries $-(R + r\cos\varphi)\cos\theta$ and $-r\cos\varphi\cos\theta$. The indices are $2, 1, 1, 0$ at $(0,0), (0,\pi), (\pi,\pi), (\pi,0)$, with critical values $R + r$, $R - r$, $-(R - r)$, $-(R + r)$. The type is $(1, 2, 1)$.

8.2 Regular intervals

Theorem 8.4 (Regular interval). [classical; Hirsch 6.2.2] *Let $f\colon M \to [a, b]$ be smooth on a compact manifold with boundary, with no critical points and $f(\partial M) = \{a, b\}$.*

Then there is a diffeomorphism $M \cong f^{-1}(a) \times [a,b]$ commuting with the maps to $[a,b]$.

The proof integrates the vector field $\operatorname{grad} f / |\operatorname{grad} f|^2$, along which f increases at unit speed. The theorem is the one-parameter case of a statement about families.

Theorem 8.5 (Ehresmann). [classical] *A proper submersion $p\colon E \to B$ is a locally trivial fibre bundle.*

In the language of Chapter 5: let $\mathcal{M}_d$ be the stack on Cart whose sections over U are the proper submersions over U with d-dimensional fibres. Ehresmann's theorem says that $\mathcal{M}_d$ is locally constant in the strongest sense: every section over a Euclidean space is a product. Its shape is $\coprod_{[F]} B\operatorname{Diff}(F)$, and a proper submersion over B is classified by a map from B to this space (Table 5.1). Compact smooth manifolds have no local moduli, only monodromy. The regular interval theorem is the case of a contractible base.

8.3 Passing a critical point

Call a Morse function $f\colon M \to [a,b]$ on a compact manifold ***admissible*** if $\partial M = f^{-1}(a) \cup f^{-1}(b)$ and a, b are regular values.

Theorem 8.6 (Attaching a cell). [classical; Hirsch 6.3.1–6.3.3] *Let $f\colon M \to [a,b]$ be an admissible Morse function with exactly one critical point, of index k. Then there is a k-cell $e^k \subset M \setminus f^{-1}(b)$, attached to $f^{-1}(a)$ along its boundary, such that $f^{-1}(a) \cup e^k$ is a deformation retract of M. Smoothly, M is obtained from $f^{-1}(a) \times [0,1]$ by attaching a handle $D^k \times D^{n-k}$ along an embedding of $S^{k-1} \times D^{n-k}$ in $f^{-1}(a) \times \{1\}$.*

For a proper function $f\colon M \to \mathbb{R}$ bounded below, write $M^c = f^{-1}(-\infty, c]$. The theorem and Theorem 8.4 give the following form, which is the one that composes.

Corollary 8.7. [reformulation; cf. Hirsch 6.2.2, 6.3.1] *If $[c,c']$ contains no critical value, $M^c \to M^{c'}$ is a homotopy equivalence. If $[c-\varepsilon, c+\varepsilon]$ contains a single critical value, attained at one critical point of index k, there is a homotopy pushout square of spaces*

$$\begin{array}{ccc} S^{k-1} & \longrightarrow & M^{c-\varepsilon} \\ \downarrow & & \downarrow \\ D^k & \longrightarrow & M^{c+\varepsilon} \end{array}$$

and hence a cofibre sequence $\Sigma^\infty_+ M^{c-\varepsilon} \to \Sigma^\infty_+ M^{c+\varepsilon} \to \mathbb{S}^k$ of spectra.

The local statement behind the theorem is about the quadratic form $q(x,y) = -|x|^2 + |y|^2$ on $\mathbb{R}^k \times \mathbb{R}^{n-k}$: the set $\{q \le -\varepsilon\}$ together with the disc $\{y = 0, |x|^2 \le \varepsilon\}$ is a deformation retract of $\{q \le \varepsilon\}$, by a deformation that shrinks y and does not move x (Exercise 8.4).

Theorem 8.8. [classical; Hirsch 6.4.1–6.4.3, 6.2.4] *A compact manifold with a Morse function of type $(\nu_0, \dots, \nu_n)$ has the homotopy type of a finite CW complex with ν_k cells of dimension k. Every n-manifold has the homotopy type of a CW complex of dimension at most n. A closed manifold with a Morse function having exactly two critical points is homeomorphic to a sphere.*

So the shape of a compact manifold is a finite space, a compact object of $\mathcal{S}$, and its suspension spectrum is a finite spectrum. Finite spectra are dualizable; this is the abstract half of Theorem 6.10, the concrete half being the identification of the dual.

8.4 Inequalities

Theorem 8.9 (Morse inequalities). [classical; Morse; Hirsch 6.3.4–6.3.8] *Let f be a Morse function of type (ν_k) on a closed manifold M, and $\beta_k = \dim H_k(M; F)$ for a field F. Then*

$$\sum_k \nu_k t^k - \sum_k \beta_k t^k = (1+t)\, Q(t)$$

for a polynomial Q with non-negative integer coefficients. In particular $\nu_k \ge \beta_k$, and $\chi(M) = \sum (-1)^k \nu_k = \sum (-1)^k \beta_k$; and $\chi(M) = 0$ if $\dim M$ is odd.

Proof. By Corollary 8.7, crossing a critical point of index k changes the homology of the sublevel set in one of two ways: either β_k increases by one, or β_{k-1} decreases by one. Let q_{k-1} be the number of critical points of index k of the second kind. Each of them kills a class created earlier by a critical point of index $k-1$ of the first kind. So $\nu_k = \beta_k + q_k + q_{k-1}$, which is the identity with $Q = \sum q_k t^k$. Setting $t = -1$ gives the Euler characteristic, and replacing f by $-f$ exchanges ν_k and ν_{n-k}. □

The equality $\chi(TM) = \sum (-1)^k \nu_k$ also follows from Theorem 7.7(d) applied to the gradient, whose zero at a critical point of index k has index $(-1)^k$. For the torus of Example 8.3, $1 + 2t + t^2$ is the Poincaré polynomial and $Q = 0$. On $\mathbb{RP}^n$ there is a Morse function with one critical point of each index, which is optimal for $F = \mathbb{Z}/2$ and not for $F = \mathbb{Q}$ (Exercise 8.3).

8.5 The sublevel functor

The proof just given used more than the statement records. The assignment $c \mapsto M^c$ is a functor from the ordered set $\mathbb{R}$ to manifolds with boundary and embeddings, and its composite with the shape,

$$S_f \colon (\mathbb{R}, \leq) \to \mathcal{S}, \qquad c \mapsto \Pi(M^c),$$

is the ***sublevel functor*** of f.

Proposition 8.10. [reformulation; cf. Corollary 8.7; persistence, [ZC05]] *For a Morse function f on a closed manifold, S_f is locally constant on the complement of the critical values, and at a critical value it changes by a pushout along a sphere for each critical point. The functor $H_k(S_f; F) \colon (\mathbb{R}, \leq) \to \mathsf{Vect}_F$ is a direct sum of interval modules; the number of infinite intervals is β_k, and the number of finite intervals is the coefficient q_k of Q in Theorem 8.9.*

So the polynomial Q counts pairs of critical points that cancel in homology, and the multiset of intervals, the ***barcode***, refines the Morse inequalities to a statement about which pairs and at which levels. The barcode is a continuous function of f in the uniform norm, which the type is not. The script computes the degree-zero barcode of a function on a circle by a union–find pass and checks that the number of finite intervals is $\nu_0 - 1$.

8.6 Flow categories

A Morse function with a generic metric determines more than a filtration.

Theorem 8.11. [literature; Cohen–Jones–Segal [CJS95]] *Let f be a Morse function on a closed manifold M with a Morse–Smale gradient. There is a topological category $\mathcal{C}_f$ whose objects are the critical points and whose morphism space from p to q is the compactified space of gradient flow lines from p to q, a manifold with corners of dimension $\operatorname{ind} p - \operatorname{ind} q - 1$, with composition the inclusion of broken flow lines. Its classifying space is homeomorphic to M.*

The Morse complex is what remains of $\mathcal{C}_f$ when one keeps only the zero-dimensional morphism spaces and counts their points; the higher-dimensional ones carry the attaching maps of the cells. The transversality condition on the metric is Theorem 4.3 applied to the stable and unstable manifolds, and the corner structure is where manifolds with corners (Chapter 1) become unavoidable.

8.7 Families of functions

The set of Morse functions is open and dense in $[M, \mathbb{R}]$ but it is not a good parameter space: two Morse functions of different types cannot be joined by a path of Morse functions. Jet transversality in one parameter says what a generic path looks like.

Theorem 8.12 (Cerf). [classical; Cerf [Cer70]] *Any two Morse functions on a closed manifold are joined by a path f_t that is Morse except at finitely many times, at each of which exactly one of the following occurs: two critical points have the same value; or there is a single degenerate critical point, with local form $x_1^3 - (t - t_0)x_1 - x_2^2 - \cdots - x_{k+1}^2 + x_{k+2}^2 + \cdots + x_n^2$, at which a pair of critical points of indices k and $k+1$ is born or dies.*

The count behind the theorem: in the space of 2-jets, the jets of degenerate critical points have codimension $n + 1$, so a one-parameter family, of total dimension $n + 1$, meets them at isolated points. On the side of handle decompositions the two events are the reordering or sliding of handles and the cancellation of a pair. So the theorem says that any two handle decompositions of a manifold are related by these moves. It is a presentation by generators and relations, one categorical level up from Theorem 8.6.

The space of functions with only Morse and birth–death singularities is connected by Cerf's theorem, but it is still not contractible. Igusa found the decoration that makes it so.

Theorem 8.13. [literature; Igusa [Igu87]; Lurie [Lur09]; Eliashberg–Mishachev [EM12]] *The space of framed functions on a compact manifold M (functions with only Morse and birth–death singularities, with a framing of the negative eigenspace of the Hessian at each critical point) is contractible.*

Igusa proved that the space is $(\dim M - 1)$-connected; contractibility is due to Lurie and, by geometric methods, to Eliashberg and Mishachev. By Principle 6.6 the theorem says that a framed function is a choice of the harmless kind, so that the handle decomposition it induces is as good as canonical. That is the form in which Morse theory enters the proof of the cobordism hypothesis (Chapter 9).

Notes

Milnor's *Morse Theory* and *Lectures on the h-Cobordism Theorem* are the classical accounts; Hirsch's chapter follows the same lines with more attention to the differentiability class. Persistence modules and their decomposition are in [ZC05] and in work of Crawley-Boevey. The flow category is the starting point

of Floer homotopy theory. Cerf's theorem is the one-parameter beginning of a hierarchy governed by the classification of singularities of functions; Igusa's work on higher parameters and the notion of framed function are from the 1980s.

Exercises

Exercise 8.1. Verify Example 8.3.

Exercise 8.2. Deduce from the polynomial identity of Theorem 8.9 the inequalities $\nu_k - \nu_{k-1} + \cdots \pm \nu_0 \geq \beta_k - \beta_{k-1} + \cdots \pm \beta_0$ for every k. Show that a Morse function on a closed connected surface of genus g has at least $2g+2$ critical points.

Exercise 8.3. Let $\lambda_0 < \cdots < \lambda_n$ and $f([x]) = \sum \lambda_i x_i^2 / \sum x_i^2$ on $\mathbb{RP}^n$. Show that f is a Morse function with one critical point of each index $0, \ldots, n$. Find Q for $F = \mathbb{Z}/2$ and for $F = \mathbb{Q}$.

Exercise 8.4. For $q(x,y) = -|x|^2 + |y|^2$ on $\mathbb{R}^k \times \mathbb{R}^{n-k}$ and $\varepsilon > 0$, construct a deformation retraction of $\{q \leq \varepsilon\}$ onto $\{q \leq -\varepsilon\} \cup \{y = 0,\ |x|^2 \leq \varepsilon\}$.

Exercise 8.5. For $f_t(x) = x^3 - tx$ on $\mathbb{R}$, find the critical points and their indices for each t, and describe the sublevel functor of f_t restricted to a large interval $[-L, L]$ before and after $t = 0$.

Chapter 9

Cobordism

In the source, cobordism occupies a short chapter: the groups are defined, the Pontryagin–Thom construction is carried out, and Thom's theorem identifying cobordism groups with homotopy groups of Thom spaces is proved in the range the earlier chapters allow. It is the place where the three tools of the book (classifying maps, tubular neighbourhoods, transversality) are used together. It is also the place from which the higher-categorical view of manifolds grew. This chapter states the classical results and then the two modern ones that extend them: the homotopy type of the cobordism category, and the cobordism hypothesis.

9.1 Cobordism groups

Definition 9.1. Closed n-manifolds M_0, M_1 are ***cobordant*** if there is a compact $(n+1)$-manifold W with $\partial W \cong M_0 \sqcup M_1$. The set of classes is $\mathfrak{N}_n$. For oriented manifolds one requires W oriented with $\partial W \cong M_1 \sqcup -M_0$, and the set of classes is Ω_n.

Theorem 9.2. [classical; Thom; Hirsch 7.1.1] *Disjoint union makes $\mathfrak{N}_n$ and Ω_n abelian groups, and product makes $\mathfrak{N}_*$ and Ω_* graded rings, commutative in the graded sense. Every element of $\mathfrak{N}_*$ has order 2.*

Transitivity of the relation uses Theorem 6.5; the inverse of $[M]$ is $[-M]$, with $M \times I$ as the null-cobordism of $M \sqcup -M$. In low dimensions $\mathfrak{N}_0 = \mathbb{Z}/2$, $\Omega_0 = \mathbb{Z}$, $\mathfrak{N}_1 = \Omega_1 = 0$, and $\mathfrak{N}_2 = \mathbb{Z}/2$, $\Omega_2 = 0$ by the classification of surfaces (Chapter 11).

9.2 The Pontryagin–Thom theorem

Let $M \subset \mathbb{R}^{n+k}$ be a closed n-manifold. The collapse of Chapter 6, followed by the map of Thom spaces induced by the classifying map of the normal bundle (Chapter 5), is a map

$$S^{n+k} \longrightarrow \mathrm{Th}(\nu) \longrightarrow \mathrm{Th}(\gamma_k),$$

where γ_k is the tautological bundle over $\mathrm{Gr}_k(\mathbb{R}^s)$ for s large.

Theorem 9.3 (Pontryagin–Thom). [classical; Pontryagin [Pon55]; Thom [Tho54]; Hirsch 7.2.3] *For $f\colon S^{n+k} \to \mathrm{Th}(\gamma_k)$ transverse to the zero section, $f \mapsto [f^{-1}(\mathrm{Gr}_k(\mathbb{R}^s))]$ defines a homomorphism $\pi_{n+k}\big(\mathrm{Th}(\gamma_k \to \mathrm{Gr}_k(\mathbb{R}^s))\big) \to \mathfrak{N}_n$ that is surjective if $k > n$ and $s \geq k+n$ and bijective if $k > n+1$ and $s \geq k+n+1$; the construction above is its inverse. Stably,*

$$\mathfrak{N}_n \cong \pi_n(MO), \qquad \Omega_n \cong \pi_n(MSO), \qquad \Omega_n^{\mathrm{fr}} \cong \pi_n(\mathbb{S}),$$

and in general $\Omega_n^\theta \cong \pi_n(M\theta)$ for a stable tangential structure $\theta\colon B \to BO$, where $M\theta$ is the Thom spectrum of θ.

Structure of the proof. Well-definedness: the embedding and the tubular neighbourhood are contractible choices (Theorems 6.3 and 6.7), and a cobordism embedded in $\mathbb{R}^{n+k} \times I$ collapses to a homotopy. Inverse: a map $f\colon S^{n+k} \to \mathrm{Th}(\gamma_k)$ can be made transverse to the zero section (Theorem 4.3), and $M = f^{-1}(\mathrm{Gr}_k)$ is a closed n-manifold with normal bundle pulled back from γ_k; a homotopy made transverse gives a cobordism (Lemma 4.4). The dimension restrictions come from Theorems 1.14 and 5.4. □

The theorem is Principle 4.17 for the cospan $S^{n+k} \to \mathrm{Th}(\gamma_k) \leftarrow \mathrm{Gr}_k$: every map to a Thom space has a derived preimage of the zero section, which has a cobordism class, and the class determines the map up to homotopy.

9.3 Thom's computation

Theorem 9.4 (Thom). [classical; [Tho54]] *MO is a wedge of suspensions of the Eilenberg–MacLane spectrum $H\mathbb{Z}/2$, and*

$$\mathfrak{N}_* \cong \mathbb{Z}/2\,[x_i :\ i \geq 2,\ i \neq 2^j - 1], \qquad |x_i| = i,$$

with x_{2i} the class of $\mathbb{RP}^{2i}$. A closed manifold is a boundary if and only if all its Stiefel–Whitney numbers vanish.

So $\dim_{\mathbb{Z}/2} \mathfrak{N}_n$ is the number of partitions of n into parts not of the form $2^j - 1$: for $n = 0, \dots, 10$ these are $1, 0, 1, 0, 2, 1, 3, 1, 5, 3, 8$. For a partition $n = i_1 + \cdots + i_r$ the ***Stiefel–Whitney number*** is $\langle w_{i_1} \cdots w_{i_r}(TM), [M]\rangle \in \mathbb{Z}/2$. It vanishes on a boundary $M = \partial W$ because $TW|_M = TM \oplus \mathbb{R}$, so the class being evaluated is restricted from W, and the fundamental class of a boundary maps to zero in the homology of W.

Example 9.5. The total Stiefel–Whitney class of $\mathbb{RP}^n$ is $(1+a)^{n+1}$ with a the generator of H^1. For $\mathbb{RP}^2$, $w = 1 + a + a^2$, and $w_2[\mathbb{RP}^2] = w_1^2[\mathbb{RP}^2] = 1$: not a boundary. For $\mathbb{RP}^3$, $w = (1+a)^4 = 1$: all numbers vanish and $\mathbb{RP}^3$

bounds. In dimension 4 the numbers $(w_1^4, w_1^2w_2, w_2^2, w_1w_3, w_4)$ are $(1,0,0,0,1)$ for $\mathbb{RP}^4$ and $(0,0,1,0,1)$ for $\mathbb{RP}^2 \times \mathbb{RP}^2$. The two manifolds are independent in $\mathfrak{N}_4 = (\mathbb{Z}/2)^2$; they are x_4 and x_2^2. The script computes these numbers and the ranks above.

Theorem 9.6. [classical; Thom; Hirsch 7.2.4] *Ω_n is finite if n is not divisible by 4, and Ω_{4k} has rank equal to the number of partitions of k. Rationally, $\Omega_* \otimes \mathbb{Q} = \mathbb{Q}[\mathbb{CP}^2, \mathbb{CP}^4, \dots]$.*

9.4 Thom spectra as colimits

The spectra MO, MSO, $M\theta$ were defined by Thom spaces of bundles over Grassmannians. There is a description in which no bundle appears.

Theorem 9.7. [literature; Ando–Blumberg–Gepner–Hopkins–Rezk [ABGHR14]] *Let $\mathrm{Pic}(\mathbb{S})$ be the ∞-groupoid of invertible spectra. A stable vector bundle on a space X determines a map $X \to B\mathrm{O} \xrightarrow{J} \mathrm{Pic}(\mathbb{S})$, sending a point to the one-point compactification of the fibre, and its Thom spectrum is the colimit of the composite functor $X \to \mathrm{Pic}(\mathbb{S}) \subset \mathsf{Sp}$. In particular $MO = \mathrm{colim}(B\mathrm{O} \to \mathsf{Sp})$ is the homotopy quotient of the sphere spectrum by the stable orthogonal group, and it is an E_∞-ring because J is an infinite loop map. For an E_∞-ring R, E_∞-ring maps $MO \to R$ correspond to null-homotopies of $B\mathrm{O} \to B\,\mathrm{GL}_1(R)$ as infinite loop maps.*

For a trivial bundle the diagram is constant and the colimit is $\Sigma^\infty_+ X$, which is Proposition 6.9. The description makes the Thom isomorphism a formal consequence of an orientation, a trivialization of the diagram after smashing with R. And it gives a reading of Theorem 9.3: framed cobordism is the sphere spectrum, and unoriented cobordism is what is left when one divides by all changes of framing.

9.5 Cobordism categories

The groups $\mathfrak{N}_n$ remember that a cobordism exists. The cobordism category remembers the cobordisms.

Definition 9.8. Cob_d is the ∞-category whose objects are closed $(d-1)$-manifolds and whose space of morphisms from M_0 to M_1 is $\coprod_{[W]} B\,\mathrm{Diff}(W \text{ rel } \partial W)$, the union over diffeomorphism classes of cobordisms. Composition is gluing (Theorem 6.5). Disjoint union makes it symmetric monoidal. With a tangential structure $\theta\colon B \to B\mathrm{O}(d)$ on the cobordisms one writes Cob_d^θ.

A concrete model takes objects to be closed submanifolds of $\mathbb{R}^\infty$ and morphisms to be submanifolds of $[a_0, a_1] \times \mathbb{R}^\infty$ that are products near the ends; by Theorem 6.7 the embeddings add nothing, and by Theorem 6.4 neither do the product conditions. The set of components of the classifying space $B\mathsf{Cob}_d$ is the set of objects modulo the relation generated by morphisms, which is $\mathfrak{N}_{d-1}$.

Theorem 9.9 (Galatius–Madsen–Tillmann–Weiss). [literature; [GMTW09]] *Let $MTO(d)$ be the Thom spectrum of the virtual bundle $-\gamma_d$ over $B\mathrm{O}(d)$. Then*

$$B\mathsf{Cob}_d \;\simeq\; \Omega^{\infty-1} MTO(d),$$

and likewise $B\mathsf{Cob}_d^\theta \simeq \Omega^{\infty-1} MT\theta(d)$ for every tangential structure θ.

On π_0 this is a form of Theorem 9.3. The proof is an application of the concordance classification (Theorem 5.5): the cobordism category is the shape of a sheaf whose sections over U are families, parametrized by U, of d-dimensional submanifolds of $\mathbb{R} \times \mathbb{R}^\infty$ that are proper over $\mathbb{R}$, and the shape of that sheaf is computed by a parametrized Pontryagin–Thom construction. So the theorem is the Pontryagin–Thom theorem for the whole category of cobordisms and not only for its set of components. For oriented surfaces, combined with Harer's stability theorem, it gives the Madsen–Weiss theorem (Theorem 11.8).

9.6 Field theories and the cobordism hypothesis

Definition 9.10. A d-dimensional ***topological field theory*** with values in a symmetric monoidal ∞-category $\mathcal{C}$ is a symmetric monoidal functor $Z\colon \mathsf{Cob}_d \to \mathcal{C}$.

An invertible theory, one whose values are invertible objects and morphisms, factors through the groupoid completion of Cob_d, so by Theorem 9.9 invertible theories are classified by maps out of the spectrum $\Sigma MTO(d)$. For general theories one needs to know Cob_d itself, by generators and relations. Morse theory provides them: by Theorem 8.6 every cobordism is a composite of handle attachments, and by Theorems 8.12 and 8.13 the relations among such composites are known and the choices involved are contractible. The cleanest statement is obtained after extending the category downwards, allowing manifolds with corners of all codimensions, to a symmetric monoidal (∞, d)-category Bord_d.

Theorem 9.11 (Cobordism hypothesis). [literature; conjectured by Baez–Dolan [BD95]; Lurie [Lur09]; see also [GP21]] *Let $\mathcal{C}$ be a symmetric monoidal (∞, d)-category. Evaluation at*

the point is an equivalence between symmetric monoidal functors $\mathsf{Bord}_d^{\mathrm{fr}} \to \mathcal{C}$*, from the framed bordism category, and the* ∞*-groupoid of fully dualizable objects of* $\mathcal{C}$*.*

Lurie's paper gives a detailed outline of a proof; the case $d = 2$ is worked out by generators and relations in [SP09], and a proof in a geometric formulation has been given by Grady and Pavlov. For $d = 1$ the statement is that the oriented one-dimensional cobordism category is the free symmetric monoidal ∞-category on a dualizable object. The value of the theory on the circle is then the trace of the identity, $\chi(X)$ in the sense of Chapter 7; for $\mathcal{C} = \mathsf{Sp}$ and $X = \Sigma^\infty_+ M$ it is the Euler characteristic of M. The Euler characteristic of a closed manifold is the partition function, on the circle, of the one-dimensional field theory it generates.

9.7 Derived cobordism

By Theorem 4.16 the cobordism ring of compact derived manifolds is $\mathfrak{N}_*$ again. The two enlargements of this chapter are therefore complementary. Passing from the groups to the category Cob_d adds information: the homotopy types of diffeomorphism groups. Passing from manifolds to derived manifolds adds none at the level of cobordism classes, and that is its use: fibre products, fixed-point sets and zero loci have classes without being put in general position first.

Notes

Stong's *Notes on Cobordism Theory* is the standard reference for the classical computations. The identification of Thom spectra with colimits is from [ABGHR14]. Cobordism categories as topological categories go back to Segal's definition of conformal field theory and to Tillmann; [GMTW09] determined their homotopy type. The cobordism hypothesis was formulated in [BD95]; besides [Lur09], see Ayala and Francis for an approach through factorization homology.

Exercises

Exercise 9.1. Let W be a compact odd-dimensional manifold with boundary. Show that $\chi(\partial W) = 2\chi(W)$. Deduce that the Euler characteristic modulo 2 is a ring homomorphism $\mathfrak{N}_* \to \mathbb{Z}/2$, and that $\mathbb{RP}^{2m}$ is not a boundary.

Exercise 9.2. Compute $\dim \mathfrak{N}_n$ for $n \leq 10$ from Theorem 9.4.

Exercise 9.3. Verify the Stiefel–Whitney numbers in Example 9.5.

Exercise 9.4. Show that $\mathfrak{N}_1 = 0$ and that the Klein bottle is a boundary. Using the classification of surfaces, conclude that $\mathfrak{N}_2 = \mathbb{Z}/2$, generated by $\mathbb{RP}^2$.

Exercise 9.5. Show directly that the Pontryagin–Thom collapse of a single point of $\mathbb{R}^k$ with the standard framing is a map $S^k \to S^k$ of degree 1, and that reversing one framing vector gives degree -1. Conclude $\Omega_0^{\mathrm{fr}} \cong \mathbb{Z}$.

Chapter 10

Isotopy

Hirsch's eighth chapter is about paths in spaces of embeddings: an isotopy of a compact submanifold extends to an isotopy of the ambient manifold; embeddings of discs are unique up to isotopy; manifolds glued by isotopic maps are diffeomorphic. Each statement is about π_0 or about lifting paths. Each has a form that concerns the whole homotopy type of a space of embeddings, and those forms are where the subject went after 1976. This chapter states them and ends with the functor calculus that organizes spaces of embeddings in high codimension. The companion volume uses this chapter more than any other.

10.1 Isotopies and vector fields

Definition 10.1. An ***isotopy*** of embeddings of V in M is a plot $I \to \mathrm{Emb}(V, M)$: a smooth map $F\colon V \times I \to M$ with every F_t an embedding. A ***diffeotopy*** of M is an isotopy $F\colon M \times I \to M$ of diffeomorphisms with $F_0 = \mathrm{id}$. Embeddings are ***isotopic*** if joined by an isotopy, and ***ambient isotopic*** if one is carried to the other by the end of a diffeotopy.

So isotopy is concordance of points of the smooth set $\mathrm{Emb}(V, M)$ in the sense of Definition 1.9. (The word concordance is used differently in the classical literature on embeddings, for embeddings of $V \times I$ in $M \times I$ that need not preserve levels; we avoid that usage.)

Theorem 10.2. [classical; Hirsch 8.1.1] *A time-dependent vector field G on M of bounded velocity, for instance one with compact support, generates a unique diffeotopy F with $\partial_t F_t(x) = G(F_t(x), t)$. Every diffeotopy with compact support arises in this way from its velocity field.*

So paths in $\mathrm{Diff}_c(M)$ starting at the identity are the same thing as paths in the vector space $\mathfrak{X}_c(M)$ of compactly supported vector fields, which is the statement that $\mathrm{Diff}_c(M)$ is a Lie group with that Lie algebra [KM97]. The hypothesis on the velocity cannot be dropped: the field $x^2\partial_x$ on $\mathbb{R}$ has solutions $x_0/(1 - x_0 t)$, which leave every compact set in finite time.

10.2 Isotopy extension

Theorem 10.3 (Isotopy extension). [classical; Thom, Palais, Cerf; Hirsch 8.1.3] *Let $V \subset M$ be a compact submanifold and $F\colon V \times I \to M$ an isotopy with image in $M \setminus \partial M$. Then F extends to a diffeotopy of M with compact support.*

The proof extends the velocity field of F, which is defined along the track of the isotopy in $M \times I$, to a compactly supported field on $M \times I$ using a tubular neighbourhood of the track, and integrates. The extension is a contractible choice. Carrying parameters through the argument gives the stronger statement.

Theorem 10.4 (Restriction is a fibration). [classical; Palais [Pal60]; Cerf; Lima] *Let $V \subset M$ be a compact submanifold. The restriction map $\mathrm{Diff}_c(M) \to \mathrm{Emb}(V, M)$ is a locally trivial fibre bundle with fibre the group $\mathrm{Diff}_c(M \text{ rel } V)$ of diffeomorphisms fixing V. More generally, for compact $V \subset V'$ the restriction $\mathrm{Emb}(V', M) \to \mathrm{Emb}(V, M)$ is a fibration.*

Theorem 10.3 is the path-lifting property of this map; Theorem 10.4 says that all homotopies lift. The consequence is a long exact sequence of homotopy groups. For V a point it reads

$$\cdots \to \pi_1 \mathrm{Diff}(M) \to \pi_1(M) \to \pi_0 \mathrm{Diff}(M \text{ rel } x) \to \pi_0 \mathrm{Diff}(M) \to \pi_0(M),$$

and the middle map sends a loop to the diffeomorphism that pushes the base point around it. For compact V the quotient of $\mathrm{Emb}(V, M)$ by $\mathrm{Diff}(V)$ is the space of submanifolds of M diffeomorphic to V, and the theorem says that the group $\mathrm{Diff}_c(M)$ acts on it with orbits that are unions of components.

Compactness is essential. Let $F_t\colon \mathbb{R} \to \mathbb{R}^3$ be a long knot whose knotted part sits at distance $1/(1-t)$ along the line, for $t < 1$, and let F_1 be the straight line. This is an isotopy in the sense of Definition 10.1, smooth because on every compact set it is eventually constant, from a knotted line to an unknotted one; it cannot be ambient (Exercise 10.4). In the language of Chapter 3, it is a plot of the mapping object that is not a path in the strong topology (Proposition 3.8).

10.3 Embeddings of discs

Theorem 10.5 (Discs). [classical; Cerf, Palais; Hirsch 8.3.1 on π_0] *For an n-manifold M, the map $\mathrm{Emb}(D^n, M) \to \mathrm{Fr}(TM)$, $f \mapsto (f(0), df_0)$, to the frame bundle is a homotopy equivalence. More generally $\mathrm{Emb}(D^k, M)$ is equivalent to the bundle of k-frames in TM, and $\mathrm{Emb}(\coprod_j D^n, M)$ to the pullback of $\mathrm{Fr}(TM)^j$ to the configuration space $\mathrm{Conf}_j(M)$.*

The proof is the Alexander-type shrinking of Theorem 6.3: $f_t(x) = f(tx)$ rescaled deforms an embedding to its linear part (Exercise 10.3). On components this is Hirsch's theorem that two embeddings of D^k in a connected M are isotopic, provided, when $k = n$ and M is orientable, they induce the same orientation: the bundle of k-frames is connected for $k < n$, and $\mathrm{Fr}(TM)$ has one or two components according as M is non-orientable or orientable. With Theorem 6.5 it makes the connected sum of connected oriented manifolds well defined.

For $M = \mathbb{R}^n$ the theorem identifies the spaces of embeddings of several discs in one:

$$\mathrm{Emb}\big(\textstyle\coprod_j \mathbb{R}^n, \mathbb{R}^n\big) \;\simeq\; \mathrm{Conf}_j(\mathbb{R}^n) \times \mathrm{O}(n)^j.$$

These spaces, with composition of embeddings, form the framed little n-discs operad; restricting to embeddings that are rectilinear, or equivalently equipped with a trivialization of the derivative, gives the operad E_n. Write Disc_n for the symmetric monoidal ∞-category of finite disjoint unions of copies of $\mathbb{R}^n$ and their embeddings. Every n-manifold M defines a presheaf $E_M = \mathrm{Emb}(-, M)$ on it, and Theorem 10.5 says that this presheaf is assembled from the configuration spaces of M and its tangent bundle.

10.4 Diffeomorphisms of spheres

Theorem 10.6. [classical; Hirsch 8.3.3; Smale; Hatcher; Cerf; Kervaire–Milnor] *The inclusion* $\mathrm{O}(n+1) \to \mathrm{Diff}(S^n)$ *is a homotopy equivalence for* $n = 1, 2, 3$. *For* $n \geq 5$ *the group* $\pi_0 \mathrm{Diff}^+(S^n)$ *is isomorphic to the group* Θ_{n+1} *of exotic* $(n+1)$*-spheres, and is in general not trivial.*

Hirsch proves the statement on π_0 for $n = 1$: every diffeomorphism of the circle is isotopic to the identity or to a reflection. The case $n = 2$ is Smale's, $n = 3$ is Hatcher's, and for $n = 4$ the inclusion is known not to be an equivalence. The contrast with Theorem 10.5 is instructive. Embeddings of a disc into a manifold are governed by linear algebra in every dimension; diffeomorphisms of a disc fixing the boundary are not, and the difference is the subject of pseudoisotopy theory.

10.5 Immersions

For immersions the passage from differential conditions to homotopy theory is complete.

Theorem 10.7 (Smale–Hirsch). [classical; Smale [Sma59]; Hirsch [Hir59]] *Let* $\dim V < \dim M$, *or* $\dim V = \dim M$ *with* V *having no closed component. Then the differential*

$$\mathrm{Imm}(V, M) \longrightarrow \mathrm{Mono}(TV, TM)$$

to the space of fibrewise injective bundle maps is a weak homotopy equivalence.

In the diagram at the end of Chapter 3, this says that for the open relation $\mathcal{R} \subset J^1(V, M)$ of injective 1-jets the inclusion of holonomic sections into all sections induces an equivalence of shapes. Gromov's formulation explains why. The presheaf $U \mapsto \mathrm{Imm}(U, M)$ on the open subsets of V takes unions of open sets along handles to homotopy pullbacks, by a restriction theorem like Theorem 10.4 that holds for immersions without any compactness hypothesis on the complement; so it is a homotopy sheaf, determined by its values on discs; and on a disc $\mathrm{Imm}(D^m, M) \simeq \mathrm{Emb}(D^m, M)$ is the bundle of m-frames by Theorem 10.5. A homotopy sheaf with those values is the sheaf of sections of a bundle, namely $\mathrm{Mono}(TV, TM)$.

Corollary 10.8. [classical; Whitney–Graustein; Smale] *Regular homotopy classes of immersions $S^1 \to \mathbb{R}^2$ are classified by the rotation number, an integer. All immersions $S^2 \to \mathbb{R}^3$ are regularly homotopic; in particular the sphere can be turned inside out.*

Proof. Up to homotopy $\mathrm{Mono}(TS^1, T\mathbb{R}^2)$ is the free loop space of the circle of unit vectors, whose components are indexed by degree. For the sphere the relevant group is π_2 of the space of 2-frames in $\mathbb{R}^3$, which is $\pi_2(\mathrm{SO}(3)) = 0$. □

The rotation numbers of the circle, the figure eight and the limaçon $r = 1 + 2\cos\theta$ are 1, 0 and 2; the script computes them by accumulating the angle of the tangent.

10.6 Embeddings, codimension, and calculus

Embeddings do not satisfy an h-principle. Whether two embeddings are isotopic depends on the codimension, and the two extremes are classical.

Theorem 10.9. [classical; Whitney; Wu; Haefliger] *Let V be a compact m-manifold. If $n \geq 2m + 2$, any two embeddings $V \to \mathbb{R}^n$ are isotopic. If V is connected, $m \geq 2$ and $n = 2m + 1$, the same holds.*

Proof of the first statement. Join the embeddings by the straight-line homotopy and regard it as a family $F\colon V \times I \to \mathbb{R}^n$. By the multijet transversality theorem one may perturb F, fixing the ends, so that the set $\{(x, y, t) : x \neq y,\ F_t(x) = F_t(y)\}$, of expected dimension $2m + 1 - n < 0$, is empty, and so that the set where dF_t drops rank, of expected dimension $(m + 1) - (n - m + 1) < 0$, is empty. Then every F_t is an embedding. □

This is the argument of Corollary 4.8 with one more parameter. In codimension 2 nothing of the kind holds: $\pi_0 \mathrm{Emb}(S^1, \mathbb{R}^3)$ is the set of knot types. Between the extremes there is a systematic theory.

Theorem 10.10 (Manifold calculus). [literature; Weiss [Wei99]; Goodwillie–Weiss [GW99]; Goodwillie–Klein [GK15]] *For a presheaf of spaces F on the open subsets of an m-manifold V that takes isotopy equivalences to equivalences, let $T_k F(U)$ be the homotopy limit of $F(W)$ over the open $W \subset U$ diffeomorphic to a disjoint union of at most k discs. Then $T_k F$ is the universal approximation to F by a presheaf that is a homotopy sheaf for covers in which every set of at most k points lies in some member. For $F = \mathrm{Emb}(-, M)$ with $\dim M = n$: $T_1 F = \mathrm{Imm}(-, M)$, and the map $\mathrm{Emb}(V, M) \to T_k \mathrm{Emb}(V, M)$ is $(k(n-m-2)+1-m)$-connected. In particular, if $n - m \geq 3$ then $\mathrm{Emb}(V, M) \simeq \mathrm{holim}_k T_k \mathrm{Emb}(V, M)$.*

So the Smale–Hirsch theorem is the linear approximation, and in codimension at least three the higher approximations converge. The limit has a description that does not mention V's open sets.

Theorem 10.11. [literature; Boavida de Brito–Weiss [BW13]] $\mathrm{holim}_k T_k \mathrm{Emb}(V, M)$ *is the space of derived maps $E_V \to E_M$ between the presheaves on discs defined by V and M.*

In codimension at least three, then, the space of embeddings of V in M is determined by the configuration spaces of the two manifolds, with their tangential data and the maps among them: by how finitely many points sit in each. In codimension two the tower does not converge to the embedding space, but its stages are still invariants of knots, related to those of finite type.

10.7 Factorization homology

Presheaves on discs classify embeddings; copresheaves on discs give invariants of manifolds. Let A be an algebra over the operad of Theorem 10.5 in a symmetric monoidal ∞-category, that is, a symmetric monoidal functor on Disc_n. Its ***factorization homology*** over an n-manifold M is the colimit $\int_M A = \mathrm{colim}\, A(U)$ over the discs U embedded in M.

Theorem 10.12. [literature; Ayala–Francis [AF15]; Lurie] $M \mapsto \int_M A$ *is symmetric monoidal and satisfies excision: if $M = M_- \cup_{M_0 \times \mathbb{R}} M_+$ is a decomposition along a collared hypersurface, then $\int_M A \simeq \int_{M_-} A \otimes_{\int_{M_0 \times \mathbb{R}} A} \int_{M_+} A$. These properties characterize it.*

Excision is where Theorems 6.4 and 6.5 enter: a collar of M_0 is what makes $\int_{M_0 \times \mathbb{R}} A$ an algebra and the two sides modules over it. For $n = 1$ and $M = S^1$ the invariant is Hochschild homology, which for the algebra of functions on a manifold returns the differential forms of Proposition 7.11.

Notes

The fibration theorem is due independently to Palais and to Cerf, with a short proof by Lima. For the homotopy types of diffeomorphism groups see

Smale, Hatcher, and the survey literature on pseudoisotopy; the failure in dimension four is due to Watanabe. Gromov's *Partial Differential Relations* and the book of Eliashberg and Mishachev treat the h-principle. Manifold calculus is surveyed by Munson; the configuration-space description is [BW13]. Factorization homology is developed in [AF15] and in Lurie's *Higher Algebra*.

Exercises

Exercise 10.1. Find the diffeotopy of $\mathbb{R}^2$ generated by $G(x,t) = (-x_2, x_1)$, and the velocity field of the diffeotopy $F_t(x) = e^t x$. Show that $x^2\partial_x$ does not generate a diffeotopy of $\mathbb{R}$.

Exercise 10.2. Compute the rotation numbers of the unit circle, the figure eight $(\sin t, \sin t \cos t)$ and the limaçon $r = 1 + 2\cos\theta$. Which pairs are regularly homotopic?

Exercise 10.3. Show that $\mathrm{Emb}(D^n, \mathbb{R}^n) \to \mathbb{R}^n \times \mathrm{GL}_n(\mathbb{R})$, $f \mapsto (f(0), df_0)$, is a homotopy equivalence.

Exercise 10.4. Show that the isotopy of long knots described after Theorem 10.4 is smooth, and that it is not induced by a diffeotopy of $\mathbb{R}^3$ if the knot is non-trivial.

Exercise 10.5. Show from Theorem 10.9 that any two embeddings of a circle in $\mathbb{R}^4$ are isotopic, and explain where the proof fails in $\mathbb{R}^3$.

Chapter 11

Surfaces

The source ends by classifying compact surfaces, using everything before it: Morse functions to cut a surface into pieces, isotopy and gluing to recognise the pieces and put them back, the Euler characteristic and orientability to tell the results apart. It is the one dimension above 1 in which the programme of the book can be carried to the end. The higher-categorical statements of the previous three chapters can also be carried to the end in dimension two, and this chapter does so. The classification theorem turns out to be the statement about components; the statements about higher homotopy are theorems of Earle and Eells and of Madsen and Weiss.

11.1 The classification

Let Σ_g be the closed orientable surface of genus g, the sphere with g handles, and N_h the sphere with $h \geq 1$ crosscaps.

Theorem 11.1 (Classification of surfaces). [classical; Hirsch 9.3.5, 9.3.10, 9.3.11] *Two compact connected surfaces are diffeomorphic if and only if they have the same Euler characteristic, the same number of boundary components, and are both orientable or both non-orientable. Every closed connected surface is diffeomorphic to exactly one Σ_g, with $\chi = 2 - 2g$, or N_h, with $\chi = 2 - h$. Removing b open discs lowers χ by b.*

Hirsch's proof takes a Morse function, orders its critical points, and inducts on the number of saddles. The pieces between consecutive critical levels are recognised by two lemmas: a connected surface carrying a Morse function with critical points only of index 0 and 2 and boundary a level set is a disc or an annulus (Hirsch 9.2), and a connected surface with a single critical point, a saddle, is a disc with two holes if it is orientable and a Möbius band with one hole if it is not (Hirsch 9.3.1–9.3.3; Exercise 11.4). The reassembly uses Theorems 6.5, 10.5 and 10.6: gluing along circles depends only on the isotopy class of the gluing map, and a diffeomorphism of the circle is isotopic to the identity or a reflection.

11.2 The monoid of surfaces

Connected sum is well defined on connected surfaces up to diffeomorphism, by Theorem 10.5 and the fact that every surface admits a diffeomorphism reversing the orientation of a small disc up to isotopy when it is non-orientable, and that Σ_g admits an orientation-reversing diffeomorphism. Write $T = \Sigma_1$ and $P = \mathbb{RP}^2$.

Corollary 11.2. [classical; Hirsch 9.1.7] *Under connected sum, the diffeomorphism classes of closed connected surfaces form the commutative monoid with unit S^2, generators T and P, and the single relation $T\#P = P\#P\#P$.*

A surface is often presented by a word: a polygon with $2n$ sides labelled by n letters, each occurring twice, with exponents giving the direction of identification. The quotient has one face and n edges, so $\chi = V - n + 1$ with V the number of vertex classes, and it is orientable exactly when every letter occurs once with each exponent. The script computes V by a union–find pass and identifies $aba^{-1}b^{-1}$ as the torus, $abab$ as the projective plane, $abab^{-1}$ as the Klein bottle, and so on (Exercise 11.1).

11.3 The classification as a presentation

Let 2Cob be the symmetric monoidal category whose objects are finite disjoint unions of circles and whose morphisms are diffeomorphism classes of compact oriented cobordisms: the homotopy category of $\mathsf{Cob}_2^{\mathrm{or}}$ of Definition 9.8. By Theorem 8.6 every cobordism is a composite of the four elementary ones given by a single critical point: the disc as a cobordism $\varnothing \to S^1$ (index 0) and $S^1 \to \varnothing$ (index 2), and the pair of pants as a cobordism from two circles to one and from one to two (index 1).

Theorem 11.3. [classical; folklore; Abrams [Abr96]; Kock [Koc04]] 2Cob *is generated as a symmetric monoidal category by*

$$\eta \colon \varnothing \to S^1, \qquad \epsilon \colon S^1 \to \varnothing, \qquad \mu \colon S^1 \sqcup S^1 \to S^1, \qquad \delta \colon S^1 \to S^1 \sqcup S^1,$$

subject to the relations: (μ, η) is a commutative monoid, (δ, ϵ) is a cocommutative comonoid, and $(\mu \sqcup \mathrm{id})(\mathrm{id} \sqcup \delta) = \delta\mu = (\mathrm{id} \sqcup \mu)(\delta \sqcup \mathrm{id})$.

Where the classification enters. Generation is Morse theory. Using the relations, every connected composite of generators from m circles to n circles can be brought to the normal form $\delta^{n-1}(\mu\delta)^g\mu^{m-1}$, with g the number of times a δ is followed by a μ that rejoins its two outputs. Two cobordisms with the same normal form are equal in the presented category. Conversely, two

connected oriented cobordisms with the same m, n and g are diffeomorphic by Theorem 11.1, since g is the genus and $m+n$ the number of boundary components. So the functor from the presented category to $\mathsf{2Cob}$ is faithful as well as full. □

The classification theorem is thus exactly the completeness of the relations. The same structure is visible one level up: by Theorem 8.12, two Morse functions on the same cobordism are joined by a path with isolated births, deaths and crossings, and each such event is one of the relations of the theorem.

11.4 Two-dimensional field theories

A commutative ***Frobenius algebra*** over a field k is a finite-dimensional commutative algebra A with a linear form $\varepsilon\colon A \to k$ such that $(a,b) \mapsto \varepsilon(ab)$ is non-degenerate.

Theorem 11.4. [classical; Dijkgraaf; Abrams [Abr96]; [Koc04]] *The category of symmetric monoidal functors $\mathsf{2Cob} \to \mathsf{Vect}_k$ is equivalent to the category of commutative Frobenius algebras over k and their isomorphisms, by $Z \mapsto Z(S^1)$. The comultiplication is dual to the multiplication under the pairing, and for the closed surface of genus g,*

$$Z(\Sigma_g) = \varepsilon(w^g), \qquad w = \mu\delta(1) = \sum_i e_i e_i^{\#},$$

where (e_i) is a basis and $(e_i^{\#})$ the dual basis for the pairing.

Example 11.5. (a) $A = k[x]/(x^2)$ with $\varepsilon(1) = 0$, $\varepsilon(x) = 1$: the cohomology of the sphere. The dual basis of $(1, x)$ is $(x, 1)$, so $w = 2x$ and $Z(S^2) = 0$, $Z(T^2) = 2$, $Z(\Sigma_g) = 0$ for $g \geq 2$. The handle element is the Euler class.

(b) $A = \mathbb{C}[G]$ for a finite abelian group G of order n, with $\varepsilon(g) = \delta_{g,e}/n$. The dual basis of (g) is $(n\,g^{-1})$, so $w = n^2 e$ and $Z(\Sigma_g) = \varepsilon(n^{2g}e) = n^{2g-1}$. This is $\#\operatorname{Hom}(\pi_1\Sigma_g, G)/\#G$, the number of principal G-bundles on Σ_g counted with weight one over the order of the automorphism group.

In every case $Z(T^2) = \dim A$: the torus is the trace of the identity of $Z(S^1)$, as in Theorem 9.11 for $d = 1$. The script checks the Frobenius relation and these values from the structure constants.

11.5 Extended theories

A field theory in the sense above assigns nothing to a point, and circles cannot be cut. Allowing 1-manifolds with boundary and surfaces with corners gives a symmetric monoidal bicategory Bord_2.

Theorem 11.6. [literature; Schommer-Pries [SP09]] *The oriented bordism bicategory* $\mathrm{Bord}_2^{\mathrm{or}}$ *has a finite presentation by generators and relations, obtained from the classification of the generic singularities of maps from surfaces to the plane and of one-parameter families of them. Consequently, extended oriented two-dimensional field theories with values in the bicategory of algebras, bimodules and bimodule maps are classified by separable symmetric Frobenius algebras.*

The proof is the method of Theorem 11.3 with $\mathbb{R}$ replaced by $\mathbb{R}^2$: a generic map to the plane has folds and cusps where a Morse function has critical points, and the jet transversality theorem in one and two parameters supplies the relations. For framed theories the theorem confirms Theorem 9.11 in dimension two: the fully dualizable algebras are the separable ones.

11.6 Moduli

By Theorem 8.5 the stack of bundles of closed connected surfaces has shape

$$\mathcal{M}_2 \;\simeq\; \coprod_{g\geq 0} B\,\mathrm{Diff}(\Sigma_g) \;\sqcup\; \coprod_{h\geq 1} B\,\mathrm{Diff}(N_h),$$

and Theorem 11.1 is the computation of $\pi_0\mathcal{M}_2$. The next question is the homotopy type of each component.

Theorem 11.7. [classical; Smale; Earle–Eells [EE69]; Gramain] *The identity component of* $\mathrm{Diff}(\Sigma_g)$ *is contractible for* $g \geq 2$*, is homotopy equivalent to the torus for* $g = 1$*, and to* $\mathrm{SO}(3)$ *for* $g = 0$*. The identity component of* $\mathrm{Diff}(N_h)$ *is contractible for* $h \geq 3$*, a circle for the Klein bottle, and* $\mathrm{SO}(3)$ *for* $\mathbb{RP}^2$*.*

So for $g \geq 2$ the component $B\,\mathrm{Diff}^+(\Sigma_g)$ is the classifying space $B\Gamma_g$ of the mapping class group $\Gamma_g = \pi_0\,\mathrm{Diff}^+(\Sigma_g)$: a surface bundle is determined by its monodromy. It is also the shape of the moduli stack of Riemann surfaces of genus g, since the space of complex structures on a surface is contractible (a choice in the sense of Principle 6.6).

Theorem 11.8 (Madsen–Weiss). [literature; [MW07]; Harer; [GMTW09]] *The cohomology of* $B\Gamma_g$ *is independent of* g *in a range of degrees growing linearly with* g*, and the stable cohomology is that of a component of* $\Omega^\infty MTSO(2)$*. Rationally,*

$$H^*(B\Gamma_\infty;\mathbb{Q}) \cong \mathbb{Q}[\kappa_1,\kappa_2,\dots], \qquad |\kappa_i| = 2i.$$

The second statement is Theorem 9.9 for oriented surfaces together with the stability theorem; the classes κ_i are the characteristic classes of surface bundles defined by Mumford, Morita and Miller, and the theorem was Mumford's conjecture.

11.7 What the classification is the π_0 of

Table 11.1 collects the chapter. In each row the first entry is a statement of the source or a direct consequence, and the second is the statement about a space or a category of which it is the shadow on components.

classical statement	of which it is π_0 or the homotopy category
closed surfaces are Σ_g, N_h	$\mathcal{M}_2 \simeq \coprod B\,\mathrm{Diff}$; Theorem 11.7
$\mathfrak{N}_2 = \mathbb{Z}/2$, $\Omega_2 = 0$	$B\mathsf{Cob}_2 \simeq \Omega^{\infty-1}MTO(2)$; Theorem 9.9
generators and relations for 2Cob	presentation of Bord_2; Theorems 9.11 and 11.6
a diffeomorphism of S^1 is isotopic to an isometry	$\mathrm{Diff}(S^1) \simeq \mathrm{O}(2)$
connected sum is well defined	$\mathrm{Emb}(D^2, M) \simeq \mathrm{Fr}(TM)$; Theorem 10.5
stable range for $H_*(B\Gamma_g)$	$\mathbb{Z} \times B\Gamma_\infty^+ \simeq \Omega^\infty MTSO(2)$; Theorem 11.8

Table 11.1: Surfaces: classical statements and the homotopy types behind them.

Notes

The classification of surfaces is from the nineteenth century (Möbius, Jordan, Dyck); proofs by Morse theory are in Hirsch and in Gramain's notes. The equivalence between two-dimensional field theories and Frobenius algebras is folklore from the late 1980s; [Koc04] is a careful account. The unoriented analogue is due to Turaev and Turner. For mapping class groups see Farb and Margalit; for the Madsen–Weiss theorem and its descendants, Galatius and Randal-Williams.

Exercises

Exercise 11.1. Identify the closed surfaces given by the words $aba^{-1}b^{-1}$, $abab$, $abab^{-1}$, $aabb$, $abca^{-1}b^{-1}c^{-1}$ and $abcabc$.

Exercise 11.2. Deduce $T\#P \cong P\#P\#P$ from Theorem 11.1, using $\chi(M\#N) = \chi(M) + \chi(N) - 2$.

Exercise 11.3. Verify Example 11.5, and show that for a semisimple commutative Frobenius algebra $A = \bigoplus_i k\,e_i$ with $\varepsilon(e_i) = \theta_i$ one has $Z(\Sigma_g) = \sum_i \theta_i^{1-g}$.

Exercise 11.4. Let W be a compact connected surface with a Morse function $f\colon W \to [a,b]$, $\partial W = f^{-1}\{a,b\}$, having exactly one critical point, of index 1.

Show that $\chi(W) = -1$, and that W is a disc with two holes if it is orientable and a Möbius band with one hole if it is not.

Exercise 11.5. Show from the relations of Theorem 11.3 that $Z(T^2) = \dim Z(S^1)$ for every two-dimensional field theory with values in vector spaces.

Part IV

Assessment

Chapter 12

What the Language Did

The two earlier volumes of this series ended with an assessment of how much of their higher-categorical vocabulary was doing work. The same question is asked here, and the answer is different in both directions: more of the vocabulary is essential, and less of the mathematics is touched by it.

12.1 Where the proofs live

Principle 12.1 (Four inputs). [observation] *The theorems of the source cited in this book are derived there from four analytic facts and the topology of function spaces: the inverse function theorem; the existence of partitions of unity; the Morse–Sard theorem; and the existence, uniqueness and smooth dependence of solutions of ordinary differential equations. The first gives Theorem 1.12 and tubular neighbourhoods. The second gives approximation, collars, and every passage from local to global. The third gives transversality and so every genericity statement. The fourth gives the regular interval theorem, isotopy extension, and the handle description of a critical point.*

None of the four is proved, shortened, or explained by anything in this book. The categorical statements begin where the four inputs end. Theorem 4.12 characterizes derived manifolds, but that the transverse fibre products exist is the inverse function theorem, and that they are dense is Sard's theorem. Theorem 5.5 classifies concordance classes, but its proof uses partitions of unity in an essential way. Theorem 10.4 says that restriction is a fibration, and the lifting is done by integrating a vector field. What the language provides is the form of the conclusions, and the form matters for what can be done next.

12.2 The source as a book about components

Table 12.1 is the summary of the book. Read down the left column and one has Hirsch's table of contents; read down the right and one has a good part of geometric topology since. In every row the left-hand statement is what one can prove with one parameter of genericity (a path between two choices) and the right-hand statement is what one gets by carrying all parameters. The source's method, described in the Preface, already contains the idea: an invariant is independent of choices because two choices are joined by a generic path. The

theorem of the source	the statement of which it is the shadow on π_0
homotopic maps pull back isomorphic bundles; classification (4.2.4, 4.3.4)	$\Pi[M, B\,\mathrm{GL}_k] \simeq \mathrm{Map}(M, B\mathrm{O}(k))$ (Theorem 5.5)
tubular neighbourhoods and collars are unique up to isotopy (4.5.3, 4.6)	the spaces of them are contractible (Theorems 6.3 and 6.4)
Hopf's theorem (5.1.10)	$S^n \to K(\mathbb{Z}, n)$ is $(n+1)$-connected; Pontryagin–Thom in M
invariance of intersection numbers (5.2.1)	derived fibre products have cobordism classes (Theorem 4.16)
χ by vector fields equals χ by homology (6.3.8)	a trace computed before and after a monoidal functor (Theorem 7.9)
regular interval theorem (6.2.2)	families of compact manifolds are classified by $\coprod B\,\mathrm{Diff}$ (Theorem 8.5)
Thom's theorem (7.2.3)	$B\mathrm{Cob}_d \simeq \Omega^{\infty-1} MTO(d)$ (Theorem 9.9)
isotopy extension (8.1.3)	$\mathrm{Diff}_c(M) \to \mathrm{Emb}(V, M)$ is a fibre bundle (Theorem 10.4)
embeddings of discs are isotopic (8.3.1)	$\mathrm{Emb}(D^n, M) \simeq \mathrm{Fr}(TM)$ (Theorem 10.5)
classification of surfaces (9.3.11)	$\mathcal{M}_2 \simeq \coprod B\,\mathrm{Diff}$, with Theorems 11.7 and 11.8

Table 12.1: The main theorems of the source as statements about components.

higher statement is that the space of choices is contractible because spheres of choices bound generic discs.

12.3 What is higher here

In the volume on game theory the ∞-categorical structure was largely decorative, and in the volume on information theory it was concentrated in one place (the bar construction behind entropy). Here it is needed in at least four places where no 1-categorical substitute exists.

First, derived fibre products. The last row of Table 1.1 has no solution in any ordinary category of ringed spaces, and Theorem 4.12 characterizes the ∞-category that solves it.

Second, moduli. The stack $B\,\mathrm{GL}_k$ and the stack of manifold bundles are not sheaves of sets, and their shapes $B\mathrm{O}(k)$ and $\coprod B\,\mathrm{Diff}(W)$ are not discrete.

Third, composition of cobordisms, which is associative only up to the contractible choices of Chapter 6; the cobordism category is an ∞-category in

an unavoidable way, and its homotopy type is the content of Theorem 9.9.

Fourth, embedding spaces, whose description in Theorems 10.10 and 10.11 is by homotopy limits over categories of discs.

In each case the higher statement was a theorem of someone else. This book contributes arrangement: the observation that Hirsch's nine chapters are, in order, the π_0 of these theories, with the dictionary made explicit and the status of each translation marked.

12.4 What the language did not do

It proved no theorem of differential topology. It did not make the analytic inputs of Principle 12.1 easier. It did not internalize genericity: the Baire property of the strong topology has no formulation in terms of the sheaf $[M, N]$ that we know of (Remark 3.14), and the density theorems are stated here exactly as in the source.

It is blind, at the level of generality used here, to the phenomena that distinguish the smooth category from the topological one. Exotic spheres appear in Theorem 10.6 as $\pi_0 \mathrm{Diff}(S^n)$ and nowhere else; nothing in the categorical formalism predicts them. The same formalism applied to topological or piecewise-linear manifolds gives the same tables with different entries, and the differences among the entries are the hard part of the subject.

And several of the statements quoted are much harder than the theorems they generalize. One should not come away thinking that Theorem 9.9 is a reformulation of Thom's theorem. It contains Thom's theorem as its statement about components, and its proof uses new ideas.

12.5 Directions

Open Problem 12.2 (Internal genericity)**.** [programme] Find a property of a subobject $A \subset [M, N]$ of the mapping object, stated in terms of plots, that is implied by residuality in the strong topology, implies the conclusion of the parametric transversality theorem for every sufficiently rich plot, and is closed under countable intersections.

Open Problem 12.3 (Synthetic transversality)**.** [programme] In a model of synthetic differential geometry, or in differential cohesive homotopy type theory, the transverse fibre products are those preserved by passage to tangent complexes (Theorem 4.15). Give a synthetic proof of Lemma 4.4 and isolate the axiom that plays the role of Sard's theorem.

Open Problem 12.4 (Families)**.** [programme] Principle 4.17 concerns a single diagram. For a family of diagrams parametrized by a manifold B, the limits

form a derived manifold over B, and the points of B where the fibre is not a manifold form a closed set of positive codimension in good cases. Describe the stratification of B by the type of the singular fibre in terms of the tangent complex, in the generality of Theorem 4.12.

The third problem is the one the companion volume takes up in a special case. A neural network architecture is a smooth family of maps parametrized by its weights; the decision boundaries, level sets and fixed-point sets of the network are fibres of fibre products over the parameter space; and the questions asked about them in practice (when is the boundary a manifold, how does its topology change in training, what is the set of parameters fitting the data) are questions about that stratification. The theorems of this book that are used there are Theorems 1.12 and 4.5 and Lemma 4.4, the openness results of Chapter 3, Morse theory in the form of Corollary 8.7 and Theorem 8.12, and the isotopy theory of Chapter 10.

Appendix A

Categorical Toolkit

This appendix fixes the meaning of the categorical terms used in the text. It is a list of definitions with references and proves nothing.

Spaces and ∞-categories

A *space* is an ∞-groupoid: a homotopy type, presented for instance by a Kan complex or a CW complex. Spaces form an ∞-category $\mathcal{S}$. An *∞-category* has objects and, for each pair of objects, a space of morphisms, with composition associative up to coherent homotopy; quasi-categories and complete Segal spaces are equivalent models. An ordinary category is an ∞-category whose mapping spaces are discrete. Limits and colimits in an ∞-category are homotopy limits and colimits in any model; in particular a pushout in $\mathcal{S}$ is a homotopy pushout. The *classifying space* $B\mathcal{C}$ of an ∞-category is the space obtained by inverting all morphisms. A category in which composition is defined after a choice from a contractible space, such as the cobordism category, presents an ∞-category.

Sheaves on Euclidean spaces

Cart is the category of the spaces $\mathbb{R}^n$ and smooth maps, with the coverage of Definition 1.5. A sheaf of sets on it is a smooth set; a sheaf of spaces (a functor $\mathsf{Cart}^{\mathrm{op}} \to \mathcal{S}$ satisfying descent for covers) is a smooth ∞-groupoid. The ∞-topos $\mathbf{H}$ of these has a chain of adjoint functors

$$\Pi \;\dashv\; \mathrm{Disc} \;\dashv\; \Gamma \;\dashv\; \mathrm{coDisc}\colon \quad \mathbf{H} \rightleftarrows \mathcal{S},$$

where Γ is evaluation at the point, Disc is the constant sheaf, and Π is the shape [Sch13]. For $X \in \mathbf{H}$ and a manifold M, a *section of X over M* is a map $M \to X$ in $\mathbf{H}$; the mapping object $[M, X]$ has sections over U the sections of X over $U \times M$. The delooping BG of a sheaf of groups G is the sheaf of groupoids with one object and automorphisms G, sheafified; maps $M \to BG$ are principal G-bundles.

Dualizable objects and traces

Let $(\mathcal{C}, \otimes, \mathbf{1})$ be a symmetric monoidal category or ∞-category. An object X is *dualizable* if there are an object $X^\vee$ and maps $\eta\colon \mathbf{1} \to X \otimes X^\vee$, $\epsilon\colon X^\vee \otimes X \to \mathbf{1}$ such that the composites $X \to X \otimes X^\vee \otimes X \to X$ and $X^\vee \to X^\vee \otimes X \otimes X^\vee \to X^\vee$ are identities. The *trace* of $f\colon X \to X$ is $\epsilon \circ \mathrm{swap} \circ (f \otimes 1) \circ \eta \in \mathrm{End}(\mathbf{1})$, and the *Euler characteristic*, or dimension, of X is $\mathrm{tr}(\mathrm{id}_X)$ [DP80, PS14]. In vector spaces the dualizable objects are the finite-dimensional ones; in spectra they are the finite spectra. An object of a symmetric monoidal (∞, d)-category is *fully dualizable* if it is dualizable and the evaluation and coevaluation maps have adjoints, which have adjoints, and so on through dimension $d-1$ [Lur09].

Spectra and Thom spectra

The ∞-category Sp of spectra is the stabilization of pointed spaces; it is symmetric monoidal under the smash product, with unit the sphere spectrum $\mathbb{S}$, and $\pi_n\mathbb{S}$ is the n-th stable homotopy group of spheres. There are adjoint functors $\Sigma^\infty_+\colon \mathcal{S} \to \mathsf{Sp}$ and $\Omega^\infty\colon \mathsf{Sp} \to \mathcal{S}$, and $\Omega^{\infty-1}E$ means $\Omega^\infty\Sigma E$. The Thom spectrum of a stable vector bundle ξ over X has k-th space the Thom space of the rank-k part of ξ over the corresponding piece of X; equivalently it is the colimit of Theorem 9.7. MO is the Thom spectrum of the universal stable bundle over $B\mathrm{O}$, and $MTO(d)$ that of the virtual bundle $-\gamma_d$ over $B\mathrm{O}(d)$.

C^∞-rings

A C^∞-ring is a product-preserving functor $\mathsf{Cart} \to \mathsf{Set}$ (Definition 1.7); a simplicial C^∞-ring is a product-preserving functor $\mathsf{Cart} \to \mathcal{S}$ (Definition 4.11). The free C^∞-ring on n generators is $C^\infty(\mathbb{R}^n)$. A C^∞-ring is *finitely presented* if it is $C^\infty(\mathbb{R}^n)/(f_1, \ldots, f_k)$; a simplicial one is finitely presented if it is a finite colimit of free ones, and the opposite of the ∞-category of these is DMfd (Theorem 4.12). Hadamard's lemma, that a smooth function vanishing on $\{x_1 = 0\}$ is divisible by x_1, is what makes quotients of C^∞-rings by coordinate functions behave as in algebra.

Notation

Mfd, Cart, Open	manifolds; Euclidean spaces; open subsets of Euclidean spaces
SmSet, Dflg, $\mathbf{H}$	smooth sets; diffeological spaces; smooth ∞-groupoids
C^∞Ring, DMfd	C^∞-rings; derived manifolds
$[M, N]$, Π	mapping object; shape
Emb, Imm, Diff	embeddings, immersions, diffeomorphisms (as smooth sets or their shapes)
$f \pitchfork g$, $M \times^h_N P$	transverse; derived fibre product
$\mathrm{Th}(\xi)$, M^{-TM}	Thom space; Thom spectrum of the stable normal bundle
$\mathfrak{N}_n$, Ω_n, Cob_d, Bord_d	cobordism groups; cobordism categories
Σ_g, N_h	closed surfaces, orientable of genus g and non-orientable with h crosscaps

Appendix B

Concordance with Hirsch

The table lists the sections of M. W. Hirsch, *Differential Topology* (Graduate Texts in Mathematics 33), with the place in this book where their content is treated and the form it takes there. Section numbers are those of the source; "2.1" means Chapter 2, Section 1.

Source	Here	Form
1.1–1.2 Differential structures; maps and tangent bundles	Chapter 1, Chapter 2	the category Mfd and its four enlargements; T as a Weil functor
1.3 Embeddings and immersions	§§1.6–1.7	transverse fibre products; Mfd as an idempotent completion
1.4 Manifolds with boundary	§1.8	neat submanifolds; the half-line as a smooth set
2.1 The weak and strong topologies	§§3.1, 3.4	open conditions as D-open subobjects of $[M, N]$
2.2 Approximations	§3.1	density; smoothing of structures (Theorem 1.2)
2.3 Approximations on ∂-manifolds and pairs	§1.8	not treated separately
2.4 Jets and the Baire property	§§2.5, 3.5, 3.6	jets compose; genericity is not internal to the topos
2.5 Analytic approximations	—	not treated
3.1 The Morse–Sard theorem	§4.1	stated; the single analytic source of genericity
3.2 Transversality	§§4.2–4.6	fibrewise lemma; derived manifolds; cobordism classes of derived fibre products
4.1–4.3 Vector bundles; classification	Chapter 5	the stack $B\,\mathrm{GL}_k$; concordance classification; bundles as idempotents
4.4 Oriented vector bundles	§5.5	tangential structures
4.5 Tubular neighbourhoods	§6.1	contractible space of choices
4.6 Collars and tubular neighbourhoods of neat submanifolds	§6.2	gluing; why cobordism categories are ∞-categories

Source	Here	Form
4.7 Analytic differential structures	—	not treated
5.1 Degrees of maps	§§7.1–7.2	Hopf's theorem as Pontryagin–Thom in M and as a Postnikov statement
5.2 Intersection numbers and the Euler characteristic	§§7.3–7.6	classes of derived intersections; traces; the diagonal
5.3 Historical remarks	—	—
6.1 Morse functions	§8.1	generic points of $[M, \mathbb{R}]$
6.2 Differential equations and regular level surfaces	§8.2	Ehresmann; families have no local moduli
6.3 Passing critical levels and attaching cells	§§8.3–8.5	homotopy pushouts; the sublevel functor; barcodes
6.4 CW-complexes	§8.3	compact manifolds are finite spaces
7.1 Cobordism and transversality	§9.1	cobordism groups
7.2 The Thom homomorphism	§§9.2–9.5	Thom spectra as colimits; the cobordism category
8.1 Extending isotopies	§§10.1–10.2	restriction is a fibration
8.2 Gluing manifolds together	§6.2	contractible choices
8.3 Isotopies of disks	§§10.3–10.4	$\mathrm{Emb}(D^n, M) \simeq \mathrm{Fr}(TM)$; the little discs operad
9.1 Models of surfaces	§11.2	the monoid of surfaces
9.2 Characterization of the disk	§11.1	used in the classification
9.3 The classification of compact surfaces	§§11.1–11.7	presentation of the cobordism category; field theories; moduli

Topics treated here that are not in the source: Weil algebras and tangent categories (Chapter 2); the mapping object and its shape (Chapter 3); derived manifolds (Chapter 4); Atiyah duality and traces (Chapters 6 and 7); flow categories, Cerf theory and framed functions (Chapter 8); the cobordism category and the cobordism hypothesis (Chapter 9); the Smale–Hirsch theorem, manifold calculus and factorization homology (Chapter 10); field theories and moduli of surfaces (Chapter 11).

Appendix C

Computations

The script `verify-differential-topology.py` that accompanies this book uses only the Python standard library and runs in a few seconds. It performs 71 checks, each printing `PASS` or `FAIL`, and exits with status zero if and only if all pass. The checks recompute the numerical examples in the text and test finite or numerical instances of structural statements. They are evidence that the examples are stated correctly; they do not prove the theorems.

Chapter	What is checked
1	the differential of $A \mapsto A^{\mathsf{T}}A$ is onto at orthogonal matrices, $n = 2, 3, 4$ (Example 1.13); signed zero counts of $x^2 - \varepsilon$; a smooth idempotent has rank equal to trace along its fixed set (Theorem 1.15); the stereographic transition is an involution
2	the chain rule in dual numbers against finite differences, in one and two variables; the expansion in $\mathbb{R}[\varepsilon, \eta]/(\varepsilon^2, \eta^2)$ and its symmetry (Proposition 2.4); the vertical lift; associativity of composition of 2-jets in one and several variables (Proposition 2.9); the reverse chain rule and the transpose relation (Proposition 2.11)
3	Example 3.3: g_n equals the identity on $[-n+1, n-1]$, has derivative -1 somewhere, and stays outside a strong neighbourhood; a degenerate critical point of $x - \beta(x - n)$ (Exercise 3.5)
4	the dimension $\rho(m + n - \rho)$ of the tangent space to the rank-ρ matrices in four cases (Proposition 4.7); the linear algebra of Lemma 4.4 on random instances, including non-transverse ones; the family $x^2 - v$; the Koszul complex of x^2 in a polynomial model; the Euler characteristic of the tangent complex; injectivity of random projections of a circle from $\mathbb{R}^5$ to $\mathbb{R}^3$ (Corollary 4.8)
5	Example 5.7: the idempotent of the tautological line and its monodromy; the idempotent of TS^2; the count of line bundles on the torus and Klein bottle (Exercise 5.5)
6	the Jacobian $1 + t$ of the normal exponential map of the circle; the focal radius b^2/a of the ellipse (Exercise 6.3); convexity of inner products; the two linear homotopies of the shift trick and the failure of the naive one (Exercise 6.2)

Chapter	What is checked
7	the degree of z^k by preimages and by winding integral; the degree of the antipodal map; intersection numbers on the torus (Example 7.5); fixed points and Lefschetz numbers of six toral automorphisms (Example 7.10); Poincaré–Hopf indices on S^2 and S^3; Euler characteristics and mod 2 Betti numbers of triangulations of S^2, $\mathbb{RP}^2$ and the torus; the Gauss linking integral (Example 7.12)
8	the critical points and indices of Example 8.3; the Morse polynomial; the birth–death family; the degree-zero barcode of random functions on a circle (Proposition 8.10); Exercise 8.3; the deformation of Exercise 8.4
9	the ranks of $\mathfrak{N}_n$ for $n \leq 10$; the Stiefel–Whitney numbers of Example 9.5; the arithmetic of Exercise 9.1; the degree of the collapse of a framed point (Exercise 9.5)
10	a time-dependent flow on $\mathbb{R}^2$ is injective with positive Jacobian; the rotation flow; blow-up of $x^2\partial_x$; the rotation numbers of Corollary 10.8; the shrinking of Exercise 10.3
11	vertex counts, Euler characteristics and orientability for seven polygon words (Exercise 11.1); $T\#P$ and $P\#P\#P$; the Frobenius algebras of Example 11.5: handle elements, $Z(\Sigma_g)$ for $g \leq 3$, the Frobenius relation, and $Z(T^2) = \dim A$

Numerical tolerances are stated in the script. The random seed is fixed, so the output is reproducible.

Appendix D

Solutions to the Exercises

Solutions to all exercises follow, by chapter. Where a solution relies on a computation, the computation is also carried out by `verify-differential-topology.py`.

Chapter 1

1.1. In Mfd: a cone is (S, u) with $u\colon S \to \mathbb{R}$ smooth and $u^2 = 0$, so $u = 0$ and the cone is the unique map $S \to \mathrm{pt}$. The same argument applies to every $S = \mathbb{R}^n$, so the sheaf-theoretic fibre product has exactly one plot of each dimension and is the terminal smooth set. In $C^\infty\mathrm{Ring}$ the pushout of $C^\infty(\mathbb{R}) \xleftarrow{x \mapsto x^2} C^\infty(\mathbb{R}) \xrightarrow{\mathrm{ev}_0} \mathbb{R}$ is $C^\infty(\mathbb{R})/(x^2)$, and Hadamard's lemma $g = g(0) + g'(0)x + x^2 h$ shows that $g \mapsto g(0) + g'(0)\varepsilon$ is an isomorphism onto $\mathbb{R}[\varepsilon]/(\varepsilon^2)$. A C^∞-ring map $\phi\colon C^\infty(M) \to \mathbb{R}[\varepsilon]$ has the form $\phi(g) = \phi_0(g) + \phi_1(g)\varepsilon$. The component ϕ_0 is an $\mathbb{R}$-algebra map to $\mathbb{R}$, hence evaluation at a point p (every character of $C^\infty(M)$ is a point evaluation, M being second countable). Multiplicativity of ϕ gives $\phi_1(gh) = g(p)\phi_1(h) + h(p)\phi_1(g)$, so ϕ_1 is a derivation at p, a tangent vector. Conversely a tangent vector v at p gives $g \mapsto g(p) + v(g)\varepsilon$, which respects all smooth operations by the chain rule.

1.2. The proof in the text shows that each $x \in \mathrm{Fix}(e)$ has a neighbourhood U on which e has constant rank $k = \operatorname{tr} de_x$. By the constant rank theorem there are charts near x and $e(x) = x$ in which e is $(u, v) \mapsto (u, 0)$ with $u \in \mathbb{R}^k$; shrinking U, the set $e(U)$ is a k-dimensional embedded submanifold near x, and $\mathrm{Fix}(e) \cap U \subset e(U)$. Conversely $e(U) \cap U' \subset \mathrm{Fix}(e)$ for a smaller neighbourhood U', because $e(e(y)) = e(y)$. So $\mathrm{Fix}(e)$ is an embedded submanifold near x; it is closed as the equalizer of e and id in a Hausdorff space. Smoothness of e into $\mathrm{Fix}(e)$ follows since the submanifold is embedded. For the example take $M = \mathbb{R} \sqcup \mathbb{R}^2$ and e the map which is the identity on $\mathbb{R}$ and $(x, y) \mapsto (0, 0)$ on $\mathbb{R}^2$; then $\mathrm{Fix}(e) = \mathbb{R} \sqcup \{0\}$. On a connected M the set $\mathrm{Fix}(e) = e(M)$ is connected, so the dimension is constant.

1.3. $f(A) = A^{\mathsf{T}}A$ is polynomial, hence smooth, with values in the vector space Sym_n of dimension $\frac{1}{2}n(n+1)$. Its differential at A is $df_A(H) = A^{\mathsf{T}}H + H^{\mathsf{T}}A$. If $A^{\mathsf{T}}A = I$ and S is symmetric, then $H = \frac{1}{2}AS$ gives $\frac{1}{2}A^{\mathsf{T}}AS + \frac{1}{2}S^{\mathsf{T}}A^{\mathsf{T}}A = S$.

So I is a regular value and $\dim \mathrm{O}(n) = n^2 - \tfrac{1}{2}n(n+1) = \tfrac{1}{2}n(n-1)$. The tangent space at I is $\ker df_I = \{H : H + H^{\mathsf{T}} = 0\}$.

1.4. A plot of $[M, N]$ of dimension n is by definition a smooth map $F\colon \mathbb{R}^n \times M \to N$, and the function it induces on points is $s \mapsto F(s, -)$, which determines F; so $[M, N]$ is concrete and its points are $\mathrm{Hom}(\mathrm{pt} \times M, N) = C^\infty(M, N)$. For the adjunction, when $S = \mathbb{R}^n$ the bijection is the definition. A general smooth set is a colimit of representables, $S = \mathrm{colim}_i \mathbb{R}^{n_i}$; the functor $- \times M$ preserves colimits because SmSet is a topos, so $\mathrm{Hom}(S \times M, N) = \lim_i \mathrm{Hom}(\mathbb{R}^{n_i} \times M, N) = \lim_i [M, N](\mathbb{R}^{n_i}) = \mathrm{Hom}(S, [M, N])$.

1.5. Let $f(x) = g(y) = z$. The tangent space to Δ_N at (z, z) is $\{(w, w)\}$. Transversality of $f \times g$ to Δ_N at (x, y) says that every $(a, b) \in T_zN \oplus T_zN$ is $(df_x u + w,\, dg_y v + w)$ for some u, v, w. Subtracting, this holds if and only if every $a - b$ is of the form $df_x u - dg_y v$, that is, if and only if $df_x(T_xM) + dg_y(T_yP) = T_zN$: given such a decomposition of $a - b$, set $w = a - df_x u$.

Chapter 2

2.1. For open subsets of Euclidean spaces, $T(U \times V) = U \times V \times \mathbb{R}^m \times \mathbb{R}^n \cong TU \times TV$ by reordering coordinates, and the reordering commutes with the maps $T(f \times g)$ and $Tf \times Tg$ since $D(f \times g) = Df \oplus Dg$. The isomorphism is therefore natural and glues over atlases. For the second statement, $Z = f^{-1}(z)$ is a submanifold with $T_xZ = \ker df_x$ by Theorem 1.12. A vector $(x, v) \in TM$ satisfies $Tf(x, v) = 0_z$ exactly when $f(x) = z$ and $df_x v = 0$, that is, when $(x, v) \in TZ$. One checks that 0_z is a regular value of Tf: in coordinates $D(Tf)_{(x,v)}(\xi, \nu) = (Df_x\xi,\, D^2f_x(v, \xi) + Df_x\nu)$, which is onto when Df_x is. So the equality is one of submanifolds.

2.2. An A-point of $\mathbb{R}^n$ is $x + v\,t + \tfrac{1}{2}a\,t^2$ with $t^3 = 0$, the 2-jet at 0 of the curve $t \mapsto x + vt + \tfrac{1}{2}at^2$; a map f acts by Taylor expansion, which is the composition of 2-jets, so T_AM is the bundle of 2-jets of curves and its fibre has dimension $2n$. The map $x \mapsto \varepsilon + \eta$ is a homomorphism since $(\varepsilon + \eta)^3 = 3\varepsilon^2\eta + 3\varepsilon\eta^2 = 0$, and it is injective: $1,\ \varepsilon + \eta,\ (\varepsilon + \eta)^2 = 2\varepsilon\eta$ are independent. On points it sends $x + vt + \tfrac{1}{2}at^2$ to $x + v\varepsilon + v\eta + a\varepsilon\eta$, that is, to (x, v, v, a) in the coordinates of Table 2.1. The fixed set of the flip is $\{v = w\}$, and $T(p)$ and p_T are the two projections $(x, v, w, a) \mapsto (x, v)$ and (x, w), so the condition $T(p) = p_T$ is again $v = w$. The image is exactly $\{(x, v, v, a)\}$, of fibre dimension $2n$, and the map is an embedding.

2.3. Write $f(x) = Ax + \tfrac{1}{2}B(x, x) + \dots$ and $g(y) = Cy + \tfrac{1}{2}E(y, y) + \dots$ with B, E symmetric bilinear. Then $g(f(x)) = CAx + \tfrac{1}{2}\big(C\,B(x, x) + E(Ax, Ax)\big) +$

$O(|x|^3)$, so

$$(C,E)\circ(A,B) = \big(CA,\ C\circ B + E\circ(A\times A)\big).$$

For a third jet (F,G), $(F,G)\circ\big((C,E)\circ(A,B)\big) = \big(FCA,\ FCB + FE(A\times A) + G(CA\times CA)\big)$ and $\big((F,G)\circ(C,E)\big)\circ(A,B) = \big(FCA,\ FCB + (FE + G(C\times C))(A\times A)\big)$, which agree.

2.4. A point of $TT\mathbb{R}^n = T_{W\otimes W}\mathbb{R}^n$ is $x + v\varepsilon + w\eta + a\varepsilon\eta$. A homomorphism $\phi\colon A \to B$ acts on A-points by applying ϕ to the coefficients. Addition: a point of $T_{W\times_{\mathbb{R}}W}$ is $x + v_1\varepsilon_1 + v_2\varepsilon_2$, and $\varepsilon_i \mapsto \varepsilon$ sends it to $x + (v_1+v_2)\varepsilon$. Vertical lift: $x + v\varepsilon \mapsto x + v\varepsilon\eta$, which is $(x,0,0,v)$. Flip: exchanging ε and η sends (x,v,w,a) to (x,w,v,a). Each map is natural because it is induced by a homomorphism (Theorem 2.3(iii)); directly, naturality of ℓ is the identity $f(x+v\varepsilon\eta) = f(x) + Df_x(v)\varepsilon\eta$, since $(\varepsilon\eta)^2 = 0$.

2.5. Let $(f,f^\sharp)$, $(g,g^\sharp)$, $(h,h^\sharp)$ be composable. Both bracketings have forward part hgf and backward part $(x,u') \mapsto f^\sharp(x, g^\sharp(f(x), h^\sharp(gf(x),u')))$, as one sees by unfolding the definition twice; so composition is associative. For $F = f_k\cdots f_1$ with scalar output, Proposition 2.11 gives $R[F](x,1) = R[f_1](x_0, R[f_2](x_1,\dots R[f_k](x_{k-1},1)))$ with $x_0 = x$ and $x_i = f_i(x_{i-1})$. The forward pass computes and stores $x_0,\dots,x_{k-1}$; the backward pass applies the transposed Jacobians $(Df_i)^{\mathsf{T}}_{x_{i-1}}$ to a single covector, for $i = k,\dots,1$. The result is the gradient of F at x.

Chapter 3

3.1. The support of $\beta(\cdot - n)$ is $[n-1, n+1]$, so $g_n = \mathrm{id}$ on $[-n+1, n-1]$ and every weak neighbourhood of id, which constrains maps on a compact set only, contains g_n for large n. Since β' attains its maximum 1 at some x^*, $g_n'(n + x^*) = -1 < 0$ while $g_n' = 1$ outside the support, so g_n is not monotone. For the strong topology, $|g_n(n) - n| = 2\beta(0) > \beta(0)e^{-n^2}$, so no g_n lies in the stated strong neighbourhood of the identity. Not closed: $h_n(x) = x/n$ are diffeomorphisms converging weakly to the constant map 0.

3.2. Fix t_0. We claim there are $\delta > 0$ and a compact K with $f_t = f_{t_0}$ off K for $|t - t_0| < \delta$. If not, for each n there are t_n with $|t_n - t_0| < 1/n$ and $x_n \notin [-n,n]$ with $f_{t_n}(x_n) \neq f_{t_0}(x_n)$; passing to a subsequence, the x_n are distinct and tend to infinity. Choose a continuous $\varepsilon\colon \mathbb{R} \to (0,\infty)$ with $\varepsilon(x_n) < |f_{t_n}(x_n) - f_{t_0}(x_n)|$ for all n. The strong neighbourhood $\{g : |g - f_{t_0}| < \varepsilon\}$ contains no f_{t_n}, contradicting continuity at t_0. Cover $[0,1]$ by finitely many such intervals and take the union of the compact sets. Conversely, if $g = f$ off a compact K, the straight-line path $f + t(g-f)$ is strongly continuous: it is constant off K, and on K uniform convergence suffices. So the path component of f is as stated.

3.3. Let $F\colon \mathbb{R}^n \times M \to N$ be smooth and $s_0 \in \mathbb{R}^n$. For a compact $K \subset M$ in a chart, F and its first derivative in the M-direction are uniformly continuous on $\bar{B}(s_0, 1) \times K$, so $\sup_{x\in K}\big(|F_s(x) - F_{s_0}(x)| + \|D_xF_s(x) - D_xF_{s_0}(x)\|\big) \to 0$ as $s \to s_0$; this is convergence in C^1_W. For the example let $F_s(x) = (\gamma_s(x), 0)$ with $\gamma_s(x) = x - 2\beta(x - 1/s)$ for $s > 0$ and $\gamma_s(x) = x$ for $s \le 0$, with β as in Example 3.3. This is smooth in (s, x) because for $|x| < R$ and $0 < s < 1/(R+1)$ the bump is not seen. F_0 is an embedding and F_s is not injective for small $s > 0$ (Example 3.3), so the set of parameters giving embeddings is not open at $s = 0$.

3.4. Reflexivity: the constant plot $\mathbb{R} \to \mathrm{pt} \to X$. Symmetry: compose with $t \mapsto 1 - t$. Transitivity: let $p, q\colon \mathbb{R} \to X$ be plots with $p(1) = q(0)$. Choose a smooth non-decreasing $\lambda\colon \mathbb{R} \to \mathbb{R}$ with $\lambda(t) = 0$ for $t \le \frac{1}{3}$ and $\lambda(t) = 1$ for $t \ge \frac{2}{3}$. Then $p\lambda$ is a plot, constant equal to $p(1)$ on $(\frac{2}{3}, \infty)$, and $q\lambda(\cdot - 1)$ is a plot, constant equal to $q(0)$ on $(-\infty, \frac{4}{3})$. On the overlap $(\frac{2}{3}, \frac{4}{3})$ of the open sets $(-\infty, \frac{4}{3})$ and $(\frac{2}{3}, \infty)$, which cover $\mathbb{R}$, the two plots restrict to the same constant plot. By the sheaf condition (after pulling back along diffeomorphisms of the intervals with $\mathbb{R}$) they glue to a plot $r\colon \mathbb{R} \to X$ with $r(0) = p(0)$ and $r(2) = q(1)$; reparametrize by $t \mapsto 2t$.

3.5. Dense: given f and a weak neighbourhood, which constrains f on some $[-R, R]$, the functions $f_a(x) = f(x) + ax$ are Morse for almost every a, because f_a is Morse exactly when $-a$ is a regular value of f', and Sard's theorem applies to f'; and $f_a \to f$ on $[-R, R]$ in every C^r norm as $a \to 0$. Not open: $f(x) = x$ is Morse. With β as in Example 3.3, let $g_n(x) = x - \beta(x - n)$. At the point x^* where β' attains its maximum 1 we have $\beta''(x^*) = 0$, so $g_n'(n + x^*) = 0$ and $g_n''(n + x^*) = 0$: a degenerate critical point. And $g_n \to f$ weakly.

Chapter 4

4.1. Let $F\colon \mathbb{R}^n \times M \to \mathbb{R}^n$, $F(v, x) = f(x) + v$. Its differential in the v-direction is the identity, so F is a submersion and in particular $F \pitchfork A$. By Theorem 4.5, $F_v = f + v$ is transverse to A for almost every v. If M and A are curves in $\mathbb{R}^3$ then $\dim M + \dim A < 3$, and a map of a 1-manifold transverse to a 1-dimensional submanifold of $\mathbb{R}^3$ cannot meet it: at a common point the images of the two tangent spaces would have to span $\mathbb{R}^3$.

4.2. Let X_0 have rank ρ. After multiplying on both sides by invertible matrices, which are diffeomorphisms of $\mathbb{R}^{m\times n}$ preserving rank, we may take $X_0 = \left(\begin{smallmatrix} I_\rho & 0 \\ 0 & 0 \end{smallmatrix}\right)$. Write a nearby matrix in blocks $X = \left(\begin{smallmatrix} P & Q \\ R & S \end{smallmatrix}\right)$ with P invertible. Row and column operations give $\operatorname{rk} X = \rho + \operatorname{rk}(S - RP^{-1}Q)$. So near X_0 the set of rank-ρ matrices is the zero set of $\Phi(X) = S - RP^{-1}Q$, a map to $\mathbb{R}^{(m-\rho)\times(n-\rho)}$ whose

differential in the S-direction is the identity. It is a submersion, and the zero set is a submanifold of codimension $(m-\rho)(n-\rho)$.

4.3. Resolve $\mathbb{R}$ as a $C^\infty(\mathbb{R})$-module by the Koszul complex $C^\infty(\mathbb{R}) \xrightarrow{x} C^\infty(\mathbb{R})$; it is exact on the left because x is not a zero divisor, and the cokernel is $\mathbb{R}$ by Hadamard's lemma. This resolution is also a cofibrant model as a simplicial C^∞-ring (it is free on one generator in degree 1 over $C^\infty(\mathbb{R})$). Tensoring with $\mathbb{R}$ along evaluation at 0 gives $\mathbb{R} \xrightarrow{0} \mathbb{R}$: the homotopy groups are $\mathbb{R}$ in degrees 0 and 1, as in Example 4.14(b). For the second, resolve $\mathbb{R}$ in the same way and tensor with $C^\infty(\mathbb{R})$ along $x \mapsto x^2$: the complex is $C^\infty(\mathbb{R}) \xrightarrow{x^2} C^\infty(\mathbb{R})$. If $x^2 g = 0$ then $g = 0$ away from 0 and so everywhere; the map is injective, $\pi_1 = 0$, and $\pi_0 = C^\infty(\mathbb{R})/(x^2) = \mathbb{R}[\varepsilon]/(\varepsilon^2)$, as in Example 4.14(a).

4.4. The critical points of f_a are the solutions of $\nabla f(x) = -a$, and the Hessian of f_a is that of f, which is the Jacobian of the map $\nabla f\colon U \to \mathbb{R}^m$. So f_a is Morse if and only if $-a$ is a regular value of ∇f. By Theorem 4.1, applied to the smooth map ∇f between manifolds of the same dimension, almost every a qualifies.

4.5. Suppose $g\colon D^n \to D^n$ is continuous without fixed points. Then $|g(x) - x| \geq 2\delta > 0$ by compactness. By Theorem 3.4 there is a smooth $h\colon D^n \to \mathbb{R}^n$ with $|h - g| < \delta$; then $h_1 = h/(1+\delta)$ maps D^n to itself, is smooth, and has $|h_1 - g| < 2\delta$, so h_1 has no fixed point. Let $\rho(x)$ be the point where the ray from $h_1(x)$ through x leaves D^n: $\rho(x) = x + t(x)\,(x - h_1(x))$ with $t(x) \geq 0$ the larger root of $|x + t(x - h_1(x))|^2 = 1$. The discriminant of this quadratic is positive because $x \neq h_1(x)$ and $|x| \leq 1$, so t is smooth, and $t = 0$ when $|x| = 1$. Thus ρ is a smooth retraction of D^n onto S^{n-1}, contradicting Corollary 4.2.

Chapter 5

5.1. The fibre over the line ℓ_θ spanned by $u(\theta) = (\cos\theta, \sin\theta)$ is ℓ_θ itself. A nowhere-zero section would be $s(\theta) = \lambda(\theta)u(\theta)$ with λ continuous and non-zero on $[0,\pi]$, and since $u(\pi) = -u(0)$ while $s(\pi) = s(0)$ we need $\lambda(\pi) = -\lambda(0)$, so λ changes sign; contradiction. The total space is $\{(\theta, t)\} = [0,\pi] \times \mathbb{R}$ with $(0,t)$ glued to $(\pi,-t)$, the open Möbius band.

5.2. Since e has constant rank, near any point one can choose k columns of e that are independent; they are smooth sections of the image spanning it, so the image ξ is a subbundle of $M \times \mathbb{R}^s$, with complement the image of $1 - e$. A section of $M \times \mathbb{R}^s$ is an element $v \in C^\infty(M)^s$; it lies in ξ if and only if $ev = v$, so $\Gamma(\xi) = e\,C^\infty(M)^s$, and $C^\infty(M)^s = e\,C^\infty(M)^s \oplus (1-e)\,C^\infty(M)^s$.

5.3. $TS^n = \{(x,v) : |x| = 1,\ \langle x, v\rangle = 0\}$ and the normal bundle of $S^n \subset \mathbb{R}^{n+1}$ is trivialized by the section $x \mapsto x$. So $TS^n \oplus \mathbb{R} \cong TS^n \oplus \nu = T\mathbb{R}^{n+1}|_{S^n}$, which is trivial. The map $B\mathrm{O}(n) \to B\mathrm{O}(n+1)$ classifies $\xi \mapsto \xi \oplus \mathbb{R}$, so the composite $S^n \to B\mathrm{O}(n) \to B\mathrm{O}(n+1)$ classifies a trivial bundle and is null-homotopic by Theorem 5.4; hence so is the composite to $B\mathrm{O}$.

5.4. If $\omega_1 - \omega_0 = dh$, let $\lambda\colon \mathbb{R} \to [0,1]$ be smooth with $\lambda(0) = 0$, $\lambda(1) = 1$, and put $\Omega = \omega_0 + d(\lambda(t)h)$ on $M \times \mathbb{R}$. It is closed and restricts to ω_0 and ω_1. Conversely, let $\Omega = \alpha_t + g_t\,dt$ be closed on $M \times \mathbb{R}$, with α_t a path of 1-forms on M and g_t of functions. The dt-component of $d\Omega = 0$ reads $\partial_t \alpha_t = d_M g_t$, so $\alpha_1 - \alpha_0 = d\int_0^1 g_t\,dt$.

5.5. A line bundle has transition functions in $\mathrm{GL}_1(\mathbb{R})$, homotopy equivalent to $\mathbb{Z}/2$, so $B\mathrm{O}(1) = \mathbb{RP}^\infty = K(\mathbb{Z}/2,1)$ and $[M, K(\mathbb{Z}/2,1)] = H^1(M;\mathbb{Z}/2) = \mathrm{Hom}(\pi_1 M, \mathbb{Z}/2)$ for connected M. For the torus $\pi_1 = \mathbb{Z}^2$ and there are 4 homomorphisms. For the Klein bottle $\pi_1 = \langle a, b \mid abab^{-1}\rangle$ has abelianization $\mathbb{Z} \oplus \mathbb{Z}/2$, and again $\mathrm{Hom}(-, \mathbb{Z}/2) = (\mathbb{Z}/2)^2$ has 4 elements. The determinant of the tangent bundle is trivial for the torus. For the Klein bottle, presented as the square with a the pair of sides glued by a reflection and b the pair glued by a translation, the orientation reverses along the loop that crosses the reflected sides, so $\det T$ is the line bundle with w_1 non-zero on that loop and zero on the other.

Chapter 6

6.1. If g_0, g_1 are positive definite symmetric bilinear forms then so is $(1-t)g_0 + tg_1$, since $(1-t)g_0(v,v) + tg_1(v,v) > 0$ for $v \neq 0$. Every $A \in \mathrm{GL}_k(\mathbb{R})$ has a unique polar decomposition $A = QP$ with Q orthogonal and $P = (A^{\mathsf{T}}A)^{1/2}$ positive definite symmetric, depending continuously on A. The path $A_t = Q\big((1-t)P + tI\big)$ stays in GL_k by convexity, fixes $\mathrm{O}(k)$, and ends in $\mathrm{O}(k)$; so $\mathrm{O}(k)$ is a deformation retract. A homotopy equivalence of topological groups that is a homomorphism induces an equivalence of classifying spaces.

6.2. Let $e_0, e_1\colon M \to \mathbb{R}^q \subset \mathbb{R}^\infty$ be embeddings. Let $S, S'\colon \mathbb{R}^\infty \to \mathbb{R}^\infty$ be the linear injections placing a vector in the odd, respectively even, coordinates. The linear map $(1-t)\mathrm{id} + tS$ is injective for $t \in [0,1]$: an element of its kernel would be an eigenvector of S with eigenvalue $-(1-t)/t \leq 0$, and S has none (if $Sx = \lambda x$ with $\lambda \neq 0, 1$ then $x_2 = 0$, $x_3 = x_2/\lambda = 0$, and so on, and $x_1 = \lambda x_1$ forces $x_1 = 0$; and S is injective). An injective linear map composed with an embedding of a compact manifold is an embedding, so $(1-t)e_1 + tSe_1$ is an isotopy from e_1 to Se_1; similarly from e_0 to $S'e_0$. Finally $(1-t)Se_1 + tS'e_0$ is an embedding for every t: for $t < 1$ the odd coordinates are $(1-t)e_1$, an injective

immersion, and for $t > 0$ the even coordinates are te_0. For non-compact M one uses closed embeddings and the same formulas. The naive homotopy can fail: for $e_1 = -e_0$ the path $(1-2t)e_0$ is constant at $t = \frac{1}{2}$.

6.3. Parametrize by arc length, with $\gamma' = T$ and $n' = -\kappa T$ (Frenet, with n the inward normal and $\kappa > 0$ for a convex curve traversed counterclockwise). Then $\partial_s = (1 - t\kappa)T$ and $\partial_t = n$, so the determinant is $1 - t\kappa(s)$ up to the sign of the frame. The map is a local diffeomorphism where $t\kappa(s) \neq 1$, hence for $|t| < 1/\kappa_{\max}$. For the ellipse $(a\cos u, b\sin u)$ the curvature is $ab/(a^2\sin^2 u + b^2\cos^2 u)^{3/2}$, with maximum a/b^2 at the ends of the major axis. So $r = b^2/a$. (Global injectivity may fail earlier in general; for the ellipse it does not on the inward side before $t = b^2/a$, and it never fails on the outward side.)

6.4. The one-point compactification of a product of locally compact spaces is the smash product of the compactifications, which gives the first identity; the second and third are the cases $\eta = \mathbb{R}^j$ over a point, where $\mathrm{Th} = S^j$, and $\xi = 0$, where $\mathrm{Th} = M_+$. For the last, $\mathbb{RP}^{n+1} \setminus \{[0 : \cdots : 0 : 1]\}$ is the total space of the tautological bundle over $\mathbb{RP}^n$ by $[x : t] \mapsto ([x],\ t\,x/|x|^2)$, where $t\,x/|x|^2$ is a vector in the line $[x]$ that is unchanged by $(x, t) \mapsto (\lambda x, \lambda t)$; this is a diffeomorphism, and the remaining point is the point at infinity.

6.5. Extend f radially: $F(rx) = rf(x)$ for $x \in S^{n-1}$, $0 \leq r \leq 1$. This is a homeomorphism of D^n (the Alexander trick), smooth away from the origin. The map $\Sigma_f = D^n \cup_f D^n \to D^n \cup_{\mathrm{id}} D^n = S^n$ that is the identity on the first disc and F on the second is well defined and bijective, continuous, and hence a homeomorphism. If f extends to a diffeomorphism G of D^n, the same construction with G in place of F is smooth on each disc and compatible with the gluing; it is a diffeomorphism for the smooth structures of Theorem 6.5 after adjusting G near the boundary to respect collars, which is possible by uniqueness of collars (Theorem 6.4).

Chapter 7

7.1. For $k \neq 0$ every point of S^1 is a regular value of $z \mapsto z^k$, with $|k|$ preimages, at each of which the derivative $\theta \mapsto k\theta$ has the sign of k; so the degree is k. (For $k = 0$ the map is constant.) The antipodal map is the restriction of -1 on $\mathbb{R}^{n+1}$, a composite of $n+1$ reflections in coordinate hyperplanes, each of which restricts to a diffeomorphism of S^n of degree -1 (one preimage, orientation reversed). So its degree is $(-1)^{n+1}$. If v is a nowhere-zero vector field on S^n, normalized to unit length, then $H(x, t) = \cos(\pi t)\,x + \sin(\pi t)\,v(x)$ is a homotopy on S^n from the identity to the antipodal map, so $1 = (-1)^{n+1}$ and n is odd.

7.2. A point $[x]$ is fixed if and only if $(A-1)x \in \mathbb{Z}^2$. The map $A-1$ is an injective endomorphism of $\mathbb{Z}^2$ with image of index $|\det(A-1)|$, so the solutions $x \in (A-1)^{-1}\mathbb{Z}^2$ form a lattice containing $\mathbb{Z}^2$ with index $|\det(A-1)|$; modulo $\mathbb{Z}^2$ there are that many. The differential of the torus map at any point is A, so each fixed point is non-degenerate with index $\operatorname{sign}\det(1-A)$. The sum is $\det(1-A) = 1 - \operatorname{tr} A + \det A$, which is $L(A)$ since A acts on $H_1 = \mathbb{Z}^2$ by A and on $H_2 = \Lambda^2 H_1$ by $\det A$.

7.3. Cyclicity: by the triangle identities, the dual map $g^\vee \colon Y^\vee \to X^\vee$ of $g \colon X \to Y$ satisfies $(g \otimes 1)\eta_X = (1 \otimes g^\vee)\eta_Y$ and $\epsilon_Y(1 \otimes g) = \epsilon_X(g^\vee \otimes 1)$. For $f \colon Y \to X$ one slides g around the loop: $\operatorname{tr}(gf) = \epsilon_Y \sigma(gf \otimes 1)\eta_Y = \epsilon_Y \sigma(g \otimes 1)(f \otimes 1)\eta_Y$, and moving g through ϵ as $g^\vee$ and then through η back as g on the other side yields $\epsilon_X \sigma(fg \otimes 1)\eta_X = \operatorname{tr}(fg)$. Multiplicativity: $X \otimes Y$ is dualizable with dual $Y^\vee \otimes X^\vee$ and structure maps built from those of X and Y; the trace diagram for $f \otimes g$ is the tensor product of the two trace diagrams after rearranging by symmetries, and $\operatorname{End}(\mathbf{1})$ is commutative. For a graded vector space $V = \bigoplus V_i$ the coevaluation is $1 \mapsto \sum e_j \otimes e^j$ over a homogeneous basis, and the symmetry sends $e_j \otimes e^j$ to $(-1)^{|e_j|\,|e^j|} e^j \otimes e_j = (-1)^{|e_j|} e^j \otimes e_j$, since $|e^j| = -|e_j|$. Evaluating gives $\operatorname{tr}(f) = \sum_j (-1)^{|e_j|} \langle e^j, f e_j \rangle = \sum_i (-1)^i \operatorname{tr}(f|_{V_i})$.

7.4. On $S^n \subset \mathbb{R}^{n+1}$ let $v(x) = e_{n+1} - \langle x, e_{n+1} \rangle x$, the projection of the constant field e_{n+1}. It vanishes at the poles $\pm e_{n+1}$. Near the south pole, in the chart $y \mapsto (y, -\sqrt{1-|y|^2})$, v is $y \mapsto y$ to first order, of index $+1$. Near the north pole it is $y \mapsto -y$, of index $(-1)^n$. So $\chi(S^n) = 1 + (-1)^n$. Homologically $H_i(S^n)$ is $\mathbb{Q}$ for $i = 0, n$ and zero otherwise, giving the same.

7.5. The disc $D = \{x^2 + y^2 \le 1,\ z = 0\}$ is bounded by K. The circle L is $\{(1 + \cos t, 0, \sin t)\}$; it meets the plane $z = 0$ at $t = 0$, the point $(2, 0, 0)$ outside D, and at $t = \pi$, the point $(0, 0, 0)$ inside D. So L meets D once, transversally (its tangent there is $(0, 0, -1)$), and the linking number is ± 1, the sign depending on the orientations: with D oriented by the normal e_3 and L by increasing t, it is -1.

Chapter 8

8.1. $f_\theta = -(R + r\cos\varphi)\sin\theta$ and $f_\varphi = -r\sin\varphi\cos\theta$. Since $R + r\cos\varphi > 0$, $f_\theta = 0$ forces $\sin\theta = 0$; then $\cos\theta = \pm 1$ and $f_\varphi = 0$ forces $\sin\varphi = 0$. The second derivatives are $f_{\theta\theta} = -(R + r\cos\varphi)\cos\theta$, $f_{\varphi\varphi} = -r\cos\varphi\cos\theta$, $f_{\theta\varphi} = r\sin\varphi\sin\theta = 0$ at the critical points. At $(0,0)$: $(-(R+r), -r)$, index 2. At $(0, \pi)$: $(-(R-r), +r)$, index 1. At (π, π): $(R - r, -r)$, index 1. At $(\pi, 0)$: $(R + r, r)$, index 0. All are non-degenerate because $R > r > 0$.

8.2. Write $M(t) = \sum \nu_k t^k$, $P(t) = \sum \beta_k t^k$, so $M - P = (1+t)Q$. Dividing by $1+t$ as a formal power series, $Q(t) = (M(t) - P(t)) \sum_j (-1)^j t^j$, whose coefficient of t^k is $\sum_{i \le k} (-1)^{k-i}(\nu_i - \beta_i)$. Non-negativity of the coefficients of Q is exactly the stated inequality. For a closed connected orientable surface of genus g, with $F = \mathbb{Q}$, $\beta = (1, 2g, 1)$, so $\sum \nu_k \ge \sum \beta_k = 2g + 2$. For a non-orientable surface N_h use $F = \mathbb{Z}/2$: $\beta = (1, h, 1)$ and the bound is $h + 2$.

8.3. In the chart $x_i = 1$ with coordinates $(x_j)_{j \ne i}$, $f = \left(\lambda_i + \sum_{j \ne i} \lambda_j x_j^2\right) / \left(1 + \sum_{j \ne i} x_j^2\right) = \lambda_i + \sum_{j \ne i} (\lambda_j - \lambda_i) x_j^2 + O(|x|^4)$. So the origin of each chart is a non-degenerate critical point of index $\#\{j : \lambda_j < \lambda_i\} = i$. There are no others: a critical point of f on $\mathbb{RP}^n$ lifts to a critical point of the quadratic form on S^n, an eigenvector of $\operatorname{diag}(\lambda_i)$, and the eigenvalues are distinct. So $\nu_k = 1$ for $0 \le k \le n$. With $F = \mathbb{Z}/2$, $\beta_k = 1$ for all k and $Q = 0$. With $F = \mathbb{Q}$, $\beta_0 = 1$, $\beta_n = 1$ if n is odd, and the others vanish; for n even $M - P = t + \cdots + t^n = (1+t)(t + t^3 + \cdots + t^{n-1})$ and for n odd $M - P = t + \cdots + t^{n-1} = (1+t)(t + t^3 + \cdots + t^{n-2})$.

8.4. Define $\mu(x, y) = \min\left(|y|, \sqrt{\max(|x|^2 - \varepsilon, 0)}\right)$, a continuous function, and let $r_t(x, y) = (x,\ y_t)$ where $y_t = y$ if $y = 0$ and otherwise $y_t = \left((1-t)|y| + t\mu(x, y)\right) y/|y|$. Then r_t is continuous, $r_0 = \mathrm{id}$, and $|y_t| \le |y|$, so $q(r_t(x, y)) \le q(x, y) \le \varepsilon$ and the deformation stays in $\{q \le \varepsilon\}$. At $t = 1$: if $|x|^2 \le \varepsilon$ then $\mu = 0$ and $r_1(x, y) = (x, 0)$, a point of the disc; if $|x|^2 > \varepsilon$ then $|y_1|^2 = \min(|y|^2, |x|^2 - \varepsilon)$ and $q(r_1(x, y)) \le -\varepsilon$. Points of the target are fixed for all t: on the disc $y = 0$, and on $\{q \le -\varepsilon\}$ we have $|y|^2 \le |x|^2 - \varepsilon$ so $\mu = |y|$.

8.5. $f_t' = 3x^2 - t$ and $f_t'' = 6x$. For $t < 0$ there are no critical points. For $t = 0$ there is one, at 0, degenerate. For $t > 0$ there are two: $x = -\sqrt{t/3}$ with $f'' < 0$, a local maximum (index 1) with value $\frac{2t}{3}\sqrt{t/3}$, and $x = +\sqrt{t/3}$ with $f'' > 0$, a local minimum (index 0) with value $-\frac{2t}{3}\sqrt{t/3}$. On $[-L, L]$ with L large, f_t is increasing near the ends. For $t < 0$ the sublevel sets are intervals $[-L, c']$, all contractible: the functor is constant after the minimum $f_t(-L)$. For $t > 0$ a second component is born at the level of the local minimum and merges with the first at the level of the local maximum: the degree-zero barcode acquires one finite interval, of length $\frac{4t}{3}\sqrt{t/3}$, which shrinks to nothing as $t \to 0^+$.

Chapter 9

9.1. Let $DW = W \cup_{\partial W} W$ be the double, a closed manifold of odd dimension, so $\chi(DW) = 0$ (Theorem 8.9). By additivity, $\chi(DW) = 2\chi(W) - \chi(\partial W)$. Hence $\chi(\partial W) = 2\chi(W)$ is even. If $\dim W$ is even, ∂W is closed of odd dimension and $\chi(\partial W) = 0$. So χ mod 2 vanishes on boundaries; it is additive under disjoint union and multiplicative under products (Theorem 7.7), hence a ring

homomorphism on $\mathfrak{N}_*$. $\chi(\mathbb{RP}^{2m}) = 1$, as $\mathbb{RP}^{2m}$ has one cell in each dimension $0, \dots, 2m$.

9.2. The allowed parts are $2, 4, 5, 6, 8, 9, 10, \dots$ (excluding $1, 3, 7$). Partitions: $n = 0$: the empty one; $n = 1$: none; $n = 2$: 2; $n = 3$: none; $n = 4$: 4, 2+2; $n = 5$: 5; $n = 6$: 6, 4+2, 2+2+2; $n = 7$: 5+2; $n = 8$: 8, 6+2, 4+4, 4+2+2, 2+2+2+2; $n = 9$: 9, 5+4, 5+2+2; $n = 10$: 10, 8+2, 6+4, 6+2+2, 5+5, 4+4+2, 4+2+2+2, 2+2+2+2+2. The counts are $1, 0, 1, 0, 2, 1, 3, 1, 5, 3, 8$.

9.3. $w(\mathbb{RP}^n) = (1+a)^{n+1}$ with $a^{n+1} = 0$, coefficients modulo 2. $n = 2$: $(1+a)^3 = 1 + a + a^2$, so $w_1 = a$, $w_2 = a^2$, and $w_1^2 = w_2 = a^2$ evaluates to 1. $n = 3$: $(1+a)^4 = 1 + a^4 = 1$. $n = 4$: $(1+a)^5 = 1 + a + a^4$ since $\binom{5}{2}, \binom{5}{3}$ are even; so $w_1 = a$, $w_2 = w_3 = 0$, $w_4 = a^4$, giving $(w_1^4, w_1^2 w_2, w_2^2, w_1 w_3, w_4) = (1, 0, 0, 0, 1)$. For $\mathbb{RP}^2 \times \mathbb{RP}^2$, $w = (1 + a + a^2)(1 + b + b^2)$: $w_1 = a + b$, $w_2 = a^2 + ab + b^2$, $w_3 = a^2 b + ab^2$, $w_4 = a^2 b^2$, and the numbers are the coefficients of $a^2 b^2$ modulo 2: $w_1^4 = (a+b)^4$ gives $6 \equiv 0$; $w_1^2 w_2$ gives $1 + 2 + 1 \equiv 0$; w_2^2 gives $2 + 1 \equiv 1$; $w_1 w_3$ gives $2 \equiv 0$; w_4 gives 1.

9.4. A closed 1-manifold is a finite union of circles, each the boundary of a disc. The Klein bottle is the boundary of the solid Klein bottle, the non-trivial D^2-bundle over S^1 (the mapping torus of a reflection of D^2). By Theorem 11.1 a closed connected surface is Σ_g or N_h. Σ_g bounds a handlebody. N_h is a connected sum of h copies of $\mathbb{RP}^2$, and connected sum is cobordant to disjoint union (attach a 1-handle to the product cobordism), so $[N_h] = h[\mathbb{RP}^2]$; this is zero for h even since every element has order 2, and equal to $[\mathbb{RP}^2] \neq 0$ for h odd by Exercise 9.1.

9.5. A tubular neighbourhood of the origin in $\mathbb{R}^k$ with the standard framing is the open unit ball B with the identification $B \cong \mathbb{R}^k$, $x \mapsto x/(1 - |x|^2)$, which is the identity to first order at 0. The collapse $S^k = \mathbb{R}^k \cup \infty \to \mathbb{R}^k \cup \infty = S^k$ is this diffeomorphism on B and constant at ∞ outside. The origin is a regular value with exactly one preimage, at which the differential is the identity; so the degree is 1. Reversing one framing vector composes the identification with a reflection, and the degree becomes -1. A framed 0-manifold in $\mathbb{R}^k$ is a finite set of points with signs, the collapse has degree the sum of the signs, and by Theorem 7.3 the degree classifies maps $S^k \to S^k$; so $\Omega_0^{\mathrm{fr}} \cong \pi_k(S^k) \cong \mathbb{Z}$.

Chapter 10

10.1. The equation $\dot{x} = (-x_2, x_1)$ is linear with solution $F_t(x) = (x_1 \cos t - x_2 \sin t,\ x_1 \sin t + x_2 \cos t)$, rotation by angle t. For $F_t(x) = e^t x$, $\partial_t F_t(x) = e^t x = F_t(x)$, so $G(y, t) = y$. For $\dot{x} = x^2$, the solution through $x_0 > 0$ is

$x_0/(1-x_0t)$, which is unbounded as $t \to 1/x_0$; so for every $t > 0$ the time-t map is undefined at $x_0 \geq 1/t$, and there is no diffeotopy.

10.2. The rotation number is the degree of $t \mapsto \gamma'(t)/|\gamma'(t)|$. Circle: the tangent turns once; 1. Figure eight: $\gamma' = (\cos t, \cos 2t)$; the curve has the symmetry $t \mapsto t + \pi$, $\gamma \mapsto (-x, y)$, under which the contributions of the two lobes to the total turning cancel; 0. Limaçon: in complex notation $\gamma = (1 + 2\cos\theta)e^{i\theta} = e^{i\theta} + e^{2i\theta} + 1$, so $\gamma' = ie^{i\theta}(1 + 2e^{i\theta})$; the factor $e^{i\theta}$ has winding number 1 and $1 + 2e^{i\theta}$ winds once around 0 since $|2e^{i\theta}| > 1$; total 2. By Corollary 10.8 no two of the three are regularly homotopic.

10.3. A homotopy inverse sends (p, A) to the embedding $x \mapsto p + Ax$. One composite is the identity. For the other, given an embedding f define, for $0 < t \leq 1$, $f_t(x) = f(0) + \big(f(tx) - f(0)\big)/t$, and $f_0(x) = f(0) + df_0(x)$. Each f_t is an embedding (a rescaling of the restriction of f to a smaller disc), $f_1 = f$, and by Taylor's theorem $f_t(x) = f(0) + df_0(x) + t\,r(x,t)$ with r smooth, so the family is smooth down to $t = 0$ and depends continuously on f. It preserves $(f(0), df_0)$ throughout. So the space of embeddings deformation retracts to the affine ones.

10.4. Parametrize so that $F_t(x) = (x, 0, 0)$ unless $|x - 1/(1-t)| < 1$, where a fixed knotted arc is inserted, for $t < 1$; put $F_1(x) = (x, 0, 0)$. On the open set $\{(x,t) : x < R\}$ we have $F_t(x) = (x, 0, 0)$ as soon as $1/(1-t) > R + 1$, so F is smooth near every point with $t = 1$; it is smooth elsewhere by construction, and each F_t is an embedding. If a diffeotopy Φ_t of $\mathbb{R}^3$ satisfied $\Phi_1 \circ F_0 = F_1$, then Φ_1 would be a diffeomorphism of $\mathbb{R}^3$ carrying the knotted line to the straight line, hence a homeomorphism of the complements. The complement of the straight line has fundamental group $\mathbb{Z}$, and the complement of a long knot has the group of the corresponding closed knot, which is not abelian if the knot is non-trivial.

10.5. Here $m = 1$ and $n = 4 = 2m + 2$, so Theorem 10.9 applies. In $\mathbb{R}^3$ the dimension count for double points of a one-parameter family gives $2m + 1 - n = 0$: a generic path of maps $S^1 \to \mathbb{R}^3$ has isolated times at which the curve crosses itself transversally, and these crossings are exactly the events that change the knot type. They cannot be removed by perturbation because a zero-dimensional transverse intersection is stable.

Chapter 11

11.1. For a word of length $2n$, $\chi = V - n + 1$, and the surface is orientable if and only if each letter occurs with both exponents. $aba^{-1}b^{-1}$: all four corners are identified, $V = 1$, $\chi = 0$, orientable: the torus. $abab$: $V = 2$,

$\chi = 1$, non-orientable: $\mathbb{RP}^2$. $abab^{-1}$: $V = 1$, $\chi = 0$, non-orientable: the Klein bottle. $aabb$: $V = 1$, $\chi = 0$, non-orientable: the Klein bottle again, as $P\#P$. $abca^{-1}b^{-1}c^{-1}$: $V = 2$, $\chi = 2 - 3 + 1 = 0$, orientable: the torus. $abcabc$: $V = 3$, $\chi = 3 - 3 + 1 = 1$, non-orientable: $\mathbb{RP}^2$. (The vertex counts are those returned by the script.)

11.2. $\chi(M\#N) = \chi(M) + \chi(N) - 2$ because a disc is removed from each (lowering each χ by 1) and the results are glued along a circle, of Euler characteristic 0. So $\chi(T\#P) = 0 + 1 - 2 = -1$ and $\chi(P\#P\#P) = 1 + 1 + 1 - 4 = -1$. Both are closed, connected and non-orientable (a connected sum is orientable only if both summands are). By Theorem 11.1 they are diffeomorphic, to N_3.

11.3. (a) The pairing matrix in the basis $(1, x)$ is $\left(\begin{smallmatrix} 0 & 1 \\ 1 & 0 \end{smallmatrix}\right)$, so $1^\# = x$ and $x^\# = 1$, and $w = 1 \cdot x + x \cdot 1 = 2x$. Then $\varepsilon(w^0) = \varepsilon(1) = 0$, $\varepsilon(w) = 2$, and $w^2 = 4x^2 = 0$. (b) $\varepsilon(g \cdot h) = \delta_{gh,e}/n$, so $g^\# = n\,g^{-1}$ and $w = \sum_g g \cdot ng^{-1} = n^2 e$; hence $\varepsilon(w^g) = n^{2g}\varepsilon(e) = n^{2g-1}$. A homomorphism $\pi_1\Sigma_g \to G$ to an abelian group is a homomorphism from $H_1 = \mathbb{Z}^{2g}$, and there are n^{2g} of them; dividing by n gives the same number. Semisimple case: $e_i e_j = \delta_{ij} e_i$, so the pairing is diagonal with entries θ_i, the dual basis is $e_i^\# = e_i/\theta_i$, and $w = \sum_i e_i/\theta_i$. Then $w^g = \sum_i e_i/\theta_i^g$ and $\varepsilon(w^g) = \sum_i \theta_i^{1-g}$.

11.4. By Corollary 8.7, W is obtained up to homotopy from $f^{-1}(a)$, a union of circles with $\chi = 0$, by attaching a 1-cell; so $\chi(W) = -1$. Both $f^{-1}(a)$ and $f^{-1}(b)$ are non-empty: the minimum and maximum of f are not attained at the only critical point, a saddle, so they are attained on the boundary. Hence W has $b \geq 2$ boundary components. If W is orientable, $-1 = 2 - 2g - b$ gives $(g, b) = (0, 3)$, as $(1, 1)$ is excluded: a sphere with three holes. If not, $-1 = 2 - h - b$ with $h \geq 1$, $b \geq 2$ gives $(h, b) = (1, 2)$: a projective plane with two holes, which is a Möbius band with one hole.

11.5. The torus is the composite $\varnothing \xrightarrow{\eta} S^1 \xrightarrow{\delta} S^1 \sqcup S^1 \xrightarrow{\mu} S^1 \xrightarrow{\epsilon} \varnothing$. Let $\beta = \epsilon\mu$, the pairing, and $\gamma = \delta\eta$, the copairing. The Frobenius relation and the unit and counit laws give the zigzag identity $(\beta \sqcup \mathrm{id})(\mathrm{id} \sqcup \gamma) = \mathrm{id}$: indeed $(\epsilon \sqcup \mathrm{id})(\mu \sqcup \mathrm{id})(\mathrm{id} \sqcup \delta)(\mathrm{id} \sqcup \eta) = (\epsilon \sqcup \mathrm{id})\,\delta\,\mu\,(\mathrm{id} \sqcup \eta) = \mathrm{id}$. So $Z(\gamma)$ and $Z(\beta)$ exhibit $A = Z(S^1)$ as its own dual, with $Z(\gamma)$ the coevaluation and $Z(\beta)$ the evaluation, and by commutativity $Z(T^2) = Z(\beta)Z(\gamma) = \mathrm{tr}(\mathrm{id}_A) = \dim A$.

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