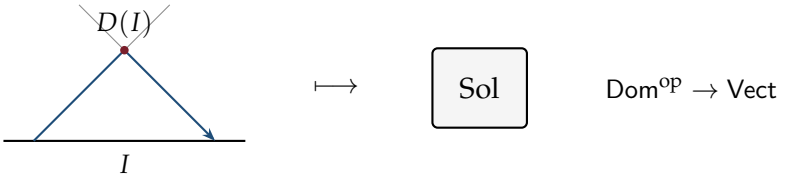
Higher Categorical Semantics

Partial Differential Equations in Higher Categories

Waves, diffusions and potentials as functors of domain, symmetry and time



Anonymous

2026

Version 1.0, 18 November 2026.

This volume is a treatment, in categorical language, of the material of Walter A. Strauss, *Partial Differential Equations: An Introduction* (second edition, Wiley, 2008). It reproduces none of that book's text, figures, tables or exercises; where a numerical example of the source is recomputed, this is said. An appendix gives a section-level concordance.

Every numbered statement carries a status tag: [classical] a classical result; [literature; source] a result of the cited literature; [reformulation; cf. source] a known result restated in the language of this volume; [observation] an observation of this volume, proved here; [programme] a proposal. Tags of the form [V6] name a check in the companion script `verify-pde.py`; tags of the form [source, §3.2] point to the source.

This book was drafted with the assistance of an AI system (Anthropic's Claude), which also wrote and ran the companion script. The bibliography was assembled from memory, except for the entries marked as checked, and should be verified before it is relied on.

Preface

Strauss's *Partial Differential Equations* [Str08] is an introduction for undergraduates built on three equations, of waves, of diffusion and of equilibrium, and on a small number of methods: characteristics, reflection, separation of variables, Green's functions, transforms. It is explicit, physically motivated, and careful to say what it does not prove.

This volume retells that material with the question of the series in mind: which constructions are *natural*, in the sense that a map of domains, of equations or of function spaces induces a map of solutions? A differential equation is treated as an object, its solutions as the values of a functor, and the methods of the source as instances of functoriality in one of a few directions.

- *Restriction.* Solutions form a sheaf. For the wave equation it is determined by data on a time slice (Theorem 3.3); for Laplace's equation it has cohomology on an annulus (Proposition 6.5).
- *Reflection.* A boundary condition on a chamber of a reflection group is a character of the group (Theorem 3.5), and in general a Lagrangian plane of boundary values (Theorem 5.1). The method of images and Fourier's method are exchanged by Pontryagin duality (Theorem 4.6).
- *Time.* Waves are a functor from time to symplectic maps, diffusions a functor from time to Markov kernels, and the source's table of contrasts follows from that (Chapter 4).
- *Symmetry.* Eigenspaces are representations (Theorem 9.1); the special functions are what the rotation groups produce; and domains can have the same spectrum because two permutation representations are isomorphic (Theorem 10.7).
- *Gluing.* Dirichlet-to-Neumann maps and transfer matrices compose as Lagrangian relations (Theorem 7.7, Theorem 13.3).
- *Discretization.* A scheme is a square that almost commutes; its stability is positivity for diffusion, causality for waves, and compatibility with the de Rham complex for elliptic problems (Chapter 12).
- *Morphisms of equations.* The solvable nonlinear examples of the source are images of linear equations (Theorem 14.4, Theorem 14.5).

Quantitative statements accompany the qualitative ones throughout: amplification factors, exponents, constants, eigenvalues, rates, counts of terms. All of them are recomputed by the companion script, which performs 107 checks.

How much of this is "higher" is assessed in Chapter 15. In brief: most of it is ordinary category theory and representation theory; cohomology enters in degree one; and gauge fields are the one place where the solution functor must take values in higher groupoids (Remark 13.2). Nothing here is a new theorem about partial differential equations, and no existence theorem is proved. A companion paper [PDEML] applies the framework to machine learning.

The volume belongs to a series with [GIHC], on game theory, [ESCHC], on information theory, [DTHC], on differential topology, and [EGHC], on evolutionary games, and uses the second and third.

Contents

I Equations 1

1 What an Equation Is 2
- 1.1 Equations in jets . . . 2
- 1.2 Linear equations and their modules . . . 3
- 1.3 The symbol and the type . . . 4
- 1.4 First-order equations are quotients . . . 5

2 Three Archetypes and Well-Posedness 7
- 2.1 Conservation and constitution . . . 7
- 2.2 Data as a restriction map . . . 8
- 2.3 Uniqueness by energy . . . 9
- 2.4 Which problem for which type . . . 9

II One Space Dimension 11

3 Waves 12
- 3.1 Two transports glued along the constants . . . 12
- 3.2 Causality . . . 12
- 3.3 Energy and the symplectic form . . . 13
- 3.4 Reflections . . . 14
- 3.5 Sources . . . 15

4 Diffusions 17
- 4.1 A semigroup of Markov kernels . . . 17
- 4.2 Symmetry and the Gaussian . . . 18
- 4.3 Irreversibility . . . 18
- 4.4 Waves and diffusions compared . . . 19
- 4.5 Images, again . . . 19
- 4.6 Sources . . . 21

5 Boundary Conditions and Fourier Series 22
- 5.1 Diagonalizing the generator . . . 22
- 5.2 A boundary condition is a Lagrangian plane . . . 22

5.3 Why sines and cosines . . . 24
5.4 Completeness, and in which category . . . 25
5.5 Inhomogeneous boundary values . . . 26

III Space **27**

6 Harmonic Functions **28**
6.1 The sheaf and its two principles . . . 28
6.2 Naturality . . . 28
6.3 From the boundary: Poisson and Brownian motion . . . 29
6.4 The annulus has cohomology . . . 30
6.5 Corners . . . 30

7 Green's Identities and Green's Functions **32**
7.1 Stokes' theorem, twice . . . 32
7.2 Dirichlet's principle . . . 32
7.3 The kernel of the inverse . . . 33
7.4 Half-space and ball . . . 33
7.5 Gluing . . . 34

8 Waves and Quanta in Space **36**
8.1 Cones . . . 36
8.2 Kirchhoff, Huygens and descent . . . 36
8.3 Singularities travel along rays . . . 37
8.4 Diffusion, and Schrödinger's equation . . . 38
8.5 The hydrogen atom . . . 38

9 Symmetry and Special Functions **40**
9.1 Eigenspaces are representations . . . 40
9.2 The drum . . . 40
9.3 The sphere . . . 41
9.4 Angular momentum . . . 42
9.5 Nodes . . . 42

IV Operators **44**

10 Eigenvalues **45**
10.1 Minimax, and what it makes functorial . . . 45
10.2 Completeness . . . 46
10.3 How many eigenvalues . . . 46

10.4 The trace of the semigroup . . . 47
10.5 Can one hear the shape? . . . 47

11 Distributions and Transforms **50**
11.1 Distributions . . . 50
11.2 Fundamental solutions . . . 50
11.3 The Fourier transform . . . 51
11.4 Source functions . . . 51
11.5 The Laplace transform . . . 52

12 Discretization **54**
12.1 Squares that almost commute . . . 54
12.2 Diffusion . . . 54
12.3 Waves . . . 55
12.4 Laplace's equation . . . 56
12.5 Finite elements . . . 56
12.6 Discretize the complex . . . 57

V Physics and Nonlinearity **58**

13 Fields **59**
13.1 Maxwell's equations . . . 59
13.2 Fluids and sound . . . 60
13.3 Scattering . . . 60
13.4 Equations of particles . . . 62
13.5 Fields as functors . . . 62

14 Nonlinear Equations **64**
14.1 Shocks . . . 64
14.2 A morphism of equations . . . 65
14.3 Solitons . . . 66
14.4 Variational problems . . . 67
14.5 Bifurcation . . . 67
14.6 Water waves . . . 68

15 Assessment **70**
15.1 What the language did . . . 70
15.2 What it did not do . . . 71
15.3 How much of this is higher . . . 72
15.4 Programme . . . 72

A Toolkit 74

B Concordance with the Source 76

C Computations 79

D Solutions to the Exercises 82

References 87

Part I

Equations

Chapter 1

What an Equation Is

The source begins by solving a few equations and only then asks what kind of thing has been solved [source, §1.1–1.2, §1.6]. We reverse the order, because the answer decides what the rest of the volume can say. A differential equation will be an object of a category. Its solutions will be the values of a functor. The statements of the following chapters are then of one form: such-and-such a map of domains, of equations or of function spaces induces a map of solutions.

1.1 Equations in jets

Let $M \subseteq \mathbb{R}^n$ be open and let u be a real function on M. The ***k*-jet** of u at x is the list of its derivatives of order at most k there; jets form a bundle $J^k M \to M$ with coordinates $(x, u_\alpha)_{|\alpha| \le k}$, and u determines a section $j^k u$.

Definition 1.1. A ***differential equation of order*** k on M is a subset $\mathcal{E} \subseteq J^k M$, typically the zero set of a function $F(x, u, u_i, u_{ij}, \dots)$. A ***solution*** is a function u whose jet $j^k u$ takes values in $\mathcal{E}$.

Differentiating $F = 0$ gives equations of higher order that every solution also satisfies; adjoining all of them gives the ***prolongation*** $\mathcal{E}^\infty \subseteq J^\infty M$. A point of $\mathcal{E}^\infty$ over x is a ***formal solution*** at x: a Taylor series that satisfies the equation to all orders.

Theorem 1.2 (Equations are coalgebras)**.** [literature; [Mar86, KhSc17]] *The infinite jet construction is a comonad J^∞ on bundles over M. Its coalgebras are the (formally integrable) differential equations in the sense of Vinogradov, and a morphism of coalgebras is a differential operator that carries solutions to solutions. In synthetic differential geometry J^∞ is base change along the unit $M \to \Im M$ of the infinitesimal shape, and in the cases of interest equations over M are bundles over $\Im M$.*

We use two consequences and none of the machinery. There is a category PDE whose morphisms are differential substitutions; the Cole–Hopf and Miura transformations of Chapter 14 are morphisms in it. And solutions are functorial: $\mathcal{E} \mapsto \mathrm{Sol}(\mathcal{E})$ sends a morphism of equations to a map of solution sets. Since solutions restrict to open subsets and glue, $\mathrm{Sol}(\mathcal{E})$ is a sheaf on M.

1.2 Linear equations and their modules

Let $\mathcal{D}$ be the ring of linear differential operators on M with smooth coefficients. A linear equation $Pu = 0$ is recorded by the left $\mathcal{D}$-module $\mathcal{D}/\mathcal{D}P$, and for any left $\mathcal{D}$-module $\mathcal{F}$ of functions

$$\mathrm{Sol}(P;\mathcal{F}) = \mathrm{Hom}_{\mathcal{D}}(\mathcal{D}/\mathcal{D}P, \mathcal{F}) = \{u \in \mathcal{F} : Pu = 0\}.$$

This is functorial in both variables. In $\mathcal{F}$ it is covariant: smooth functions sit inside distributions, and classical solutions inside weak ones (Chapter 11). In the equation it is contravariant: if $QR = SP$ then $u \mapsto Ru$ maps $\mathrm{Sol}(P)$ to $\mathrm{Sol}(Q)$.

Proposition 1.3 (Factorization). [reformulation] *For operators P, Q there is an exact sequence*

$$0 \longrightarrow \mathrm{Sol}(Q) \longrightarrow \mathrm{Sol}(PQ) \xrightarrow{Q} \mathrm{Sol}(P).$$

If P and Q commute, then $\mathrm{Sol}(P) + \mathrm{Sol}(Q) \subseteq \mathrm{Sol}(PQ)$, with equality as soon as Q maps $\mathrm{Sol}(P)$ onto itself: for $u \in \mathrm{Sol}(PQ)$ choose $w \in \mathrm{Sol}(P)$ with $Qw = Qu$, and then $u - w \in \mathrm{Sol}(Q)$.

The wave operator is the case $P = \partial_t - c\partial_x$, $Q = \partial_t + c\partial_x$, checked on polynomials [E5]; d'Alembert's solution in Chapter 3 is the equality $\mathrm{Sol}(PQ) = \mathrm{Sol}(P) + \mathrm{Sol}(Q)$.

In a diagram: for commuting P and Q the square of inclusions

$$\begin{array}{ccc} \mathrm{Sol}(P) \cap \mathrm{Sol}(Q) & \hookrightarrow & \mathrm{Sol}(P) \\ \downarrow & & \downarrow \\ \mathrm{Sol}(Q) & \hookrightarrow & \mathrm{Sol}(PQ) \end{array}$$

is a pullback of vector spaces, and it is also a pushout exactly when $\mathrm{Sol}(P) + \mathrm{Sol}(Q) = \mathrm{Sol}(PQ)$. For the wave operator the corner is the constants, and d'Alembert's theorem says that the square is bicartesian: a solution is a right-moving and a left-moving wave, glued along what they have in common.

For constant coefficients $\mathcal{D} = \mathbb{R}[\partial_1, \dots, \partial_n]$ is a polynomial ring, $P = p(\partial)$, and the module is $\mathbb{R}[\xi]/(p)$: the equation *is* the hypersurface $V(p) = \{p(\xi) = 0\} \subseteq \mathbb{C}^n$. Each point $\xi \in V(p)$ gives the exponential solution $e^{\xi \cdot x}$.

Theorem 1.4 (Fundamental principle). [literature; [Ehr70, Pal70]] *On a convex domain every smooth solution of a constant-coefficient equation $p(\partial)u = 0$ is an integral, over $V(p)$, of exponential solutions with polynomial coefficients.*

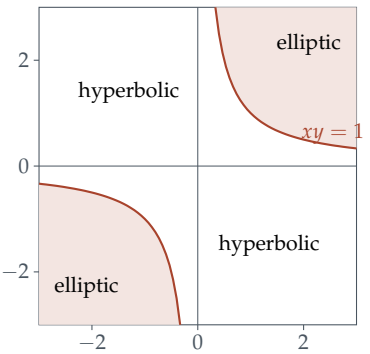
Separation of variables is thus not a trick that happens to work: for constant coefficients it is the general solution. The source's Fourier method (Chapter 5) is the special case in which boundary conditions select a discrete set of points of $V(p)$.

1.3 The symbol and the type

Write a second-order operator as $P = \sum_{i,j} a_{ij}(x)\partial_i\partial_j + \text{lower order}$, with a_{ij} symmetric. Its ***principal symbol*** is the quadratic form $\sigma_P(x,\xi) = \sum a_{ij}(x)\xi_i\xi_j$.

Theorem 1.5 (Type). [classical; source, Theorem 1 of §1.6] *Under a change of coordinates $y = \varphi(x)$ the symbol transforms by congruence with the Jacobian, so σ_P is a function on the cotangent bundle and its inertia (p, q, z) at each point is invariant. The operator is **elliptic** at x if σ_P is definite, **hyperbolic** if it is non-degenerate with one sign differing from the rest, **parabolic** if it is semidefinite with a one-dimensional kernel. In two variables these are the cases $a_{12}^2 < a_{11}a_{22}$, $>$ and $=$.*

Invariance was tested on three hundred random symbols and changes of variables, and the classification of the source's variable-coefficient example by the sign of $1 - xy$ was reproduced. [E1, E2]



The type of $yu_{xx} - 2u_{xy} + xu_{yy}$ in the plane: hyperbolic where $xy < 1$, elliptic where $xy > 1$, parabolic on the two branches of $xy = 1$. The type is a function on the domain and may change across it.

The ***characteristic variety*** $\mathrm{Char}(P) = \{(x,\xi) : \xi \neq 0,\ \sigma_P(x,\xi) = 0\}$ is the real part of the hypersurface above, taken to leading order. It is empty exactly for elliptic operators; for the wave operator it is the light cone and for the heat operator a line. [E5]

Remark 1.6 (Algebra does not see the type). [observation] The homogeneous polynomial solutions of degree k of a non-zero homogeneous second-order operator in n variables with constant coefficients form a space of dimension $\binom{n+k-1}{k} - \binom{n+k-3}{k-2}$, whatever the type: the operator maps polynomials of degree k onto those of degree $k-2$. [E4] The count is 2 in the plane and $2k+1$ in space, for Laplace, wave and degenerate operators alike. Elliptic, hyperbolic and parabolic equations have the same formal solutions; they differ in which formal solutions converge and in how solutions depend on data. The type is a statement about the *real* points of the characteristic variety, and its consequences are analytic. This is why the three archetypes need three chapters, and

why the categorical language must include topology on spaces of functions to say anything about them.

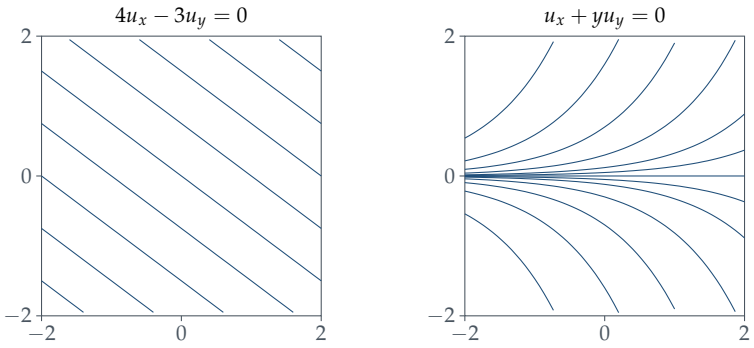
Example 1.7 (Kovalevskaya)**.** For $u_t = u_{xx}$ the Taylor coefficients in t are determined by those in x: $\partial_t^j u = \partial_x^{2j} u$. With polynomial data x^k the series terminates, in the heat polynomials, which solve the equation exactly. With the analytic datum $1/(1+x^2)$ the coefficients of t^j at $x = 0$ are $(-1)^j (2j)!/j!$, successive ratios are $2(2j+1)$, and the series diverges for every $t \neq 0$. [E6] The Cauchy–Kovalevskaya theorem gives convergent solutions from analytic data on a non-characteristic surface; $t = 0$ is characteristic for the heat equation.

1.4 First-order equations are quotients

A first-order linear equation $a(x,y)u_x + b(x,y)u_y = 0$ says that u is constant along the integral curves of the vector field $X = (a, b)$, its ***characteristics***.

Proposition 1.8 (Solutions are functions on the leaf space)**.** [reformulation; cf. source, §1.2] *If $q\colon M \to Q$ is a submersion whose fibres are the characteristics of X, then pullback $q^*\colon C^\infty(Q) \to \mathrm{Sol}(X)$ is an isomorphism. A curve $\Gamma \subset M$ that meets every characteristic exactly once and transversally is a section of q, and restriction $\mathrm{Sol}(X) \to C^\infty(\Gamma)$ is then an isomorphism.*

The source's two examples are $q = -3x - 4y$ for $4u_x - 3u_y = 0$ and $q = e^{-x}y$ for $u_x + yu_y = 0$. [E3] The second half of the proposition is the first well-posedness statement of the subject, and its failure is instructive: the line $y = 0$ is itself a characteristic of $u_x + yu_y = 0$, every solution is constant on it, and the problem with data $u(x,0) = \varphi(x)$ has no solution unless φ is constant and infinitely many if it is. Data must be given on a section of the quotient.



Characteristics of the source's two first-order examples: the level sets of $q = -3x - 4y$ and of $q = e^{-x}y$. A solution is a function on the space of these curves.

Notes

The geometric theory of differential equations in jet spaces is due to Lie, Cartan, Spencer and Vinogradov; for the categorical form see [Vino1] and, for the comonad, [Mar86, KhSc17]; a gap in the synthetic treatment concerning limits of jet bundles is closed in [GKSS26]. $\mathcal{D}$-modules are treated in [Cou95, HTT08]. The fundamental principle is in [Ehr70, Pal70]. The classification by type in the source follows the classical reduction of quadratic forms.

Exercises

Exercise 1.1. Find the type of $u_{xx} - 4u_{xy} + 4u_{yy} = 0$, of $u_{xx} + 2u_{xy} + 5u_{yy} = 0$ and of $u_{xy} = 0$.

Exercise 1.2. Find a function q whose level sets are the characteristics of $u_x + 2xy^2 u_y = 0$, and hence the general solution.

Exercise 1.3. Verify Remark 1.6 in two variables directly, by exhibiting two independent harmonic polynomials of each degree $k \geq 1$ and two independent solutions of $u_{xx} = 0$ of each degree.

Exercise 1.4. Let $P = \partial_t - c\partial_x$ and $Q = \partial_t + c\partial_x$ act on smooth functions on the plane. Show that $\mathrm{Sol}(PQ) = \mathrm{Sol}(P) + \mathrm{Sol}(Q)$.

Chapter 2

Three Archetypes and Well-Posedness

Three equations carry the source: the wave equation, the diffusion equation and Laplace's equation. This chapter says where they come from, in a form that shows which of their properties are forced, and then states what it means for a problem to be correctly set.

2.1 Conservation and constitution

Each derivation in the source [source, §1.3] has two steps. The first is a balance: what is in a region changes only by what crosses its boundary,

$$\partial_t \rho + \operatorname{div} j = 0. \tag{2.1}$$

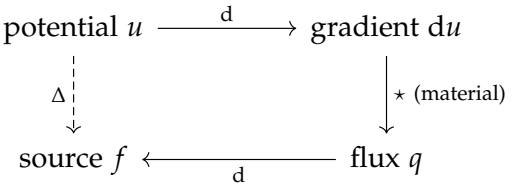
The second is a material law expressing the flux j through the unknown.

phenomenon	conserved	material law	equation
transport	mass	$j = c\rho$	$u_t + cu_x = 0$
diffusion	mass, heat	$j = -k\nabla u$ (Fick, Fourier)	$u_t = k\Delta u$
vibration	momentum	stress $= T\nabla u$ (Hooke)	$\rho u_{tt} = T\Delta u$
equilibrium	—	either of the last two, steady	$\Delta u = 0$

Proposition 2.1 (Topological and metric parts). [reformulation; cf. [Ton75, Bos98]] *The balance* (2.1) *is the statement that a differential form is closed: it uses only the exterior derivative, which commutes with pullback along every smooth map. The material law is a Hodge star: it uses a metric, and commutes only with isometries. The Laplacian is the composite* $\Delta = -\mathrm{d}^*\mathrm{d}$, *with* $\mathrm{d}^* = \pm \star\, \mathrm{d}\, \star$.

On a graph the same factorization is exact arithmetic: with D the incidence matrix, the Laplacian is $D^{\mathsf{T}}WD$ for a diagonal matrix W of conductances, and the columns of $D^{\mathsf{T}}D$ sum to zero, which is conservation of mass. [W3] Two things follow at once. The symmetries of the three archetypes are the isometries, because the only non-natural ingredient is the metric (Chapters 6 and 9). And a discretization should discretize d exactly and $\star$ approximately (Chapter 12).

With signs suppressed, the proposition is the square



in which the horizontal arrows are natural for all smooth maps and the right-hand one only for isometries. Every linear equation of the volume is a way round a square of this shape, sometimes with time as one of the variables.

2.2 Data as a restriction map

An equation has many solutions; a problem adds conditions [source, §1.4]. Initial conditions prescribe u, and for second order in time also u_t, on $\{t = 0\}$. Boundary conditions prescribe u (Dirichlet), the normal derivative (Neumann) or a combination $\partial_n u + au$ (Robin) on the boundary of the spatial domain. In every case there is a linear map

$$r\colon \operatorname{Sol}(P; X) \longrightarrow Y$$

from solutions in a chosen function space X to data in a chosen space Y.

Definition 2.2 (Hadamard)**.** The problem (P, r, X, Y) is ***well-posed*** if r is an isomorphism of topological vector spaces: surjective (existence), injective (uniqueness), with continuous inverse (stability).

The definition is the source's [source, §1.5] with one addition that the categorical phrasing makes unavoidable: stability is a property of a morphism in a category of topological vector spaces, and it depends on the category. A problem may be well-posed for one pair (X, Y) and ill-posed for another.

Example 2.3 (Kernel and cokernel)**.** For the Neumann problem $-\Delta u = f$, $\partial_n u = 0$, the map from u to f is not an isomorphism: constants are in the kernel and a solution exists only if $\int f = 0$. It is Fredholm of index zero, with an exact sequence $0 \to \mathbb{R} \to X \to Y \to \mathbb{R} \to 0$. On a path with six vertices the rank is 5 and the system is solvable exactly for sources of mean zero. [W5] The source's underdetermined and overdetermined problems are those with non-zero kernel and cokernel; its first example, $u'' + u = 0$ with $u(0) = u(L) = 0$, has a kernel exactly when $\sin L = 0$. [W6]

Example 2.4 (Hadamard)**.** The functions $u_n(x, y) = n^{-1} e^{-\sqrt{n}} \sin nx \sinh ny$ are harmonic, vanish on $y = 0$, and have normal derivative $e^{-\sqrt{n}} \sin nx$ there, which tends to zero with all its derivatives. At $y = 1$ their size is $e^{-\sqrt{n}} \sinh n / n$.

n	25	100	400
size of the data	6.7×10^{-3}	4.5×10^{-5}	2.1×10^{-9}
size of the solution at $y = 1$	9.7×10^{6}	6.1×10^{36}	1.3×10^{162}

The Cauchy problem for Laplace's equation has at most one solution, and its inverse is not continuous in any topology of finitely many derivatives. [W1]

Example 2.5 (Time's arrow)**.** For $u_t = u_{xx}$ on $(0, \pi)$ the mode $\sin nx$ evolves by the factor $e^{-n^2 t}$. Forwards this is a contraction. Backwards over a time 0.1 the modes $n = 1, 10, 30$ are amplified by 1.1, 2.2×10^4 and 1.2×10^{39}. [W2] The backward problem is solvable on a dense set of data and its solution map is unbounded: the heat flow is a semigroup that is not a group (Chapter 4).

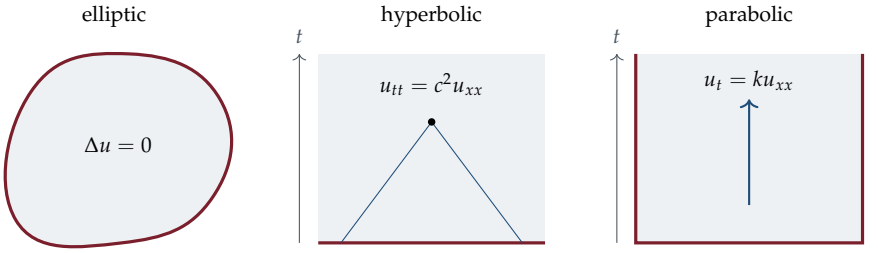
2.3 Uniqueness by energy

Theorem 2.6 (Two energy laws)**.** [classical; source, §2.2–2.3] *For the wave equation $u_{tt} = c^2 u_{xx}$ on the line, with data of compact support, the energy $E = \frac{1}{2}\int (u_t^2 + c^2 u_x^2)\, dx$ is constant. For the diffusion equation $u_t = k u_{xx}$ with Dirichlet or Neumann conditions, $\frac{d}{dt}\frac{1}{2}\int u^2\, dx = -k \int u_x^2\, dx \le 0$. In both cases a solution with zero data is zero.*

On a grid of twenty points the wave energy drifted by 5.5×10^{-12} in relative terms over five time units, and the square norm under the diffusion flow decreased at every step. [W4] The two laws are the first appearance of the dichotomy of Chapter 4: a conserved quadratic form makes the wave flow a group of isometries; a decreasing one makes the heat flow a semigroup of contractions.

2.4 Which problem for which type

type	archetype	well-posed problem	solution map
elliptic	$\Delta u = 0$	values on the boundary	a projection (minimizer)
hyperbolic	$u_{tt} = c^2 \Delta u$	u, u_t at one time	a group
parabolic	$u_t = k \Delta u$	u at one time, forwards	a semigroup



Where the data live. Elliptic: on the whole boundary. Hyperbolic: two functions on an initial slice, each point depending on an interval of it. Parabolic: one function on the initial slice and on the sides, in one direction of time only.

Mismatched pairings fail. Initial data for Laplace's equation give Example 2.4; final data for the heat equation give Example 2.5; boundary values for the wave equation on a square give non-uniqueness (Exercise 2.1). The rest of the volume takes the three rows in turn, and Chapter 15 returns to the table.

Notes

The separation of field equations into topological balance laws and metric constitutive laws is Tonti's [Ton75]; in the language of differential forms it is the basis of compatible discretization [Bos98, AFW06]. Well-posedness is Hadamard's notion; his example is in [Had23]. Fredholm theory for elliptic boundary problems is treated in [Eva10].

Exercises

Exercise 2.1. Show that the Dirichlet problem for $u_{tt} = u_{xx}$ on the square $[0, \pi]^2$ in the (x, t)-plane does not have unique solutions.

Exercise 2.2. Let D be the incidence matrix of a connected graph. Show that the range of $D^{\mathsf{T}} D$ is the space of vectors of mean zero.

Exercise 2.3. An initial temperature on $(0, \pi)$ is to be recovered from a measurement at time 0.1 whose error is 10^{-10} in each mode. Which is the first mode in which the error of the reconstruction exceeds 1?

Exercise 2.4. For $u_t = k u_{xx}$ on $(0, l)$ with the Robin conditions $u_x(0) = a_0 u(0)$ and $u_x(l) = -a_l u(l)$, compute $\frac{d}{dt} \frac{1}{2} \int u^2$. For which signs of a_0, a_l is it certain to be non-positive?

Part II

One Space Dimension

Chapter 3

Waves

The wave equation on the line, $u_{tt} = c^2 u_{xx}$, is the one archetype that can be solved in closed form by looking at it. This chapter extracts four structures from the solution: a decomposition of the solution sheaf, a causal structure, a symplectic form, and an action of reflection groups. Each reappears later in a setting where no formula is available.

3.1 Two transports glued along the constants

Theorem 3.1 (d'Alembert). [classical; source, §2.1] *Let $q_\pm(x,t) = x \pm ct$. Every smooth solution on the plane is $u = f \circ q_+ + g \circ q_-$, and (f,g) is unique up to $(f+\kappa, g-\kappa)$:*

$$0 \longrightarrow \mathbb{R} \longrightarrow C^\infty(\mathbb{R}) \oplus C^\infty(\mathbb{R}) \xrightarrow{q_+^* + q_-^*} \mathrm{Sol} \longrightarrow 0$$

is exact. The solution with $u = \varphi$, $u_t = \psi$ at $t = 0$ is $u(x,t) = \frac{1}{2}[\varphi(x+ct) + \varphi(x-ct)] + \frac{1}{2c}\int_{x-ct}^{x+ct} \psi$.

Proof. By Proposition 1.3 and Proposition 1.8: the two factors are transport operators with leaf spaces $q_\pm$, each maps the solutions of the other onto themselves, and their common solutions are the constants. The formula is the solution of two linear equations for f' and g'. [V1] □

Corollary 3.2 (Parallelogram identity). [classical] *A function of the form $f \circ q_+ + g \circ q_-$ satisfies $u(A) + u(C) = u(B) + u(D)$ for the vertices A, B, C, D, in order, of every parallelogram whose sides are characteristics. Conversely a continuous function with this property is of that form.*

The identity holds to rounding error for exact solutions. [V2] It uses no derivative, so it defines solutions that are merely continuous, or less: the first weak formulation of the volume. Restricted to a grid with $c\,\Delta t = \Delta x$ it is an exact numerical scheme (Chapter 12).

3.2 Causality

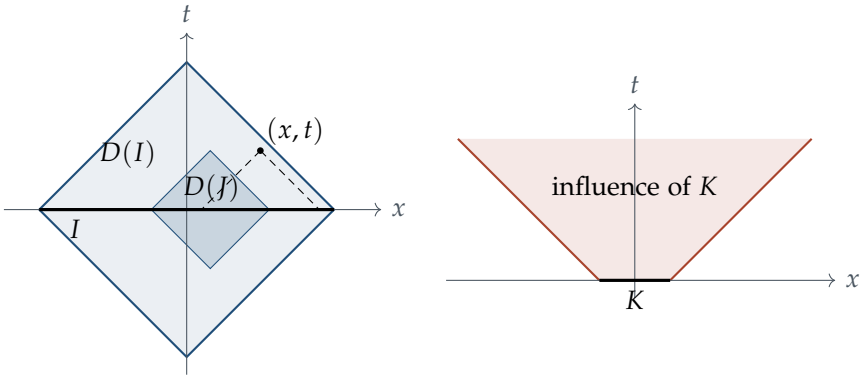
For an interval I of the x-axis let $D(I)$ be its ***domain of dependence***: the points (x,t) with $[x - c|t|, x + c|t|] \subseteq I$, a diamond with waist I.

Theorem 3.3 (Time slices). [reformulation; cf. source, §2.2] *Restriction to the waist is an isomorphism*

$$\mathrm{Sol}(D(I)) \xrightarrow{\sim} C^\infty(I) \oplus C^\infty(I), \qquad u \mapsto (u, u_t)|_{t=0},$$

natural for inclusions of intervals. Data outside I do not affect the solution in $D(I)$, and data in I affect the solution only in the ***domain of influence*** *of I.*

Changing the data outside the interval of dependence of a point left the solution there unchanged to the last bit; changing them inside did not. [V3] Two remarks. The assignment $I \mapsto \mathrm{Sol}(D(I))$ is a functor on the poset of intervals, and the theorem says it is determined by a much smaller amount of information than a general sheaf on the plane would be; this is the property that field theory axiomatizes (Chapter 13). And the theorem has no counterpart for diffusion, where the solution at any point depends on all the data (Chapter 4).



Left: the diamond $D(I)$, the smaller diamond of a subinterval J, and the interval on which the value at a point depends. Right: the region that data in K can affect. Slopes are $\pm 1/c$.

3.3 Energy and the symplectic form

Proposition 3.4. [classical] *For solutions u, v with spatially compact support,*

$$\Omega(u, v) = \int_{\mathbb{R}} (u\, v_t - u_t\, v)\, dx$$

does not depend on t. It is a symplectic form on Sol*; translations in space and time preserve it; and the energy of Theorem 2.6 is $E(u) = -\frac{1}{2}\Omega(u, \partial_t u)$, the Hamiltonian of time translation.*

Proof. $\partial_t(uv_t - u_t v) = c^2(uv_{xx} - u_{xx}v) = c^2\partial_x(uv_x - u_x v)$, which integrates to zero. For the last statement integrate $u\, u_{tt} = c^2 u\, u_{xx}$ by parts. □

Both quantities were computed at several times and agree to eight digits. [V4, V8] The wave flow is thus a one-parameter group of linear symplectic maps. This is the whole of "reversibility" and "no loss of information" in the source's comparison table.

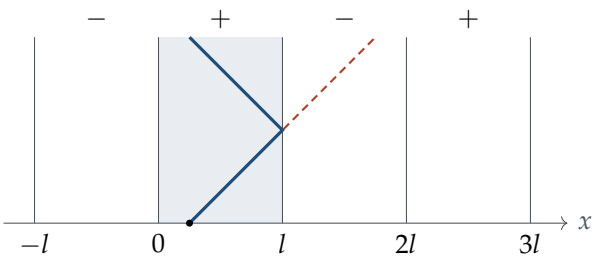
3.4 Reflections

The reflection $\rho\colon x \mapsto -x$ preserves the equation, so $\mathbb{Z}/2$ acts on $\mathrm{Sol}(\mathbb{R})$, which splits into even and odd solutions.

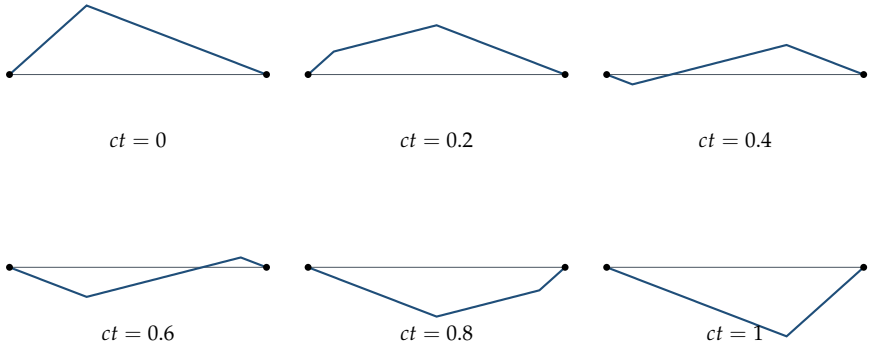
Theorem 3.5 (Boundary conditions as characters)**.** [reformulation; cf. source, §3.2] *Restriction to $x > 0$ is an isomorphism from odd solutions on the line to solutions on the half-line with the Dirichlet condition at 0, and from even solutions to solutions with the Neumann condition, for the wave equation. More generally, let W be a group generated by reflections of $\mathbb{R}^n$ with fundamental chamber C, let the homogeneous wave, heat or Laplace equation be invariant under W, and let $\chi\colon W \to \{\pm 1\}$ be a character. Weak solutions on $\mathbb{R}^n$ with $u \circ w = \chi(w)u$ restrict isomorphically to solutions on C with the Dirichlet condition on the walls whose reflection has $\chi = -1$ and the Neumann condition on those with $\chi = +1$.*

Sketch. An odd function vanishes on the mirror and an even one has vanishing normal derivative there. Conversely, the extension by $u \circ w = \chi(w)u$ solves the equation in every chamber. At an odd wall the zero trace matches the two values and the normal derivatives agree. At an even wall the values already agree and the zero normal derivative makes the fluxes match. Thus no distribution supported on a wall appears in the equation, as in Chapter 11. □

For the half-line this gives the source's two-region formula. [V5] For an interval $(0, l)$ the group is the infinite dihedral group generated by the reflections in 0 and l, with chamber $(0, l)$. The product of the two reflections is translation by $2l$, on which the Dirichlet–Dirichlet character is $+1$: solutions are $2l$-periodic in x and therefore $2l/c$-periodic in t. The source's formulas, one for each diamond of space-time according to the number of reflections at each end, are the terms of the sum over the group; the sum agrees with the Fourier series of Chapter 5. [V6]



Unfolding. A characteristic reflected at a wall of the chamber $(0, l)$ (solid) is a straight characteristic on the line (dashed), read in a translate of the chamber and weighted by the value of the character there; the signs shown are those of the Dirichlet–Dirichlet character.



A string of unit length plucked at $x = 0.3$, at six times in the first half period, computed from the sum over images. The corner splits in two, each half is reflected with a change of sign, and at $ct = 1$ the shape is the initial one turned through half a turn.

Remark 3.6 (How far the method goes). The source remarks that the reflection method cannot be generalized to two or three dimensions and does not work for diffusion [source, §3.2]. For the diamond-by-diamond formulas this is so. In the form of Theorem 3.5 the method applies to the reflection-invariant wave, heat and Laplace equations (Chapter 4) and to every domain that is a chamber of a reflection group: half-spaces, slabs, quadrants, wedges of angle π/m, rectangles and boxes, and the equilateral, half-square and half-equilateral triangles. This construction requires the domain to tile space by reflections.

3.5 Sources

Theorem 3.7 (Duhamel). [classical; source, §3.4, Theorem 1] *The solution of $u_{tt} - c^2 u_{xx} = f$ with zero data at $t = 0$ is $u(x,t) = \frac{1}{2c}\iint_{\Delta(x,t)} f$, the integral over the triangle of dependence of (x,t).*

For $f = \cos x$ this is $\cos x\,(1 - \cos ct)/c^2$. [V7] The theorem defines the ***retarded*** operator $G^+ \colon f \mapsto u$, whose value at a point depends only on f in the past cone. Reversing time gives the advanced operator G^-, and the difference $G = G^+ - G^-$ maps sources to solutions.

Theorem 3.8 (Solutions are radiated). [literature; [BGP07]] *With $P = \partial_t^2 - c^2\partial_x^2$, the sequence*

$$0 \longrightarrow C_c^\infty \xrightarrow{P} C_c^\infty \xrightarrow{G} \mathrm{Sol}_{sc} \longrightarrow 0$$

is exact, where Sol_{sc} denotes solutions with compactly supported Cauchy data. In these terms, and with suitable sign conventions, $\Omega(Gf, Gh) = \iint f\, Gh$.

Every solution is the field of some source, and two sources give the same field exactly when they differ by P of a compactly supported function. The theorem holds on every globally hyperbolic space-time, where there is no d'Alembert formula; it is the form in which the four structures of this chapter survive.

Notes

The exposition follows the source's Chapter 2 and Sections 3.2 and 3.4. For the wave equation on Lorentzian manifolds, Green operators and the exact sequence, see [BGP07]. Reflection groups and their chambers are treated in [Hum90]; the triangles that tile the plane by reflection are Lamé's, see [McC11].

Exercises

Exercise 3.1. Prove the first half of Corollary 3.2.

Exercise 3.2. For the half-line with the Neumann condition and data $u = \varphi$, $u_t = 0$, write the solution in the region $0 < x < ct$.

Exercise 3.3. A string on $(0, l)$ is fixed at 0 and free at l. Show from Theorem 3.5 that every motion has period $4l/c$.

Exercise 3.4. Verify Theorem 3.7 for $f(x,t) = \cos x$ by computing the integral.

Chapter 4

Diffusions

The diffusion equation $u_t = ku_{xx}$ has an explicit solution on the line as well, but the structures it carries are different in kind from those of Chapter 3: order where there was a symplectic form, a semigroup where there was a group, probability where there was mechanics.

4.1 A semigroup of Markov kernels

Let $S(x,t) = (4\pi kt)^{-1/2} e^{-x^2/4kt}$, the source function, and $(T_t\varphi)(x) = \int S(x-y,t)\varphi(y)\,dy$.

Theorem 4.1. [classical; source, §2.4 and §3.5] *For bounded continuous φ, $u = T_t\varphi$ is the bounded solution of the diffusion equation with $u \to \varphi$ as $t \to 0$. Moreover*

(a) $T_{t+s} = T_t T_s$ *and* $T_0 = \mathrm{id}$;

(b) $T_t\varphi \geq 0$ *if* $\varphi \geq 0$, *and* $T_t 1 = 1$;

(c) $\inf\varphi \leq T_t\varphi \leq \sup\varphi$.

Properties (a) and (b) say that $t \mapsto T_t$ is a functor from the monoid $(\mathbb{R}_{\geq 0}, +)$, regarded as a category with one object, to the category Stoch of Markov kernels of [ESCHC]. Property (c), the maximum principle on the line, is a consequence of (b): an average lies between the extremes. The kernel solves the equation, has unit mass and composes correctly to twelve digits; a unit step diffuses into $\frac{1}{2}(1 + \mathrm{erf}(x/\sqrt{4kt}))$. [D1, D2]

Theorem 4.2 (Maximum principle). [classical; source, §2.3] *A solution in a space-time rectangle attains its maximum and its minimum on the initial side or the lateral sides. Hence the Dirichlet problem has at most one solution and depends continuously on its data in the uniform norm.*

The discrete form is exact: the explicit difference scheme is a substochastic matrix precisely when $k\Delta t/\Delta x^2 \leq \frac{1}{2}$, and then it obeys the maximum principle; beyond that it has negative entries and the sawtooth grows. [D3] Stability of that scheme, the first example in the source's chapter on computation, is membership in the Markov category (Chapter 12).

Remark 4.3 (Generators). [classical; [Paz83]] A strongly continuous semigroup of contractions on a Banach space has a closed, densely defined generator $A = \lim (T_t - 1)/t$, and the Hille–Yosida theorem characterizes the operators that arise: those with $\|(\lambda - A)^{-1}\| \le 1/\lambda$ for $\lambda > 0$. The diffusion equation is the statement that the generator of (T_t) is $k\,\partial_x^2$. The resolvent is the Laplace transform of the semigroup (Chapter 11).

4.2 Symmetry and the Gaussian

The equation is invariant under translations, the reflection $x \mapsto -x$, and the scaling $(x, t) \mapsto (ax, a^2 t)$ [source, §2.4]. The last is not an isometry; it exists because the equation is of first order in time and second in space.

Proposition 4.4. [classical] *The source function is, up to a constant factor, the unique solution of the form $t^{-1/2} g(x/\sqrt{t})$: the solution of unit mass fixed by the scaling action.* [D7]

Convergence of $T_t \varphi$ to a multiple of S as $t \to \infty$, for integrable φ, is the statement that this fixed point attracts; it is the central limit theorem in the language of [ESCHC].

4.3 Irreversibility

Theorem 4.5 (Monotone quantities). [classical] *Along the diffusion flow of a probability density u:*

(a) $\frac{d}{dt} \int u^2 = -2k \int u_x^2 \le 0$;

(b) $\frac{d}{dt} \left(- \int u \log u \right) = k \int u_x^2 / u \ge 0$ *(entropy increases at the rate of the Fisher information);*

(c) for two solutions, every relative entropy $D(u \,\|\, v)$ and every f-divergence is non-increasing.

Moreover $T_t \varphi$ is smooth for every $t > 0$, and positive everywhere if $\varphi \ge 0$ is not zero.

Statement (c) is the data-processing inequality: it holds for every morphism of Stoch [ESCHC]. The invertible morphisms of Stoch are the deterministic bijections, and T_t is not one of them; that is the categorical form of irreversibility. Entropy and square norm were tested under a doubly stochastic step on fifty random densities. [D4] With $k = 1$ the indicator function of $[-1, 1]$ gives the positive value 2.3×10^{-90} at $x = 10$ after time 0.1: nothing travels, and everything is everywhere at once. [D8]



Left: the source function $S(x,t)$ with $k=1$ at three times; the area under each curve is one. Right: the same kernel applied to a unit step. Every curve on the right is strictly between 0 and 1 at every point: the maximum principle, and infinite speed.

4.4 Waves and diffusions compared

The source closes its second chapter with a table of seven contrasts [source, §2.5]. All seven follow from three.

	waves	diffusions
time acts by	a group	a semigroup
what is preserved	a symplectic form; energy	positivity; total mass
target category	linear symplectic maps	Markov kernels
speed of propagation	finite: Theorem 3.3	infinite: positivity of T_t
singularities	transported along characteristics	erased: T_t smooths
backwards in time	well-posed: inverses exist	ill-posed: Example 2.5
maximum principle	no	yes: Theorem 4.2
as $t \to \infty$	energy constant	decay to the fixed point
information	kept	lost: Theorem 4.5(c)

The upper three rows are structure; the lower six are the source's, and each is a consequence. A functor from time to symplectic maps is automatically invertible and conserves a quadratic form. A functor from time to Markov kernels is automatically monotone and contracts every divergence. The Schrödinger equation (Chapter 8) sits between the two: a group, like waves, generated by the operator of diffusion.

4.5 Images, again

Theorem 3.5 applies without change, since the equation is invariant under reflections.

Theorem 4.6 (Kernels on the half-line and the interval). [reformulation; cf. source, §3.1] *The Dirichlet and Neumann kernels on the half-line are* $S(x-y,t) \mp S(x+y,t)$. *On* $(0,l)$ *with Dirichlet conditions,*

$$\begin{aligned} K(x,y,t) &= \sum_{m\in\mathbb{Z}} \big[S(x-y-2ml,t) - S(x+y-2ml,t)\big] \\ &= \frac{2}{l}\sum_{n\geq 1} e^{-k(n\pi/l)^2 t} \sin\frac{n\pi x}{l}\sin\frac{n\pi y}{l}. \end{aligned}$$

The first expression is the sum over the infinite dihedral group weighted by the sign character; the second is the sum over eigenfunctions of Chapter 5. Their equality is the Poisson summation formula. The two sums are complementary:

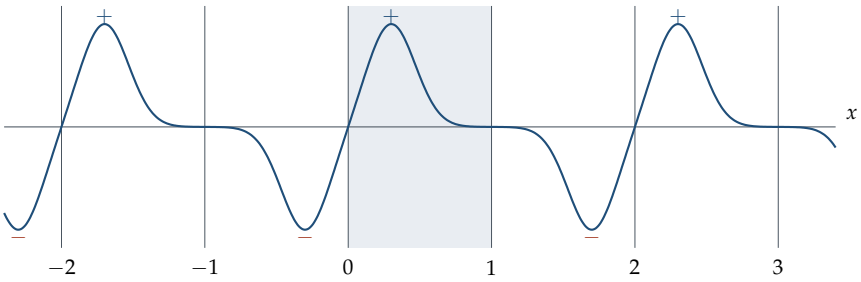
terms needed for accuracy 10^{-10} ($k=0.7$, $l=1$)	$t=0.001$	$t=0.05$	$t=1$
shells of images	0	1	4
eigenfunctions	56	8	1

[D6] For short times a particle has not yet met the walls and one or two images suffice; for long times it has forgotten where it started and one mode suffices. On the half-line the odd kernel applied to the constant 1 gives $\operatorname{erf}(x/\sqrt{4kt})$, the probability that a Brownian particle started at x has not yet reached the wall; the even kernel reflects it. [D5] So the reflection method does work for diffusion, in the form of a sum over the group, and its dual is the method that replaces it in the source.

Poisson summation is itself a naturality statement. For the quotient $\mathbb{R}\to\mathbb{R}/2l\mathbb{Z}$ and the dual inclusion of the lattice $\frac{\pi}{l}\mathbb{Z}$ in the line of frequencies, the square

$$\begin{array}{ccc} \mathcal{S}(\mathbb{R}) & \xrightarrow{\text{sum over } 2l\mathbb{Z}} & C^\infty(\mathbb{R}/2l\mathbb{Z}) \\ \Big\downarrow{\scriptstyle \text{Fourier transform}} & & \Big\downarrow{\scriptstyle \text{Fourier coefficients}} \\ \mathcal{S}(\mathbb{R}) & \xrightarrow[\text{restrict to } \frac{\pi}{l}\mathbb{Z}]{} & s(\mathbb{Z}) \end{array}$$

commutes, up to the constant $1/2l$ fixed by the normalizations: summing over a subgroup on one side is restricting to its annihilator on the other. Going round the top with the Gaussian $S(\cdot,t)$ gives the sum over images, and round the bottom the sum over eigenfunctions, since the transform of $S(\cdot,t)$ is $e^{-k\xi^2 t}$.



The Dirichlet kernel of the interval $(0,1)$, shaded, as a sum over images, for a source at $y = 0.3$ and $kt = 0.014$: positive copies at $y + 2m$, negative ones at $-y + 2m$. The sum is odd about every integer, so it vanishes at both ends of the interval.

4.6 Sources

Theorem 4.7 (Duhamel). [classical; source, §3.3] *The solution of $u_t - ku_{xx} = f$ with $u(\cdot,0) = \varphi$ is $u(t) = T_t\varphi + \int_0^t T_{t-s} f(s)\,ds$.*

The formula is that of variation of constants and holds for every semigroup; for the wave equation written as a first-order system it is Theorem 3.7. With $f = \sin x$ and $\varphi = 0$ it gives $(1 - e^{-t})\sin x$. [D9]

Notes

The source treats diffusion in its Sections 2.3–2.5, 3.1, 3.3 and 3.5. Semigroup theory is in [Paz83]. The identity between the rate of entropy and the Fisher information is de Bruijn's, see [CoTho6]; monotonicity of divergences under Markov kernels is the subject of [ESCHC]. The image and eigenfunction forms of the kernel on an interval are two expressions of a Jacobi theta function.

Exercises

Exercise 4.1. Derive the Gaussian from the ansatz $u = t^{-1/2} g(x/\sqrt{t})$.

Exercise 4.2. A particle diffuses with $k = 1$ from $x = 1$ on the half-line with an absorbing wall at 0. Find the probability that it has not been absorbed by time 1.

Exercise 4.3. Prove (a) and (b) of Theorem 4.5.

Exercise 4.4. Estimate, as functions of kt/l^2 and the tolerance ϵ, the number of image shells and of eigenfunctions needed in Theorem 4.6, and compare with the table.

Chapter 5

Boundary Conditions and Fourier Series

On a bounded interval there is no explicit kernel, and the source turns to Fourier's method: separate the variables, solve an eigenvalue problem in x, and superpose [source, Ch. 4–5]. Three questions organize the chapter. What is a boundary condition? Why sines and cosines? In what sense does the series converge?

5.1 Diagonalizing the generator

Let $A = -d^2/dx^2$ on $(0, l)$ with boundary conditions fixed. The diffusion and wave equations are $u_t = -kAu$ and $u_{tt} = -c^2 Au$. If $AX = \lambda X$, then $e^{-k\lambda t}X$ solves the first and $\cos(c\sqrt{\lambda}\,t)X$, $\sin(c\sqrt{\lambda}\,t)X$ solve the second. Fourier's method is the statement that A can be diagonalized, so that the semigroup and the group of the last two chapters become multiplication operators:

$$T_t = e^{-ktA}, \qquad \text{wave group} = \begin{pmatrix} \cos ct\sqrt{A} & \sin(ct\sqrt{A})/c\sqrt{A} \\ -c\sqrt{A}\sin ct\sqrt{A} & \cos ct\sqrt{A} \end{pmatrix}.$$

For Dirichlet conditions the eigenpairs are $(n\pi/l)^2$, $\sin(n\pi x/l)$, $n \geq 1$; for Neumann conditions $(n\pi/l)^2$, $\cos(n\pi x/l)$, $n \geq 0$. [F1]

5.2 A boundary condition is a Lagrangian plane

For functions f, g on $[a, b]$, Green's second identity in one dimension reads

$$\int_a^b (-f''g + fg'')\,dx = \left[-f'g + fg'\right]_a^b =: \omega\big(\beta(f), \beta(g)\big),$$

where $\beta(f) = (f(a), f'(a), f(b), f'(b)) \in \mathbb{R}^4$ is the vector of boundary values of f. The form ω is symplectic on the space $\mathbb{R}^4$ of boundary values. A pair of independent linear boundary conditions is a plane $V \subset \mathbb{R}^4$.

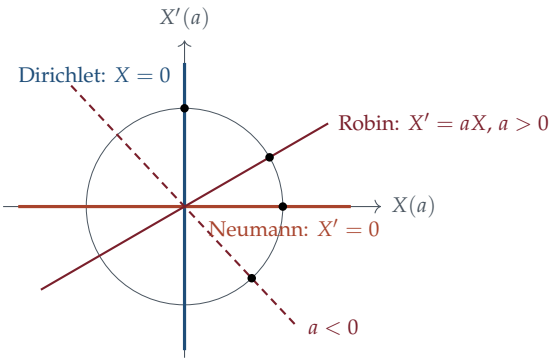
Theorem 5.1. [reformulation; cf. source, §5.3; [EvMa99, KoSc99]] *The boundary conditions V are symmetric in the source's sense if and only if V is Lagrangian: $\omega|_V = 0$. In that case eigenfunctions for distinct eigenvalues are orthogonal and all eigenvalues are real (the source's Theorems 1 and 2). The self-adjoint realizations of $-d^2/dx^2$ on $[a, b]$ are*

in bijection with the Lagrangian planes of $(\mathbb{R}^4, \omega)$*, a compact manifold of dimension three.*

Proof. The first statement is the definition, since the source's condition is $\omega(\beta(f), \beta(g)) = 0$ for all f, g satisfying the conditions, and a plane on which a symplectic form on $\mathbb{R}^4$ vanishes is Lagrangian. For eigenfunctions X_1, X_2 the identity gives $(\lambda_1 - \lambda_2) \int X_1 X_2 = \omega(\beta(X_1), \beta(X_2)) = 0$. □

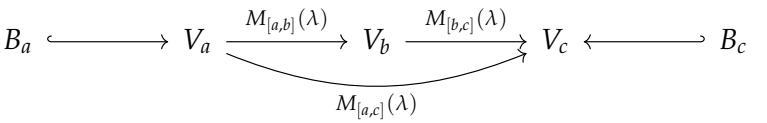
condition	plane V spanned by	ω on the two vectors
Dirichlet	$(0,1,0,0)$, $(0,0,0,1)$	0
Neumann	$(1,0,0,0)$, $(0,0,1,0)$	0
Robin	$(1,a_0,0,0)$, $(0,0,1,-a_l)$	0
periodic	$(1,0,1,0)$, $(0,1,0,1)$	0
$X(a) = X(b)$, $X'(a) = 2X'(b)$	$(1,0,1,0)$, $(0,2,0,1)$	-1

[F3] The last row is the source's example of conditions for which the method fails. The first three are *local*: V is a sum of a line at a and a line at b, and the lines at one endpoint form a circle, with Dirichlet and Neumann two of its points and the Robin constant a coordinate on the rest. The periodic condition is non-local: it is the graph of the identity between the two ends, which is what it means to glue them.



The boundary conditions at one end of an interval are the lines through the origin in the plane of (X, X')*, a circle's worth. The Dirichlet and Neumann conditions are two points of it and the Robin conditions fill in the rest; a separated condition for the interval is a point of the product of two such circles.*

Remark 5.2 (Intervals compose). For fixed λ the solutions of $-X'' = \lambda X$ on $[a, b]$ have boundary values forming a Lagrangian plane in $\mathbb{R}^4$, the graph of the transfer matrix from $(X(a), X'(a))$ to $(X(b), X'(b))$, which has determinant one. Joining $[a, b]$ to $[b, c]$ multiplies transfer matrices, that is, composes Lagrangian relations. An eigenvalue is a λ for which the plane of the equation meets the plane of the boundary conditions. This compositional picture returns for networks in Chapter 7 and for scattering in Chapter 13.



Here $V_x = \mathbb{R}^2$ holds the pair $(X(x), X'(x))$ and a separated boundary condition is a line B in it. The number λ is an eigenvalue exactly when the line $M_{[a,c]}(\lambda)B_a$ is the line B_c: the shooting method, read as a composite of relations.

The Robin condition. With $X' - a_0 X = 0$ at 0 and $X' + a_l X = 0$ at l, the quadratic form of A is $\int X'^2 + a_0 X(0)^2 + a_l X(l)^2$, which is positive when both constants are positive (radiation) and can be negative when they are negative (absorption). The source finds a zero eigenvalue exactly when $a_0 + a_l = -a_0 a_l l$ [source, §4.3]. For $a_0 = a_l = -1$ that is $l = 2$: there is one negative eigenvalue for $l < 2$ and two for $l > 2$; at $l = 1$ it is -2.3821. [F2] A negative eigenvalue $-\gamma^2$ means that the diffusion equation has the growing solution $e^{k\gamma^2 t}X(x)$: the boundary feeds the interior faster than diffusion can spread it.

5.3 Why sines and cosines

Take periodic conditions on $(-l, l)$: the domain is the circle $\mathbb{T} = \mathbb{R}/2l\mathbb{Z}$, a compact abelian group. The operator A commutes with translations, so its eigenspaces are representations of $\mathbb{T}$, and the irreducible ones are the characters $e^{in\pi x/l}$, $n \in \mathbb{Z}$.

Theorem 5.3 (Fourier series as duality)**.** [classical] *The characters of $\mathbb{T}$ form the dual group $\mathbb{Z}$, and the map $f \mapsto (\hat{f}(n))_{n\in\mathbb{Z}}$ is a unitary isomorphism $L^2(\mathbb{T}) \to \ell^2(\mathbb{Z})$ that carries every translation-invariant operator to a multiplication operator; A becomes multiplication by $(n\pi/l)^2$.*

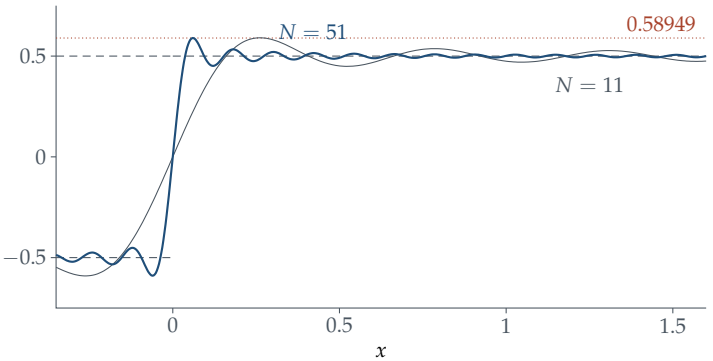
The reflection $x \mapsto -x$ acts on $\mathbb{T}$ and exchanges n and $-n$. Its odd part is spanned by the sines and its even part by the cosines, and by Theorem 3.5 these are the Dirichlet and Neumann problems on $(0, l)$: the interval is the circle of twice its length modulo reflection. The sine and cosine series of the source [source, §5.1–5.2] are the two isotypic components of the full Fourier series. On the finite group $\mathbb{Z}/n$ the whole statement is a computation: the characters diagonalize the periodic difference Laplacian with eigenvalues $4\sin^2(\pi k/n)$, and their even and odd parts under reflection are the eigenvectors of the Neumann and Dirichlet matrices. [F7] For the function x on $(0, l)$ Parseval's identity gives $\sum n^{-2} = \pi^2/6$. [F4]

5.4 Completeness, and in which category

The source distinguishes uniform, mean-square and pointwise convergence and proves a theorem for each [source, §5.4].

Theorem 5.4. [classical; source, §5.4, Theorems 2–6] *For a symmetric boundary condition:* (a) *the eigenfunction expansion of every $f \in L^2$ converges to f in mean square, and Parseval's equality holds;* (b) *if f is C^2 and satisfies the boundary conditions the convergence is uniform;* (c) *for the classical series, if f and f' are piecewise continuous the series converges at each point to the mean of the one-sided limits. The partial sum is the best approximation in mean square among combinations of the first N eigenfunctions.*

Statement (a) says that the coefficient map is an isomorphism in the category of Hilbert spaces. It is not an isomorphism of the Banach space of continuous functions onto a sequence space: there are continuous functions whose Fourier series diverge at a point. Completeness, like well-posedness (Definition 2.2), is a statement about a morphism in a specified category. The Gibbs phenomenon is the visible form of the difference: near a unit jump the partial sums overshoot to $\frac{1}{\pi}\int_0^{\pi}\frac{\sin\theta}{\theta}d\theta = 0.58949$ where the function is $\frac{1}{2}$, an excess of 8.95 per cent of the jump that does not decrease with N (the source's "9 percent"); the peaks at $N = 51, 201, 801$ are 0.58955, 0.58949, 0.58949. [F5] The series converges in L^2 and not uniformly.



Partial sums of the Fourier series of a function with a unit jump at 0. The first peak moves towards the jump as N grows and its height does not decrease: it is 0.58955 for $N = 51$ and tends to 0.58949.

Smoothness of f is decay of $\hat{f}$: the coefficients of a function with a jump fall like $1/n$, with a corner like $1/n^2$, and for an analytic function geometrically. [F6] This is the dictionary by which regularity of solutions is read off from the multipliers $e^{-k\lambda_n t}$ (smoothing) and $\cos c\sqrt{\lambda_n}\,t$ (no smoothing).

5.5 Inhomogeneous boundary values

If the boundary values are not zero the solutions no longer form a vector space but an affine space over the solutions of the homogeneous problem. The source's method of shifting the data [source, §5.6] chooses an origin: any function w with the right boundary values, after which $u - w$ solves a homogeneous problem with a source. For $u_t = u_{xx}$ with $u(0) = 0$, $u(1) = 1$ the natural choice is the steady state $w = x$, and the solution from zero initial data is $x + \sum_n \frac{2(-1)^n}{n\pi} e^{-n^2\pi^2 t} \sin n\pi x$. [F8]

Notes

Boundary conditions as Lagrangian subspaces go back to von Neumann's extension theory; for ordinary differential operators see [EvMa99], and for networks of intervals [KoSc99]. Fourier analysis on locally compact abelian groups is in [Rud62]. The convergence theorems and the Gibbs phenomenon are treated as in the source; for a continuous function with a divergent Fourier series see [Kato4].

Exercises

Exercise 5.1. Verify the last column of the table of boundary conditions.

Exercise 5.2. Find the eigenvalues and eigenfunctions for the Dirichlet condition at 0 and the Neumann condition at l, and recover Exercise 3.3.

Exercise 5.3. Derive the criterion $a_0 + a_l = -a_0 a_l l$ for a zero eigenvalue of the Robin problem.

Exercise 5.4. Given $\int_0^\pi \frac{\sin\theta}{\theta} d\theta = 1.85194$, find the overshoot of the partial sums of a function with a jump of height J.

Part III

Space

Chapter 6

Harmonic Functions

Laplace's equation has no time, and its theory is a theory of how a function on a domain is controlled by its boundary values and by the shape of the domain [source, Ch. 6]. Harmonic functions form a sheaf; this chapter describes its symmetries, its global sections in terms of the boundary, and its cohomology.

6.1 The sheaf and its two principles

For U open in $\mathbb{R}^n$ let $\mathcal{H}(U)$ be the space of harmonic functions on U. Restrictions of harmonic functions are harmonic and local harmonic functions glue, so $\mathcal{H}$ is a sheaf, the solution sheaf of Δ.

Theorem 6.1. [classical; source, §6.1 and §6.3] (Mean value.) *A continuous function is harmonic if and only if its value at each point equals its average over every sphere centred there and contained in the domain.* (Maximum principle.) *A harmonic function on a bounded connected domain, continuous up to the boundary, attains its maximum and minimum only on the boundary unless it is constant. Hence the Dirichlet problem has at most one solution, and it depends continuously on the boundary values in the uniform norm.*

For a harmonic cubic the circle average equals the central value to twelve digits; for x^2 it exceeds it by $r^2/2$. On a grid, a discrete harmonic function with random boundary values stays strictly between their extremes. [H1, H2]

6.2 Naturality

Theorem 6.2 (Symmetries of the Laplacian). [classical]

(a) $\Delta(u \circ g) = (\Delta u) \circ g$ *for every rigid motion* g*, and the second-order operators with this property are* $a\Delta + b$*.*

(b) In the plane, $\Delta(u \circ f) = |f'|^2 (\Delta u) \circ f$ *for every holomorphic* f*. Hence* $\mathcal{H}$ *is a sheaf on Riemann surfaces: harmonicity is conformally invariant in dimension two.*

(c) In dimension $n \geq 3$ *the Kelvin transform* $u \mapsto |x|^{2-n} u(x/|x|^2)$ *preserves harmonic functions.*

Statements (a) and (b) are polynomial identities, verified exactly for a rotation with rational entries and for $f(z) = z^2$; a shear fails (a). [H4] By Proposition 2.1 this is the expected answer: the Laplacian is built from d, which is natural for everything, and a Hodge star, which is natural for isometries, and in the plane the star on 1-forms happens to depend only on the conformal class. The source's computation of the Laplacian in polar coordinates [source, §6.1] is (a) used to choose coordinates adapted to the rotation group; the separated solutions $r^n \cos n\theta$, $r^n \sin n\theta$ are the real and imaginary parts of z^n, and (b) explains why conformal mapping solves plane potential problems.

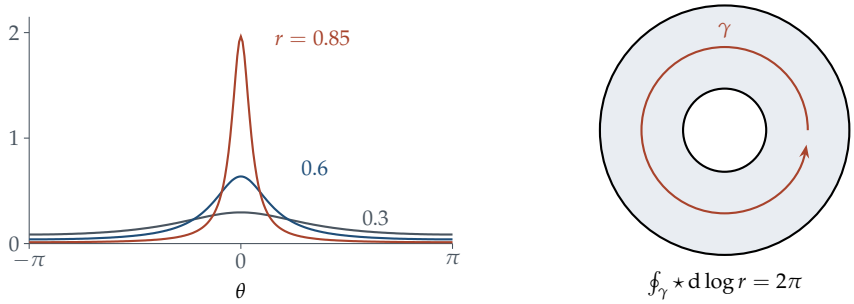
6.3 From the boundary: Poisson and Brownian motion

Theorem 6.3 (Poisson's formula). [classical; source, §6.3, Theorem 1] *For continuous h on the circle of radius a,*

$$u(r,\theta) = \frac{a^2 - r^2}{2\pi} \int_0^{2\pi} \frac{h(\phi)\, d\phi}{a^2 - 2ar\cos(\theta - \phi) + r^2}$$

is the harmonic function in the disk with boundary values h.

The kernel is positive and has total mass one. [H3] So the solution operator of the Dirichlet problem is a Markov kernel from the domain to its boundary: to each interior point x it assigns a probability measure on the boundary, the ***harmonic measure*** seen from x. The mean value property, the maximum principle and the continuity of the solution in the data are all properties of Markov kernels.



Left: the Poisson kernel of the unit disk at three radii; as $r \to 1$ it concentrates at the boundary point nearest to the observer. Right: on an annulus the conjugate differential of $\log r$ has a non-zero period around the hole, so $\log r$ is not the real part of a holomorphic function (Proposition 6.5).

Theorem 6.4 (Kakutani). [classical; source, §6.1, Example 4; [Kak44]] *The harmonic measure from x is the distribution of the point at which Brownian motion started at x first reaches the boundary: $u(x) = \mathbb{E}\, h(B_\tau)$.*

On a 7×7 grid the probability that a random walk leaves through the top side solves the five-point equation, agrees with the limit of the walk to ten digits, and equals $\frac{1}{4}$ at the centre. [H7] Composition of kernels gives the consistency of harmonic measures for nested domains: to exit the large domain, first exit the small one. In the language of [ESCHC], the Dirichlet problem is a morphism in Stoch, and the heat semigroup of Chapter 4 and the Poisson kernel are two views of one process.

6.4 The annulus has cohomology

In the plane, a harmonic function u on a disk is the real part of a holomorphic function: it has a ***harmonic conjugate*** v with $\mathrm{d}v = \star\,\mathrm{d}u = -u_y\,dx + u_x\,dy$.

Proposition 6.5. [classical] *On a plane domain U the form $\star\,\mathrm{d}u$ is closed for harmonic u, and u has a harmonic conjugate on U if and only if its class in $H^1(U;\mathbb{R})$ vanishes. On an annulus this class is the period $\oint \star\,\mathrm{d}u$ around the hole. The period of $\log r$ is 2π; the periods of $r^{\pm n}\cos n\theta$ and $r^{\pm n}\sin n\theta$ are zero.*

The source's solution of the Dirichlet problem in an annulus [source, §6.4] has, besides the terms that appear for the disk, the functions $r^{-n}\cos n\theta$, $r^{-n}\sin n\theta$ and $\log r$. The first two families are the real parts of z^{-n}, holomorphic in the annulus. The last is not the real part of anything: it generates the cokernel of $\mathrm{Re}\colon \mathcal{O}(U) \to \mathcal{H}(U)$, which is $H^1(U;\mathbb{R}) = \mathbb{R}$. The three periods were computed numerically as 2π, 0, 0. [H5] This is the first cohomology class of the volume that is not zero, and its discrete counterpart is computed in Chapter 12.

6.5 Corners

Proposition 6.6. [classical; source, §6.4] *In a wedge of opening β the harmonic functions vanishing on both edges are spanned near the vertex by $r^{n\pi/\beta}\sin(n\pi\theta/\beta)$. The gradient of the first behaves like $r^{\pi/\beta-1}$: it is bounded if $\beta \le \pi$ and unbounded if $\beta > \pi$.*

At a re-entrant corner of angle $3\pi/2$ the exponent is $\frac{2}{3}$ and the gradient grows like $r^{-1/3}$. [H6] Interior regularity of harmonic functions is unconditional; regularity at the boundary is a property of the boundary. The exponent π/β is the reason that finite element methods lose accuracy on an L-shaped domain (Chapter 12), and it is again conformal naturality: $z \mapsto z^{\pi/\beta}$ opens the wedge to a half-plane.

Notes

The source's Chapter 6 is followed throughout. For conformal invariance and the Kelvin transform see [ABR01]; for Brownian motion and harmonic measure [Kak44, MoPe10]; for corner singularities [Gri85]. The Dirichlet problem in rectangles and cubes [source, §6.2] is separation of variables, taken up in Chapter 9.

Exercises

Exercise 6.1. For a quadratic polynomial u in the plane, show that the average over a circle of radius r exceeds the value at the centre by $\frac{r^2}{4}\Delta u$.

Exercise 6.2. Verify Theorem 6.2(b) for $f(z) = z^2$ and $u = x^2$.

Exercise 6.3. Compute $\oint \star\, \mathrm{d}u$ around the unit circle for $u = \log r$ and for $u = \operatorname{Re}(1/z)$.

Exercise 6.4. Compare the behaviour of the gradient of the first singular function at the re-entrant corner of an L-shaped room and at one of its convex corners.

Chapter 7

Green's Identities and Green's Functions

The source derives its remaining general facts about harmonic functions from two integral identities and then inverts the Laplacian by a kernel [source, Ch. 7]. Both steps have a structural reading, and the second has a compositional one: boundary-value problems can be glued.

7.1 Stokes' theorem, twice

Proposition 7.1 (Green's identities). [classical; source, §7.1–7.2] *For functions on a bounded domain D with smooth boundary,*

$$\int_D v\,\Delta u + \int_D \nabla v \cdot \nabla u = \int_{\partial D} v\,\partial_n u,$$
$$\int_D (u\,\Delta v - v\,\Delta u) = \int_{\partial D} (u\,\partial_n v - v\,\partial_n u).$$

Both are Stokes' theorem for the form $v \star \mathrm{d}u$. On a grid the first is the factorization of Proposition 2.1: the Dirichlet Laplacian is $D^{\mathsf{T}}D$. [G1] The right-hand side of the second is a symplectic form on the space of pairs (boundary values, normal derivatives), the same form as in Theorem 5.1.

Corollary 7.2 (The Dirichlet-to-Neumann map is symmetric). [reformulation] *The pairs $(u|_{\partial D}, \partial_n u|_{\partial D})$ for u harmonic in D form a Lagrangian subspace of the boundary data. It is the graph of the operator $\Lambda_D \colon h \mapsto \partial_n u$, where u is the harmonic function with boundary values h; hence Λ_D is symmetric, and by the first identity non-negative.*

7.2 Dirichlet's principle

Theorem 7.3. [classical; source, §7.1] *Among functions with given boundary values, the harmonic one has the least energy $E(w) = \frac{1}{2}\int_D |\nabla w|^2$. More precisely, if u is harmonic and z vanishes on the boundary, then $E(u+z) = E(u) + E(z)$.*

The identity is the Pythagorean theorem for the energy inner product: harmonic functions are orthogonal to functions vanishing on the boundary. [G2] So the solution operator of the Dirichlet problem is an orthogonal projection. The

same holds in any subspace, which is the Rayleigh–Ritz or Galerkin method [source, §7.1, Exercise 7]: the approximate solution in a trial space is the projection of the true one, and its error is orthogonal to the trial space. [G7] Finite elements are this statement with a particular trial space (Chapter 12).

7.3 The kernel of the inverse

Definition 7.4 (source, §7.3)**.** The ***Green's function*** of D with pole x_0 is the function $G(x, x_0)$ that is harmonic in D away from x_0, vanishes on ∂D, and differs from $-1/(4\pi|x - x_0|)$ by a function harmonic near x_0 (in three dimensions).

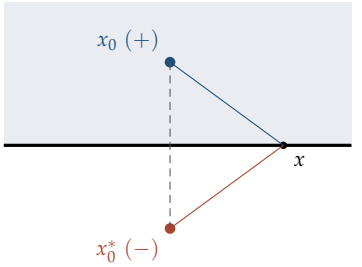
Theorem 7.5. [classical; source, §7.3] (a) *The solution of $\Delta u = f$ in D, $u = h$ on ∂D is $u(x_0) = \int_{\partial D} h\, \partial_n G + \int_D f G$.* (b) *$G(x, x_0) = G(x_0, x)$.* (c) *$G \le 0$ in D, and $\partial_n G \ge 0$ on ∂D with total mass one.*

In operator terms, $-G$ is the kernel of the inverse of $-\Delta$ with Dirichlet conditions; (b) says the inverse is symmetric because the operator is; (c) says the inverse preserves positivity, which is the maximum principle, and identifies $\partial_n G$ with the harmonic measure of Theorem 6.3. On eight grid points the inverse of the Dirichlet Laplacian is the symmetric positive matrix $\min(i, j)(9 - \max(i, j))/9$, the discrete form of the kernel $x(1 - y)$ for $x < y$ on the unit interval. [G3, G8]

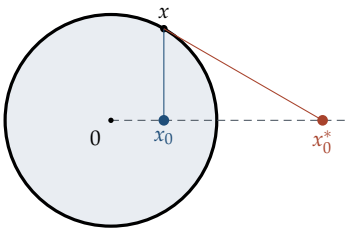
7.4 Half-space and ball

Theorem 7.6. [classical; source, §7.4] *For the half-space $\{z > 0\}$, $G(x, x_0) = -\frac{1}{4\pi|x - x_0|} + \frac{1}{4\pi|x - x_0^*|}$ with x_0^* the mirror image of x_0. For the ball of radius a, $G(x, x_0) = -\frac{1}{4\pi|x - x_0|} + \frac{a}{|x_0|}\frac{1}{4\pi|x - x_0^*|}$ with $x_0^* = a^2 x_0/|x_0|^2$, and $\partial_n G = \dfrac{a^2 - |x_0|^2}{4\pi a|x - x_0|^3}$ on the sphere.*

The first is Theorem 3.5 for the reflection in a plane with the sign character. The second is the same after a conformal change: inversion in the sphere exchanges the ball with its exterior, and the factor $a/|x_0|$ is the weight of the Kelvin transform of Theorem 6.2(c). Both kernels vanish on the boundary and have normal derivatives of total mass one; the Poisson kernel of the ball returns the value r_0 of the harmonic function z at height r_0. [G4, G5]



half-space: equal distances on the wall



ball: $|x - x_0^*|/|x - x_0| = a/|x_0|$ on the sphere

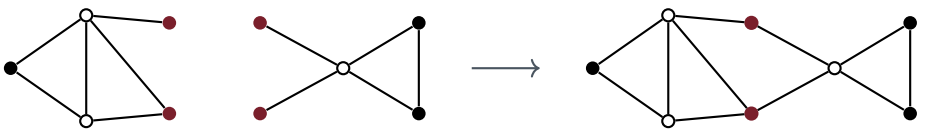
The two image constructions. In the half-space the image is the mirror point with the opposite charge. In the ball of radius a it is the inverse point $x_0^* = a^2 x_0/|x_0|^2$ with charge $-a/|x_0|$: on the sphere the ratio of the two distances is constant (here $a = 1.4$, $|x_0| = 0.7$, ratio 2), which is what makes the sum vanish there.

7.5 Gluing

Let a ***network*** be a finite graph with positive conductances on its edges and a chosen set of ***terminals***; a potential is harmonic if Kirchhoff's law holds at every non-terminal vertex. Eliminating the interior vertices from the Laplacian (a Schur complement, or Kron reduction) gives the matrix Λ that sends terminal potentials to terminal currents: the discrete Dirichlet-to-Neumann map.

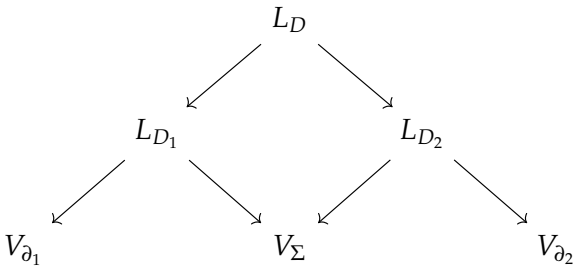
Theorem 7.7 (Black-boxing is a functor)**.** [literature; [BaFo18]] *Networks with terminals are the morphisms of a category in which composition glues terminals. Sending a network to the graph of its Dirichlet-to-Neumann map is a functor to the category* LagRel *of symplectic vector spaces and Lagrangian relations. In particular the response of a glued network is determined by the responses of its parts.*

For ten pairs of random networks glued along two shared terminals, the response of the whole agreed with the composite of the responses of the parts to ten digits; conductances 2 and 3 in series give $6/5$. [G6] The continuous statement is the same with Corollary 7.2 in place of the matrix: if $D = D_1 \cup_\Sigma D_2$, the Lagrangian subspace of D is the composite of those of D_1 and D_2 over the interface Σ. Domain decomposition methods in numerical analysis are this functoriality used as an algorithm: solve on the pieces, then solve an equation on the interface.



D_1 D_2 $D_1 \cup_\Sigma D_2$

Gluing two networks along the shared terminals Σ (in colour). Filled nodes are terminals, open ones are interior. After gluing, the nodes of Σ are interior, and Kirchhoff's law there is the condition that the currents from the two sides cancel.



Each V is a space of pairs (potential, current) on a piece of boundary, with the sign of the current reversed on one side of Σ so that what leaves D_1 enters D_2. The top object is the image of the pullback: pairs of solutions on the two parts that agree on the interface in value and in flux, which is to say solutions on the whole.

Remark 7.8. Theorem 5.1, Corollary 7.2 and Theorem 7.7 are one statement at three levels: boundary conditions, solutions and parts of a domain are all Lagrangian relations on spaces of boundary data, and a boundary-value problem is solved by composing them. An elliptic problem is a morphism from the empty boundary to the boundary data, and its solution is the composite with the boundary condition. This is as close as the elliptic theory comes to the time-evolution functors of the hyperbolic and parabolic theories.

Notes

The source's Chapter 7 is followed for the identities, Dirichlet's principle and Green's functions. Black-boxing of passive linear networks as a functor to Lagrangian relations is due to Baez and Fong [BaFo18]; Kron reduction is surveyed in [DoBu13]. For Dirichlet-to-Neumann operators and domain decomposition see [QuVa99].

Exercises

Exercise 7.1. Deduce the symmetry of Λ_D from Green's second identity.

Exercise 7.2. Compute the Kron reduction of two conductances g_1, g_2 in series onto the two end terminals.

Exercise 7.3. Show that $-d^2/dx^2$ on $(0,1)$ with Dirichlet conditions has the inverse kernel $x(1-y)$ for $x < y$, and expand it in eigenfunctions.

Exercise 7.4. Show that for $|x| = a$ one has $|x - x_0^*| = (a/|x_0|)\,|x - x_0|$, and deduce that the Green's function of the ball vanishes on the sphere.

Chapter 8

Waves and Quanta in Space

In three dimensions the wave equation keeps its causal structure and gains a property it did not have on the line; the diffusion kernel becomes a product; and a third evolution equation, Schrödinger's, turns out to be both of the others at once [source, Ch. 9].

8.1 Cones

Theorem 8.1 (Energy and causality). [classical; source, §9.1] *For $u_{tt} = c^2\Delta u$ in space the energy $\frac{1}{2}\int (u_t^2 + c^2|\nabla u|^2)$ is conserved, and the value of u at (x_0, t_0) depends only on the data in the ball $|x - x_0| \le ct_0$.*

The proof integrates the energy identity over a truncated cone. The cone is the characteristic variety of Chapter 1, $\tau^2 = c^2|\xi|^2$, seen in space-time. As in Theorem 3.3, restriction to a spacelike surface is an isomorphism from solutions in its domain of dependence to pairs of data [source, §9.3, Theorem 2].

A function $f(t - \gamma(x))$ of a phase solves the equation for every profile f exactly when the phase satisfies the eikonal equation $|\nabla\gamma| = 1/c$; these are the surfaces the source calls characteristic. A plane phase with $|\nabla\gamma| = 1/c$ passes the test and one with the wrong slope fails it. [S1]

8.2 Kirchhoff, Huygens and descent

Theorem 8.2 (Kirchhoff). [classical; source, §9.2] *In three dimensions the solution with data $u = \varphi$, $u_t = \psi$ is*

$$u(x,t) = \frac{1}{4\pi c^2 t}\int_{|y-x|=ct} \psi\, dS + \partial_t\Big[\frac{1}{4\pi c^2 t}\int_{|y-x|=ct} \varphi\, dS\Big].$$

It depends only on the data on *the sphere of radius ct.*

This is ***Huygens's principle***: a disturbance of bounded extent is heard for a bounded time. The reason is visible for radial solutions: $(ru)_{tt} = c^2(ru)_{rr}$, so ru solves the one-dimensional equation and $u = (F(r - ct) + G(r + ct))/r$. [S1]

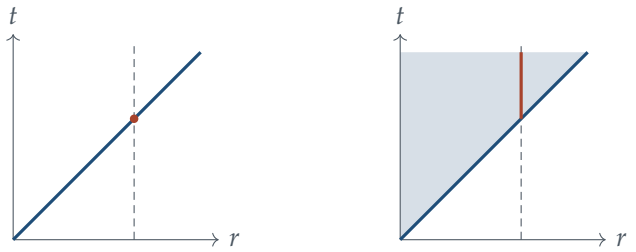
In two dimensions the source obtains the solution by ***descent***: a function of (x, y) is a function of (x, y, z) that does not depend on z.

Proposition 8.3 (Descent loses sharpness). [reformulation; cf. source, §9.2] *Solutions in the plane are the solutions in space invariant under translation in z. The pushforward of the uniform measure on a sphere of radius R under projection to the plane has the density $\left(2\pi R\sqrt{R^2-\rho^2}\right)^{-1}$ on the disk, so*

$$u(x,t) = \frac{1}{2\pi c}\int_{|y-x|<ct} \frac{\psi(y)\,dy}{\sqrt{c^2t^2-|y-x|^2}} + \cdots$$

depends on the data in the whole disk, with a tail of order $1/t$.

For a source in the unit ball heard at distance 5 with $c = 1$, the three-dimensional signal is zero at $t = 3$, non-zero at $t = 5$ and exactly zero again at $t = 6.5$; the two-dimensional one never stops, and $t\,u(t)$ is 1.29, 1.002, 1.0000 times its limit at $t = 8$, 80, 800. [S2] Sharp propagation holds in odd dimensions from three upwards and fails in even ones; the source's remark that three is the best of all possible dimensions is the remark that it is the smallest of these. The structural point is that the property is not inherited by invariants under a non-compact group: the plane is the quotient of space by a line, and the shadow of a sphere is a disk.



The support of the response to an impulse at the origin. A listener at fixed distance (dashed) hears an instant in three dimensions and a tail that never ends in two.

8.3 Singularities travel along rays

Theorem 8.4 (Propagation of singularities). [literature; [Hor71, DuHo72]] *For a solution u of $Pu = 0$ with P of real principal type, the wave front set $\mathrm{WF}(u) \subseteq T^*M$ is contained in $\mathrm{Char}(P)$ and is a union of bicharacteristics, the integral curves of the Hamiltonian vector field of the principal symbol.*

For the wave equation the bicharacteristics project to light rays, and the theorem contains the source's statement that only characteristic surfaces can carry singularities [source, §9.3, Theorem 3]. For an elliptic operator $\mathrm{Char}(P)$ is empty and solutions are smooth. For the heat equation the characteristic directions are dt and singularities do not propagate at finite speed at all. The

wave front set lives on the cotangent bundle; that singularities have a direction as well as a position is the starting point of microlocal analysis, where the solution sheaf is studied on T^*M [KaSc90].

8.4 Diffusion, and Schrödinger's equation

In n dimensions the diffusion kernel is the product of one-dimensional kernels [source, §9.4]: the semigroup on $L^2(\mathbb{R}^n) = L^2(\mathbb{R})^{\otimes n}$ is $T_t \otimes \cdots \otimes T_t$. Everything in Chapter 4 holds unchanged.

The free Schrödinger equation $iu_t = -\frac{1}{2}\Delta u$ is formally the diffusion equation at imaginary time.

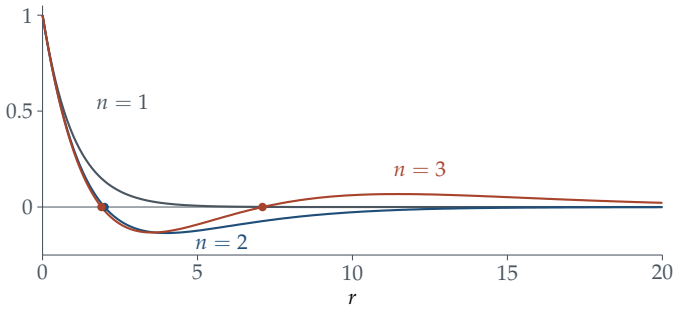
Theorem 8.5 (One semigroup, two boundary values). [classical; [ReSi75]] *Let H be a non-negative self-adjoint operator. Then $z \mapsto e^{-zH}$ is a holomorphic semigroup of contractions on $\operatorname{Re} z > 0$, strongly continuous up to the imaginary axis. Its restriction to $z = t > 0$ is the heat semigroup; its restriction to $z = it$ is the unitary group solving $iu_t = Hu$.*

The kernel $(2\pi it)^{-1/2} e^{ix^2/2t}$ solves the equation and equals the diffusion kernel evaluated at time it. A Gaussian packet keeps its norm exactly while its mean square width grows as $(1 + t^2)/2$. [S3] In the table of Chapter 4 Schrödinger's equation takes the first row from waves (a group, here unitary) and the generator from diffusions; it has infinite speed of propagation and no maximum principle, and it neither transports singularities sharply nor erases them, but disperses them. For the harmonic oscillator $-v'' + x^2 v = \lambda v$ the eigenvalues are $1, 3, 5, \ldots$ [S5]

8.5 The hydrogen atom

Theorem 8.6. [classical; source, §9.5 and §10.7] *The equation $-\Delta v - \frac{2}{r} v = \lambda v$ in space has the square-integrable solutions $v = w(r) e^{-r/n} Y(\theta, \phi)$ with eigenvalues $\lambda = -1/n^2$, $n = 1, 2, \ldots$; the eigenvalue $-1/n^2$ has multiplicity n^2.*

The radial series terminates exactly when the exponent is the reciprocal of an integer, and the polynomials for $n = 1, 2, 3$ are 1, $1 - \frac{1}{2}r$, $1 - \frac{2}{3}r + \frac{2}{27}r^2$, as in the source's table. [S4] The multiplicity is $\sum_{l<n}(2l + 1)$: rotational symmetry accounts for the factor $2l + 1$ (Chapter 9) and does not explain why different l share an eigenvalue.



The radial functions $w_n(r)e^{-r/n}$ of hydrogen for $n = 1, 2, 3$ and angular momentum zero. The n-th has $n-1$ nodes (marked), and its tail is longer by the factor n: the eigenvalue $-1/n^2$ is read off the rate of decay.

Remark 8.7 (The hidden symmetry). [literature; [Foc35]] The bound states of the Coulomb problem carry an action of SO(4), and those of energy $-1/n^2$ form its irreducible representation on harmonic polynomials of degree $n-1$ in four variables, whose dimension is n^2 by Remark 1.6. [E4] The "accidental" degeneracy is forced by a symmetry that is not a symmetry of space.

Notes

The source's Chapter 9 is followed. For Huygens's principle in general dimension see [Eva10]; for wave front sets and propagation [Hor71, DuHo72]; for holomorphic semigroups [ReSi75]. The SO(4) symmetry of the hydrogen atom is Fock's [Foc35].

Exercises

Exercise 8.1. Show that the radial solutions of the wave equation in space are $(F(r-ct)+G(r+ct))/r$.

Exercise 8.2. Derive the density in Proposition 8.3 and hence the two-dimensional formula from Kirchhoff's.

Exercise 8.3. For $iu_t = -\frac{1}{2}u_{xx}$ with $u(x,0) = e^{-x^2/2}$, show that $u = (1+it)^{-1/2}e^{-x^2/2(1+it)}$ and that the mean square width of $|u|^2$ is $(1+t^2)/2$.

Exercise 8.4. Find the radial polynomial for $n = 3$ from the recursion $\frac{k(k+1)}{2}a_k = (\frac{k}{3}-1)a_{k-1}$, and count the states with eigenvalue $-\frac{1}{9}$.

Chapter 9

Symmetry and Special Functions

The source returns to Fourier's method for domains in the plane and in space, and meets Bessel and Legendre functions [source, Ch. 10]. The organizing fact is one sentence: the Laplacian commutes with the symmetries of the domain, so its eigenspaces are representations of the symmetry group.

9.1 Eigenspaces are representations

Theorem 9.1. [reformulation; cf. source, §10.1] *Let G be a group of isometries of a bounded domain D preserving the boundary condition. Then G acts on each eigenspace of the Laplacian. Consequently $L^2(D)$ is the orthogonal sum of its isotypic components, each preserved by the Laplacian; and an eigenvalue whose eigenspace contains an irreducible representation of dimension d has multiplicity at least d.*

The Laplacian of a square grid commutes with the quarter turn and its second eigenvalue is double, the symmetry group of the square having a two-dimensional irreducible representation; on a 5×6 grid, with only reflections, it is simple. [Y6]

Theorem 9.2 (Products). [classical; source, §11.5, Theorem 2] $L^2(D_1 \times D_2) = L^2(D_1) \otimes L^2(D_2)$ *and* $\Delta_{D_1 \times D_2} = \Delta_{D_1} \otimes 1 + 1 \otimes \Delta_{D_2}$. *The eigenfunctions of a product are the products of eigenfunctions, and its eigenvalues the sums.* [Y4]

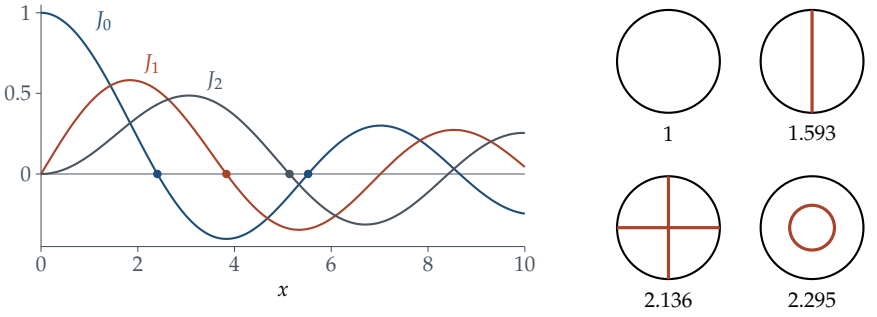
Separation of variables is therefore two different things. In Cartesian coordinates it is the tensor product. In polar and spherical coordinates it is the decomposition of Theorem 9.1 under a rotation group, followed by an ordinary differential equation in the radius. The special functions are what the second kind produces.

9.2 The drum

For the disk of radius a the group is $\mathrm{O}(2)$. The irreducible representations of the rotations are the characters $e^{in\theta}$; in the n-th isotypic component the eigenvalue equation becomes Bessel's equation $R'' + \frac{1}{r}R' + (\lambda - \frac{n^2}{r^2})R = 0$, with the solutions $J_n(\sqrt{\lambda}r)$ regular at the centre.

Theorem 9.3. [classical; source, §10.2 and §10.5] *The Dirichlet eigenvalues of the disk are* $(z_{nm}/a)^2$*, with* z_{nm} *the m-th positive zero of* J_n*, and eigenfunctions* $J_n(z_{nm}r/a)\cos n\theta$*,* $J_n(z_{nm}r/a)\sin n\theta$*. Eigenvalues with* $n \geq 1$ *are double.*

The first zeros are 2.4048 ($n = 0$), 3.8317 ($n = 1$), 5.1356 ($n = 2$), 5.5201 ($n = 0$), so the frequencies of a drum are in the ratios $1 : 1.593 : 2.136 : 2.295$ and not $1 : 2 : 3 : 4$. [Y3]



Left: the Bessel functions J_0, J_1, J_2 *with the four smallest zeros* 2.405, 3.832, 5.136, 5.520. *Right: the nodal sets of the corresponding modes of a circular drum, labelled by frequency relative to the fundamental. The angular index counts nodal diameters and is the weight of the rotation group; the last mode is the second radial one, with a nodal circle at* 2.405/5.520 *of the radius.*

Remark 9.4 (What a Bessel function is). [classical] The plane wave e^{iy} is an eigenfunction of the Laplacian with eigenvalue -1. Its component in the n-th character of the rotation group is $J_n(r)e^{in\theta}$:

$$e^{ir\sin\theta} = \sum_{n\in\mathbb{Z}} J_n(r)e^{in\theta}, \qquad J_n(r) = \frac{1}{2\pi}\int_0^{2\pi} e^{i(r\sin\tau - n\tau)}\,d\tau.$$

Since the projection onto an isotypic component commutes with the Laplacian, $J_n(r)e^{in\theta}$ is again an eigenfunction, which is Bessel's equation. The companion script computes J_n from this integral. Bessel functions are matrix coefficients of the group of rigid motions of the plane [Vil68].

9.3 The sphere

Let $\mathcal{H}_l$ be the space of harmonic homogeneous polynomials of degree l in three variables.

Theorem 9.5 (Spherical harmonics). [classical; source, §10.3, Theorem 1] $\dim \mathcal{H}_l = 2l + 1$*;* $\mathcal{H}_l$ *is an irreducible representation of* $\mathrm{SO}(3)$*; its restriction to the unit sphere is the eigenspace of the spherical Laplacian with eigenvalue* $-l(l+1)$*; and* $L^2(S^2)$ *is the orthogonal sum of these spaces.*

Proof of the first and third statements. Δ maps polynomials of degree l onto those of degree $l-2$, so $\dim \mathcal{H}_l = \binom{l+2}{2} - \binom{l}{2} = 2l+1$. With $E = \sum x_i \partial_i$ the Euler operator and L_1, L_2, L_3 the rotation fields, there is the operator identity $r^2\Delta = E(E+1) - L^2$, where $L^2 = -(L_1^2 + L_2^2 + L_3^2)$. On $\mathcal{H}_l$ the left side vanishes and $E = l$, so $L^2 = l(l+1)$; and $-L^2$ is the spherical Laplacian. $\square$

The dimension count, the identity, the commutation relations $[L_1, L_2] = -L_3$ and the commutation of the L_i with Δ are checked in exact arithmetic. [Y1] The last statement of the theorem is the Peter–Weyl theorem for the homogeneous space $S^2 = \mathrm{SO}(3)/\mathrm{SO}(2)$. In each $\mathcal{H}_l$ there is one function invariant under rotations about the axis, the ***zonal harmonic*** $P_l(\cos\theta)$; the Legendre polynomials satisfy $((1-s^2)P_l')' + l(l+1)P_l = 0$, $P_l(1) = 1$, and are orthogonal on $[-1,1]$ with square norm $2/(2l+1)$. [Y2] For the ball the radial equation is Bessel's with half-integer order [source, §10.3].

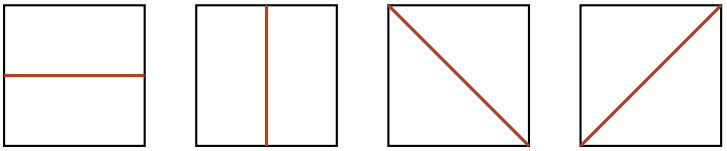
9.4 Angular momentum

The operators L_i generate the rotations, and in quantum mechanics are the components of angular momentum up to a factor [source, §10.7]. The algebra they span is $\mathfrak{so}(3)$; L^2 is its Casimir element, which acts on an irreducible representation by a scalar; the scalar is $l(l+1)$ and the dimension $2l+1$. That is all of the angular analysis of a rotationally symmetric problem, the hydrogen atom of Theorem 8.6 included: the potential enters only the radial equation.

9.5 Nodes

Theorem 9.6 (Courant). [classical; source, §10.4, Theorem 1] *The first eigenfunction has no zeros in the domain. The zero set of the n-th eigenfunction divides the domain into at most n pieces; in one dimension, into exactly n.*

On a path of twelve points the k-th eigenvector changes sign $k-1$ times. On a 5×6 grid the first eight eigenvectors have $1, 2, 2, 4, 3, 3, 6, 4$ nodal domains. The square has the eigenvalues $2, 5, 5, 8, 10, 10$ of the source's list. [Y5] Symmetry forces some nodes: by Theorem 3.5 an eigenfunction that transforms by the sign character under a reflection vanishes on the mirror, and the Dirichlet eigenfunctions of a chamber of a reflection group are exactly the antisymmetric eigenfunctions of the whole.



Nodal lines in the eigenspace of eigenvalue 5 *on the square of side* π*. The eigenspace is the two-dimensional representation of the symmetry group of the square, and the node depends on the vector chosen in it:* $a \sin x \sin 2y + b \sin 2x \sin y = 2 \sin x \sin y \, (a \cos y + b \cos x)$.

Notes

The source's Chapter 10 is followed. For special functions as matrix coefficients of group representations see [Vil68]; for spherical harmonics [StWe71]; for nodal domains [CoHi53], and for their discrete analogue [DGLS01].

Exercises

Exercise 9.1. For the square of side π with Dirichlet conditions, describe the action of the symmetries of the square on the eigenspace of $\lambda = 5$. What is the multiplicity of $\lambda = 50$, and is that eigenspace irreducible?

Exercise 9.2. Compute $\dim \mathcal{H}_l$ in n variables and check the cases $n = 2, 3, 4$ against Remark 1.6.

Exercise 9.3. Deduce Bessel's equation for J_n from Remark 9.4.

Exercise 9.4. List the four lowest frequencies of a circular drum of radius a with wave speed c, with their multiplicities.

Part IV

Operators

Chapter 10

Eigenvalues

For a general domain there are no formulas for the eigenvalues, and the source turns to what can be said without them: variational characterizations, completeness, and asymptotics [source, Ch. 11]. We add the question the asymptotics provoke, whether the spectrum determines the domain, because its answer is a statement about representations.

10.1 Minimax, and what it makes functorial

Theorem 10.1 (Minimax). [classical; source, §11.1–11.2 and §11.6] *For the Dirichlet problem on a bounded domain D,*

$$\lambda_n(D) = \min_{\substack{W \subset H_0^1(D) \\ \dim W = n}} \max_{w \in W \setminus 0} \frac{\int |\nabla w|^2}{\int w^2}.$$

In particular the Rayleigh–Ritz values in any n-dimensional trial space are upper bounds for $\lambda_1, \dots, \lambda_n$.

On $(0,1)$ the trial function $x(1-x)$ gives 10, and adding $x^2(1-x)^2$ gives 9.86975, against $\pi^2 = 9.86960$. [L1] In finite dimensions the theorem is the interlacing of the eigenvalues of a matrix with those of its compressions. [L2]

The formula depends on the domain only through the space of trial functions, and it is monotone in that space.

Corollary 10.2 (Monotonicity). [classical; source, §11.6, Theorems 4 and 5] *If $D \subseteq D'$ then $\lambda_n(D) \geq \lambda_n(D')$ for every n: Dirichlet eigenvalues are a functor from domains and inclusions to decreasing sequences. For one domain the Neumann eigenvalues satisfy $\mu_n \leq \lambda_n$. The Neumann eigenvalues are* not *monotone in the domain.*

Proof. Extension by zero embeds $H_0^1(D)$ isometrically in $H_0^1(D')$, so the minimum is over a larger family. For the second statement $H_0^1(D) \subseteq H^1(D)$. For the third, a rectangle of sides 1.3 and 0.1 fits inside the unit square along its diagonal, and its first non-zero Neumann eigenvalue $\pi^2/1.69 = 5.84$ is smaller than that of the square, $\pi^2 = 9.87$. □

An L-shaped region in a 6×6 grid has every Dirichlet eigenvalue above the corresponding one of the square. [L3] The asymmetry has the same source as in

Theorem 3.5: Dirichlet conditions are compatible with extension by zero and Neumann conditions with nothing of the kind. The spaces $H^1(D)$ are related by restriction, which does not preserve the Rayleigh quotient.

10.2 Completeness

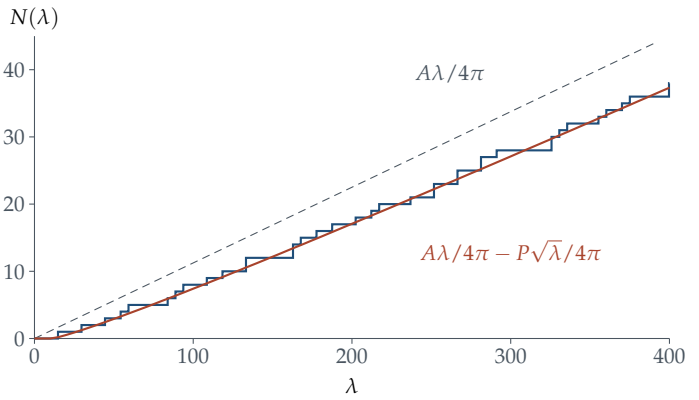
Theorem 10.3 (Spectral theorem for the Laplacian). [classical; source, §11.3–11.5] *On a bounded domain, with Dirichlet, Neumann or Robin conditions, there is an orthonormal basis of $L^2(D)$ consisting of eigenfunctions, with eigenvalues $\lambda_n \to \infty$.*

The source proves this from the minimax principle. In general the spectral theorem says that a self-adjoint operator is unitarily equivalent to multiplication by a real function on a measure space, and its functional calculus $f \mapsto f(A)$ is a homomorphism of algebras. Every operator met so far is a function of one operator: the heat semigroup e^{-tA}, the wave group $\cos(t\sqrt{A})$, the resolvent $(s + A)^{-1}$, the Green's operator A^{-1}, the Schrödinger group e^{-itA}. In finite dimensions this is the resolution of the identity by eigenvectors. [L9]

10.3 How many eigenvalues

Theorem 10.4 (Weyl). [classical; source, §11.6, Theorem 1] *For the Dirichlet problem, $\lambda_n / n \to 4\pi / A$ in the plane, with A the area, and $\lambda_n^{3/2} / n \to 6\pi^2 / V$ in space, with V the volume.*

The source's proof brackets the domain between unions of rectangles and uses Corollary 10.2. For a $1 \times \sqrt{2}$ rectangle the count $N(\lambda)$ of eigenvalues below λ is 208, 869, 3531 at $\lambda = 2000, 8000, 32000$, and the two-term expression $A\lambda/4\pi - P\sqrt{\lambda}/4\pi$, with P the perimeter, gives 207.9, 865.9, 3532.5. [L4]



The counting function of the Dirichlet eigenvalues of the $1 \times \sqrt{2}$ rectangle (steps, 38 eigenvalues below 400), Weyl's leading term (dashed), and the two-term expression with the perimeter correction. The area is heard first and the perimeter second.

10.4 The trace of the semigroup

The function $Z(t) = \sum_n e^{-\lambda_n t}$ is the trace of the heat semigroup of Chapter 4: $Z(t) = \int_D K(x, x, t)\, dx$.

Theorem 10.5 (Hearing area, perimeter and holes). [literature; [Kac66, McSi67]] *For a plane domain with smooth boundary and h holes, $Z(t) = \dfrac{A}{4\pi t} - \dfrac{P}{8\sqrt{\pi t}} + \dfrac{1-h}{6} + o(1)$ as $t \to 0$. For a polygon the constant term is $\sum_i (\pi^2 - \theta_i^2)/24\pi\theta_i$ over the interior angles θ_i.*

Proposition 10.6 (The rectangle, exactly). [classical] *For an $a \times b$ rectangle, $Z(t) = \dfrac{ab}{4\pi t} - \dfrac{2(a+b)}{8\sqrt{\pi t}} + \dfrac{1}{4} + O(e^{-c/t})$.*

Proof. By Theorem 9.2, $Z = z_a z_b$ with $z_a(t) = \sum_{m \geq 1} e^{-\pi^2 m^2 t/a^2}$, and Poisson summation gives

$$z_a(t) = \frac{a}{2\sqrt{\pi t}} - \frac{1}{2} + \frac{a}{\sqrt{\pi t}} \sum_{k \geq 1} e^{-(2ak)^2/4t}. \qquad \square$$

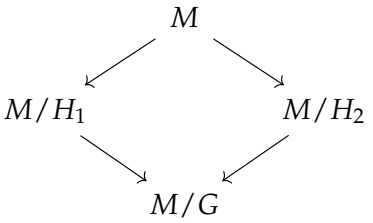
The two sides of the last identity agree to fourteen digits, and for the rectangle the three-term expression is exact to 10^{-15} at $t = 0.02$. [L5, L6] The identity is Theorem 4.6 on the diagonal: the left side is the sum over eigenfunctions and the right side the sum over images, now indexed by the closed billiard paths of lengths $2ak$. It is the smallest instance of a trace formula, spectrum on one side and closed geodesics on the other. The constant $\frac{1}{4}$ is four corners at $\frac{1}{16}$ each; in the smooth case the constant $(1-h)/6$ is the Euler characteristic divided by six. A topological invariant can be heard.

10.5 Can one hear the shape?

Theorem 10.7 (Sunada). [literature; [Sun85]] *Let a finite group G act freely by isometries on a compact manifold M, and let H_1, H_2 be subgroups that meet every conjugacy class of G in the same number of elements. Then M/H_1 and M/H_2 have the same Laplace spectrum.*

Proof. The condition says that the permutation representations $\mathbb{Q}[G/H_1]$ and $\mathbb{Q}[G/H_2]$ are isomorphic. For any representation V of G the invariants are $V^{H_i} = \mathrm{Hom}_G(\mathbb{Q}[G/H_i], V)$, naturally in V. Apply this to each eigenspace of the Laplacian on M: the eigenspaces of M/H_1 and M/H_2 have equal dimensions. $\square$

Isospectrality is here an isomorphism in a category of representations that is not induced by an isometry. Gordon, Webb and Wolpert used the method to produce plane domains with the same Dirichlet spectrum that are not congruent [GWW92]; the answer to Kac's question is no.



Both middle spaces are covered by M and cover M/G; no map between them is asserted, and in general there is none. What relates them is an isomorphism one level up, between the functors $V \mapsto V^{H_1}$ and $V \mapsto V^{H_2}$ on representations of G.

The smallest case can be computed in full. In the group of order 32 of affine maps $x \mapsto ux + a$ of $\mathbb{Z}/8$, the subgroups $\{(0,1),(0,3),(0,5),(0,7)\}$ and $\{(0,1),(4,3),(4,5),(0,7)\}$ meet each of the eleven conjugacy classes equally and are not conjugate. The two quotient graphs on eight vertices, for the generators $(0,3),(0,5),(1,7)$, have the same characteristic polynomial and are not isomorphic: their loops are distributed differently. [L8] Without any group, an exhaustive search finds no two connected graphs on at most five vertices with the same Laplacian spectrum and different degree sequences, and exactly two such spectra on six vertices; the first is shared by a graph with degrees $(2,2,2,2,2,4)$ and one with degrees $(1,2,2,3,3,3)$. [L7]



Two graphs with the Laplacian characteristic polynomial $x^6 - 14x^5 + 73x^4 - 176x^3 + 192x^2 - 72x$. One can hear the number of spanning trees, which is $72/6 = 12$ for both ($3 \cdot 4$ on the left, $2^2 \cdot 3$ on the right), and cannot hear the degrees.

Notes

The source's Chapter 11 is followed for the variational principles, completeness and Weyl's law. The heat trace expansion is in [Kac66, McSi67]. Sunada's theorem is [Sun85], and the plane domains are in [GWW92]; the group of order 32 is Gassmann's example, see [Bro88].

Exercises

Exercise 10.1. Compute the Rayleigh quotient of $x(1-x)$ on $(0,1)$.

Exercise 10.2. Use Corollary 10.2 to bound the first Dirichlet eigenvalue of the unit disk between those of an inscribed and a circumscribed square, and compare with z_{01}^2.

Exercise 10.3. Derive the two-term count for an $a \times b$ rectangle by counting lattice points in a quarter of an ellipse.

Exercise 10.4. Verify that the constant term of Theorem 10.5 for a polygon gives $\tfrac{1}{4}$ for a rectangle, and find it for an equilateral triangle.

Chapter 11

Distributions and Transforms

Several objects used so far were not functions: the initial state of the source function, the pole of a Green's function, a solution with a corner. The source legitimizes them late [source, Ch. 12]. In the language of Chapter 1 the step is a change of the second argument of the solution functor.

11.1 Distributions

A ***test function*** on an open set U is a smooth function of compact support, and a ***distribution*** is a continuous linear functional on test functions [source, §12.1]. Distributions restrict to smaller open sets and glue, so they form a sheaf $\mathcal{D}'$, and differential operators act on them by duality: $(\partial f, \varphi) = -(f, \partial\varphi)$. So $\mathcal{D}'$ is a sheaf of modules over the differential operators that contains the smooth functions, and $\mathrm{Sol}(P; \mathcal{D}')$ is defined: these are the ***weak solutions***.

Example 11.1. The delta function is the limit of the source functions, $(S(\cdot, t), \varphi) = \varphi(0) + t\varphi''(0) + O(t^2)$ for $k = 1$. The derivative of the step function is δ, and the second derivative of $|x|$ is 2δ. [T1, T2] The function $\frac{1}{2}[\varphi(x + ct) + \varphi(x - ct)]$ is a weak solution of the wave equation for every locally integrable φ; this is Corollary 3.2 stated in $\mathcal{D}'$.

Theorem 11.2 (Kernels). [classical; [Sch57]] *Continuous linear maps from test functions on X to distributions on Y correspond bijectively to distributions on $X \times Y$.*

This is the tensor–hom adjunction for the spaces in question; it holds because they are nuclear. It is the reason that every solution operator in this volume is given by a kernel: the source function, the Green's function, the Poisson kernel, the Kirchhoff kernel. One limit of the theory should be recorded now. There is no associative product on $\mathcal{D}'$ extending the product of continuous functions [Sch54]. Linear equations with smooth coefficients act on distributions; nonlinear ones do not, and Chapter 14 must find other notions of weak solution.

11.2 Fundamental solutions

A ***fundamental solution*** of P is a distribution E with $PE = \delta$; then $u = E * f$ solves $Pu = f$ for f of compact support [source, §12.2].

Theorem 11.3. [classical; [Mal56, Ehr54]] *Every non-zero differential operator with constant coefficients has a fundamental solution.*

operator	fundamental solution	singular support
Δ in $\mathbb{R}^3$	$-1/4\pi r$	the origin
$\partial_t - k\partial_x^2$	$S(x,t)$ for $t>0$, else 0	the origin
$\partial_t^2 - c^2\partial_x^2$	$1/2c$ for $\lvert x\rvert < ct$, else 0	the cone $\lvert x\rvert = ct$
$\partial_t^2 - c^2\Delta$ in $\mathbb{R}^3$	$\delta(r-ct)/4\pi c\, r$	the cone $r = ct$

The first line is checked against a test function. [T3] The last column is the type of Theorem 1.5 once more: an elliptic or parabolic fundamental solution is smooth away from the pole, so that all solutions are smooth where the right-hand side is; a hyperbolic one is singular along the characteristic cone, which is Theorem 8.4. The last line is Huygens's principle: the support, and not only the singular support, is the cone.

11.3 The Fourier transform

Theorem 11.4. [classical; source, §12.3] *The Fourier transform $F(k) = \int f(x)e^{-ikx}dx$ is an isomorphism of the Schwartz space with itself and of the tempered distributions with themselves; it is unitary on L^2 up to the factor 2π; and it exchanges ∂_x with multiplication by ik and multiplication by x with $i\partial_k$.*

These properties are checked numerically. [T4] Structurally: the line is its own Pontryagin dual, as the circle and the integers were dual in Theorem 5.3; and the transform is an automorphism of the Weyl algebra generated by x and ∂ with $[\partial, x] = 1$, since $[ik, i\partial_k] = 1$. An equation with constant coefficients becomes an equation of order zero, $p(ik)\hat{u} = \hat{f}$, and solving it is division by a polynomial. The zero set of that polynomial is the characteristic variety of Chapter 1, and the difficulty of the division there is the whole difficulty.

The duality is functorial for homomorphisms of groups: the dual of the quotient $\mathbb{R} \to \mathbb{R}/2l\mathbb{Z}$ is the inclusion of the lattice $\frac{\pi}{l}\mathbb{Z} \subset \mathbb{R}$. Applied to a function this is the Poisson summation formula, and applied to the heat kernel it is Theorem 4.6 and Proposition 10.6. The image method and Fourier's method are dual in the literal sense.

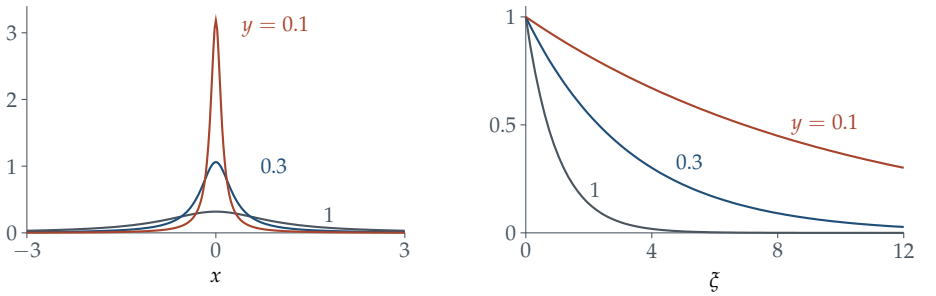
11.4 Source functions

With the transform the source functions are computed and not guessed [source, §12.4]. For diffusion, $\hat{S}_t = -k^2\hat{S}$, $\hat{S}(k,0) = 1$, so $\hat{S} = e^{-k^2t}$ and S is the Gaussian.

[T5] For Laplace's equation in the half-plane $y > 0$ the same computation gives $\hat{u}(k, y) = e^{-|k|y}\hat{h}(k)$ and the Poisson kernel

$$P_y(x) = \frac{1}{\pi}\frac{y}{x^2 + y^2} = \frac{1}{\pi}\operatorname{Im}\frac{1}{x - iy}.$$

Two things are visible in this formula. First, $P_y * P_{y'} = P_{y+y'}$: the Dirichlet problem in a half-space is also a semigroup, in the variable normal to the boundary, generated by $-\sqrt{-\partial_x^2}$; it is a Markov semigroup, the Cauchy process, and harmonic measure is its kernel. Second, δ is the boundary value of a holomorphic function: $P_y \to \delta$ as $y \to 0$, with error of order y. [T7]



Left: the Poisson kernel $P_y(x) = y/\pi(x^2 + y^2)$ of the half-plane, tending to the delta function as $y \to 0$. Right: its Fourier transform $e^{-y|\xi|}$, tending to the constant 1. Convergence of distributions on one side is pointwise convergence of multipliers on the other.

Remark 11.5 (Hyperfunctions). [literature; [Sat59]] Sato defines a generalized function on the line as a pair of holomorphic functions above and below it, modulo functions holomorphic across it: the space is $\mathcal{O}(\mathbb{C} \setminus \mathbb{R})/\mathcal{O}(\mathbb{C})$, the first cohomology of the plane with support in the line and coefficients in $\mathcal{O}$. Distributions embed in it, δ as the class of $-1/2\pi iz$. This is the second place in the volume, after Proposition 6.5, where a cohomology group is the object of study and not a tool; in several variables it is a higher cohomology group, and the theory of linear equations built on it is derived from the start [KaSc90].

11.5 The Laplace transform

Proposition 11.6. [classical; source, §12.5; [Paz83]] *For a contraction semigroup with generator A and $s > 0$, $(s - A)^{-1} = \int_0^\infty e^{-st} T_t \, dt$.*

So the Laplace transform in time, the source's last technique, turns an evolution problem into a family of stationary ones, and the three presentations of one object, semigroup, generator and resolvent, determine one another. [T6] Together with Theorem 8.5 this closes a circle: elliptic problems $(s - A)u = f$, parabolic problems $u_t = Au$ and, for $A \leq 0$ self-adjoint, hyperbolic problems $u_{tt} = Au$ are three functions of one operator.

Notes

The source's Chapter 12 is followed. For distributions and the kernel theorem see [Sch57, Hor83]; for the impossibility of multiplication [Sch54]; for fundamental solutions [Mal56, Ehr54]; for hyperfunctions [Sat59, KaSc90].

Exercises

Exercise 11.1. Show that $|x|'' = 2\delta$ in the sense of distributions.

Exercise 11.2. Find fundamental solutions of d^2/dx^2 and of $-d^2/dx^2 + 1$ on the line.

Exercise 11.3. Derive the source function of the diffusion equation from $\hat{S} = e^{-k^2 t}$.

Exercise 11.4. Show that the Poisson kernels of the half-plane form a semigroup under convolution.

Chapter 12

Discretization

The source devotes a chapter to computation and opens it with a warning: a scheme that looks reasonable can produce nonsense [source, §8.1]. The categorical view of a numerical method is that it is a functor on a small category of grids that approximately commutes with the solution functor; the warning is that approximate commutation of one step says nothing about many.

12.1 Squares that almost commute

Let $E(t)$ be the solution operator of a well-posed linear evolution problem, R_h the restriction of functions to a grid of mesh h, and E_h one step of a scheme with time step Δt tied to h.

$$\begin{array}{ccc} X & \xrightarrow{E(\Delta t)} & X \\ {\scriptstyle R_h}\downarrow & & \downarrow{\scriptstyle R_h} \\ X_h & \xrightarrow[E_h]{} & X_h \end{array}$$

The scheme is ***consistent*** of order p if the square commutes up to $O(\Delta t\, h^p)$ on smooth functions; ***stable*** if the powers E_h^n are bounded uniformly for $n\Delta t \le T$; ***convergent*** if $E_h^n R_h - R_h E(n\Delta t) \to 0$.

Theorem 12.1 (Lax equivalence). [classical; [LaRi56]] *For a consistent scheme, stability is equivalent to convergence.*

Idea. $E_h^n R_h - R_h E^n = \sum_{j<n} E_h^{n-1-j}(E_h R_h - R_h E)E^j$: the defect of one square, transported by the remaining steps. Stability is exactly what keeps the transport from amplifying it. □

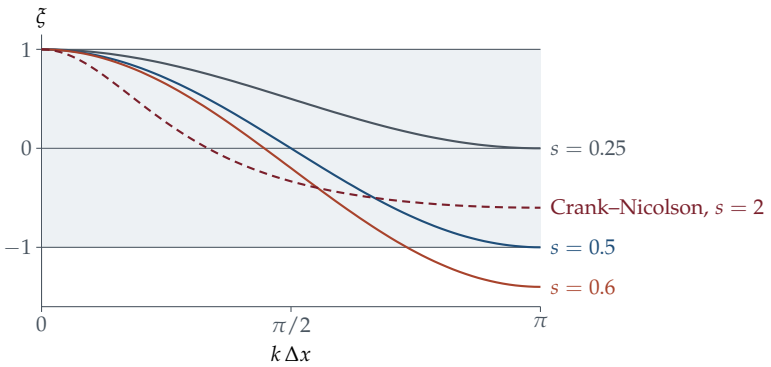
The centred three-level scheme for the heat equation is consistent of second order and its amplification factor exceeds one for every mesh ratio, 1.48 at ratio 0.1: it converges for no data but the trivial. [N10]

12.2 Diffusion

For the explicit scheme with $s = k\Delta t/\Delta x^2$ the amplification factor of the mode e^{ikx} is $1 - 2s(1 - \cos k\Delta x)$ [source, §8.2].

Theorem 12.2. [classical; source, §8.2] *The explicit scheme is stable if and only if $s \le \frac{1}{2}$, which is also the condition for its matrix to have non-negative entries.*

At $s = 0.52$ the computed growth per step equals the predicted 1.0684; at $s = 0.5$ nothing grows. [N1] The second clause is Theorem 4.2: stability, here, is that the scheme is a morphism of the Markov category, and the source's observation that "none of the coefficients are negative" predicts the right condition for that reason. The error is of second order in Δx, and of fourth order at the special ratio $s = \frac{1}{6}$, where the leading error terms in time and space cancel: halving the mesh divides the error by 4.06 and by 16.1. [N2] The Crank–Nicolson scheme, the Cayley transform of the generator, has amplification factor of modulus at most one for every s and is of second order in time. [N3]



The factor by which one step of the explicit scheme multiplies the mode e^{ikx}, for three values of $s = k\,\Delta t/\Delta x^2$, and of the Crank–Nicolson scheme at $s = 2$. Stability is confinement to the band $|\xi| \le 1$; at $s = 0.6$ the shortest waves leave it.

12.3 Waves

Theorem 12.3 (Courant–Friedrichs–Lewy). [classical; source, §8.3; [CFL28]] *The centred scheme for the wave equation is stable if and only if $c\,\Delta t \le \Delta x$. If $c\,\Delta t > \Delta x$ the scheme cannot converge, whatever its other properties: its domain of dependence is strictly inside the true one.*

The source writes the condition as $s = c^2\Delta t^2/\Delta x^2 \le 1$ and gives the same explanation. With $c\,\Delta t = 2\Delta x$ the scheme returns bit-identical values at $(0, 1)$ for two data sets that differ only near $x = 1$, at every refinement, while the exact solutions differ by $1/2e$. [N5] This is Theorem 3.3 read as a constraint: a discretization must respect the causal structure of the solution functor, in the sense that the discrete domains of dependence contain the continuous ones. At equality the scheme is the parallelogram identity of Corollary 3.2 and reproduces exact solutions to rounding error; at $s = 1.1$ the sawtooth

grows by the predicted factor 1.8633 per step. [N4] A scheme that preserves the symplectic form of Proposition 3.4 is constructed in Chapter 13.



The Courant–Friedrichs–Lewy condition. The value at the top grid point is computed from the grid points in the shaded triangle; the solution there depends on the data in the interval cut out by the characteristics (dashed). If that interval is not inside the triangle, the scheme never sees data on which the answer depends.

12.4 Laplace's equation

The five-point scheme says that each value is the mean of its four neighbours: the discrete mean value property, with the random walk of Theorem 6.4 behind it [source, §8.4]. The classical iterations solve it by running a process to equilibrium.

Proposition 12.4. [classical; [You50]] *On an $N \times N$ grid the Jacobi iteration contracts the error by $\cos(\pi/N)$ per sweep, the Gauss–Seidel iteration by $\cos^2(\pi/N)$, and successive over-relaxation with the optimal parameter $\omega = 2/(1+\sin(\pi/N))$ by $\omega - 1$.*

For $N = 16$ a random error is reduced by 10^{-6} in 585, 298 and 41 sweeps. [N6] Jacobi iteration is the explicit diffusion scheme at $s = \frac{1}{4}$ run to its steady state, as the source notes; Gauss–Seidel is "about twice as fast", exactly so asymptotically; and the choice of ω, which the source leaves as an art, has for this model problem the closed form above.

12.5 Finite elements

By Theorem 7.3 the Galerkin solution in a space of trial functions is the energy projection of the true solution [source, §8.5]. For piecewise linear functions on a uniform mesh of the interval the stiffness matrix is the difference Laplacian divided by h; the Galerkin solution is exact at the nodes, because the Green's function $x(1-y)$ is itself piecewise linear; and the energy error halves with h. [N7] On a uniform triangulation of the square, linear elements give the five-point scheme. [N8] The method earns its keep on domains where differences do not fit, and there the corner exponents of Proposition 6.6 limit the rate.

12.6 Discretize the complex

Proposition 2.1 suggests discretizing the two factors of the Laplacian separately: the exterior derivative exactly, on cochains of a mesh, and the Hodge star approximately.

Theorem 12.5 (Finite element exterior calculus). [literature; [AFW06]] *On a triangulated domain the Whitney forms are a finite-dimensional subcomplex of the de Rham complex, isomorphic to simplicial cochains, and there are projections onto it that are bounded and commute with* d. *Consequently the discrete Hodge Laplacian is stable and convergent in every degree, and its harmonic forms have the dimensions of the cohomology of the domain.*

Stability here is a consequence of naturality: the discretization is a map of complexes. On a triangulated annulus with 12 vertices, 24 edges and 12 triangles, $\mathrm{dd} = 0$ holds in integer arithmetic and the Betti numbers are $(1, 1, 0)$. [N9] The single harmonic 1-cochain is the discrete image of the class of $\mathrm{d}\theta$, whose conjugate is $\log r$ (Proposition 6.5); a scheme that does not respect the complex can compute a spurious number of such modes.

Notes

The source's Chapter 8 is followed for the schemes and their stability analysis. The equivalence theorem is [LaRi56]; the stability condition for waves is [CFL28]; the theory of over-relaxation is Young's [You50]. Finite element exterior calculus is due to Arnold, Falk and Winther [AFW06].

Exercises

Exercise 12.1. Derive the amplification factor of the explicit diffusion scheme and the condition $s \le \frac{1}{2}$.

Exercise 12.2. Show that the explicit scheme has truncation error of fourth order in Δx when $s = \frac{1}{6}$.

Exercise 12.3. For the wave scheme the amplification factors satisfy $\xi^2 - 2(1 + p)\xi + 1 = 0$ with $p = s(\cos k\Delta x - 1)$. Show that both roots have modulus one for all k exactly when $s \le 1$.

Exercise 12.4. Compute the Euler characteristic of the triangulated annulus of the text and deduce its first Betti number.

Part V

Physics and Nonlinearity

Chapter 13

Fields

The source's chapter on physics [source, Ch. 13] is where the equations stop being scalar. Its topics are electromagnetism, fluids, scattering and the equations of particles. Each exhibits a structure that the scalar theory only hinted at, and one of them is the first object of this volume that is higher-categorical in earnest.

13.1 Maxwell's equations

In the source's units the equations are

$$\partial_t E = c\,\operatorname{curl} B - 4\pi J, \qquad \partial_t B = -c\,\operatorname{curl} E, \qquad \operatorname{div} E = 4\pi\rho, \qquad \operatorname{div} B = 0.$$

On space-time let $F = E \wedge dt + B$ be the field 2-form. Up to constants and signs fixed by conventions, the second and fourth equations are $\mathrm{d}F = 0$ and the first and third are $\mathrm{d} \star F = \star J$: two topological equations and one metric relation, the pattern of Proposition 2.1. Conservation of charge, $\partial_t \rho + \operatorname{div} J = 0$, is $\mathrm{dd} = 0$ applied to $\star F$. (The source's text gives this equation with the opposite sign of div J; its Exercise 1 of §13.1 has it as here.) In vacuum each component satisfies the wave equation, and the transverse plane wave $E = (0, f(x-ct), 0)$, $B = (0, 0, f(x-ct))$ is a solution. [P2]

Since $\mathrm{d}F = 0$, locally $F = \mathrm{d}A$ for a potential A, determined up to $A \mapsto A + \mathrm{d}\chi$.

$$\begin{array}{ccccccc} \underset{\text{gauge } \chi}{\Omega^0} & \xrightarrow{\mathrm{d}} & \underset{\text{potential } A}{\Omega^1} & \xrightarrow{\mathrm{d}} & \underset{\text{field } F}{\Omega^2} & \xrightarrow{\mathrm{d}} & \Omega^3 \\ & & & & \downarrow \star & & \\ \Omega^4 & \xleftarrow{\mathrm{d}} & \underset{\text{current } \star J}{\Omega^3} & \xleftarrow{\mathrm{d}} & \Omega^2 & & \end{array}$$

The top row is the de Rham complex of space-time and is natural for every smooth map; the bottom row is the same complex read from degree two; the one vertical arrow is the only place where the metric, and with it the speed of light, enters.

Proposition 13.1 (Fields form a groupoid). [reformulation] *Potentials on a space-time M are the objects of a groupoid whose morphisms are gauge transformations. The field strength is a functor from it to closed 2-forms. Flat potentials modulo gauge are classified by $H^1(M;\mathbb{R})$ modulo integral classes, their holonomies; and a closed 2-form is the field strength of a potential on a line bundle exactly when its periods are integral.*

On a cubical decomposition of the 3-torus, curl grad $= 0$ and div curl $= 0$ hold in integer arithmetic, the field strength of a random potential is unchanged by a random gauge transformation, and the Betti numbers are $(1, 3, 3, 1)$: there are three independent closed potentials that are not gradients, the discrete Aharonov–Bohm fields. [P1]

Remark 13.2 (Where the theory becomes higher). [literature; [Fre00, HoSi05]] Gauge-equivalent potentials are the same field, but in more than one way: a gauge transformation is data. So the fields on M are not a set but a groupoid, and the assignment $M \mapsto \{\text{line bundles with connection on } M\}$ is not a sheaf of sets but a stack; its isomorphism classes form the differential cohomology group $\hat{H}^2(M)$, an extension of integral classes by forms. For a field whose potential is a 2-form, gauge transformations have gauge transformations of their own and the fields form a 2-groupoid. The solution functor of Chapter 1 takes values in sets only for equations without gauge symmetry; for Maxwell's equations and their relatives it takes values in higher groupoids. Nothing earlier in the volume needed this, and nothing in the source's treatment, which works with E and B, sees it.

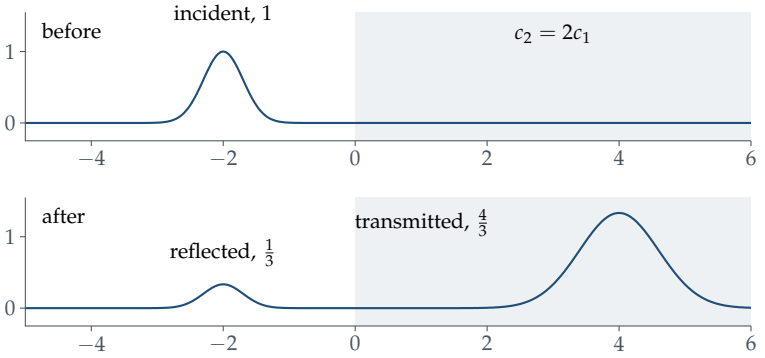
13.2 Fluids and sound

The Euler equations of a compressible fluid, $\partial_t \rho + \operatorname{div}(\rho v) = 0$ and $\partial_t v + (v \cdot \nabla) v = -\rho^{-1} \nabla p(\rho)$, are a conservation law and a constitutive law again [source, §13.2]. Their linearization at a state of rest is the wave equation with $c^2 = p'(\rho_0)$: in one dimension the linearized system has characteristic speeds $\pm c$, numerically ± 1.2272 for $p = \rho^{1.4}$ at $\rho_0 = 1.2$. [P3] Linearization is functorial: the linearized equation at a solution is the tangent space of the solution set there, and acoustics is the tangent of fluid dynamics at rest. For an incompressible fluid Arnold showed that the motions are the geodesics of the group of volume-preserving diffeomorphisms [Arn66].

13.3 Scattering

A wave meets an obstacle and is partly reflected. For a string made of two materials with speeds c_1, c_2 the reflected and transmitted amplitudes are

$(c_2 - c_1)/(c_2 + c_1)$ and $2c_2/(c_2 + c_1)$ [source, §13.3]; the solution and its slope are continuous at the junction and the energy fluxes balance. [P4]



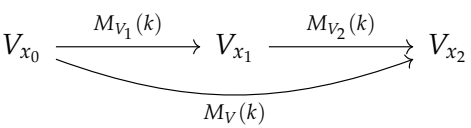
A pulse on a string whose speed doubles at $x = 0$ (shaded side). The reflected pulse has amplitude $(c_2 - c_1)/(c_2 + c_1) = \frac{1}{3}$ and the transmitted one $2c_2/(c_2 + c_1) = \frac{4}{3}$, stretched by the ratio of speeds. The displacement and its slope are continuous at the junction throughout.

For the equation $-\psi'' + V\psi = k^2\psi$ with V of compact support, let $M_V(k)$ be the matrix taking (ψ, ψ') on the left of the support to (ψ, ψ') on the right.

Theorem 13.3 (Transfer is a functor). [reformulation; cf. source, §13.4, Theorem 1] *$M_V(k)$ has determinant one, and if V is V_1 followed by V_2 then $M_V = M_{V_2} M_{V_1}$. The reflection and transmission coefficients computed from M_V satisfy $|R|^2 + |T|^2 = 1$.*

Proof. The Wronskian of two solutions is constant, which is the first statement, and it is the symplectic form of Theorem 5.1 on boundary values. Composition is uniqueness of solutions of the initial value problem. The identity is the constancy of the Wronskian of ψ and $\bar{\psi}$, the source's argument. □

For a potential of four constant pieces the product of the four matrices agreed with a direct numerical integration across the whole to eight digits, and the identity held to twelve, at three energies. [P5] This is Remark 5.2 with the eigenvalue parameter in the continuous spectrum: an interval with a potential is a symplectic map between boundary values, and concatenation is composition.

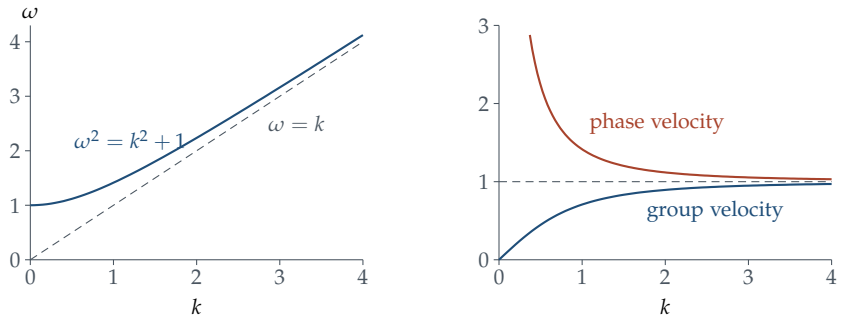


When $V \to 0$ at infinity the operator $-d^2/dx^2 + V$ has, besides finitely many eigenvalues, the ***continuous spectrum*** $[0, \infty)$, with generalized eigenfunctions that are bounded and not square-integrable [source, §13.4]. The spectral theorem of Theorem 10.3 then requires the distributions of Chapter 11: the

eigenfunction expansion is an integral, and for $V = 0$ it is the Fourier transform.

13.4 Equations of particles

The Klein–Gordon equation $u_{tt} - c^2\Delta u + m^2 u = 0$ has plane waves with $\omega^2 = c^2k^2 + m^2$; its group velocity is below c and its phase velocity above, with product c^2. [P6]



The Klein–Gordon equation with $c = m = 1$. Left: the dispersion relation, a hyperbola asymptotic to the light cone. Right: the group velocity k/ω and the phase velocity ω/k; their product is 1 at every wave number.

Proposition 13.4 (Dirac). [classical; source, §13.5] *The source's four matrices satisfy $\gamma^\mu\gamma^\nu + \gamma^\nu\gamma^\mu = 2\eta^{\mu\nu}$ with $\eta = \operatorname{diag}(1,-1,-1,-1)$. Hence $(\gamma^\mu\partial_\mu)^2 = \partial_t^2 - \Delta$, and every component of a solution of the Dirac equation solves the Klein–Gordon equation.*

[P7] The wave operator has no first-order square root with scalar coefficients, since its symbol is an indefinite quadratic form. It has one with coefficients in the Clifford algebra of that form, which is the universal algebra in which the form becomes a square. This is Proposition 1.3 for the hyperbolic operator in more than one space dimension.

13.5 Fields as functors

The four structures of Chapter 3 are available for the Klein–Gordon equation on any globally hyperbolic space-time.

Theorem 13.5 (Locally covariant fields). [literature; [BFV03, BGP07]] *The assignment $M \mapsto (\mathrm{Sol}_{sc}(M), \Omega)$ is a functor from globally hyperbolic space-times and causal isometric embeddings to symplectic vector spaces. Solutions on causally separated regions are Ω-orthogonal, and an embedding whose image contains a Cauchy surface induces an isomorphism. Composing with the functor from symplectic spaces to algebras of canonical commutation relations gives a quantum field theory.*

So the free quantum field is the solution functor of a hyperbolic equation, followed by quantization: the exact sequence of Theorem 3.8, the causality of Theorem 3.3 and the symplectic form of Proposition 3.4 are its axioms. A numerical scheme can respect this structure: the leapfrog step for a discretized Klein–Gordon equation satisfies $M^{\mathsf{T}} J M = J$ to rounding error, and the Euler step does not. [P8] In the formalism of factorization algebras the same data are organized by open sets of space-time, and the solution space is replaced by its derived version, a complex whose zeroth cohomology is Sol [CoGw17].

Notes

The source's Chapter 13 is followed for the equations themselves. For differential forms in electromagnetism see [Bos98]; for differential cohomology and gauge fields [Fre00, HoSi05]; for the geometry of the Euler equations [Arn66]; for scattering theory [ReSi75]; for locally covariant field theory [BFV03, BGP07] and factorization algebras [CoGw17].

Exercises

Exercise 13.1. Derive the conservation of charge from the source's form of Maxwell's equations.

Exercise 13.2. Verify the energy balance $R^2 + (c_1/c_2)T^2 = 1$ for the string of two materials.

Exercise 13.3. Find the transfer matrix of the potential $V = g\,\delta(x)$ and the transmission probability $|T|^2$.

Exercise 13.4. Show that the group and phase velocities of the Klein–Gordon equation have product c^2.

Chapter 14

Nonlinear Equations

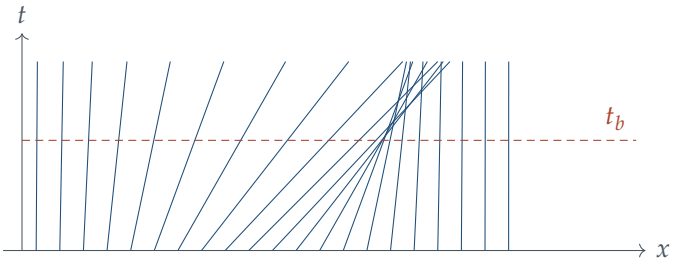
The source ends with a sample of nonlinear phenomena: shocks, solitons, variational problems, bifurcation and water waves [source, Ch. 14]. Linear structure is gone, and with it most of the functors of the previous chapters. What remains, and what explains the two solvable examples, are morphisms of equations in the sense of Theorem 1.2.

14.1 Shocks

For $u_t + a(u)u_x = 0$ the solution is constant along characteristics, which are straight lines of slope determined by the data, as in Proposition 1.8 [source, §14.1]. When they cross the quotient is no longer a manifold and the smooth solution ends.

Proposition 14.1. [classical] *For $u_t + uu_x = 0$ with data φ, the smooth solution exists exactly until $t_b = -1/\min \varphi'$.*

For $\varphi = e^{-x^2}$, $t_b = \sqrt{e/2} = 1.1658$. [X1] The source's explicit solution for $\varphi = x^2$ is verified, with its domain $1 + 4tx \geq 0$. [X2]

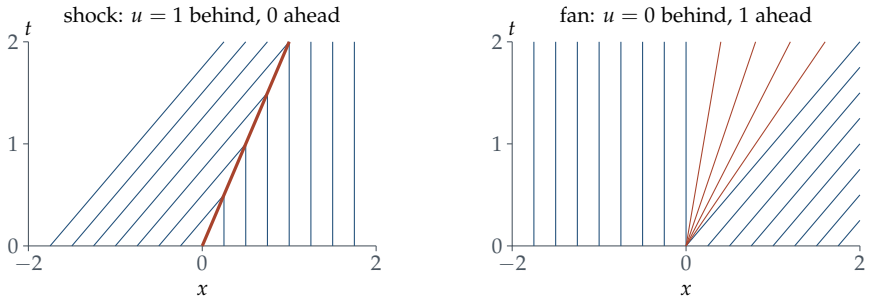


Characteristics of $u_t + uu_x = 0$ for the data e^{-x^2}: straight lines $x = x_0 + \varphi(x_0)t$. They first meet at $t_b = 1.166$; above that level the map from (x_0, t) to (x, t) is no longer injective.

After t_b one asks for weak solutions. Distributions are not available, since u^2 must be formed (Chapter 11); instead the equation is written as a conservation law $u_t + A(u)_x = 0$ and tested against smooth functions. A piecewise smooth weak solution has jumps moving with the Rankine–Hugoniot speed $s = (A(u^+) - A(u^-))/(u^+ - u^-)$.

Proposition 14.2 (Weak solutions belong to the conservation form). [classical] *The equations $u_t + (\frac{1}{2}u^2)_x = 0$ and $(u^2)_t + (\frac{2}{3}u^3)_x = 0$ have the same smooth solutions and different weak ones: a jump from 1 to 0 moves at speed $\frac{1}{2}$ under the first and $\frac{2}{3}$ under the second.*

An isomorphism of equations on smooth solutions need not extend to weak solutions; the weak solution functor is defined on balance laws, in the sense of Proposition 2.1, and not on the ideals they generate. The same distinction decides numerical methods: for the step from 1 to 0 a scheme in conservation form places the shock at 1.005 after two time units, against the exact 1, and the equally consistent non-conservative scheme leaves it at the origin. [X3]



The two Riemann problems for $u_t + uu_x = 0$. Left: characteristics run into the shock, which moves at the Rankine–Hugoniot speed $\frac{1}{2}$. Right: characteristics leave a gap, which the entropy solution fills with the fan $u = x/t$; a shock placed there would have characteristics coming out of it and is the solution that vanishing viscosity rejects.

Weak solutions are not unique: the step from 0 to 1 admits both a stationary-looking shock and the fan x/t. The ***entropy condition*** $a(u^-) > s > a(u^+)$ selects the fan, as does every conservative monotone scheme. [X3]

Theorem 14.3 (Vanishing viscosity). [classical; [Hop50, Ole57, Kru70]] *As $\varepsilon \to 0$ the solutions of $u_t + uu_x = \varepsilon u_{xx}$ converge to a weak solution of the inviscid equation satisfying the entropy condition, and that solution is unique.*

14.2 A morphism of equations

Theorem 14.4 (Cole–Hopf). [classical; [Hop50, Col51]] *If $\varphi > 0$ solves the heat equation $\varphi_t = \varepsilon\varphi_{xx}$, then $u = -2\varepsilon\varphi_x/\varphi$ solves $u_t + uu_x = \varepsilon u_{xx}$, and every solution arises so, from φ unique up to a constant factor.*

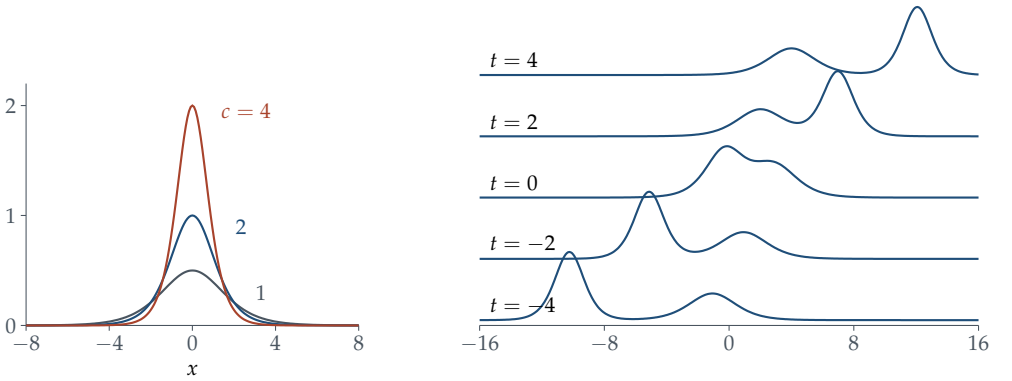
In the category PDE this is a morphism from the heat equation to the viscous Burgers equation, the quotient by scaling. For three superposed exponential solutions of the heat equation the residual of the image is 10^{-12}. The image of $1 + e^{-(x-t/2)/2\varepsilon}$ is the viscous profile $(1 + e^{(x-t/2)/2\varepsilon})^{-1}$, which travels at

the Rankine–Hugoniot speed $\frac{1}{2}$ and steepens to the shock as $\varepsilon \to 0$. [X4] A nonlinear equation is solved here by exhibiting it as the image of a linear one; the theorem above is then a statement about the heat semigroup.

$$\begin{array}{ccc} \varphi(\cdot,0) & \xrightarrow{\text{heat semigroup}} & \varphi(\cdot,t) \\ \Big\downarrow{\scriptstyle -2\varepsilon\,\partial_x \log} & & \Big\downarrow{\scriptstyle -2\varepsilon\,\partial_x \log} \\ u(\cdot,0) & \xrightarrow[\text{Burgers flow}]{} & u(\cdot,t) \end{array}$$

14.3 Solitons

The Korteweg–de Vries equation $u_t + u_{xxx} + 6uu_x = 0$ has the travelling waves $\frac{c}{2}\operatorname{sech}^2\big(\frac{\sqrt{c}}{2}(x-ct)\big)$, of mass $2\sqrt{c}$ [source, §14.2]. [X5] Two of them pass through each other and emerge unchanged except for a shift: for the exact two-soliton solution with wave numbers $k_1 = 1.6$, $k_2 = 1$ the fast one is advanced by $\frac{2}{k_1}\log\frac{k_1+k_2}{k_1-k_2} = 1.8329$ and the slow one retarded by 2.9327. [X6]



Left: solitary waves $\frac{c}{2}\operatorname{sech}^2\big(\frac{\sqrt{c}}{2}x\big)$ of the Korteweg–de Vries equation; taller ones are narrower and faster. Right: the exact two-soliton solution with $k_1 = 1.6$, $k_2 = 1$ at five times, each curve raised for clarity. The taller wave overtakes the shorter; both emerge with their shapes, the taller advanced by 1.83 and the shorter retarded by 2.93 relative to free motion.

Theorem 14.5 (Inverse scattering). [literature; [GGKM67, Lax68]; source, §14.2, Theorem 1] *Let $u(\cdot,t)$ solve the equation and let $L(t) = -\partial_x^2 - u(\cdot,t)$. Then $L(t)$ is unitarily equivalent to $L(0)$: its eigenvalues $-\kappa_n^2$ do not depend on t, and its scattering data evolve by $T_t = 0$, $R_t = 8ik^3R$, $\dot{c}_n = 4\kappa_n^3 c_n$.*

For the two-soliton potential the two eigenvalues are -0.6400 and -0.2500, that is $-k_i^2/4$, at $t = 0$ and at $t = 1.5$. [X7] The theorem is a second morphism of equations, deeper than the first: the map from potentials to scattering

data of Theorem 13.3 carries the nonlinear flow to a linear one with constant coefficients, as the Fourier transform does for linear equations. The conserved eigenvalues are the solitons, and the factors $e^{4\kappa^3 t}$ are their motion.

$$\begin{array}{ccc} u(\cdot,0) & \xrightarrow{\text{scattering}} & (R,\kappa_n,c_n)(0) \\ \text{KdV flow}\downarrow & & \downarrow R\mapsto e^{8ik^3t}R,\ c_n\mapsto e^{4\kappa_n^3 t}c_n \\ u(\cdot,t) & \xrightarrow[\text{scattering}]{} & (R,\kappa_n,c_n)(t) \end{array}$$

14.4 Variational problems

A function that minimizes $\int \mathcal{L}(x,u,\nabla u)\,dx$ satisfies the Euler–Lagrange equation $\partial\mathcal{L}/\partial u = \operatorname{div}(\partial\mathcal{L}/\partial\nabla u)$ [source, §14.3]. Dirichlet's principle is the case $\mathcal{L} = \frac{1}{2}|\nabla u|^2$. For the area of a surface of revolution the equation is $uu'' = 1 + u'^2$, solved by the catenary, with first integral $u/\sqrt{1+u'^2}$. [X8]

Theorem 14.6 (Noether). [classical; [Noe18, Olv93]] *A one-parameter group of symmetries of the action gives a conservation law of the Euler–Lagrange equation, and conversely.*

Time translation gives the energy of Theorem 2.6. The symplectic form of Proposition 3.4 is of the same origin: it is the integral over a time slice of a closed form on space-time built from two variations. In the jet-space language of Chapter 1, conservation laws are cohomology classes of degree $n-1$ in the horizontal complex of the equation [And92], and the Euler–Lagrange equation is the critical locus of a function, which in the presence of gauge symmetry (Remark 13.2) must be taken in the derived sense [CoGw17].

14.5 Bifurcation

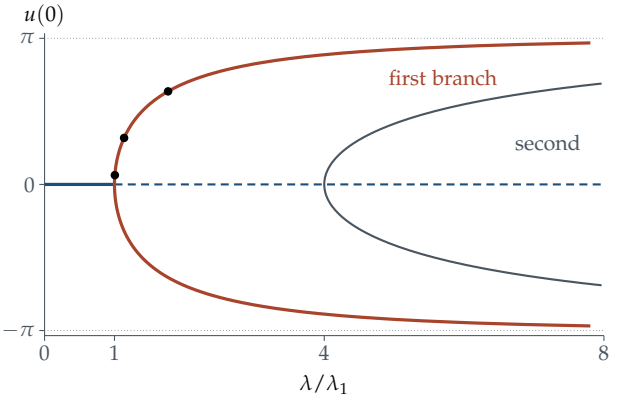
Theorem 14.7 (Buckling). [classical; source, §14.4; [CrRa71]] *For $u'' + \lambda \sin u = 0$ on $(0,l)$ with $u' = 0$ at both ends, the trivial solution is the only one for $\lambda \le \pi^2/l^2$. At each $\lambda_n = n^2\pi^2/l^2$ a pair of branches of non-trivial solutions leaves it, and on the first branch the amplitude $A = u(0)$ satisfies*

$$\lambda/\lambda_1 = \left(\tfrac{2}{\pi}K(\sin\tfrac{A}{2})\right)^2 = 1 + \tfrac{1}{8}A^2 + \cdots,$$

with K the complete elliptic integral.

The values of λ/λ_1 at amplitudes 0.2, 1, 2 are 1.0050, 1.1371, 1.7660: the branch bends to the right, as in the source's figure. [X9] The bifurcation points are

the eigenvalues of the linearization, as they must be by the implicit function theorem; that a branch really leaves each of them follows from the simplicity of the eigenvalue [CrRa71], or from the change of the index of the trivial solution, which is the degree argument of [DTHC, Ch. 7]. The symmetry $u \mapsto -u$ is why the branches come in pairs.



The bifurcation diagram of the buckling problem: the amplitude $u(0)$ against the load. The trivial solution is the only one up to λ_1 and persists, unstable, beyond it (dashed). The first pair of branches is $\lambda/\lambda_1 = (\frac{2}{\pi}K(\sin\frac{A}{2}))^2$, with the three computed points marked; the second leaves at $4\lambda_1$. All branches bend to the right and approach $\pm\pi$, the rod folded back on itself.

14.6 Water waves

The motion of water under gravity is a free boundary problem: Laplace's equation for the velocity potential in a domain whose upper boundary is the unknown [source, §14.5]. In the formulation of Zakharov and of Craig and Sulem the unknowns are the surface elevation and the potential on it, and the dynamics is driven by the Dirichlet-to-Neumann operator of Corollary 7.2 for the moving domain [CrSu93]. Linearizing at still water of depth h gives the dispersion relation $\omega^2 = gk\tanh kh$: waves travel at $\sqrt{gh}$ when they are long and at $\sqrt{g/k}$ when they are short. [X10] The Korteweg–de Vries equation is the first correction to the long-wave limit, which is how the soliton was first seen on a canal.

Notes

The source's Chapter 14 is followed for the examples. For conservation laws and entropy solutions see [Lax73, Eva10]; for the Cole–Hopf transformation [Hop50, Col51]; for inverse scattering [GGKM67, Lax68]; for Noether's theorem and the variational bicomplex [Olv93, And92]; for bifurcation from a

simple eigenvalue [CrRa71]; for the Hamiltonian formulation of water waves [CrSu93].

Exercises

Exercise 14.1. Find the breaking time for $u_t + uu_x = 0$ with $\varphi = 1/(1+x^2)$.

Exercise 14.2. Verify the two shock speeds of Proposition 14.2.

Exercise 14.3. Prove the first statement of Theorem 14.4.

Exercise 14.4. Verify that $f = \frac{c}{2}\operatorname{sech}^2(\frac{\sqrt{c}}{2}\xi)$ satisfies $-cf + f'' + 3f^2 = 0$.

Chapter 15

Assessment

15.1 What the language did

Eight things, roughly in the order of how much they changed our reading of the source.

Boundary conditions became objects. A boundary condition is a character of a reflection group when the domain is a chamber (Theorem 3.5) and a Lagrangian plane of boundary values in general (Theorem 5.1). The first explains when the method of images works and extends it to diffusion, where the source says it fails; the second explains the source's "symmetric" conditions and puts Dirichlet, Neumann, Robin and periodic conditions in one three-dimensional family.

Two methods turned out to be one. The sum over images and the sum over eigenfunctions are exchanged by Pontryagin duality (Theorem 4.6, Theorem 11.4), and the same exchange computes the heat trace (Proposition 10.6). The source presents Fourier's method as the replacement for a method that does not generalize; they are dual.

The comparison of waves and diffusions was reduced to three lines. Group or semigroup; symplectic form or positivity; and the six remaining rows of the source's table follow (Chapter 4).

Stability of schemes was identified with structure. For diffusion it is membership in the Markov category (Theorem 12.2); for waves it is respect for the causal structure (Theorem 12.3); for elliptic problems it is being a map of complexes (Theorem 12.5).

Problems could be glued. Dirichlet-to-Neumann maps, transfer matrices and boundary conditions compose as Lagrangian relations (Theorem 7.7, Theorem 13.3).

The type was located. It is invisible to the algebra of formal solutions and lives in the real points of the characteristic variety (Remark 1.6); it reappears as the singular support of the fundamental solution (Theorem 11.3) and as the kind of functor that time is.

Solvable nonlinear equations were explained by morphisms. Cole–Hopf and the scattering transform are maps of equations onto linear ones (Theorem 14.4, Theorem 14.5).

Isospectrality became an isomorphism of representations that is not an isometry (Theorem 10.7).

15.2 What it did not do

It proved no theorem of analysis. Existence of solutions, in every chapter, rests on estimates that the language does not supply: the maximum principle, energy inequalities, compactness. The definition of well-posedness as an isomorphism in a category of topological vector spaces (Definition 2.2) says where the difficulty is and does not reduce it. The machinery cited in Chapter 1, the jet comonad and $\mathcal{D}$-modules, was used only to say what a morphism of equations is. Microlocal analysis and hyperfunctions, where sheaf theory does real work for linear equations, are described and not used. The nonlinear chapter treats two integrable examples and one scalar conservation law; for systems of conservation laws, for the Euler and Navier–Stokes equations and for the full water wave problem the categorical viewpoint has, so far as we can see, nothing to offer beyond the remarks made. The observations tagged as such are short and may well be known in other forms.

15.3 How much of this is higher

Source	Structure used here	Level
Ch. 1	solution functor; symbol; quotient by characteristics	1-categorical
Ch. 2–3	group and Markov semigroup; reflection groups; causal functor	1-categorical
Ch. 4–5	Lagrangian planes; Pontryagin duality	1-categorical; relations compose
Ch. 6–7	sheaf of harmonic functions; Lagrangian relations	cohomology in degree 1; a category of cospans
Ch. 8	commuting squares; cochain maps	1-categorical; chain level
Ch. 9	causal functor; descent; a holomorphic semigroup	1-categorical; microlocal sheaves cited
Ch. 10	representations of symmetry groups	1-categorical
Ch. 11	monotone functors; trace of a semigroup; permutation representations	1-categorical
Ch. 12	sheaf of distributions; hyperfunctions	local cohomology
Ch. 13	gauge groupoids; functorial field theory	stacks; derived solution spaces
Ch. 14	morphisms of equations; conservation laws	1-categorical; horizontal cohomology

The subject at the level of the source is ordinary category theory with a good deal of representation theory. Higher structure enters at four points, and only the last is unavoidable. Cohomology in degree one appears for the annulus, for comparison of discretizations, and for flat gauge fields. Networks and domains with boundary compose as cospans, which form a bicategory or double category when the gluing is tracked up to isomorphism. Hyperfunctions and the solution complexes of $\mathcal{D}$-modules are derived objects; the cokernel in Example 2.3 is the first Ext group of the theory. And gauge fields form stacks, so that for the equations of Chapter 13 the solution functor does not take values in sets.

15.4 Programme

[programme] (1) Characterize the boundary-value problems solvable by images as those on quotients of space by reflection groupoids, including orbifold covers, and decide whether mixed and Robin conditions have a place there. (2)

Formulate the equivalence theorem of Theorem 12.1 as a statement about lax natural transformations between functors on a category of grids, with stability as the condition for the colimit to exist. (3) Extend Theorem 7.7 from networks to domains with geometry, as a functor from a category of cobordisms to Lagrangian relations, and relate it to the gluing formulas for determinants of Laplacians. (4) Find a universal property that selects entropy solutions among weak solutions of a balance law, as Theorem 14.3 selects them by a limit.

Notes

The companion paper [PDEML] applies the framework to machine learning: padding as a boundary condition, the score in diffusion models as the datum that makes the backward heat equation well-posed, the spectral gap as the rate of oversmoothing in graph networks, and the type of an equation as the rank of its solution operator. The volume belongs to a series with [GIHC, ESCHC, DTHC, EGHC].

Appendix A

Toolkit

General notions used in the text, in the order in which they are needed.

Sheaves. A sheaf on a space assigns a set, or vector space, to each open subset, with restriction maps, such that compatible local sections glue uniquely. Solutions of a differential equation form a sheaf, as do smooth functions and distributions. Its cohomology in degree one measures the failure of local solutions of an auxiliary problem to be global.

Modules and solutions. For a ring $\mathcal{D}$ of differential operators and a left ideal generated by P, maps of left modules $\mathcal{D}/\mathcal{D}P \to \mathcal{F}$ are the elements of $\mathcal{F}$ killed by P. Exactness properties of $\mathrm{Hom}_{\mathcal{D}}(-, \mathcal{F})$ turn factorizations of operators into relations between solution spaces.

Quadratic forms. A real quadratic form has an inertia: the numbers of positive, negative and zero squares in any diagonalization. It is invariant under change of basis (Sylvester).

Groups, actions, characters. A group acting on a space acts on functions by $(g \cdot u)(x) = u(g^{-1}x)$. For a homomorphism χ to $\{\pm 1\}$ the functions with $g \cdot u = \chi(g)u$ form the χ-isotypic part. A representation is irreducible if it has no invariant subspace; an operator commuting with the group preserves isotypic parts and acts on each irreducible one by a scalar (Schur).

Semigroups and Markov kernels. A one-parameter semigroup is a family T_t, $t \geq 0$, with $T_{t+s} = T_t T_s$. A Markov kernel assigns to each point a probability measure, measurably; kernels compose, and they are the morphisms of the category Stoch. A semigroup of Markov kernels is positive and preserves constants.

Symplectic spaces and Lagrangian relations. A symplectic form is a non-degenerate antisymmetric bilinear form; a subspace is Lagrangian if the form vanishes on it and it has half the dimension. A Lagrangian relation from V to

W is a Lagrangian subspace of $\overline{V} \oplus W$, with the sign of the form reversed on V; graphs of symplectic maps are examples, and relations compose.

Duality of abelian groups. The characters of a locally compact abelian group form its dual group; the dual of the circle is $\mathbb{Z}$, the line is self-dual, and the dual of a quotient is a subgroup. The Fourier transform is the isomorphism of L^2 of a group with L^2 of its dual.

Complexes. A complex is a sequence of maps with $\mathrm{dd} = 0$; its cohomology is kernel modulo image. The de Rham complex of forms and the cochain complex of a triangulation have the same cohomology, and a map of complexes induces a map on cohomology.

Groupoids and stacks. A groupoid is a category in which every morphism is invertible. A stack assigns a groupoid to each open set, with gluing up to specified isomorphism; fields with gauge symmetry form stacks.

Appendix B

Concordance with the Source

The table lists the sections of Walter A. Strauss, *Partial Differential Equations: An Introduction*, second edition [Stro8], with the place in this volume where their content is treated. The source was read from a text extraction of its PDF: the contents, every statement labelled as a theorem, definition or principle, located by search, and selected passages of every chapter. It was not read in full, its exercises were consulted only where the text refers to them, and figures were seen only through the surrounding text, with one page inspected as an image to confirm a sign.

Source	Here	Form
1.1 What is a PDE?	Definition 1.1, Theorem 1.2	jets; a category of equations
1.2 First-order linear equations	Proposition 1.8	quotient by the characteristics
1.3 Flows, vibrations, diffusions	Proposition 2.1	balance law plus material law
1.4 Initial and boundary conditions	Chapter 2	data as a restriction map
1.5 Well-posed problems	Definition 2.2, Example 2.4	an isomorphism of topological vector spaces
1.6 Types of second-order equations	Theorem 1.5, Remark 1.6	inertia of the symbol
2.1 The wave equation	Theorem 3.1, Corollary 3.2	two pullbacks; parallelogram identity
2.2 Causality and energy	Theorem 3.3, Proposition 3.4	functor on intervals; symplectic form
2.3 The diffusion equation	Theorem 4.2, Theorem 2.6	as stated
2.4 Diffusion on the whole line	Theorem 4.1, Proposition 4.4	Markov semigroup; scaling
2.5 Comparison of waves and diffusions	Chapter 4	three structural rows
3.1 Diffusion on the half-line	Theorem 4.6	images; survival probability
3.2 Reflections of waves	Theorem 3.5, Remark 3.6	characters of reflection groups

Source	Here	Form
3.3 Diffusion with a source	Theorem 4.7	variation of constants
3.4 Waves with a source	Theorem 3.7, Theorem 3.8	Green operators; exact sequence
3.5 Diffusion revisited	Theorem 4.1, Theorem 4.5	smoothing; monotone quantities
4.1–4.2 Separation of variables	Chapter 5	diagonalizing the generator
4.3 The Robin condition	Theorem 5.1	a line in the plane of end values
5.1–5.2 Coefficients; even, odd, periodic	Theorem 5.3	duality; isotypic components
5.3 Orthogonality	Theorem 5.1	Lagrangian planes
5.4–5.5 Completeness; Gibbs	Theorem 5.4	isomorphism of Hilbert spaces only
5.6 Inhomogeneous boundary conditions	Chapter 5	affine space of solutions
6.1 Laplace's equation	Theorem 6.1, Theorem 6.2	sheaf; naturality
6.2 Rectangles and cubes	Theorem 9.2	tensor products
6.3 Poisson's formula	Theorem 6.3, Theorem 6.4	Markov kernel to the boundary
6.4 Circles, wedges, annuli	Proposition 6.5, Proposition 6.6	first cohomology; corner exponents
7.1–7.2 Green's identities	Proposition 7.1, Theorem 7.3, Corollary 7.2	Stokes; projection; Lagrangian graph
7.3 Green's functions	Theorem 7.5	kernel of the inverse
7.4 Half-space and sphere	Theorem 7.6	reflection; inversion
8.1 Opportunities and dangers	Theorem 12.1	squares that almost commute
8.2 Approximations of diffusions	Theorem 12.2	stability as positivity
8.3 Approximations of waves	Theorem 12.3	stability as causality
8.4 Approximations of Laplace's equation	Proposition 12.4	iterations and their rates
8.5 Finite element method	Chapter 12, Theorem 12.5	energy projection; complexes
9.1 Energy and causality	Theorem 8.1	cones
9.2 The wave equation in space-time	Theorem 8.2, Proposition 8.3	Huygens; descent
9.3 Rays, singularities, sources	Theorem 8.4	wave front sets
9.4 Diffusion and Schrödinger	Theorem 8.5	one holomorphic semigroup

Source	Here	Form
9.5 The hydrogen atom	Theorem 8.6, Remark 8.7	hidden symmetry
10.1 Fourier's method, revisited	Theorem 9.1, Theorem 9.2	eigenspaces as representations
10.2, 10.5 Drumhead; Bessel functions	Theorem 9.3, Remark 9.4	characters of the rotation group
10.3, 10.6 Ball; Legendre functions	Theorem 9.5	harmonic polynomials
10.4 Nodes	Theorem 9.6	as stated
10.7 Angular momentum	Chapter 9	Casimir element
11.1–11.2 Minima; computation	Theorem 10.1	minimax
11.3–11.5 Completeness	Theorem 10.3	functional calculus
11.6 Asymptotics	Corollary 10.2, Theorem 10.4	monotone functor; Weyl's law
12.1 Distributions	Chapter 11, Theorem 11.2	change of solution space; kernels
12.2 Green's functions, revisited	Theorem 11.3	fundamental solutions
12.3 Fourier transforms	Theorem 11.4	duality of the line
12.4 Source functions	Chapter 11	Poisson semigroup
12.5 Laplace transform techniques	Proposition 11.6	resolvent
13.1 Electromagnetism	Proposition 13.1, Remark 13.2	forms; gauge groupoid
13.2 Fluids and acoustics	Chapter 13	linearization
13.3–13.4 Scattering; continuous spectrum	Theorem 13.3	transfer matrices compose
13.5 Equations of elementary particles	Proposition 13.4, Theorem 13.5	Clifford relations; field functor
14.1 Shock waves	Proposition 14.1, Proposition 14.2, Theorem 14.4	weak solutions; a morphism of equations
14.2 Solitons	Theorem 14.5	isospectral flow
14.3 Calculus of variations	Theorem 14.6	conservation laws
14.4 Bifurcation theory	Theorem 14.7	as stated, with the branch computed
14.5 Water waves	Chapter 14	dispersion relation only
Appendix A.1–A.5	—	assumed
—	Theorem 7.7	added: gluing of boundary problems
—	Theorem 10.5, Proposition 10.6, Theorem 10.7	added: heat trace; isospectrality

Appendix C

Computations

The script `verify-pde.py` that accompanies this volume uses only the Python standard library, fixes its random seed, and runs in under ten seconds. It performs 107 checks, each printing `PASS` or `FAIL` with a one-sentence statement of what is checked and the computed values, and exits with status zero if and only if all pass. The checks recompute the numerical statements of the text and test finite or numerical instances of the structural ones. They are evidence that examples are stated correctly; they do not prove the theorems. Polynomial identities and ranks are computed in exact rational arithmetic; integrals by Simpson's rule; derivatives by fourth-order differences; eigenvalues by Jacobi rotations or Sturm bisection.

The curves in the figures were computed from the formulas of the text by a separate plotting script and are embedded in the source as coordinates. Where a figure shows a number that the text states (the Gibbs peak, the two-soliton amplitude, the points on the buckling branch), the plotting script asserts agreement with the value checked here before drawing it.

Checks	Chapter	What is computed
E1–E6	1	invariance of the type under changes of variables; the source's variable-coefficient example; solutions by characteristics; dimensions of polynomial solutions for three types; factorization of the wave operator; heat polynomials and Kovalevskaya's divergent series
W1–W6	2	Hadamard's example with sizes; backward amplification; the Laplacian as $D^{\mathsf{T}}D$; the two energy laws; the Neumann alternative; a boundary problem with a kernel
V1–V8	3	d'Alembert's formula; the parallelogram identity; causality; energy; reflection on the half-line; the sum over the dihedral group against the Fourier series; Duhamel's formula; the symplectic form
D1–D9	4	the kernel as a Markov semigroup; the diffused step; the explicit scheme as a Markov matrix; entropy and square norm; half-line kernels; the two expansions on an interval with term counts; scaling; positivity at a distance; Duhamel's formula

Checks	Chapter	What is computed
F1–F8	5	orthogonality; Robin eigenvalues and the zero criterion; Lagrangian planes; Parseval; the Gibbs constant; decay of coefficients; characters of $\mathbb{Z}/n$ and their even and odd parts; shifted data
H1–H7	6	mean values; the discrete maximum principle; Poisson's formula; naturality for rotations and conformal maps; periods on the annulus; the corner exponent; hitting probabilities
G1–G8	7	summation by parts; Dirichlet's principle; the discrete Green's function; half-space and ball by images; gluing of networks; Galerkin orthogonality; the kernel on an interval
S1–S5	8	radial and plane waves, the eikonal condition; Huygens in three dimensions and the tail in two; the Schrödinger kernel and a spreading packet; hydrogen; the oscillator
Y1–Y6	9	harmonic polynomials and the rotation algebra; Legendre polynomials; Bessel zeros; spectra of products; nodal counts; multiplicities forced by symmetry
L1–L9	10	Rayleigh–Ritz bounds; interlacing; monotonicity and its failure for Neumann conditions; Weyl's law; the heat trace of a rectangle; a trace formula; cospectral graphs by exhaustive search; the Sunada pair of order 32; completeness
T1–T7	11	the delta function as a limit; derivatives by duality; the fundamental solution in space; the Fourier transform; the source function from the transform; resolvent as Laplace transform; boundary values
N1–N10	12	instability at $s = 0.52$; orders two and four; Crank–Nicolson; the exact wave scheme and its instability; causality; three iterations; finite elements in one and two dimensions; Betti numbers of a triangulated annulus; a consistent unstable scheme
P1–P8	13	the cubical 3-torus and gauge invariance; a plane electromagnetic wave; the sound speed; the string of two materials; transfer matrices; Klein–Gordon; Dirac matrices; a symplectic integrator
X1–X10	14	breaking time; the source's explicit solution; conservative and non-conservative schemes; Cole–Hopf; one and two solitons with phase shifts; isospectrality; the catenary; the buckling branch; water waves

Three points of contact with the source's own numbers and statements were found. Its Gibbs constant "0.59" and "9 percent" are 0.58949 and 8.95 per cent (F5). Its remark that the reflection method does not work for diffusion is true of the formulas it gives and not of the sum over the reflection group,

which agrees with the eigenfunction series to 10^{-10} (D6). And the sign of the continuity equation in the text of its Section 13.1 differs from that in its Exercise 1 of the same section, which is the correct one.

Appendix D

Solutions to the Exercises

Solutions to all exercises follow, by chapter.

Chapter 1

1.1. With $d = a_{12}^2 - a_{11}a_{22}$: $d = 4 - 4 = 0$, parabolic; $d = 1 - 5 < 0$, elliptic; $a_{11} = a_{22} = 0$, $a_{12} = \frac{1}{2}$, $d > 0$, hyperbolic.

1.2. Along a characteristic $dy/dx = 2xy^2$, so $-1/y = x^2 + C$. Take $q = x^2 + 1/y$; the general solution is $u = f(x^2 + 1/y)$ on each of $y > 0$ and $y < 0$.

1.3. The real and imaginary parts of $(x + iy)^k$ are harmonic and independent. For $u_{xx} = 0$ take y^k and xy^{k-1}. In both cases the space of all homogeneous polynomials has dimension $k + 1$ and the operator maps it onto the space of degree $k - 2$, of dimension $k - 1$.

1.4. $\mathrm{Sol}(P)$ consists of the functions $h(x + ct)$, and $Q\,h(x + ct) = 2c\,h'(x + ct)$. Every smooth function of one variable is a derivative, so Q maps $\mathrm{Sol}(P)$ onto itself and Proposition 1.3 applies.

Chapter 2

2.1. The functions $\sin nx \sin nt$ solve the equation and vanish on the whole boundary of the square.

2.2. $D^{\mathsf{T}}D$ is symmetric, so its range is the orthogonal complement of its kernel. If $D^{\mathsf{T}}Dv = 0$ then $|Dv|^2 = 0$, so v is constant along every edge and, the graph being connected, constant.

2.3. The error in mode n is $10^{-10}e^{0.1n^2}$, which exceeds 1 when $n^2 > 10 \ln 10^{10} = 230.3$: the mode $n = 16$.

2.4. $\frac{d}{dt}\frac{1}{2}\int u^2 = k\int uu_{xx} = k[uu_x]_0^l - k\int u_x^2 = -ka_l u(l)^2 - ka_0 u(0)^2 - k\int u_x^2$. It is non-positive for all solutions when $a_0, a_l \geq 0$.

Chapter 3

3.1. In coordinates $(\xi,\eta) = (x+ct, x-ct)$ the vertices are (ξ,η), $(\xi+h,\eta)$, $(\xi+h,\eta+k)$, $(\xi,\eta+k)$, and both sums equal $f(\xi)+f(\xi+h)+g(\eta)+g(\eta+k)$.

3.2. With the even extension, $v(x,t) = \frac{1}{2}[\varphi(ct+x)+\varphi(ct-x)]$ for $0<x<ct$.

3.3. The character is -1 on the reflection in 0 and $+1$ on the reflection in l, hence -1 on their product, the translation by $2l$. So the extended solution satisfies $u(x+2l,t) = -u(x,t)$ and has period $4l$ in x; by Theorem 3.1 the functions f and g have period $4l$, and u has period $4l/c$ in t.

3.4. $\int_{x-c(t-s)}^{x+c(t-s)} \cos y\,dy = 2\cos x \sin c(t-s)$, and $\frac{1}{2c}\int_0^t 2\cos x \sin c(t-s)\,ds = \cos x\,(1-\cos ct)/c^2$.

Chapter 4

4.1. With $\xi = x/\sqrt{t}$, the equation becomes $-\frac{1}{2}g - \frac{1}{2}\xi g' = kg''$, that is $(kg' + \frac{1}{2}\xi g)' = 0$. For a solution decaying at infinity $kg' = -\frac{1}{2}\xi g$ and $g = Ce^{-\xi^2/4k}$.

4.2. By Theorem 4.6 it is $\operatorname{erf}(x/\sqrt{4kt}) = \operatorname{erf}(\frac{1}{2}) = 0.5205$.

4.3. (a) $\frac{d}{dt}\int u^2 = 2k\int uu_{xx} = -2k\int u_x^2$. (b) Since $\int u_t = 0$, $\frac{d}{dt}(-\int u\log u) = -\int u_t \log u = -k\int u_{xx}\log u = k\int u_x^2/u$.

4.4. The m-th shell of images is at distance at least $(2m-1)l$, so it is negligible when $e^{-(2m-1)^2l^2/4kt} < \epsilon$: about $\frac{1}{2} + \sqrt{kt\ln(1/\epsilon)}/l$ shells. The n-th mode is negligible when $e^{-kn^2\pi^2t/l^2} < \epsilon$: about $(l/\pi)\sqrt{\ln(1/\epsilon)/kt}$ modes. For the table these give 0.6 and 58, 1.4 and 8.2, 4.5 and 1.8.

Chapter 5

5.1. With $\omega(u,v) = -u_4v_3 + u_3v_4 + u_2v_1 - u_1v_2$: for the Robin pair every term vanishes; for $(1,0,1,0)$ and $(0,2,0,1)$ it is $1-2=-1$.

5.2. $X = \sin((n+\frac{1}{2})\pi x/l)$, $\lambda = ((n+\frac{1}{2})\pi/l)^2$, $n \ge 0$. All frequencies are odd multiples of $\pi c/2l$, so every motion has period $4l/c$.

5.3. For $\lambda = 0$, $X = A + Bx$. The conditions give $B = a_0A$ and $B = -a_l(A + Bl)$, which have a non-zero solution exactly when $a_0 = -a_l(1+a_0l)$.

5.4. $(1.85194/\pi - \frac{1}{2})J = 0.0895\,J$ on each side of the jump.

Chapter 6

6.1. Expand u about the centre. The linear terms average to zero and the quadratic terms $\frac{r^2}{2}(u_{xx}\cos^2\theta + 2u_{xy}\cos\theta\sin\theta + u_{yy}\sin^2\theta)$ average to $\frac{r^2}{4}(u_{xx} + u_{yy})$.

6.2. $u \circ f = (x^2 - y^2)^2$, whose Laplacian is $(12x^2 - 4y^2) + (12y^2 - 4x^2) = 8(x^2 + y^2)$; and $|f'|^2\Delta u = 4(x^2 + y^2)\cdot 2$.

6.3. For $\log r$, $\star\,\mathrm{d}u = (-y\,dx + x\,dy)/r^2 = \mathrm{d}\theta$, with period 2π. The function $\mathrm{Re}(1/z)$ has the single-valued conjugate $\mathrm{Im}(1/z)$, so its period is 0.

6.4. At the re-entrant corner $\beta = 3\pi/2$: the exponent is $\frac{2}{3}$ and the gradient grows like $r^{-1/3}$. At a convex corner $\beta = \pi/2$: the exponent is 2 and the gradient vanishes like r.

Chapter 7

7.1. For harmonic u, v with boundary values h, h', Green's second identity gives $\int_{\partial D}(h\,\Lambda h' - h'\,\Lambda h) = \int_D (u\Delta v - v\Delta u) = 0$.

7.2. Eliminating the middle vertex from the Laplacian with rows $(g_1, -g_1, 0)$, $(-g_1, g_1 + g_2, -g_2)$, $(0, -g_2, g_2)$ gives the conductance $g_1 - g_1^2/(g_1 + g_2) = g_1 g_2/(g_1 + g_2)$.

7.3. The kernel is linear in x on each side of y, vanishes at 0 and 1, is continuous, and its derivative drops by 1 at $x = y$: $x(1 - y)$ for $x < y$ and $y(1 - x)$ for $x > y$. In eigenfunctions, $\sum_n 2\sin(n\pi x)\sin(n\pi y)/(n\pi)^2$.

7.4. With $r_0 = |x_0|$ and $|x| = a$: $|x - x_0^*|^2 = a^2 - 2\frac{a^2}{r_0^2}x\cdot x_0 + \frac{a^4}{r_0^2} = \frac{a^2}{r_0^2}(r_0^2 - 2x\cdot x_0 + a^2) = \frac{a^2}{r_0^2}|x - x_0|^2$. The two terms of the Green's function then cancel.

Chapter 8

8.1. For radial u, $\Delta u = u_{rr} + \frac{2}{r}u_r = (ru)_{rr}/r$, so $(ru)_{tt} = c^2(ru)_{rr}$ and Theorem 3.1 applies to ru.

8.2. On the sphere of radius R the area element over the plane element $\rho\,d\rho\,d\theta$ is $R\rho\,d\rho\,d\theta/\sqrt{R^2 - \rho^2}$ on each hemisphere. Dividing twice this by $4\pi R^2$ gives the density. With $R = ct$, $\frac{1}{4\pi c^2 t}\int_{\text{sphere}}\psi\,dS = \frac{1}{2\pi c}\int_{\text{disk}}\psi\,dy/\sqrt{c^2t^2 - \rho^2}$.

8.3. With $a = 1 + it$: $u_t = i\big(-\frac{1}{2a} + \frac{x^2}{2a^2}\big)u$ and $u_{xx} = \big(-\frac{1}{a} + \frac{x^2}{a^2}\big)u$, so $iu_t = -\frac{1}{2}u_{xx}$. Then $|u|^2 = |a|^{-1}e^{-x^2\mathrm{Re}(1/a)}$ with $\mathrm{Re}(1/a) = 1/(1 + t^2)$, a Gaussian of variance $(1 + t^2)/2$.

8.4. $a_1 = -\frac{2}{3}$, $3a_2 = (\frac{2}{3} - 1)a_1 = \frac{2}{9}$ so $a_2 = \frac{2}{27}$, and $a_3 = 0$. The states are those with $l = 0, 1, 2$: $1 + 3 + 5 = 9$.

Chapter 9

9.1. The eigenspace is spanned by $\sin x \sin 2y$ and $\sin 2x \sin y$. Reflection in the diagonal exchanges them and the quarter turn about the centre sends the first to minus the second: the two-dimensional irreducible representation. For $50 = 1 + 49 = 49 + 1 = 25 + 25$ the multiplicity is 3. The function $\sin 5x \sin 5y$ spans an invariant line, and $\sin x \sin 7y \pm \sin 7x \sin y$ span two more: the eigenspace is a sum of three one-dimensional representations, and its multiplicity is arithmetic and not forced by symmetry.

9.2. $\binom{n+l-1}{l} - \binom{n+l-3}{l-2}$: 2 for $n = 2$, $2l + 1$ for $n = 3$, $(l + 1)^2$ for $n = 4$.

9.3. e^{iy} satisfies $\Delta u = -u$. Projection onto the n-th character of the rotation group commutes with Δ, so $J_n(r)e^{in\theta}$ satisfies the same equation, which in polar coordinates is $J_n'' + \frac{1}{r}J_n' + (1 - \frac{n^2}{r^2})J_n = 0$.

9.4. $cz/2\pi a$ with $z = 2.4048$ (simple), 3.8317 (double), 5.1356 (double), 5.5201 (simple).

Chapter 10

10.1. $\int_0^1 (1 - 2x)^2 = \frac{1}{3}$ and $\int_0^1 x^2(1 - x)^2 = \frac{1}{30}$; the quotient is 10.

10.2. The inscribed square has side $\sqrt{2}$ and first eigenvalue $2\pi^2/2 = 9.87$; the circumscribed one has side 2 and first eigenvalue $2\pi^2/4 = 4.93$. So $4.93 \leq \lambda_1 \leq 9.87$; in fact $\lambda_1 = 2.4048^2 = 5.78$.

10.3. The eigenvalues are $\pi^2(m^2/a^2 + n^2/b^2)$ with $m, n \geq 1$. The lattice points with $m, n \geq 0$ in the quarter ellipse number about its area, $ab\lambda/4\pi$; those on the two axes, about $(a + b)\sqrt{\lambda}/\pi$, are counted with weight one half too many each, giving the correction $-(a + b)\sqrt{\lambda}/2\pi = -P\sqrt{\lambda}/4\pi$.

10.4. Rectangle: $4 \cdot (\pi^2 - \pi^2/4)/(24\pi \cdot \pi/2) = \frac{1}{4}$. Equilateral triangle: $3 \cdot (\pi^2 - \pi^2/9)/(24\pi \cdot \pi/3) = \frac{1}{3}$.

Chapter 11

11.1. $\int |x|\varphi'' = \int_0^\infty x\varphi'' - \int_{-\infty}^0 x\varphi'' = [x\varphi' - \varphi]_0^\infty - [x\varphi' - \varphi]_{-\infty}^0 = 2\varphi(0)$.

11.2. $\frac{1}{2}|x|$, by the previous exercise; and $\frac{1}{2}e^{-|x|}$, which satisfies $E'' = E$ away from 0 and whose derivative drops by 1 at 0.

11.3. $S = \frac{1}{2\pi}\int e^{-k^2t+ikx}dk$; completing the square, $-t(k-ix/2t)^2 - x^2/4t$, gives $(4\pi t)^{-1/2}e^{-x^2/4t}$.

11.4. The transform of P_y is $e^{-|k|y}$, and $e^{-|k|y}e^{-|k|y'} = e^{-|k|(y+y')}$.

Chapter 12

12.1. Substituting $u_j^n = \xi^n e^{ijk\Delta x}$ gives $\xi = 1 - 2s(1-\cos k\Delta x) \in [1-4s, 1]$, of modulus at most one for all k exactly when $s \le \frac{1}{2}$.

12.2. The scheme equals $u_t + \frac{\Delta t}{2}u_{tt} - u_{xx} - \frac{\Delta x^2}{12}u_{xxxx}$ up to higher order. For solutions $u_{tt} = u_{xxxx}$, so the leading error is $(\frac{\Delta t}{2} - \frac{\Delta x^2}{12})u_{xxxx}$, which vanishes when $\Delta t = \Delta x^2/6$.

12.3. The product of the roots is 1. They are complex conjugate, hence of modulus one, when $(1+p)^2 \le 1$, that is $-2 \le p \le 0$; otherwise real and distinct with one of modulus greater than one. Since p ranges over $[-2s, 0]$, the condition is $s \le 1$.

12.4. $12 - 24 + 12 = 0 = b_0 - b_1 + b_2$ with $b_0 = 1$ and $b_2 = 0$, so $b_1 = 1$.

Chapter 13

13.1. $\partial_t \rho = \frac{1}{4\pi}\operatorname{div}\partial_t E = \frac{1}{4\pi}\operatorname{div}(c\ \operatorname{curl} B - 4\pi J) = -\operatorname{div} J$.

13.2. $R^2 + \frac{c_1}{c_2}T^2 = \big((c_2-c_1)^2 + 4c_1c_2\big)/(c_1+c_2)^2 = 1$.

13.3. ψ is continuous and ψ' jumps by $g\psi(0)$: the matrix is $\left(\begin{smallmatrix}1 & 0\\ g & 1\end{smallmatrix}\right)$. From $1+R = T$ and $g(1+R) + ik(1-R) = ikT$, $T = 2ik/(2ik-g)$ and $|T|^2 = 4k^2/(4k^2+g^2)$.

13.4. $\omega\, d\omega = c^2 k\, dk$, so $\frac{d\omega}{dk}\cdot\frac{\omega}{k} = c^2$.

Chapter 14

14.1. $\varphi' = -2x/(1+x^2)^2$ is least at $x = 1/\sqrt{3}$, where it equals $-9/8\sqrt{3}$; $t_b = 8\sqrt{3}/9 = 1.5396$.

14.2. $[\frac{1}{2}u^2]/[u] = \frac{1}{2}$ and $[\frac{2}{3}u^3]/[u^2] = \frac{2}{3}$ for the jump from 1 to 0.

14.3. Let $w = \log\varphi$, so $w_t = \varepsilon(w_{xx} + w_x^2)$ and $u = -2\varepsilon w_x$. Then $u_t = -2\varepsilon^2(w_{xxx} + 2w_xw_{xx}) = \varepsilon u_{xx} - uu_x$.

14.4. With $\theta = \frac{\sqrt{c}}{2}\xi$, $(\operatorname{sech}^2)'' = 4\operatorname{sech}^2 - 6\operatorname{sech}^4$, so $f'' = \frac{c^2}{2}\operatorname{sech}^2\theta - \frac{3c^2}{4}\operatorname{sech}^4\theta$, while $-cf = -\frac{c^2}{2}\operatorname{sech}^2\theta$ and $3f^2 = \frac{3c^2}{4}\operatorname{sech}^4\theta$.

References

Entries were assembled from memory, except [KhSc17] and [GKSS26], whose authors, titles and identifiers were checked online on 17 November 2026. Volume and page numbers in particular should be verified before they are relied on.

[ABR01] S. Axler, P. Bourdon, W. Ramey, *Harmonic Function Theory*, 2nd ed., Graduate Texts in Mathematics 137, Springer, 2001.

[AFW06] D. N. Arnold, R. S. Falk, R. Winther, Finite element exterior calculus, homological techniques, and applications, *Acta Numer.* 15 (2006), 1–155.

[And92] I. M. Anderson, Introduction to the variational bicomplex, in *Mathematical Aspects of Classical Field Theory*, Contemp. Math. 132, American Mathematical Society, 1992, 51–73.

[Arn66] V. I. Arnold, Sur la géométrie différentielle des groupes de Lie de dimension infinie et ses applications à l'hydrodynamique des fluides parfaits, *Ann. Inst. Fourier* 16 (1966), 319–361.

[BaFo18] J. C. Baez, B. Fong, A compositional framework for passive linear networks, *Theory Appl. Categ.* 33 (2018), 1158–1222.

[BFV03] R. Brunetti, K. Fredenhagen, R. Verch, The generally covariant locality principle: a new paradigm for local quantum field theory, *Comm. Math. Phys.* 237 (2003), 31–68.

[BGP07] C. Bär, N. Ginoux, F. Pfäffle, *Wave Equations on Lorentzian Manifolds and Quantization*, European Mathematical Society, 2007.

[Bos98] A. Bossavit, *Computational Electromagnetism*, Academic Press, 1998.

[Bro88] R. Brooks, Constructing isospectral manifolds, *Amer. Math. Monthly* 95 (1988), 823–839.

[CFL28] R. Courant, K. Friedrichs, H. Lewy, Über die partiellen Differenzengleichungen der mathematischen Physik, *Math. Ann.* 100 (1928), 32–74.

[CoGw17] K. Costello, O. Gwilliam, *Factorization Algebras in Quantum Field Theory*, vol. 1, Cambridge University Press, 2017.

[CoHi53] R. Courant, D. Hilbert, *Methods of Mathematical Physics*, vol. I, Interscience, 1953.

[Col51] J. D. Cole, On a quasi-linear parabolic equation occurring in aerodynamics, *Quart. Appl. Math.* 9 (1951), 225–236.

[CoTho6] T. M. Cover, J. A. Thomas, *Elements of Information Theory*, 2nd ed., Wiley, 2006.

[Cou95] S. C. Coutinho, *A Primer of Algebraic D-Modules*, London Math. Soc. Student Texts 33, Cambridge University Press, 1995.

[CrRa71] M. G. Crandall, P. H. Rabinowitz, Bifurcation from simple eigenvalues, *J. Funct. Anal.* 8 (1971), 321–340.

[CrSu93] W. Craig, C. Sulem, Numerical simulation of gravity waves, *J. Comput. Phys.* 108 (1993), 73–83.

[DGLS01] E. B. Davies, G. M. L. Gladwell, J. Leydold, P. F. Stadler, Discrete nodal domain theorems, *Linear Algebra Appl.* 336 (2001), 51–60.

[DoBu13] F. Dörfler, F. Bullo, Kron reduction of graphs with applications to electrical networks, *IEEE Trans. Circuits Syst. I* 60 (2013), 150–163.

[DTHC] Anonymous, *Differential Topology in Higher Categories*, Higher Categorical Semantics, 2026. Companion volume.

[DuHo72] J. J. Duistermaat, L. Hörmander, Fourier integral operators II, *Acta Math.* 128 (1972), 183–269.

[EGHC] Anonymous, *Evolutionary Game Theory in Higher Categories*, Higher Categorical Semantics, 2026. Companion volume.

[Ehr54] L. Ehrenpreis, Solution of some problems of division I, *Amer. J. Math.* 76 (1954), 883–903.

[Ehr70] L. Ehrenpreis, *Fourier Analysis in Several Complex Variables*, Wiley-Interscience, 1970.

[ESCHC] Anonymous, *Entropy, Sources, and Channels in Higher Categories*, Higher Categorical Semantics, 2026. Companion volume.

[Eva10] L. C. Evans, *Partial Differential Equations*, 2nd ed., Graduate Studies in Mathematics 19, American Mathematical Society, 2010.

[EvMa99] W. N. Everitt, L. Markus, *Boundary Value Problems and Symplectic Algebra for Ordinary Differential and Quasi-Differential Operators*, Math. Surveys Monogr. 61, American Mathematical Society, 1999.

[Foc35] V. Fock, Zur Theorie des Wasserstoffatoms, *Z. Phys.* 98 (1935), 145–154.

[Fre00] D. S. Freed, Dirac charge quantization and generalized differential cohomology, in *Surveys in Differential Geometry VII*, International Press, 2000, 129–194.

[GGKM67] C. S. Gardner, J. M. Greene, M. D. Kruskal, R. M. Miura, Method for solving the Korteweg–de Vries equation, *Phys. Rev. Lett.* 19 (1967), 1095–1097.

[GIHC] Anonymous, *Games and Information in Higher Categories*, Higher Categorical Semantics, 2026. Companion volume.

[GKSS26] G. Giotopoulos, I. Khavkine, H. Sati, U. Schreiber, Synthetic differential jet bundles are reduced, preprint, arXiv:2601.15862, 2026. Checked.

[Gri85] P. Grisvard, *Elliptic Problems in Nonsmooth Domains*, Pitman, 1985.

[GWW92] C. Gordon, D. Webb, S. Wolpert, Isospectral plane domains and surfaces via Riemannian orbifolds, *Invent. Math.* 110 (1992), 1–22.

[Had23] J. Hadamard, *Lectures on Cauchy's Problem in Linear Partial Differential Equations*, Yale University Press, 1923.

[Hop50] E. Hopf, The partial differential equation $u_t + uu_x = \mu u_{xx}$, *Comm. Pure Appl. Math.* 3 (1950), 201–230.

[Hor71] L. Hörmander, Fourier integral operators I, *Acta Math.* 127 (1971), 79–183.

[Hor83] L. Hörmander, *The Analysis of Linear Partial Differential Operators I*, Springer, 1983.

[HoSi05] M. J. Hopkins, I. M. Singer, Quadratic functions in geometry, topology, and M-theory, *J. Differential Geom.* 70 (2005), 329–452.

[HTT08] R. Hotta, K. Takeuchi, T. Tanisaki, *D-Modules, Perverse Sheaves, and Representation Theory*, Progr. Math. 236, Birkhäuser, 2008.

[Hum90] J. E. Humphreys, *Reflection Groups and Coxeter Groups*, Cambridge University Press, 1990.

[Kac66] M. Kac, Can one hear the shape of a drum?, *Amer. Math. Monthly* 73 (1966), no. 4, part II, 1–23.

[Kak44] S. Kakutani, Two-dimensional Brownian motion and harmonic functions, *Proc. Imp. Acad. Tokyo* 20 (1944), 706–714.

[KaSc90] M. Kashiwara, P. Schapira, *Sheaves on Manifolds*, Grundlehren 292, Springer, 1990.

[Kat04] Y. Katznelson, *An Introduction to Harmonic Analysis*, 3rd ed., Cambridge University Press, 2004.

[KhSc17] I. Khavkine, U. Schreiber, Synthetic geometry of differential equations: I. Jets and comonad structure, preprint, arXiv:1701.06238, 2017. Checked.

[KoSc99] V. Kostrykin, R. Schrader, Kirchhoff's rule for quantum wires, *J. Phys. A* 32 (1999), 595–630.

[Kru70] S. N. Kružkov, First order quasilinear equations in several independent variables, *Math. USSR-Sb.* 10 (1970), 217–243.

[LaRi56] P. D. Lax, R. D. Richtmyer, Survey of the stability of linear finite difference equations, *Comm. Pure Appl. Math.* 9 (1956), 267–293.

[Lax68] P. D. Lax, Integrals of nonlinear equations of evolution and solitary waves, *Comm. Pure Appl. Math.* 21 (1968), 467–490.

[Lax73] P. D. Lax, *Hyperbolic Systems of Conservation Laws and the Mathematical Theory of Shock Waves*, SIAM, 1973.

[Mal56] B. Malgrange, Existence et approximation des solutions des équations aux dérivées partielles et des équations de convolution, *Ann. Inst. Fourier* 6 (1956), 271–355.

[Mar86] M. Marvan, A note on the category of partial differential equations, in *Differential Geometry and its Applications (Brno, 1986)*, 1987, 235–244.

[McC11] B. J. McCartin, *Laplacian Eigenstructure of the Equilateral Triangle*, Hikari, 2011.

[McSi67] H. P. McKean, I. M. Singer, Curvature and the eigenvalues of the Laplacian, *J. Differential Geom.* 1 (1967), 43–69.

[MoPe10] P. Mörters, Y. Peres, *Brownian Motion*, Cambridge University Press, 2010.

[Noe18] E. Noether, Invariante Variationsprobleme, *Nachr. Ges. Wiss. Göttingen, Math.-Phys. Kl.* (1918), 235–257.

[Ole57] O. A. Oleĭnik, Discontinuous solutions of non-linear differential equations, *Uspekhi Mat. Nauk* 12 (1957), 3–73.

[Olv93] P. J. Olver, *Applications of Lie Groups to Differential Equations*, 2nd ed., Graduate Texts in Mathematics 107, Springer, 1993.

[Pal70] V. P. Palamodov, *Linear Differential Operators with Constant Coefficients*, Springer, 1970.

[Paz83] A. Pazy, *Semigroups of Linear Operators and Applications to Partial Differential Equations*, Springer, 1983.

[PDEML] Anonymous, Partial differential equations in machine learning and AI: naturality, well-posedness, and spectral rates, Higher Categorical Semantics, 2026. Companion paper.

[QuVa99] A. Quarteroni, A. Valli, *Domain Decomposition Methods for Partial Differential Equations*, Oxford University Press, 1999.

[ReSi75] M. Reed, B. Simon, *Methods of Modern Mathematical Physics*, vols. II and III, Academic Press, 1975 and 1979.

[Rud62] W. Rudin, *Fourier Analysis on Groups*, Interscience, 1962.

[Sat59] M. Sato, Theory of hyperfunctions I, *J. Fac. Sci. Univ. Tokyo Sect. I* 8 (1959), 139–193.

[Sch54] L. Schwartz, Sur l'impossibilité de la multiplication des distributions, *C. R. Acad. Sci. Paris* 239 (1954), 847–848.

[Sch57] L. Schwartz, *Théorie des distributions*, Hermann, 1950–1951; new edition 1966.

[Str08] W. A. Strauss, *Partial Differential Equations: An Introduction*, 2nd ed., Wiley, 2008. The source.

[StWe71] E. M. Stein, G. Weiss, *Introduction to Fourier Analysis on Euclidean Spaces*, Princeton University Press, 1971.

[Sun85] T. Sunada, Riemannian coverings and isospectral manifolds, *Ann. of Math.* 121 (1985), 169–186.

[Ton75] E. Tonti, *On the Formal Structure of Physical Theories*, Istituto di Matematica del Politecnico di Milano, 1975.

[Vil68] N. Ja. Vilenkin, *Special Functions and the Theory of Group Representations*, Transl. Math. Monogr. 22, American Mathematical Society, 1968.

[Vin01] A. M. Vinogradov, *Cohomological Analysis of Partial Differential Equations and Secondary Calculus*, Transl. Math. Monogr. 204, American Mathematical Society, 2001.

[You50] D. M. Young, Iterative methods for solving partial difference equations of elliptic type, *Trans. Amer. Math. Soc.* 76 (1954), 92–111; Ph.D. thesis, Harvard University, 1950.