Adeles and the Product Formula

An introductory higher-categorical account

Abstract. The adelic product formula states that the local additive Haar modules of a nonzero global field element multiply to one. This introduction builds adeles and ideles as filtered colimits of products, recovers units by a pullback, and constructs the adelic norm by regrouping the places over each place of the base field. The determinant, viewed as a symmetric monoidal functor to graded lines, gives the local-to-global norm comparison. These constructions reduce the proof for all global fields to prime factorization over $\mathbb{Q}$ and degree bookkeeping over $\mathbb{F}_q(t)$. A second proof uses the exact sequence of the additive adelic quotient and its Haar measures. The idele-class quotient and the adelic complex of a regular curve exhibit the groupoid and derived aspects of the presentation.

Keywords. Global fields, adeles, ideles, product formula, determinant norm, restricted products, adelic descent.

1 Global fields and their places

A *global field* K is a finite extension of $\mathbb{Q}$ or of $\mathbb{F}_p(t)$ for a prime p. A *place* is an equivalence class of nontrivial absolute values, where equivalence means that they induce the same topology. We write V_K for the set of places, $V_{K,\infty}$ for the finite set of archimedean places, and K_v for the completion at v. At a nonarchimedean place, K_v is a complete discretely valued field with finite residue field. Its valuation ring, maximal ideal, and residue field size are

$$\mathcal{O}_v = \{x : \mathrm{ord}_v(x) \geq 0\}, \quad \mathfrak{m}_v, \quad q_v = \#(\mathcal{O}_v/\mathfrak{m}_v),$$

with $\mathrm{ord}_v(\pi_v) = 1$ for a uniformizer π_v. At an archimedean place K_v is $\mathbb{R}$ or $\mathbb{C}$. Each pair of conjugate complex embeddings contributes one place. These are the standard place and completion theorems for global fields [1, §13.1].

We use the *normalized local module* $|\cdot|_v$ on $K_v^\times$. It is the factor by which a scalar dilates additive Haar measure. Equivalently, for $a \neq 0$ it is

$$|a|_v = \begin{cases} q_v^{-\,\mathrm{ord}_v(a)}, & v \text{ nonarchimedean}, \\ |a|_{\mathrm{usual}}, & K_v = \mathbb{R}, \\ |a|_{\mathrm{usual}}^2, & K_v = \mathbb{C}. \end{cases} \tag{1}$$

Section 3 derives all three cases from Haar measure. The square at a complex place reflects its real dimension.

The proofs use products, filtered colimits, and pullbacks in spaces or in an ∞-topos. Ordinary sets and groups are regarded as discrete objects when we discuss their algebraic structure. When topology matters, we specify it explicitly and give a sheaf model that retains it. The arithmetic input is supplied by the completion and norm theorems and by calculations on the two rational base fields.

2 Adeles and ideles

2.1 Restricted products

Let J be a set of indices, $J_\infty \subset J$ a finite subset, and $U_j \to X_j$ a family of maps in a category with products and filtered colimits. Only the indices outside J_∞ require a choice of U_j. Write $\mathcal{F}(J)$ for the poset of finite $S \supseteq J_\infty$. Define

$$\mathrm{R}_J(X,U) = \varinjlim_{S \in \mathcal{F}(J)} \left(\prod_{j \in S} X_j \times \prod_{j \notin S} U_j \right). \tag{2}$$

In an ∞-category the products and colimit are the corresponding ∞-categorical ones. The transition from S to $T \supseteq S$ uses $U_j \to X_j$ for $j \in T \setminus S$. If X_S denotes the stage in parentheses, the universal property is

$$\mathrm{Map}(\mathrm{R}_J(X,U), Y) \simeq \varprojlim_{\mathcal{F}(J)^{\mathrm{op}}} \mathrm{Map}(X_S, Y).$$

Thus a map out of the restricted product is given by maps on the stages with homotopy-coherent compatibility. For inclusions of discrete objects in spaces (2) is the usual set of tuples belonging to U_j outside a finite set. The poset is filtered by finite union, and filtered colimits of spaces preserve 0-truncation and commute with finite limits [4, Thm. 9.1.5.1].

Definition 2.1. The *adeles* and *ideles* of K are

$$\mathbb{A}_K = \prod_{v \in V_K}{}' (K_v, \mathcal{O}_v),$$
$$\mathbb{I}_K = \prod_{v \in V_K}{}' (K_v^\times, \mathcal{O}_v^\times),$$

with every archimedean place included in each finite stage. We give every stage its product topology and the union the topology for which a subset is open exactly when its intersection with each stage is open.

For example the adele stages over $\mathbb{Q}$ have the form $\mathbb{R} \times \prod_{p \in S_{\mathrm{fin}}} \mathbb{Q}_p \times \prod_{p \notin S_{\mathrm{fin}}} \mathbb{Z}_p$. An adele can have infinitely many nonzero coordinates. The finite restriction concerns the places where a coordinate fails to be integral. Similarly, an idele has nonzero coordinates everywhere and is a unit at almost every nonarchimedean place.

At a nonarchimedean place both $\mathcal{O}_v$ and $\mathcal{O}_v^\times$ are compact and open in their respective local groups. It follows

that the stages are open, the additive adele group and the idele group are locally compact, and the coordinatewise algebra operations are continuous in their respective topologies. In fact two inputs lie in one common stage, and on each stage the operations are continuous products of local operations. The unrestricted archimedean and finitely many nonarchimedean factors are locally compact, and the remaining product is compact. Coordinate projections separate points, so both groups are Hausdorff. See [2, §25.2] and [3, §26.1].

2.2 Units as an inverse graph

Proposition 2.2. *The unit group of the underlying ring $\mathbb{A}_K$ is $\mathbb{I}_K$. With topology, $\mathbb{I}_K$ is the inverse graph*

$$\{(x, y) \in \mathbb{A}_K \times \mathbb{A}_K : xy = 1\}$$

under the identification $x \mapsto (x, x^{-1})$.

Proof. For a commutative ring object, the object of units is the pullback of multiplication $R \times R \to R$ along the unit $1 \to R$. Filtered colimits commute with these finite limits in an ∞-topos [5, Prop. 6.4.5.9]. Units also commute with products, since the pullback condition is coordinatewise. Applied to the adele stages this gives the claimed unit group. In tuples, both x and x^{-1} must be integral at almost all places. In the inverse graph, intersecting with $\mathbb{A}_{K,S} \times \mathbb{A}_{K,S}$ gives the idele stage $\mathbb{I}_{K,S}$ with its product topology. These are open and cover the graph. □

Over $\mathbb{Q}$, let $u^{(p)}$ be the idele equal to p at p and to 1 everywhere else. Along the primes in increasing order it converges to 1 in the adele topology, because a basic neighborhood restricts only finitely many coordinates within the stage integral at every finite place. It stays outside the open idele stage requiring a unit at every prime. Thus the inverse graph is needed to obtain the idele topology from the adele ring.

2.3 A sheaf model for the topology

On the site of profinite sets with finite jointly surjective covering families, a space T defines the set-valued sheaf $P \mapsto \mathrm{Cont}(P, T)$, viewed as a 0-truncated object in the ∞-topos of space-valued sheaves. This is the usual condensed-set construction [6, §1]. Products are computed on sections. For either restricted product above, a continuous map from a profinite P into the union has compact image and therefore lands in one stage. Indeed, the open stages cover the image, and a finite subcover can be absorbed by taking the union of its finite indexing sets. The resulting sectionwise filtered colimit is a sheaf, since the set-valued sheaf condition for a finite covering family is a finite equalizer condition. Consequently (2) also presents both restricted products as sheaves in this model. The finite-limit proof of Proposition 2.2 then applies to these sheaves and recovers the topological inverse graph.

2.4 Regrouping places

Lemma 2.3. *Let $p : W \to V$ have finite fibers with $W_\infty = p^{-1}(V_\infty)$. There is a canonical equivalence*

$$\mathrm{R}_W(X, U) \simeq \mathrm{R}_V(X', U'),$$
$$X'_v = \prod_{p(w)=v} X_w, \qquad U'_v = \prod_{p(w)=v} U_w.$$

Proof. The stages indexed by $p^{-1}(S)$, with S finite and containing V_∞, are cofinal. For a finite stage T on the W side, the S satisfying $T \subseteq p^{-1}(S)$ form a poset with least element $p(T) \cup V_\infty$. At each such stage, the product over W is the iterated product over the fibers and then over V, by the universal property of products. The equivalences respect the transition maps. For topological spaces with open inclusions the same stagewise argument identifies the restricted product topologies, since the cofinal stages give their open sets. □

In particular, compatible fiberwise maps of pairs $(X'_v, U'_v) \to (Y_v, B_v)$ induce a map on restricted products. The regrouping takes place inside each product stage, so it works for infinite sets of places without an interchange law for filtered colimits and infinite products.

3 Norms and local modules

3.1 Determinants and completions

For a finite field extension L/F and $a \in L^\times$, the field norm is $\mathrm{N}_{L/F}(a) = \det_F(m_a)$, where m_a is multiplication by a on L as an F-vector space. This makes $\mathrm{N}_{L/F}$ a group homomorphism. The determinant is functorial on the groupoid of finite-dimensional vector spaces and linear isomorphisms. With direct sum on the source and tensor product on the target it is symmetric monoidal with values in the Picard groupoid of $\mathbb{Z}$-graded lines: $E \mapsto (\bigwedge^{\dim E} E, \dim E)$. A Picard groupoid is a symmetric monoidal groupoid whose objects are invertible under tensor product up to isomorphism. Interchanging lines of degrees m, n contributes the sign $(-1)^{mn}$. This is the sign obtained when the corresponding blocks of wedge factors are interchanged. The exterior power construction also commutes with extension of scalars [7, 8].

Proposition 3.1 (Norm under completion)**.** *Let L/F be a finite separable extension of global fields. For each place v of F there is a canonical isomorphism*

$$L \otimes_F F_v \simeq \prod_{w|v} L_w. \tag{3}$$

Under $F \hookrightarrow F_v$,

$$\mathrm{N}_{L/F}(a) = \prod_{w|v} \mathrm{N}_{L_w/F_v}(a) \qquad (a \in L^\times). \tag{4}$$

Proof. Choose a primitive element for L/F with minimal polynomial h. Over F_v, the distinct irreducible factors of h give a product of field quotients by the Chinese

remainder theorem. Each factor carries the unique extending local absolute-value class and the image of L is dense in it. For density, the image spans the factor over F_v and F is dense in F_v. Conversely the map to each completion L_w factors through one of these fields. Its image is complete and contains the dense image of L, so it is all of L_w. Thus the factors are indexed by the extensions of v to L [1, Thm. 13.5]. The map in (3) is induced by the completion maps, giving its independence of h.

After base change, multiplication by a is the direct sum of the local multiplications. Base change takes the determinant line of L to that of $L \otimes_F F_v$. The direct sum is sent to the tensor product of the local lines. The scalar on their tensor product is the product of the local scalars, giving (4). Any sign in reordering the lines cancels when conjugating a line automorphism by that reordering. □

3.2 Haar measure and the local norm

Let μ be a nonzero additive Haar measure of a local field M. A scalar $a \in M^\times$ acts by an additive homeomorphism. Uniqueness of Haar measure gives a positive factor $d_M(a)$ defined by $\mu(aE) = d_M(a)\mu(E)$ for measurable E. It is independent of the chosen measure and is multiplicative because dilations compose. In a nonarchimedean field, a unit preserves the valuation ring and a uniformizer sends it to a subgroup of index q. Taking the valuation ring to have measure 1 gives $d_M(a) = q^{-\operatorname{ord}(a)}$. Real or complex scalar multiplication has real determinant of absolute value $|a|_{\text{usual}}$ or $|a|^2_{\text{usual}}$, respectively. This proves (1).

Lemma 3.2 (Local norm compatibility)**.** *If M/E is a finite extension of local fields, then*

$$d_M(a) = d_E\big(\mathrm{N}_{M/E}(a)\big).$$

Proof. Choose an E-basis of M and use product Haar measure on $E^{[M:E]}$. For a linear automorphism of E^n the Haar dilation factor is d_E of its determinant. For an elementary shear this follows from translation invariance in one coordinate and Fubini's theorem. Coordinate permutations preserve product measure and have determinant ± 1, of module 1. Diagonal maps scale each coordinate by its own module. Gaussian elimination decomposes an invertible matrix into these transformations. Apply the result to m_a and use the independence of Haar normalization. □

For nonarchimedean M/E, with residue degree f and ramification index e, $q_M = q_E^f$. Comparing exponents in Lemma 3.2 gives

$$\operatorname{ord}_E(\mathrm{N}_{M/E}(a)) = f \operatorname{ord}_M(a). \tag{5}$$

Here f weights the norm on arbitrary elements. For a base scalar $b \in E^\times$ one has $\operatorname{ord}_M(b) = e \operatorname{ord}_E(b)$ and $d_M(b) = d_E(b)^{ef}$. In particular the local norm maps local units to local units. For $\mathbb{C}/\mathbb{R}$ it sends z to $z\overline{z}$, consistent with the squared modulus.

3.3 The adelic comparison

First we check that elements of a global field give diagonal ideles. A number field is finite separable over $\mathbb{Q}$. In positive characteristic, let $\mathbb{F}_q$ be the algebraic closure of $\mathbb{F}_p$ in K. This is finite, since for every finite constant subfield $k' \subset K$ the degree of $k'(t)/\mathbb{F}_p(t)$ over $\mathbb{F}_p(t)$ is $[k' : \mathbb{F}_p]$, bounded by $[K : \mathbb{F}_p(t)]$. A separating transcendence basis over the perfect field $\mathbb{F}_q$ gives a parameter s with $K/\mathbb{F}_q(s)$ finite separable [9]. The initial parameter t may give an inseparable extension, so we use the separating parameter in the proof.

Lemma 3.3 (Finite support)**.** *For $a \in K^\times$, both a and a^{-1} are integral at all but finitely many nonarchimedean places of K.*

Proof. Choose a rational base field F as above and a monic polynomial over F for each of a and a^{-1}. At almost all nonarchimedean places of F their coefficients are integral, by factorization of rational numbers or rational functions. They are then integral at each place of K lying above those base places. If a root of a monic polynomial with integral coefficients had negative valuation, its highest-degree term would have strictly smaller valuation than every other term, which prevents the sum from vanishing. Finitely many places of K lie above the finitely many excluded base places by Proposition 3.1. □

Thus $K^\times$ embeds by a diagonal homomorphism $\Delta_K : K^\times \hookrightarrow \mathbb{I}_K$. The diagonal ring map into $\mathbb{A}_K$ includes 0 as well. For general ideles set

$$\mu_K : \mathbb{I}_K \longrightarrow \mathbb{R}_{>0}, \qquad \mu_K(x) = \prod_{v \in V_K} |x_v|_v. \tag{6}$$

Each stage map uses only the finitely many indices of that stage. These maps are continuous homomorphisms and compatible, so the universal property gives a continuous homomorphism on the whole idele group. For a finite separable extension, Lemma 2.3 and the unit property of the local norms give the *idele norm*

$$(\mathrm{N}^{\mathbb{I}}_{L/F}(x))_v = \prod_{w|v} \mathrm{N}_{L_w/F_v}(x_w). \tag{7}$$

The norms are continuous as determinants, and a fiber over a place outside the finite stage contributes a unit. Therefore (7) is a continuous homomorphism. The two compatibilities are

$$\mu_L = \mu_F \circ \mathrm{N}^{\mathbb{I}}_{L/F}, \tag{8}$$

$$\mathrm{N}^{\mathbb{I}}_{L/F} \circ \Delta_L = \Delta_F \circ \mathrm{N}_{L/F}\,. \tag{9}$$

The first is Lemma 3.2 multiplied over fibers, using finite support. The second is Proposition 3.1.

4 The adelic product formula

Theorem 4.1. *For every global field K and $a \in K^\times$,*

$$\prod_{v \in V_K} |a|_v = 1. \tag{10}$$

Equivalently $\mu_K \circ \Delta_K$ is the trivial character.

Proof. By (8) and (9), for any finite separable extension L/F we have

$$\mu_L(\Delta_L a) = \mu_F\big(\Delta_F \, \mathrm{N}_{L/F}(a)\big). \tag{11}$$

The rational base fields used in Lemma 3.3 therefore suffice.

Over $\mathbb{Q}$, write $a = \pm \prod_p p^{n_p}$ with only finitely many n_p nonzero. The p-factor in the product is p^{-n_p} and the real factor is $\prod_p p^{n_p}$.

Over $\mathbb{F}_q(t)$, the finite places correspond to monic irreducibles $P \in \mathbb{F}_q[t]$, with residue size $q^{\deg P}$. The remaining place is infinity, with residue size q and $\mathrm{ord}_\infty(t) = -1$. Write $a = c \prod_P P^{n_P}$ for $c \in \mathbb{F}_q^\times$. Its finite product is $q^{-\sum_P n_P \deg P}$, and $\mathrm{ord}_\infty(a) = -\sum_P n_P \deg P$. Hence the factor at infinity is $q^{\sum_P n_P \deg P}$. This proves the claim in both cases and (11) completes the proof. □

Example 4.2 (A complex place and a ramified prime)**.** In $\mathbb{Q}(i)$ take $a = (2+i)/(1+i)$. Its only nonzero finite valuations are 1 at $(2+i)$ and -1 at $(1+i)$. The residue sizes are 5 and 2, as follows from the indices of the principal ideals in $\mathbb{Z}[i]$. The finite factors are $1/5$ and 2. At the single complex place, $|a|^2_{\text{usual}} = 5/2$. The product is $(1/5) \cdot 2 \cdot (5/2) = 1$. In particular, although 2 ramifies with index two, the generator $1+i$ has valuation one and local module $1/2$.

Example 4.3 (A place of degree two)**.** In $\mathbb{F}_5(t)$ take $a = (t^2+2)/(t-1)^3$. The numerator is irreducible because 3 is a nonsquare in $\mathbb{F}_5$. Its place has module 5^{-2}. The factors at $t-1$ and infinity are 5^3 and 5^{-1}, respectively. They give $5^{-2}5^3 5^{-1} = 1$.

4.1 The idele-class character

The algebraic idele-class group is $C_K = \mathbb{I}_K/\Delta_K(K^\times)$. It is the set of components of the action groupoid $\mathbb{I}_K//K^\times$: an arrow labelled a sends x to $\Delta_K(a)x$. Multiplication of objects and of arrow labels makes this groupoid a Picard groupoid. The action is free, so its isotropy groups are trivial and it is equivalent to the discrete groupoid on C_K. Theorem 4.1 says that μ_K is constant on these orbits. Equivalently μ_K sends every arrow to an identity in the discrete Picard groupoid with group of objects $\mathbb{R}_{>0}$. It thereby gives a symmetric monoidal functor on the action groupoid. With the quotient topology on C_K, the induced character is continuous by the universal property of the topological quotient.

5 An additive proof by Haar measures

The local module is defined through additive Haar measure. We can therefore prove the product formula by constructing a Haar measure on $\mathbb{A}_K$ preserved by $K^\times$. The construction uses the additive quotient $\mathbb{A}_K/K$.

Proposition 5.1. *The diagonal copy of K is closed and discrete in $\mathbb{A}_K$, and $\mathbb{A}_K/K$ is compact.*

Proof. First let F be $\mathbb{Q}$ or $\mathbb{F}_q(t)$, with $B = \mathbb{Z}$ or $\mathbb{F}_q[t]$, respectively. For $x \in \mathbb{A}_F$ choose a nonzero $d \in B$ such that dx_v is integral at every finite polynomial or prime place of B. The Chinese remainder theorem gives $b \in B$ with $b \equiv dx_v \pmod{d\mathcal{O}_v}$ at all places dividing d. Here $B/(\pi^r) \simeq \mathcal{O}_v/\pi^r\mathcal{O}_v$ for the corresponding prime π of B. Then $x - b/d$ is integral at every place of B. In the number-field base case subtract an integer to put its real component in $[0,1]$. In the function-field base case subtract its polynomial part at infinity to put its infinity component in $t^{-1}\mathbb{F}_q[[t^{-1}]]$. These operations yield representatives in a compact product of local sets. For discreteness, all coordinates at places of B may be required to be integral and the infinity coordinate to lie in $(-1,1)$ or $t^{-1}\mathbb{F}_q[[t^{-1}]]$, respectively. This open neighborhood intersects the diagonal in 0, since an element integral at every place of B lies in B.

For K/F finite separable of degree n, choose an F-basis e_i and its trace-dual basis e_i^*, using nondegeneracy of the trace pairing. Away from finitely many v, all these basis elements are integral at each $w \mid v$ by Lemma 3.3. Under (3), the coefficient of e_i in $x = (x_w)_{w|v}$ is

$$\sum_{w|v} \mathrm{Tr}_{K_w/F_v}(e_i^* x_w).$$

This follows by base change of the trace of multiplication. For integral x_w the result is integral, since traces of integral elements lie in the integrally closed ring $\mathcal{O}_v$. Conversely, integral coefficients give integral x_w by integrality of the e_i. The basis thus identifies the local lattices $\mathcal{O}_v^n$ and $\prod_{w|v} \mathcal{O}_w$ outside finitely many places. Regrouping as in Lemma 2.3, these local linear homeomorphisms give $\mathbb{A}_F^n \simeq \mathbb{A}_K$ as additive topological groups and identify F^n with K. This gives discreteness and compactness of the quotient for K. A discrete subgroup of a Hausdorff topological group is closed. □

For the short exact sequence of locally compact abelian groups

$$0 \longrightarrow K \longrightarrow \mathbb{A}_K \longrightarrow \mathbb{A}_K/K \longrightarrow 0,$$

use counting measure on K and probability Haar measure on the compact quotient. Quotient integration gives the Haar measure on $\mathbb{A}_K$ determined by these choices. For a continuous compactly supported function f, its integral is the integral over $\mathbb{A}_K/K$ of $\sum_{b \in K} f(x+b)$, where x is any representative of the coset. Multiplication by $a \in K^\times$ permutes the summation group and preserves probability

Haar measure on the quotient. Therefore it preserves the resulting Haar measure on $\mathbb{A}_K$.

On a restricted product stage, the restriction of a Haar measure is proportional to the product of local Haar measures, with the integral factors given probability measure. Choose a stage containing every place at which a is a nonunit. Multiplication by a is an automorphism of this stage and has dilation factor $\prod_v |a|_v$, computed on product sets that are unrestricted within the unit-preserving integral factors. It follows that the product is 1. This is the measure argument behind Tate's proof [10, Thm. 4.3.1].

Functoriality is explicit in this argument: the Haar measures on a locally compact abelian group form an $\mathbb{R}_{>0}$-torsor, and quotient integration combines the subgroup and quotient torsors into the middle one. The counting and probability choices give a section fixed by the action of $K^\times$.

At the level of connective algebraic K-theory, the same sequence gives a nullhomotopy for the map induced by adelic extension of scalars [11, Rem. 0.3]. For a finite-dimensional K-vector space E, tensor the additive sequence with E and give each term the topology of a finite power of K, $\mathbb{A}_K$, or $\mathbb{A}_K/K$, respectively. Change of basis acts continuously, so this defines a short exact sequence of exact functors to the exact category of second-countable locally compact abelian groups. Its short exact sequences use closed embeddings and their topological quotients. Additivity identifies the middle functor's map of K-theory spectra with the sum of the endpoint maps. The endpoints factor through countable discrete and second-countable compact groups, respectively. Countable direct sums in the first category and countable products in the second yield exact endofunctors T with $T \simeq \mathrm{id} \oplus T$. By additivity their K-theory spectra are contractible, giving the asserted nullhomotopy.

6 Curves and adelic totalization

Let X be a regular integral Noetherian scheme of dimension one, with function field K and set of closed points $|X|$. Write $\widehat{\mathcal{O}}_x$ for the completed local ring at a closed point and $K_x = \mathrm{Frac}(\widehat{\mathcal{O}}_x)$. There is one generic point, and every strict specialization chain goes from it to a closed point. The additive normalized adelic complex of $\mathcal{O}_X$ is

$$\left[K \oplus \prod_{x \in |X|} \widehat{\mathcal{O}}_x \xrightarrow{d} \prod_{x \in |X|}{}' (K_x, \widehat{\mathcal{O}}_x) \right], \tag{12}$$

in cohomological degrees 0 and 1, with $d(a, (b_x)) = (a - b_x)_x$. The terms come from the generic point, the closed points, and the two-point flags. The complex represents $R\Gamma(X, \mathcal{O}_X)$, the derived global sections [12, Ex. 2.2]. Equivalently it is the homotopy fiber of d as a map of degree-zero objects in the derived ∞-category of abelian groups.

For the dimension-one calculation, let $\mathcal{K}$ be the sheaf of rational functions. For each closed point let $i_x : \{x\} \hookrightarrow X$ be its inclusion. There is a flasque resolution

$$0 \longrightarrow \mathcal{O}_X \longrightarrow \mathcal{K} \longrightarrow \bigoplus_{x \in |X|} i_{x*}(K_x/\widehat{\mathcal{O}}_x) \longrightarrow 0.$$

At a closed point, DVR approximation identifies $K/\mathcal{O}_{X,x}$ with $K_x/\widehat{\mathcal{O}}_x$. Each rational function has only finitely many poles on a Noetherian curve. The last sheaf is the sheaf of finitely supported local principal parts on opens, because all opens are quasi-compact. Its restrictions are surjective by extension by zero. The generic stalk has no principal parts since any finite support can be deleted from a neighborhood. Taking global sections gives the complex $[K \to \bigoplus_x K_x/\widehat{\mathcal{O}}_x]$. Complex (12) maps to this complex by projecting off $\prod_x \widehat{\mathcal{O}}_x$ in degree zero and reducing coordinates modulo $\widehat{\mathcal{O}}_x$ in degree one. The kernel is a copy of $\prod_x \widehat{\mathcal{O}}_x$ mapped by minus the identity to itself and is acyclic. This identifies the two complexes.

For a smooth, proper, geometrically integral curve over $\mathbb{F}_q$, the closed points index all the places of its function field. Set

$$\mathrm{Div}(X) = \bigoplus_{x \in |X|} \mathbb{Z}[x],$$
$$\deg\Big(\sum_x n_x[x]\Big) = \sum_x n_x[\kappa(x) : \mathbb{F}_q].$$

The homomorphism $\mathrm{div}_{\mathbb{I}} : \mathbb{I}_K \to \mathrm{Div}(X)$ sends u to $\sum_x \mathrm{ord}_x(u_x)[x]$, and

$$\log \mu_K(u) = -(\log q) \deg(\mathrm{div}_{\mathbb{I}}\, u). \tag{13}$$

For diagonal a, this divisor is the principal divisor of a. Thus the product formula says exactly that a principal divisor has degree zero. Geometrically, for nonconstant a the map $a : X \to \mathbb{P}^1_{\mathbb{F}_q}$ is finite flat, and the zero and pole divisors are its fibers over 0 and ∞. Their degrees both equal the rank of the finite flat map, with lengths accounting for multiplicity even in the inseparable case [13, §42.18]. Constants have zero divisor.

There is also a groupoid version of the divisor description. The action groupoid $\mathrm{Div}(X)//K^\times$ uses translation by principal divisors. Its components are divisor classes and the automorphisms of each object are the global units $\mathbb{F}_q^\times$. This records the constant scalars that act trivially on divisors. The functor sending D to $\mathcal{O}_X(D)$ identifies it with the Picard groupoid of line bundles and their isomorphisms. An arrow $D \to D + \mathrm{div}(a)$ is sent to multiplication by a^{-1}. Every line bundle has a nonzero rational section defining a divisor, and every isomorphism between these subsheaves of the rational-function sheaf acts at the generic point by a unique rational scalar. The valuation conditions on this scalar identify exactly the arrows in the action groupoid. The identifications respect tensor products. By (13), degree defines a symmetric monoidal functor from this Picard groupoid to discrete $\mathbb{Z}$.

For $X = \operatorname{Spec} \mathcal{O}_K$ of a number field, the closed points give the finite adeles in (12). The arithmetic product formula also uses the archimedean places, with the real and complex modules of (1). More generally, the Beilinson cosimplicial adelic ring $\mathbb{A}_X^\bullet$ packages iterated completion along flags on a Noetherian scheme. Groechenig's adelic descent theorem gives $\operatorname{Perf}(X) \simeq \operatorname{Tot} \operatorname{Perf}(\mathbb{A}_X^\bullet)$, where Perf denotes the ∞-category of perfect complexes and Tot the limit of the cosimplicial diagram [14, Thm. 3.1 in the arXiv version]. The calculation above identifies the degree-zero and degree-one additive terms in the regular-curve case.

7 Conclusion

The product formula factors into three operations that respect different structures. Regrouping the restricted product gives the domain for the idele norm. The symmetric monoidal determinant identifies the global norm after completion, and additive Haar dilation fixes the local normalizations. The rational base calculations then prove triviality on the diagonal. In the curve case the same triviality has the geometric form of degree zero, while the adelic complex exhibits how the local rings and the generic field enter derived global sections.

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