Higher-Order Logic in ∞-Toposes

Extending the Lambek–Scott correspondences

Abstract. We develop an ∞-categorical counterpart to the relationships among typed calculi, topos semantics, and recursion in Lambek and Scott. The indexed slices of a Grothendieck ∞-topos interpret substitution, dependent sums and products, identities, and propositional quantifiers. We prove an indexed comparison with its ordinary topos of 0-truncated objects, and give explicit examples in which truncation loses sections or their components, including a dependent product along a subterminal map of a finite presheaf site. Further results give a local condition under which dependent products commute with truncation, classify unrestricted and discrete sorts by different presheaf topoi, and describe descent, univalence, and torsors at the level of mapping spaces. We formulate natural-number elimination by contractibility of the space of solutions and prove the failure of absolute internal Church's thesis in nondegenerate Grothendieck ∞-toposes under classical external mathematics. The distinction between ordinary and higher semantics runs through each comparison.

Keywords. Categorical logic, ∞-topos, dependent type theory, truncation, univalence, realizability.

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1 Introduction

Lambek and Scott organize categorical logic around three correspondences: typed λ-calculi and cartesian closed categories, intuitionistic type theories and toposes, and descriptions of numerical functions by arrows in categories equipped with recursion principles [1]. Their preface also points to dependent types and local cartesian closure as a distinct correspondence. A higher categorical development must explain what each of these comparisons becomes when maps have homotopies and families have nontrivial automorphisms. The resulting comparison depends on which operations preserve the truncated fragment.

We use the indexed family $X \mapsto \mathcal{E}_{/X}$ of a Grothendieck ∞-topos $\mathcal{E}$. A context is an object X, a type in context is a map to X, and a term is a section. Substitution, dependent sum and dependent product form an adjoint triple. Propositional quantifiers are obtained by restricting to subobjects, with images used for existential quantification. Theorem 2.2 compares these constructions simultaneously on the full subcategory of set-valued types over set-valued contexts. This comparison explains which rules of ordinary extensional logic the higher model recovers.

Two separate tests delimit the comparison. In Section 3, spaces and spaces over the 2-sphere have equivalent ordinary logic on set-valued types, yet differ on sections of a constant higher type. We also give two explicit failures of interchanging 0-truncation and dependent product. One failure is detected by a nonsplit extension of finite groups. The other already occurs over a subterminal context on a finite presheaf site. Thus the ordinary comparison is an agreement of operations on a specified fragment, and information needed by dependent products can be lost by truncating the input to those operations.

The later sections follow three other aspects of the source. Descent and identity expose the role of coherence in the semantics of families. Univalent universes and classifying objects express classification by equivalence rather than by a choice of names. Natural-number recursion is formulated by a universal property of spaces of solutions, and realizability explains how an effectiveness condition can change which numerical maps exist. We use the book as a thematic point of departure and work at article scale. The proofs assemble standard topos and type-theoretic constructions around specific comparison and obstruction calculations.

Conventions. All categories in the higher development are $(\infty, 1)$-categories, all limits and colimits have their homotopy-coherent meanings, and $\mathcal{S}$ denotes the category of spaces. We write $\mathrm{Map}_{\mathcal{C}}(A, B)$ for the mapping space and $\mathcal{C}^{\simeq}$ for the maximal subgroupoid. A map is n-truncated if its homotopy fibers after applying $\mathrm{Map}_{\mathcal{C}}(T, -)$ are n-truncated spaces for every T. In particular, a (-1)-truncated map is a monomorphism, and its fiber is either empty or contractible. Write $\tau^X_{\leq n}$ for relative n-truncation in the slice over X. We fix ambient universes and use bounded classifiers when discussing small families. The underlying external mathematics is standard set-theoretic mathematics. Logical principles ascribed to a model are statements of its internal language.

2 Substitution and indexed logic

An ∞-topos is an accessible left exact localization of a presheaf ∞-category of spaces. We use its local cartesian closure, image factorizations, and universal colimits [2, Ch. 6]. The terminal object in the slice $\mathcal{E}_{/X}$ is $1_X = \mathrm{id}_X$. For a type $p : A \to X$, the space of terms is $\mathrm{Map}_{\mathcal{E}_{/X}}(1_X, p)$, the homotopy fiber of $\mathrm{Map}_{\mathcal{E}}(X, A) \to \mathrm{Map}_{\mathcal{E}}(X, X)$ at id_X. This formulation includes the homotopy that makes an underlying map a section.

Simple typing can be read in the single category $\mathcal{E}$:

currying is the equivalence

$$\mathrm{Map}_{\mathcal{E}}(T \times A, B) \simeq \mathrm{Map}_{\mathcal{E}}(T, B^A).$$

The evaluation map and its adjuncts give application and abstraction. Applying π_0 shows that the homotopy category of a cartesian closed ∞-category is cartesian closed when equipped with these induced operations. A dependent context changes the base from T to the total space of a family. Pullback in slices supplies the dependence that the nonindexed exponential does not encode.

2.1 The adjoint triple

For $f : X \to Y$, composition supplies $\Sigma_f : \mathcal{E}_{/X} \to \mathcal{E}_{/Y}$ and pullback supplies f^*. The pullback property gives

$$\mathrm{Map}_{/Y}(\Sigma_f A, B) \simeq \mathrm{Map}_{/X}(A, f^*B).$$

The functor f^* preserves colimits by universality of colimits in $\mathcal{E}$, and the slices are presentable. The adjoint functor theorem therefore gives its right adjoint Π_f, characterized by

$$\mathrm{Map}_{/X}(f^*B, A) \simeq \mathrm{Map}_{/Y}(B, \Pi_f A).$$

The second formula interprets abstraction with a dependent domain as adjoint transposition. The unit and counit identify the operations that express the semantic β and η rules.

Proposition 2.1 (Base change and Frobenius)**.** *Suppose that*

$$\begin{array}{ccc} X' & \xrightarrow{g'} & X \\ {\scriptstyle f'}\downarrow & & \downarrow{\scriptstyle f} \\ Y' & \xrightarrow[g]{} & Y \end{array}$$

is a pullback square. There are natural equivalences

$$\begin{aligned} g^*\Sigma_f &\simeq \Sigma_{f'}(g')^*, \\ g^*\Pi_f &\simeq \Pi_{f'}(g')^*, \\ \Sigma_f(A \times_X f^*B) &\simeq (\Sigma_f A) \times_Y B. \end{aligned}$$

Proof. The first equivalence is the pullback pasting law applied to the total space of A. For a test object T over Y', the other Beck–Chevalley comparison follows from the chain

$$\begin{aligned} \mathrm{Map}_{/Y'}(T, g^*\Pi_f A) &\simeq \mathrm{Map}_{/Y}(\Sigma_g T, \Pi_f A) \\ &\simeq \mathrm{Map}_{/X}(f^*\Sigma_g T, A) \\ &\simeq \mathrm{Map}_{/X}(\Sigma_{g'}(f')^*T, A) \\ &\simeq \mathrm{Map}_{/X'}((f')^*T, (g')^*A) \\ &\simeq \mathrm{Map}_{/Y'}(T, \Pi_{f'}(g')^*A). \end{aligned}$$

Here $f^*\Sigma_g \simeq \Sigma_{g'}(f')^*$ is the first comparison for the transposed square. Yoneda gives the result. For Frobenius both total spaces are canonically the pullback $A \times_Y B$ over Y. The comparisons are compatible with pasting: the pullback equivalences are the ones induced by the pasted universal cones, and taking adjoint mates respects pasting. □

2.2 Propositions and ordinary recovery

Let $\mathrm{Sub}_{\mathcal{E}}(X)$ be the poset of (-1)-truncated maps to X up to equivalence. Mapping spaces of such maps in the slice are empty or contractible. Restrictions of the preceding functors to predicates give

$$\exists_f(P) = \tau^Y_{\leq -1}\Sigma_f P, \qquad \forall_f(P) = \Pi_f P.$$

The formula for $\exists_f$ is reflection into subobjects after composition. Its mapping property says that $\exists_f(P) \leq Q$ if and only if $P \leq f^*Q$. For $\forall_f$, the mapping-space adjunction shows that its value is again a monomorphism and that $f^*Q \leq P$ if and only if $Q \leq \forall_f(P)$. The image is stable under pullback, so the predicate comparisons follow from Proposition 2.1; for Frobenius at this level use that pulling back the image of $\Sigma_f P$ along $Q \hookrightarrow Y$ computes the image of $P \times_X f^*Q$ in Q. Composing the resulting mono with $Q \hookrightarrow Y$ gives intersection with Q. This describes ordinary intuitionistic quantification in an arbitrary higher context, regardless of its truncation level.

Theorem 2.2 (Indexed recovery of the ordinary semantics)**.** *Let $\mathcal{E}$ be a Grothendieck ∞-topos and let $\mathcal{C} = \mathcal{E}_{\leq 0}$ be its ordinary topos of 0-truncated objects. For $X \in \mathcal{C}$ there are natural identifications*

$$(\mathcal{E}_{/X})_{\leq 0} \simeq \mathcal{C}_{/X}, \qquad \mathrm{Sub}_{\mathcal{E}}(X) \cong \mathrm{Sub}_{\mathcal{C}}(X).$$

For a map f in $\mathcal{C}$, the identifications intertwine substitution, Σ_f, Π_f, $\exists_f$ and $\forall_f$ with their ordinary versions. They also intertwine finite limits, relative exponentials, the diagonal equality predicate, and the subobject classifier. The comparison is natural in X with respect to substitution and carries the base-change and Frobenius comparisons to the ordinary ones.

Proof. The 1-topos characterization identifies $\mathcal{E}_{\leq 0}$ with a Grothendieck topos [2, Thm. 6.4.1.5]. If X is 0-truncated and $A \to X$ is relatively 0-truncated, then $A \to 1$ is 0-truncated by closure under composition. Conversely, if both A and X are 0-truncated, the homotopy fibers of the map of discrete mapping spaces $\mathrm{Map}(T, A) \to \mathrm{Map}(T, X)$ are discrete. Full faithfulness of $\mathcal{C} \hookrightarrow \mathcal{E}$ now identifies the slices. A mono over X lies in this slice and its mapping property restricts to the ordinary mono property. Conversely, an ordinary mono has an invertible relative diagonal in $\mathcal{C}$. Finite limits in $\mathcal{C}$ are computed in $\mathcal{E}$, so the same diagonal is invertible there.

Pullback preserves truncated maps. If f lies in $\mathcal{C}$, it is 0-truncated, so its composite with a relatively 0-truncated map is again 0-truncated. The adjunction for Π_f preserves truncated objects since its mapping spaces are tested by pulling back the source. The two adjunctions therefore restrict to the displayed ordinary slices. The reflection $\tau^Y_{\leq -1}$ applied after Σ_f has the usual image universal property against the same subobjects. This proves the two claims about predicate quantifiers as well.

Finite limits and diagonals agree by the full embedding and closure under finite limits. Relative exponentials are

characterized in the slice by the right adjoint to product; equivalently they can be formed as dependent products of the pullback of the codomain along the domain. The subobject classifier Ω is 0-truncated because $\mathrm{Map}(T, \Omega)$ is the discrete space of subobjects of T. Its representing property restricts to $\mathcal{C}$, so the classifier and the power objects built from it agree. The base-change and Frobenius comparisons restrict because they are furnished by the same pullback and adjunction properties. □

Ordinary interpretation of an extensional higher-order language uses the data of this theorem: products and powers for types, subobjects for formulas, diagonals for equality, and quantifiers in context. Thus interpreting a signature whose sorts and contexts are 0-truncated gives the same terms and predicates in both models, up to the indicated identifications. This assertion concerns the semantics of that fragment. In particular, an identity type of an arbitrary higher type should not be treated as the diagonal predicate of a set.

3 What truncation discards

Write $X \mapsto cX$ for the constant-object functor $\mathcal{S} \to \mathcal{E}$. Intrinsically, $cX \simeq \mathrm{colim}_X 1$. There are two different questions about recovering higher semantics. The first asks whether the truncated doctrine determines its higher ambient category. The second asks whether a higher family can be truncated before or after a computation in the slices.

Proposition 3.1 (The ordinary fragment does not determine the higher category)**.** *The ∞-toposes $\mathcal{S}$ and $\mathcal{S}_{/S^2}$ have equivalent ordinary toposes of 0-truncated objects, and their indexed set and predicate semantics agree under this equivalence. The two ∞-toposes are inequivalent.*

Proof. Straightening identifies spaces over S^2 with families $S^2 \to \mathcal{S}$. The 0-truncated objects of the slice are the families with discrete fibers. A functor from a space to sets takes values on the subgroupoid of bijections, and factors through the fundamental groupoid of the space. As S^2 is connected and simply connected, its fundamental groupoid is terminal up to equivalence. Its category of set-valued families is therefore equivalent to **Set**. Theorem 2.2 gives the indexed comparison.

In the slice, the terminal object is id_{S^2} and cK is $S^2 \times K \to S^2$. The section space is $\mathrm{Map}(S^2, K)$. In $\mathcal{S}$ the section space of cK is K. Take $K = K(\mathbb{Z}, 2)$. The former has components $H^2(S^2; \mathbb{Z}) \cong \mathbb{Z}$, while the latter is connected. The section space of cK depends only on the category since terminal objects and K-indexed colimits do. An equivalence of the two higher categories is therefore impossible. □

Under the additional restriction to 1-localic ∞-toposes, the ordinary topos does determine the higher topos up to equivalence [2, Prop. 6.4.5.7 and Def. 6.4.5.8].

For any $f : X \to Y$ and family A over X, there is a canonical map

$$\tau^Y_{\leq 0}\Pi_f A \longrightarrow \Pi_f \tau^X_{\leq 0} A. \tag{3.1}$$

It comes from applying Π_f to the truncation unit and factoring into a 0-truncated target. The theorem about restriction implies that this map is invertible if A is already relatively 0-truncated. It does not imply that (3.1) is invertible for every A over a 0-truncated base, or for every A with connected fibers.

Example 3.2 (A coherent section obstructed by a group extension)**.** Take $f : BC_2 \to 1$ in spaces and let $A \to BC_2$ be the map $BC_4 \to BC_2$ induced by reduction modulo 2. Here C_n is the cyclic group of order n. Its homotopy fiber is BC_2, as is seen from the homotopy exact sequence and the surjectivity of $C_4 \to C_2$. Thus $\tau^{BC_2}_{\leq 0} A$ is terminal in the slice and the right side of (3.1) is contractible. The untruncated $\Pi_f A$ is the space of sections of A over BC_2. A point in this space would induce a splitting of $C_4 \to C_2$ on fundamental groups. Basepoint choices only affect conjugation, which acts trivially here. The elements of order dividing 2 in C_4 both map to 0 in C_2, so such a splitting cannot exist. The left side of (3.1) is empty. The example can also be read as a space of homotopy fixed points with nonempty connected fibers and no coherent fixed point.

Example 3.3 (Failure over a subterminal context)**.** Let D be the finite poset of nonempty proper faces of a 2-simplex, and let $C = D^{\triangleright}$ be the result of adjoining a terminal object t. Put $\mathcal{E} = \mathrm{Fun}(C^{\mathrm{op}}, \mathcal{S})$. Let U be the subterminal presheaf with value $*$ on D and value $\varnothing$ at t. For $f : U \to 1$, take $A = U \times cS^1 \to U$. Both contexts in f are 0-truncated. Relative truncation is computed objectwise, so $\tau^U_{\leq 0} A \simeq 1_U$ and $(\Pi_f \tau^U_{\leq 0} A)(t) \simeq *$.

Since t is terminal, $y(t) = 1$. Evaluate the untruncated product at t, using Yoneda and adjunction:

$$\begin{aligned}(\Pi_f A)(t) &\simeq \mathrm{Map}_{/U}(U, U \times cS^1) \\ &\simeq \mathrm{Map}_{\mathcal{E}}(U, cS^1) \\ &\simeq \lim_{D^{\mathrm{op}}} S^1 \simeq \mathrm{Map}(|ND|, S^1).\end{aligned}$$

In the last line the diagram is constant. The restriction of U to D^{op} is terminal, and U is its left Kan extension along $D^{\mathrm{op}} \hookrightarrow C^{\mathrm{op}}$: the extension has value initial at t since there are no arrows from objects of D to t in C^{op}. This explains the reduction to the smaller index. The realization $|ND|$ is the barycentric subdivision of the boundary of a triangle and is equivalent to S^1. Its maps to S^1 have integer degree, so $(\tau_{\leq 0}\Pi_f A)(t)$ has components indexed by $\mathbb{Z}$. The comparison at t takes this discrete space to $*$ and loses the degrees. Here both bases are in the ordinary fragment. The higher information lies in the family over that fragment.

3.1 Finite locally constant index

Call $f : X \to Y$ *finite locally constant* if some effective epimorphism $\coprod_i Y_i \to Y$ makes the pullback of f over each Y_i a finite coproduct of copies of 1_{Y_i}. The number of copies may vary with i.

Proposition 3.4 (Truncating products with finite locally constant index)**.** *For a finite locally constant $f : X \to Y$, all relatively n-truncations with $n \geq -1$ commute with Π_f:*

$$\tau^Y_{\leq n} \Pi_f A \;\simeq\; \Pi_f \tau^X_{\leq n} A.$$

Proof. The natural comparison factors the image of the truncation unit through the truncated source, as in (3.1). By Proposition 2.1 and stability of relative truncation under pullback, it suffices to prove the comparison after restriction to each Y_i. These restrictions detect equivalences by effective-epimorphism descent, explained in Section 4. Over Y_i the domain is a finite coproduct of copies of the terminal object. A type over that domain is a tuple of types over Y_i by disjoint and universal coproducts. Substitution sends a type to the constant tuple, so the dependent product is the finite product of the tuple.

Truncation in a slice preserves finite products. For the nonempty case, take the product of the truncation units. Each unit has trivial relative n-truncation over its target. The class of these maps is closed under composition and pullback, by the relative factorization system [2, §6.5.1]. The product of the units factors as successive pullbacks of the individual units. Its target is a product of n-truncated types and is therefore n-truncated. The product map thus exhibits the truncation by the uniqueness of this factorization. The empty product is terminal and is already truncated. This applies over each Y_i and proves the claim. □

4 Descent for contexts and families

Write $\mathcal{S}$ for the ∞-category of spaces. A Grothendieck ∞-topos $\mathcal{E}$ is an accessible left exact localization of $\mathrm{Fun}(C^{\mathrm{op}}, \mathcal{S})$ for a small ∞-category C. In the presheaf category, restriction along an arrow of C is part of the functorial data of a presheaf. In a localization, the local objects satisfy additional gluing conditions. There is also an intrinsic gluing statement, formulated in $\mathcal{E}$ itself, that applies to the dependent families of Section 2.

Theorem 4.1 (Descent for slices)**.** *Let $D : I \to \mathcal{E}$ be a small diagram with colimit X. Restriction along its colimit cone induces an equivalence*

$$\mathcal{E}_{/X} \;\simeq\; \lim_{i \in I^{\mathrm{op}}} \mathcal{E}_{/D(i)}. \tag{4.1}$$

For an arrow $i \to j$ in I, the corresponding transition functor on the right is pullback along $D(i) \to D(j)$. The limit is in Cat_∞, in an ambient size in which the slice categories are objects.

Proof. An object on the right consists of a diagram $A \to D$ whose naturality squares are pullbacks. Indeed, the equivalence identifying A_i with the restriction of A_j is equivalent to the assertion that the square for $i \to j$ is a pullback. The limit includes the higher coherences among those squares. One can formalize this description as the ∞-category of cartesian sections of the pullback of the codomain fibration along D. The inverse candidate extends this transformation to a map of colimit cones by setting $A_X = \mathrm{colim}_{i \in I} A_i$ and taking the induced map $A_X \to X$. By the descent characterization of ∞-topoi, the squares $A_i \to A_X$ over $D(i) \to X$ are pullbacks [2, Thms. 6.1.0.6 and 6.1.3.9]. Thus restriction of the candidate recovers the cartesian section. In the other direction, pulling the original colimit cone back along $A_X \to X$ gives a colimit cone by universality of colimits. Both constructions act on morphisms through functorial pullback and colimit. Their comparison equivalences come from the two universal properties. □

For $U \to X$, write $U^{[n]} = U \times_X \cdots \times_X U$ with $n+1$ factors. The simplicial object $U^{[\bullet]}$ uses projections as faces and diagonals as degeneracies. The arrow $U \to X$ is an *effective epimorphism* when the augmentation identifies $|U^{[\bullet]}|$ with X. Applying Theorem 4.1 gives

$$\mathcal{E}_{/X} \simeq \lim_{[n] \in \Delta} \mathcal{E}_{/U^{[n]}}. \tag{4.2}$$

At degree zero there is a family on U. At degree one there is an equivalence between its two pullbacks to $U \times_X U$. At degree two there is a homotopy comparing the composite of two such equivalences with the third equivalence. The totalization records the full simplicial compatibility, including degenerate simplices. In an ordinary category some of these comparisons reduce to equations. Their role in (4.2) is to specify the maps and higher simplices of the category of gluing data.

In particular, let A, B be families over X and let A_n, B_n denote their restrictions. Then

$$\mathrm{Map}_{\mathcal{E}_{/X}}(A, B) \simeq \lim_{[n] \in \Delta} \mathrm{Map}_{\mathcal{E}_{/U^{[n]}}}(A_n, B_n). \tag{4.3}$$

Mapping spaces commute with this limit of categories. One way to see this is to express a mapping space as the fiber at the two endpoints of the maximal subgroupoid of a functor category on Δ^1. Taking that functor category, taking the maximal subgroupoid, and taking the endpoint fiber all preserve the limits involved. Taking $A = \mathrm{id}_X$ in (4.3) shows that terms of B glue by the same descent formula. An equivalence of families can be tested after restriction to U: all the other components are further pullbacks, and equivalences in the limit are detected componentwise.

Example 4.2 (An effective epimorphism with nontrivial overlaps)**.** Let G be a discrete group and BG the realization of its nerve. For the basepoint $1 \to BG$, the iterated homotopy fiber products are equivalent to G^n in degree n. A tuple records the successive paths between copies of

the basepoint. Inner face maps compose adjacent paths, and inserting a constant path gives a degeneracy. Thus the Čech object is equivalent to the bar simplicial space of G, and $1 \to BG$ is an effective epimorphism. A family on BG is thereby equivalent to a space F equipped with a homotopy-coherent G-action. The space F is the fiber at the basepoint and the action is transport. The morphism space of two such families consists of coherently equivariant maps, as (4.3) describes.

Take $F = G$ with the left translation action. The family corresponds to the universal G-cover $EG \to BG$. It has sections over the basepoint. A global section would give a homotopy fixed point and hence a fixed element of the discrete fiber. For $G \neq 1$, every element of F moves under some translation, so the global section space is empty. The descent formula locates the difference at degree one, where the local term has to agree with all of its translates.

Example 4.3 (A family on a homotopy pushout)**.** The span $1 \leftarrow 1 \amalg 1 \to 1$, with the two fold maps, has pushout S^1 in $\mathcal{S}$. Its double mapping cylinder has two vertices joined by two edges. Slice descent gives

$$\mathcal{S}_{/S^1} \simeq \mathcal{S} \times_{\mathcal{S}\times\mathcal{S}} \mathcal{S},$$

where the two functors to $\mathcal{S} \times \mathcal{S}$ are diagonals. A family is specified by fibers F_0, F_1 and equivalences $e_0, e_1 : F_0 \simeq F_1$. After using e_0 to identify the fibers, the datum is a space F_0 with a self-equivalence $e_0^{-1}e_1$. The direction of the comparison corresponds to a choice of generator of $\pi_1(S^1)$. For $F_0 = \{0, 1\}$ and $e_0^{-1}e_1$ the transposition, the resulting family is the connected two-sheeted cover. For discrete fibers, the section condition is the fixed-element condition for this permutation.

Descent is a condition on gluing in the base. A Čech nerve is built from one map by repeated pullbacks. A hypercover allows new covers of matching objects in higher degrees. Equation (4.2) follows from effective epimorphy in every Grothendieck ∞-topos. Identifying the realization of every hypercover with its target is a hypercompleteness assertion [2, Thm. 6.5.3.12].

5 Identity types in the slices

Let $p : A \to X$ be a type in context X and form the diagonal $\Delta_{A/X} : A \to A \times_X A$. Regard this arrow as the family of identities over pairs of elements of A. Given a test context $T \to X$ and two terms $a, b : T \to A$ over X, the corresponding identity type over T is

$$\mathrm{Id}_{A/X}(a, b) = (a, b)^*(\Delta_{A/X}). \tag{5.1}$$

All pullbacks in (5.1) are homotopy pullbacks. In particular, the map from their total space to T includes the endpoint comparison data.

Proposition 5.1 (Generalized paths)**.** *For a, b as in (5.1), the space of sections of $\mathrm{Id}_{A/X}(a, b) \to T$ is equivalent to the path space from a to b in $\mathrm{Map}_{\mathcal{E}_{/X}}(T, A)$. The equivalence is natural under pullback of the test context.*

Proof. Writing $M = \mathrm{Map}_{\mathcal{E}_{/X}}(T, A)$, the pullback universal property identifies the section space with the homotopy fiber of $M \to M \times M$ at (a, b). For a Kan-complex model of M, the endpoint evaluation $M^{\Delta^1} \to M \times M$ is a fibration and the inclusion of constant paths $M \to M^{\Delta^1}$ is an equivalence. Its ordinary fiber therefore computes the stated homotopy fiber. Precomposition in T preserves this description. □

Example 5.2 (Parallel identities)**.** In $\mathcal{S}$, take $X = 1$, $A = BG$, and $a = b$ the basepoint. By Proposition 5.1 the identity type is $\Omega BG \simeq G$. Its cardinality is $|G|$ when G is finite. For nontrivial G, this identity type has several components, violating the monomorphism condition on the diagonal. Applying (-1)-truncation in the slice over $BG \times BG$ gives the subobject of pairs connected by a path. In an ordinary topos, mapping spaces are discrete and the same diagonal formula gives the usual equality predicate.

5.1 Path induction and coherence

Write $P \to A \times_X A$ for the family $\Delta_{A/X}$ or for an equivalent representative of it in the slice. Reflexivity is a map $r : A \to P$ over $A \times_X A$ induced by this choice, with r an equivalence on total spaces. Let $q : C \to P$ be a family of conclusions and let d be a section of $r^*C \to A$, the reflexivity case. The square

$$\begin{array}{ccc} r^*C & \longrightarrow & C \\ \downarrow & & \downarrow q \\ A & \xrightarrow{r} & P \end{array}$$

is a pullback. Since r is an equivalence, pullback gives an equivalence between the section spaces

$$\mathrm{Map}_{\mathcal{E}_{/P}}(P, C) \xrightarrow{\simeq} \mathrm{Map}_{\mathcal{E}_{/A}}(A, r^*C).$$

Its fiber at d is contractible. An element of this fiber is a section $J(d)$ of C together with a specified identification $r^*J(d) \simeq d$. This supplies the semantic elimination rule and its computation at reflexivity. Pulling any of the data back along a substitution $X' \to X$ preserves the pullback square and the equivalence r. The construction therefore applies to the substituted motive and reflexivity case as well.

The intrinsic interpretation uses equivalences with coherent comparisons, as seen in the specified identification in the fiber. The syntactic rule of intensional Martin-Löf type theory is formulated with a *judgmental* computation at reflexivity. Interpreting that form of the rule uses a presentation with chosen comprehension and stable type-former operations. In the simplicial-set model, for example, one works with Kan fibrations and a path-object factorization. The chosen reflexivity map is a trivial cofibration and the fibration representing a motive supplies a lift against it. Kapulkin and Lumsdaine construct a coherent simplicial interpretation, and Shulman's model-topos theorem supplies strict univalent interpretations presenting every Grothendieck ∞-topos [4, 5]. These presentation results address the judgmental equalities required of the model.

5.2 Dependent function extensionality

Let $p : A \to X$, $q : B \to A$, and let s, t be sections of $\Pi_p B \to X$. Denote by $\widehat{s}, \widehat{t}$ the corresponding sections of $B \to A$ under the adjunction of Section 2. There is an equivalence of families over X

$$\mathrm{Id}_{(\Pi_p B)/X}(s,t) \simeq \Pi_p\big(\mathrm{Id}_{B/A}(\widehat{s},\widehat{t})\big). \tag{5.2}$$

Indeed, over any $T \to X$, the adjunction gives an equivalence

$$\mathrm{Map}_{\mathcal{E}_{/X}}(T, \Pi_p B) \simeq \mathrm{Map}_{\mathcal{E}_{/A}}(T \times_X A, B).$$

It carries the two generalized sections to the pointwise generalized sections. Passing to path spaces and using Proposition 5.1 and the same adjunction identifies the spaces represented by the two sides of (5.2) at T. The identification is natural in T, so Yoneda gives (5.2). This argument proves the semantic equivalence at the level of the whole family, including its behavior under substitution.

6 Universes and univalence

A subobject classifier assigns to each context a space of truth values. An object classifier can assign a space of types, with paths that encode equivalences of types. To formulate a size-bounded version, fix a sufficiently large regular cardinal κ. A map $A \to X$ is *relatively κ-compact* if its pullback along every $T \to X$ with T a κ-compact object has κ-compact total space. Here κ-compactness means that mapping out of the object commutes with κ-filtered colimits. Write $(\mathcal{E}^{\kappa}_{/X})^{\simeq}$ for the maximal subgroupoid of the full subcategory of $\mathcal{E}_{/X}$ on these maps. In the ambient size convention of Theorem 4.1, the object-classifier theorem yields a universal family $\widetilde{U}_\kappa \to U_\kappa$ and equivalences natural in X

$$\mathrm{Map}_{\mathcal{E}}(X, U_\kappa) \;\simeq\; (\mathcal{E}^{\kappa}_{/X})^{\simeq}. \tag{6.1}$$

The map underlying (6.1) pulls back the universal family [2, §6.1.6, especially Thm. 6.1.6.8]. The classification claim specifies a size-bounded class of arrows. A syntactic universe also requires closure under the type formers that its formation rules specify.

Proposition 6.1 (Univalence of a classifying family)**.** *Let $u, v : T \to U_\kappa$ classify $A \to T$ and $B \to T$. The canonical map*

$$\mathrm{Path}_{\mathrm{Map}_{\mathcal{E}}(T,U_\kappa)}(u,v) \;\longrightarrow\; \mathrm{Eq}_{\mathcal{E}_{/T}}(A,B)$$

is an equivalence, where the right side is the space of equivalences from A to B over T. Equivalently, the identity family on U_κ over $U_\kappa \times U_\kappa$ represents the space of equivalences between the two pulled-back universal families.

Proof. An equivalence of spaces induces equivalences on path spaces between corresponding points. Apply this to (6.1). A path between two objects in the maximal subgroupoid of a category is an equivalence between those objects, with its higher homotopies. The equivalence in (6.1) maps a path of codes to transport in the universal family, which identifies the resulting path-space equivalence with the canonical map of the statement. The identity family represents the left side by Proposition 5.1, with base context 1. Naturality in $T \to U_\kappa \times U_\kappa$ gives the second claim. □

Proposition 6.2 (Descent for size-bounded families)**.** *For a small diagram $D : I \to \mathcal{E}$ with colimit X, restriction induces an equivalence*

$$(\mathcal{E}^{\kappa}_{/X})^{\simeq} \;\simeq\; \lim_{i \in I^{\mathrm{op}}} (\mathcal{E}^{\kappa}_{/D(i)})^{\simeq}.$$

A family $A \to X$ is relatively κ-compact if its restriction to each $D(i)$ is relatively κ-compact.

Proof. Mapping from the colimit into U_κ yields the limit of the mapping spaces from its stages. Applying (6.1) to these spaces gives the equivalence. To prove detection, take the restrictions of A and their coherences induced by pullback. The displayed equivalence glues them to a relatively κ-compact map $A' \to X$. In the full slice this family has the same descent datum as A. By Theorem 4.1, it is equivalent to A, and relative κ-compactness is invariant under equivalence. □

For an effective epimorphism $U \to X$, this proposition gives a local criterion involving only U. All the Čech stages map to U, so relative κ-compactness there propagates to the stages by pullback. With the whole descent datum specified, the classifying map $X \to U_\kappa$ is recovered from its restrictions and their comparison paths. This realizes local type codes as ordinary maps into a single ambient object, to which the colimit universal property applies.

Example 6.3 (Codes for finite types carry permutations)**.** In $\mathcal{S}$, let $u_n : 1 \to U_\kappa$ classify a discrete n-element space $\underline{n}$. Its loop space in U_κ is $\mathrm{Aut}(\underline{n}) = \Sigma_n$, a discrete group. The connected component of u_n is equivalent to $B\Sigma_n$: the groupoid of spaces equivalent to $\underline{n}$ has a full subgroupoid on that one object, and inclusion is an equivalence of groupoids. In particular the nonidentity permutation on $\underline{2}$ gives a noncontractible loop. Composing the classifying map of the nontrivial double cover $S^1 \to B\Sigma_2$ with $B\Sigma_2 \to U_\kappa$ recovers Example 4.3. The same permutation appears there as monodromy and here as an identity between universe codes.

For a discrete group G, the whole space of codes for families over BG with n-element fibers is $\mathrm{Map}_{\mathcal{S}}(BG, B\Sigma_n)$. Both arguments are nerves of ordinary groupoids, so the mapping space is the nerve of the functor groupoid. Its objects are homomorphisms $\rho : G \to \Sigma_n$, and an arrow $\rho \to \rho'$ is a permutation σ with $\rho'(g) = \sigma\rho(g)\sigma^{-1}$ for all g. In particular the loop group at ρ is the centralizer of $\rho(G)$ in Σ_n. This recovers both the classification by conjugacy and the automorphisms of each cover from the same mapping space.

The proposition-valued fragment gives the ordinary subobject classifier Ω. For each T, the maximal subgroupoid of monomorphisms into T is the discrete space of subobjects of T. To see this, a map over T into a monomorphism has either empty or contractible mapping space. If the map is an equivalence, its domain and codomain determine the same subobject. Thus $\mathrm{Map}_{\mathcal{E}}(T, \Omega)$ is 0-truncated, and paths between codes of propositions are equivalences of the corresponding subobjects. Equivalence of two propositions over T is the same as implication in both directions over T. This recovers propositional extensionality from the classifier and explains why permutation symmetries appear only once the classified families are allowed to have richer fibers [6, §3.3].

Finally, (6.1) classifies A together with its context. For a chosen $A \to X$ in the size class, the space of a code $a : X \to U_\kappa$ *together with an equivalence* $a^*\widetilde{U}_\kappa \simeq A$ is contractible, since it is the fiber of (6.1) at A. The connected component of the code in the mapping space has loop space $\mathrm{Aut}_{\mathcal{E}_{/X}}(A)$ by Proposition 6.1.

7 Syntactic categories and classifiers

A syntactic category encodes the structural operations of a calculus in a universal property. For the simply typed cartesian closed calculus, an object is a type, and a morphism $A \to B$ is a term of type B in a single variable of type A, modulo definitional equality. Products encode finite contexts, including the empty context, and composition is substitution. Abstraction and application give mutually inverse functions

$$\mathrm{Hom}(C \times A, B) \;\cong\; \mathrm{Hom}(C, B^A).$$

The inverse equations are the β- and η-laws. Given interpretations of the generating types and constants in a cartesian closed category, recursive interpretation of terms gives the corresponding structure-preserving functor. Conversely, such a functor sends the generic types and constants to an interpretation. The two constructions recover one another by induction on expressions, with uniqueness relative to the chosen cartesian closed structure. This is the universal-property formulation of the simply typed part of the syntax–semantics correspondence.

In a homotopy-coherent semantic category the displayed universal property is an equivalence of spaces. In the dependent case, types over Γ are objects of $\mathcal{E}_{/\Gamma}$, and terms of $A \to \Gamma$ form the section space $\mathrm{Map}_{\mathcal{E}_{/\Gamma}}(\mathrm{id}_\Gamma, A)$. A path of sections is a homotopy over Γ, and further paths encode higher identifications. Substitution is implemented by the functoriality of the slice construction. A presentation interpreting an ordinary judgmental equality still requires strictly stable operations at the level of the chosen syntax and its model. For example, the local-universes coherence theorem constructs split comprehension models under its stated hypotheses from weakly stable logical structure on a comprehension category [3]. Strict syntactic interpretability and an invariant universal property for the space of models address two separate questions. The following classifiers give universal properties for specific fragments and make their semantic morphisms explicit.

7.1 Finite-limit theories

Write $\mathcal{P}(C) = \mathrm{Fun}(C^{\mathrm{op}}, \mathcal{S})$ for a small ∞-category C. For ∞-topoi $\mathcal{E}$ and $\mathcal{F}$, let $\mathrm{Geom}(\mathcal{E}, \mathcal{F})$ be the space of geometric morphisms, computed as the maximal subgroupoid of the category of colimit-preserving left exact functors $\mathcal{F} \to \mathcal{E}$. These functors are inverse images of geometric morphisms since the presentable adjoint functor theorem supplies right adjoints. We write $\mathcal{D}^{\simeq}$ for the maximal subgroupoid of an ∞-category $\mathcal{D}$. Corestriction to this subgroupoid retains equivalences of models and their higher homotopies.

Proposition 7.1 (Finite-limit presheaf classifier)**.** *For every small finitely complete ∞-category C and every ∞-topos $\mathcal{E}$, restriction along the Yoneda embedding gives a natural equivalence*

$$\mathrm{Geom}(\mathcal{E}, \mathcal{P}(C)) \;\simeq\; \mathrm{Fun}^{\mathrm{lex}}(C, \mathcal{E})^{\simeq}.$$

Under the equivalence, the universal model in $\mathcal{P}(C)$ is the Yoneda embedding.

Proof. The free cocompletion theorem identifies colimit-preserving functors out of $\mathcal{P}(C)$ with functors out of C, by restriction along Yoneda [2, Theorem 5.1.5.6]. Yoneda preserves finite limits, so a left exact functor on presheaves restricts to a left exact functor on C. Conversely, the colimit-preserving extension of a left exact functor $C \to \mathcal{E}$ is left exact by [2, Proposition 6.1.5.2]. Thus the equivalence restricts to the full subcategories in question and induces an equivalence on their cores. The identity inverse image restricts to Yoneda, giving the universal model. □

The preservation requirement controls which semantic constructions are included. Finite limits suffice to specify projections, pullback squares, and equations in finite-limit theories. Exponentials and dependent products require additional preservation conditions. For a counterexample, take the point of the presheaf topos on the ordinary arrow category [1] whose inverse image is evaluation at 1. For $A = (\varnothing \to *)$ and $B = (\varnothing \to \varnothing)$, in presheaf notation with the arrow directed from the value at 1 to the value at 0, both $A(1)$ and $B(1)$ are empty, while

$$\begin{aligned}(B^A)(1) &\simeq \mathrm{Map}_{\mathcal{P}([1])}(y(1) \times A, B)\\ &\simeq \mathrm{Map}_{\mathcal{P}([1])}(A, B) \simeq \varnothing.\end{aligned}$$

Here $y(1)$ is terminal. The exponential of the evaluated objects is $*$. A classifying statement about dependent type theories with chosen rules therefore needs a category of models and morphisms designed to preserve those rules.

An accessible left exact localization can impose additional conditions on the models in the proposition. Suppose

$L\colon \mathcal{P}(C) \to \mathcal{F}$ is such a localization and i its right adjoint. Then $\mathrm{Geom}(\mathcal{E}, \mathcal{F})$ is equivalent to the subspace of those models whose colimit-preserving extensions invert every L-equivalence. Indeed, a colimit-preserving $F\colon \mathcal{P}(C) \to \mathcal{E}$ inverting those maps factors essentially uniquely through L. The induced functor $\overline{F} \simeq Fi$ preserves finite limits whenever F does, since i is a right adjoint. It also preserves colimits. A colimit of a diagram D in $\mathcal{F}$ can be calculated as $L\,\mathrm{colim}(iD)$, and applying $\overline{F}$ gives $F\,\mathrm{colim}(iD)$. This describes a restricted class of theories by the semantic maps they invert and leaves the localization itself as an explicit hypothesis.

7.2 Two classifiers for one sort

Let $\mathcal{K} \subset \mathcal{S}$ be the essentially small full subcategory generated by a point under finite colimits. Equivalently, one can obtain its objects by iterating finite homotopy colimit constructions starting with a point. At each step one may choose a small collection of objects since the diagrams have finite shape and the mapping spaces are small. Passing to the closure under equivalence gives an essentially small full subcategory. Write Fin for the category of finite sets, considered inside spaces as discrete objects.

Theorem 7.2 (Unrestricted and discrete sorts)**.** *The following equivalences of spaces are natural in $\mathcal{E}$:*

$$\mathrm{Geom}(\mathcal{E}, \mathcal{P}(\mathcal{K}^{\mathrm{op}})) \simeq \mathcal{E}^{\simeq},$$
$$\mathrm{Geom}(\mathcal{E}, \mathcal{P}(\mathrm{Fin}^{\mathrm{op}})) \simeq (\mathcal{E}_{\leq 0})^{\simeq}.$$

Both equivalences are given by evaluating the corresponding finite-limit model at a point.

Proof. For $X \in \mathcal{E}$, the cotensor X^K is the limit of the constant diagram on X indexed by the space K. Consequently,

$$\mathrm{Map}_{\mathcal{E}}(T, X^K) \;\simeq\; \mathrm{Map}_{\mathcal{S}}(K, \mathrm{Map}_{\mathcal{E}}(T, X)).$$

The right side sends colimits in K to limits, so $K \mapsto X^K$ is a left exact functor on $\mathcal{K}^{\mathrm{op}}$. It is the right Kan extension of X along the inclusion $j\colon \{*\} \to \mathcal{K}^{\mathrm{op}}$, since the indexing category $(K \downarrow j)$ is the space of maps $K \to *$ in $\mathcal{K}^{\mathrm{op}}$, namely K. If F is left exact, the unit $F \to j_* j^* F$ is an equivalence at the point. The objects where the unit is an equivalence are closed under finite limits, since both functors preserve those limits. The point generates $\mathcal{K}^{\mathrm{op}}$ under finite limits, and hence the unit is an equivalence everywhere. The counit $j^* j_* X \to X$ is an equivalence at once. This proves the first equivalence by Proposition 7.1.

For finite sets the right Kan extension of X is again given by powers $S \mapsto X^S$, now with S discrete. The same argument using products shows that evaluation at a singleton classifies product-preserving functors on $\mathrm{Fin}^{\mathrm{op}}$. To determine which are left exact, consider in Fin the pushout of the two copies of the fold $2 \to 1$. Its pushout object is 1. In a left exact model this gives

$$X \xrightarrow{\sim} X \times_{X \times X} X,$$

where the two maps to $X \times X$ are diagonals. The displayed map is the diagonal of the diagonal $X \to X \times X$. Its being an equivalence means that this latter diagonal is a monomorphism, which characterizes 0-truncated objects. Conversely, when X is 0-truncated, $\mathrm{Map}_{\mathcal{E}}(T, X)$ is discrete for every T. Powers by finite sets then send finite colimits of sets to limits after applying $\mathrm{Map}_{\mathcal{E}}(T, -)$. Limits of discrete spaces agree with limits in sets, and the mapping-space criterion gives left exactness. Applying Proposition 7.1 proves the second identification. □

The fold diagram measures precisely where the theories diverge. Its pushout in spaces is S^1: the two points of the middle object give the two edges between the two outer points in the double mapping cylinder. The unrestricted classifier therefore assigns to this pushout the cotensor $X^{S^1} \simeq X \times_{X \times X} X$, the object of free loops in X. In the discrete theory, the corresponding pushout has value 1, and left exactness requires the resulting constant-loop map $X \to X^{S^1}$ to be an equivalence. For an explicit computation in spaces, if $X = K(A, 1)$ for an abelian group A, then the components of the free loop space are indexed by A. Evaluation at a point gives a fibration with fiber $\Omega X \simeq A$, and its monodromy is conjugation, which is trivial here. Thus any nonzero A obstructs the constant-loop map from being an equivalence. The difference between the classifiers is already detected by a single finite-limit condition, even though their generic object has the same product operations.

The truncation functor $\mathcal{K} \to \mathrm{Fin}$, $K \mapsto \pi_0 K$, preserves finite colimits and lands in finite sets because finite colimits of finite sets are finite. After taking opposites and applying Proposition 7.1, it determines a geometric morphism

$$\mathcal{P}(\mathrm{Fin}^{\mathrm{op}}) \longrightarrow \mathcal{P}(\mathcal{K}^{\mathrm{op}}).$$

On model spaces, postcomposition with this morphism is exactly the inclusion of the core of the 0-truncated objects into the core of all objects. The functor from the discrete theory into the unrestricted one at the level of models is consequently realized by an explicit morphism of their classifying topoi.

7.3 Adding an inhabitation axiom

The same unrestricted sort can be constrained by an existential axiom. Let $C = \mathcal{K}^{\mathrm{op}}$, $A = y(*)$, and let $I \hookrightarrow 1$ be the image of $A \to 1$ in $\mathcal{P}(C)$. The terminal object is representable by the terminal object of C, so $I \hookrightarrow 1$ determines a sieve on that object. Take the Grothendieck topology J generated by declaring it covering. The topos $\mathrm{Shv}_J(C)$ classifies objects X such that $X \to 1$ is an effective epimorphism. In the internal language this condition is the truth of $\exists x : X$, with existence interpreted by propositional truncation.

To prove the claim, let $F\colon \mathcal{P}(C) \to \mathcal{E}$ be the inverse image corresponding to X. Inverse-image functors preserve effective epimorphisms since they preserve the formation and realization of Čech nerves. They also preserve

monomorphisms by left exactness. By uniqueness of the effective epi–mono factorization, $F(I) \hookrightarrow 1$ is the image of $X \to 1$. If F factors through sheafification, it inverts $I \hookrightarrow 1$, and so the image is all of 1.

For the converse, suppose $F(I) \hookrightarrow 1$ is an equivalence. For any sieve on a representable, its image under F is the image of the coproduct of the objects in the corresponding family of arrows under $F \circ y$. Indeed a sieve is covered by the representables mapping into it, and F preserves images and coproducts. Thus the sieves inverted by F are exactly the covering sieves of the relative canonical topology for the left exact functor $F \circ y$ [2, Proposition 6.2.4.2]. By hypothesis that topology contains the generating cover of J, so F inverts all its covering-sieve inclusions. The universal property of sheafification and the preceding localization argument give the factorization, with its space of choices contractible.

For example, take $\mathcal{E} = \mathcal{S}_{/S^1}$ and let X be the object given by the basepoint map $* \to S^1$. Its structure map is an effective epimorphism because it is surjective on components. A section would split the induced map on π_1, which is impossible. This distinguishes local inhabitation, represented by the sheaf classifier, from the additional data of a term of the type in the terminal context.

8 Classifying objects and torsors

A classifying object represents a functor on the objects of a fixed topos. This complements the preceding classifying topoi, whose universal property varies the ambient topos. Group torsors give a representability statement that displays all levels of equivalence data.

Let G be a group object in $\mathcal{E}$, presented by its reduced groupoid object $B_\bullet G$. Thus $B_0 G = 1$, $B_1 G = G$, and the Segal identifications give $B_n G \simeq G^n$. Define $BG = |B_\bullet G|$, with its map $1 \to BG$ induced from degree zero. By effectiveness of groupoid objects in an ∞-topos, $B_\bullet G$ is the Čech nerve of that map, which is an effective epimorphism. In particular $G \simeq 1 \times_{BG} 1$, compatibly with the group multiplication from the groupoid structure.

Definition 8.1. A right G-torsor over X is a coherently G-equivariant map $p\colon P \to X$, with trivial action on X, such that p is an effective epimorphism and the action comparison

$$P \times G \longrightarrow P \times_X P, \qquad (u, g) \longmapsto (u, ug)$$

is an equivalence. The action is encoded by its simplicial action groupoid with n-simplices $P \times G^n$. Write $\mathrm{Tors}_G(X)$ for the space of these torsors and their equivariant equivalences over X.

The action comparison is often called the principal bundle condition. It states that once two points in a fiber have been chosen, the space of group elements taking the first point to the second is contractible. If p has a section s, restricting the comparison along the first coordinate s identifies $X \times G$ with P by $(x, g) \mapsto s(x)g$. In particular each torsor becomes trivial after its own effective epimorphism $P \to X$, using the diagonal section of the pullback. This supplies local triviality without choosing an external topology on $\mathcal{E}$.

Theorem 8.2 (Classification at the level of spaces)**.** *There is a natural equivalence*

$$\mathrm{Map}_{\mathcal{E}}(X, BG) \;\simeq\; \mathrm{Tors}_G(X).$$

The torsor corresponding to $f\colon X \to BG$ is determined by the pullback square

$$\begin{array}{ccc} P & \longrightarrow & 1 \\ \downarrow & & \downarrow \\ X & \xrightarrow{f} & BG. \end{array}$$

Proof. Effectiveness identifies the Čech nerve of $1 \to BG$ with the reduced action groupoid $B_\bullet G$. It follows that $1 \to BG$ is a G-torsor in the slice over BG. Base change preserves effective epimorphisms and the action comparison, so pullback along f gives a torsor over X.

In the other direction, the action comparison identifies the action groupoid of P with the Čech nerve of p. In degree one this is the principal bundle condition, and the Segal maps give the equivalence in all degrees. Projection to G^n defines a simplicial map from that nerve to $B_\bullet G$. Realization gives a classifying map $X \to BG$, because the augmentation of the Čech nerve realizes to X. The simplicial map together with the augmentation gives an equivariant map from the given P to $X \times_{BG} 1$ over X. More explicitly, the realization maps give a map of the action groupoids into $X \times_{BG} B_\bullet G$, and equivariance follows from its compatibility with the projection to $B_\bullet G$.

An equivariant map between torsors over the same base is an equivalence. To see this, pull back along the source torsor. The diagonal section trivializes the source, and its image under the equivariant map trivializes the target. In these coordinates the comparison is the identity. Base change along the effective epimorphism detects equivalences by descent, proving the assertion. Thus the comparison $P \to X \times_{BG} 1$ is an equivalence.

Finally, the action groupoid of a pullback torsor for f is the base change of $B_\bullet G$ along f. Its realization recovers f since realizations are universal in $\mathcal{E}$. The comparisons are natural, being induced by pullbacks, simplicial maps and their colimits. They give inverse equivalences on the spaces, including higher homotopies. This is the internal principal bundle classification of [7, Section 3]. □

Example 8.3 (From principal bundles to higher cohomology)**.** Let A be a discrete abelian group and take $\mathcal{E} = \mathcal{S}$, with $G = K(A, n)$ for $n \geq 1$ in its loop group structure from $K(A, n+1)$. The delooping is $K(A, n+1)$, so the components of $\mathrm{Tors}_G(X)$ are $H^{n+1}(X; A)$. For a CW model of X, the higher groups at the trivial torsor satisfy

$$\pi_j \,\mathrm{Tors}_G(X) \;\cong\; H^{n+1-j}(X; A) \qquad (1 \leq j \leq n+1).$$

Indeed, based maps $S^j \to \mathrm{Map}(X, K(A, n+1))$ correspond to based maps from $S^j \wedge X_+$, and cohomology of that smash product is given by the suspension isomorphism. The higher groups vanish above $n+1$. For $X = S^{n+1}$, the component set and the group in degree $n+1$ are both A, while the other positive-degree groups are zero. Every component has the same homotopy groups by translation in the abelian group object $\mathrm{Map}(X, K(A, n+1))$. Over a point and for a discrete group G, the classification similarly gives the space BG with one component and automorphism group G. Thus the sets of equivalence classes recover familiar classifications and the spaces additionally record symmetries of each model.

9 Truth, choice, and localization

A proposition over a context X is a (-1)-truncated map to X. The higher analogue of a predicate equipped with witnesses is an arbitrary map $p : A \to X$. In the latter, the homotopy type of the fiber records both the witnesses and the identifications among them. Passing from this type to its truth value is a relative operation. It takes place in $\mathcal{E}_{/X}$, so it uses the geometry of the context as well as that of the domain.

9.1 The support of a family

Definition 9.1. The *support* $\mathrm{supp}_X(A)$ of $A \to X$ is its relative (-1)-truncation. Write the image factorization as

$$A \twoheadrightarrow \mathrm{supp}_X(A) \hookrightarrow X,$$

where the first arrow is effective epimorphic and the second is a monomorphism. If $U \hookrightarrow X$ is a proposition, write $\bot_X$ and $\top_X$ for the least and greatest propositions over X.

The support has the following elementary calculus. It applies to families whose fibers have arbitrary homotopy types.

Proposition 9.2. *Let $f : Y \to X$ be a map in an ∞-topos.*

1. *For $A \to X$ and a proposition $U \to X$, precomposition gives an equivalence*
$$\mathrm{Map}_{/X}(\mathrm{supp}_X(A), U) \simeq \mathrm{Map}_{/X}(A, U).$$
2. *There is a natural equivalence $f^* \mathrm{supp}_X(A) \simeq \mathrm{supp}_Y(f^* A)$.*
3. *The support of the slice coproduct $P \amalg Q$ is their join in $\mathrm{Sub}(X)$ when P, Q are propositions. For $B \to Y$, the support of the composite $B \to Y \to X$ is $\exists_f \mathrm{supp}_Y(B)$.*

Proof. The first assertion is orthogonality of effective epimorphisms and monomorphisms applied to $A \twoheadrightarrow \mathrm{supp}_X(A)$ and $U \hookrightarrow X$. For the second, pull back the image factorization. Both classes of maps are stable under pullback, so the new factorization has the universal property of the image. A map from $P \amalg Q$ to a proposition exists exactly when the two summands admit maps to it. This is the least-upper-bound property. For the composite, a map over X from B to a proposition U is adjoint to a map from B to f^*U over Y. Since f^*U is a proposition, this factors through $\mathrm{supp}_Y(B)$. The resulting equivalence of conditions is the defining adjunction of $\exists_f$. □

Full support is equivalent to effective epimorphicity of $A \to X$. In a sheaf model this is the local availability of witnesses, tested after base change to each stage and passage to a cover. A section $X \to A$ is a witness compatible over the entire context. These two conditions agree for a proposition A but differ for general families. In fact a mono with full support is an equivalence. For other families, the unit into the support permits elimination into propositions and produces a map into a given type only when suitably coherent additional data are available.

Example 9.3 (A disjunction with no global tag)**.** Use space-valued sheaves on $\mathbb{R}$. Let U and V be the subterminals represented by the opens $(-\infty, 1)$ and $(-1, \infty)$. They join to 1 because the two opens cover the line. Thus $U \amalg V \to 1$ has full support. A global section of this coproduct would pull the disjoint summands back to complementary opens W_U, W_V covering the line, with $W_U \subseteq U$ and $W_V \subseteq V$. By connectedness one would be the whole line, contrary to each containment. The calculation works in the ordinary sheaf topos as well: the obstruction concerns gluing the tags and already occurs for a 0-truncated family.

Relative truncation is compatible with change of base, as in Proposition 9.2. Preservation of finite limits by a truncation functor is a separate condition. In spaces, for a nontrivial discrete group G, the homotopy pullback of the two basepoints $* \to BG \leftarrow *$ is G as a discrete space. Truncating the diagram objectwise by $\tau_{\leq 0}$ first yields $* \to * \leftarrow *$, whose pullback is $*$. Thus the functor of components can preserve the meanings of ordinary predicates while changing a finite-limit construction on higher types. This applies already to the self-intersection of a single point. The same diagram also exhibits failure of left exactness of propositional truncation if BG is replaced by the discrete two-element space and the basepoints select its two distinct elements.

9.2 Boolean propositions and the strength of choice

Definition 9.4. Say that the predicate logic of $\mathcal{E}$ is *Boolean* when every subobject lattice $\mathrm{Sub}_{\mathcal{E}}(X)$ is Boolean. Write $\neg P$ for the Heyting implication $P \Rightarrow \bot_X$.

This condition is exactly excluded middle for propositions in every context. It is encoded by the universal predicate $t : 1 \hookrightarrow \Omega$, where Ω is the subobject classifier. If $t \vee \neg t = \top_\Omega$, any predicate satisfies the law by pullback

along its classifying map. Indeed pullback preserves the initial object, unions, and predicate implication. For the last assertion use preservation of finite limits together with base change for the right adjoints to substitution in the slices. The converse applies excluded middle to t itself.

Under classical external logic, the predicate logic of $\mathcal{S}$ is Boolean. A monomorphism of spaces includes a union of connected components: its fibers over points are empty or contractible, and path transport maintains this condition or exchanges the contractible fibers by equivalences. The complementary union yields its negation. There is still a difference between truth and witness selection. If G is nontrivial, the basepoint $* \to BG$ is effective epimorphic by connectedness. A section would make BG a retract of $*$ and hence contractible. This fails since $\pi_1(BG) \cong G$.

Even a choice principle restricted to sets can fail with Boolean predicate logic. Take $\mathcal{E} = \mathrm{Fun}(BG, \mathcal{S})$, the ∞-topos of G-spaces. A subobject of a G-space is an invariant union of components. To see this, classify the underlying mono by its subset of components. Each arrow of BG gives a permutation on components, and equivariance makes the subset invariant. Conversely a union of components for an invariant subset is closed under this transport, with its equivariance induced from the given G-space. The complement is invariant, proving Booleanity. The free transitive G-set G, viewed as a discrete G-space, has full support over 1 by objectwise computation of colimits and finite limits. A section of $G \to 1$ would be an invariant element of the discrete G-set. There is none for $G \neq 1$. The terminal object and the free transitive G-set both belong to the ordinary topos of G-sets, so the example also refutes that every epi of sets splits in this model.

For comparison, a space-valued sheaf topos on a space Z has $\mathrm{Sub}(1)$ identified with the frame of opens. Here $\neg U = \mathrm{int}(Z \setminus U)$ because this is the largest open disjoint from U. On $Z = \mathbb{R}$, the formula $U \vee \neg U$ for $U = (0, \infty)$ omits the origin. This gives an intuitionistic model even when the metatheory used to build the sheaf topos is classical. Internal Booleanity and the external rules used in constructing the category have different domains of application.

9.3 An explicit dense localization

A left exact localization $L : \mathcal{E} \rightleftarrows \mathcal{D} : i$ is an adjunction with fully faithful right adjoint and finite-limit-preserving left adjoint. In the presentable setting we take L accessible. Such a reflector preserves images because the factorization into an effective epimorphism and a mono is preserved. It preserves monos by transporting their diagonals through finite limits. It preserves effective epis by transporting the simplicial Čech nerve through finite limits and its realization through colimits. Thus if $A \to X$ is any family, there is an equivalence

$$L\,\mathrm{supp}_X(A) \simeq \mathrm{supp}_{LX}(LA),$$

viewed as objects over LX. A localization can send a proper support to the top support, as the following small sheaf model shows.

Example 9.5 (Localizing at the dense open of Sierpiński space)**.** Let $Z = \{0, 1\}$ with opens $\varnothing, U = \{1\}, Z$. Its space-valued sheaves are diagrams $A \to B$ of spaces, with A the value on Z and B the value on U. Every open cover of a nonempty open contains that open itself, so the only additional condition imposes a terminal value on $\varnothing$. The subterminal $U \hookrightarrow 1$ is the map of diagrams

$$(\varnothing \longrightarrow *) \;\longrightarrow\; (* \longrightarrow *).$$

A diagram $A \to B$ is local for this map exactly when $A \to B$ is an equivalence, since mapping out of the two diagrams computes sections over Z and U respectively. The local diagrams identify with $\mathcal{S}$ via evaluation at U, with inverse up to equivalence given by the constant diagram. Evaluation is the reflector: maps into $(C \xrightarrow{\mathrm{id}} C)$ are determined, up to contractible choice, by the map $B \to C$. The reflector preserves limits as limits of diagrams are computed pointwise. Consequently this geometric localization preserves all higher homotopy in the value on U. It turns $U \hookrightarrow 1$ into an equivalence, although $U \vee \neg U = U$ before localization.

The double-negation action of this reflector can be computed in arbitrary contexts. Fix $r : A \to B$, and identify one of its subobjects by a pair of unions of components $M \subseteq A$ and $N \subseteq B$ with $M \subseteq r^{-1}(N)$. In the following formulas complements inside a space are complementary unions of components. Any subobject disjoint from (M, N) has its second entry contained in $B \setminus N$, hence its first entry contained in $r^{-1}(B \setminus N)$. This latter pair is itself a subobject disjoint from (M, N). Therefore

$$\begin{aligned} \neg(M, N) &= (r^{-1}(B \setminus N), B \setminus N), \\ \neg\neg(M, N) &= (r^{-1}N, N). \end{aligned}$$

Here r^{-1} denotes the homotopy pullback of a mono. Reflecting $(M, N) \hookrightarrow (A, B)$ gives $N \hookrightarrow B$. Pulling its image under the right adjoint back along the unit of the adjunction gives precisely $(r^{-1}N, N)$. Thus double negation on predicates agrees with this localization in every context, even when A and B have nontrivial higher homotopy. The closure of a subobject and the higher data local to U can both be read directly from the localization. Compare the general treatment of topological localizations in [2, §6.2] and left exact modalities in [8].

Example 9.6 (A reflector that changes implications)**.** The same diagram category has a different accessible left exact localization. Evaluation at Z has fully faithful right adjoint $C \mapsto (C \to *)$, as a map of diagrams to this target is specified by its value on the first entry. The left adjoint, denoted K, sends both U and 0 to the empty space. Since U was dense, $\neg U = 0$ in the source, so there is a strict inequality

$$K(\neg U) = \varnothing \quad < \quad * = \neg K(U)$$

in the subobject lattice of the terminal space. This also computes a failure to preserve an exponential. The Heyting implication $U \Rightarrow 0$ is the exponential 0^U, since both operands are subterminal and the exponential has the adjunction for implication. In the target the corresponding exponential is $\varnothing^{\varnothing} \simeq *$. Evaluation at Z preserves stable images by the argument above, whereas implication uses a right adjoint and the displayed strict inequality shows precisely where its preservation can fail. This matters when a geometric localization is used to move an interpretation of logic between models: a proof that uses implication may require an additional preservation argument beyond left exactness.

10 Recursion and computability

The natural numbers object organizes the categorical study of recursion in Lambek and Scott [1]. In an ∞-topos, its elimination principle applies to families with arbitrary homotopy types. The natural numbers themselves still form a set internally. The higher structure appears in the spaces of sections and in the comparisons that express the recursion equations. This distinction also allows us to locate the additional mathematical structure necessary for a realizability interpretation.

10.1 The recursion space

Let $\mathcal{E}$ be a Grothendieck ∞-topos, write 1 for its terminal object, and put

$$N = \coprod_{n\in\mathbb{N}} 1, \qquad z = i_0, \qquad si_n = i_{n+1},$$

where i_n is the nth coproduct injection. Here $\mathbb{N}$ in the index denotes the external set and N denotes an object of $\mathcal{E}$. The relation $si_n = i_{n+1}$ denotes the comparison supplied by the coproduct property. The general relation between inductive types and homotopy-initial algebras is treated by Awodey, Gambino, and Sojakova [9].

Proposition 10.1 (Homotopy-initial natural numbers)**.** *The object N is 0-truncated. Given $a : 1 \to A$ and $b : A \to A$, the space of triples*

$$(f, \alpha, \beta), \qquad f : N \to A, \quad \alpha : fz \simeq a, \quad \beta : fs \simeq bf$$

is contractible. In particular $(N, [z, s])$ is initial among algebras for $1 \amalg (-)$ with homotopy-commuting morphisms of algebras.

Proof. Universality of coproducts gives $N \times N \simeq \coprod_{m,n\in\mathbb{N}} 1$. The pullback of the diagonal $N \to N \times N$ to the summand indexed by (m, n) is the pullback $1 \times_N 1$ of the injections i_m and i_n. By disjointness it is initial when $m \neq n$, and equivalent to 1 when $m = n$. A pullback of a diagonal along an arbitrary map to $N \times N$ decomposes into the corresponding pullbacks over these summands. Thus the diagonal is a monomorphism, which is the criterion for 0-truncatedness.

Set $K = \mathrm{Map}_{\mathcal{E}}(1, A)$. The coproduct property identifies $\mathrm{Map}_{\mathcal{E}}(N, A)$ with $\prod_{n\in\mathbb{N}} K$. In this description the desired space consists of sequences (x_n) with paths $x_0 \rightsquigarrow a$ and $x_{n+1} \rightsquigarrow bx_n$ for every n. The paths are part of the points of this space. The space of the first entry with its path to a is a based path space with variable starting point and is contractible. At each finite stage, forgetting the last entry and its path has contractible homotopy fibers, since extending the sequence is choosing a point of K together with a path from that point to a specified point of K. Each finite stage is therefore contractible. The desired space is the inverse limit of these finite stages in $\mathcal{S}$, so is itself contractible.

For an endofunctor F, take the category of algebras to be the pullback of the arrow category along $X \mapsto (FX, X)$. The mapping space from $u : FX \to X$ to $v : FY \to Y$ is then the homotopy equalizer of the maps $f \mapsto fu$ and $f \mapsto vFf$ from $\mathrm{Map}_{\mathcal{E}}(X, Y)$ to $\mathrm{Map}_{\mathcal{E}}(FX, Y)$. For $F = 1 \amalg (-)$ and the source $(N, [z, s])$, the two components of the equalizer are exactly α and β. □

Contractibility is stronger than uniqueness of the homotopy class of the underlying arrow. For instance, in $\mathcal{S}$, choose $A = S^1$, let a be its base point, and take b to be the identity. The underlying arrow space $(S^1)^{\mathbb{N}}$ has nontrivial loops. The base path and all step paths in Proposition 10.1 supply the data that make the space of solutions contractible.

10.2 Dependent recursion and parameters

For a family $p : E \to N$, an initial datum is a point $a : 1 \to E$ over z, and a successor datum is a map $d : E \to E$ over s. The latter consists of d and a chosen homotopy $pd \simeq sp$. A dependent solution is a section σ of p, a comparison $\sigma z \simeq a$ over z, and a comparison $\sigma s \simeq d\sigma$ over s.

Proposition 10.2 (Dependent elimination)**.** *The space of dependent solutions is contractible. The same construction holds in every slice $\mathcal{E}_{/\Gamma}$, with natural numbers object $\Gamma \times N \to \Gamma$, and is stable under pullback of contexts.*

Proof. Let $E_n = i_n^* E$ and $K_n = \mathrm{Map}_{\mathcal{E}}(1, E_n)$. Regard 1 with structure map i_n as the nth object of a discrete diagram in $\mathcal{E}_{/N}$. Its coproduct is id_N, because coproducts in a slice are computed on the domains. Therefore

$$\mathrm{Map}_{\mathcal{E}_{/N}}(\mathrm{id}_N, p) \simeq \prod_{n\in\mathbb{N}} K_n.$$

The successor datum and the chosen comparison for si_n induce maps $d_n : K_n \to K_{n+1}$. The base datum gives a point of K_0. In these coordinates the solution space consists of $x_n \in K_n$ with paths $x_0 \rightsquigarrow a$ and $x_{n+1} \rightsquigarrow d_n(x_n)$. Again, each finite stage is contractible by taking the path space with variable starting point, and the solution space is their inverse limit.

Apply the argument in $\mathcal{E}_{/\Gamma}$, whose terminal object is id_Γ. Universality of coproducts identifies the sum of its copies of id_Γ with $\Gamma \times N$ over Γ. A change of context preserves this coproduct and sends the recursion data and their comparisons to those for the pulled-back family. It sends a solution to a solution and identifies the distinguished solution up to the contractible space described above. □

If p is a monomorphism, the dependent conclusion is induction for a predicate on N. A proof at zero and a proof of successor closure give a section of that monomorphism, hence the predicate holds everywhere. If p has higher fibers, the same proof yields a coherent family of terms. For an ordinary example with parameters, let $a : \Gamma \to N$ and $b : \Gamma \times N \times N \to N$ be the base and step maps for primitive recursion. Over Γ, recurse with state (n, x), initial state $(0, a)$, and transition $(n, x) \mapsto (s(n), b(-, n, x))$. The first coordinate of the resulting map is the identity on the numerical index by uniqueness of recursion. The second coordinate defines r with $r(\gamma, 0) = a(\gamma)$ and $r(\gamma, s(n)) = b(\gamma, n, r(\gamma, n))$. The equality notation here is justified by the 0-truncated target. In a type-theoretic presentation, judgmental computation rules require the corresponding coherent choices of model structure.

In particular, define addition by using the first numerical input as initial value and incrementing the state at each iteration. Multiplication uses zero as initial value and adds the first input at each iteration. On an external numerals pair (m, n), these maps land in the summands indexed by $m + n$ and mn, respectively. The claim follows by ordinary induction on the second index. Conversely these values determine the arrows by the coproduct property of $N \times N$. The same reasoning extends to a primitive recursive definition built from zero, successor, projections, composition, and primitive recursion. Inverse images of geometric morphisms preserve the sums of copies of 1, the finite products, and the input maps of recursion. They therefore preserve these arithmetic operations. This preservation concerns the specified operations and their equations.

Example 10.3 (Higher-order numerical recursion)**.** Let $F = N^N$ and let $1_N = sz : 1 \to N$ be the numeral one. In context F, the generic function f is evaluation. Take $f(1_N)$ as initial value and f as transition for a recursion with target N. Transposing the resulting map $F \times N \to N$ defines an arrow $\Phi : F \to F$ satisfying

$$\Phi(f)(0) = f(1_N), \qquad \Phi(f)(s(n)) = f\big(\Phi(f)(n)\big).$$

Recursion with target F now gives $h : N \to F$ with $h(0) = s$ and $h(s(m)) = \Phi(h(m))$. Its uncurried map $a : N \times N \to N$ satisfies

$$\begin{aligned} a(0, n) &= s(n), \\ a(s(m), 0) &= a(m, 1_N), \\ a(s(m), s(n)) &= a(m, a(s(m), n)). \end{aligned}$$

These equations give the Ackermann–Péter function on external numerals, as follows by induction first on m and then on n. This function lies beyond numerical primitive recursion [10, §1.4]. The outer state F is itself a type of functions, so the categorical construction uses recursion at a higher simple type. The object F is 0-truncated by the exponential adjunction and the truncation of N. Thus this example already belongs to the ordinary fragment.

10.3 Absolute Church's thesis

Fix a standard external enumeration $(\varphi_e)_{e \in \mathbb{N}}$ of the partial recursive functions. Write $H(e, n, t, m)$ for the decidable relation saying that t is a code of a terminating computation of program e on input n with output m. For a Grothendieck ∞-topos, interpret this predicate as the subobject of N^4 obtained by taking the coproduct of the summands on which the external relation holds. The object N^4 here is the fourfold product, and N^N below is the exponential. The corresponding form of Church's thesis is the internal proposition

$$\forall f : N^N \; \exists e : N \; \forall n : N \; \exists t : N \; H(e, n, t, f(n)). \tag{10.1}$$

The numerical relation can equivalently be interpreted from its primitive recursive characteristic function. The geometric inverse image $\mathcal{S} \to \mathcal{E}$ preserves finite products and the coproducts defining numerical operations, so the two descriptions agree on every numeral tuple and hence on their coproduct. Existential quantification in (10.1) is propositional.

Proposition 10.4 (Failure for Grothendieck semantics)**.** *In the classical external metatheory, the truth value of* (10.1) *is the initial subobject of* 1 *in every Grothendieck* ∞*-topos. In a nondegenerate* ∞*-topos this proposition is false.*

Proof. There exists an external function $g : \mathbb{N} \to \mathbb{N}$ not computed by any φ_e. For example, use the characteristic function of the diagonal halting set. The map $\bar{g} : N \to N$ defined on each summand by $i_n \mapsto i_{g(n)}$ supplies an internal instance of f. Let $P \hookrightarrow N$ be the subobject of indices satisfying $\forall n \, \exists t \, H(e, n, t, \bar{g}(n))$. For each external e, choose an n at which φ_e is undefined or differs from g. With these two numerals substituted and with the output $g(n)$ fixed, the subobject of possible t is initial by the interpretation of H and the disjointness of summands. Its existential image in 1 is initial. Universal elimination shows that $i_e^* P$ is therefore also initial. Universality of coproducts now gives $P \simeq \coprod_e i_e^* P \simeq 0$, so the existential image of P in 1 is initial. The universal assertion (10.1) is contained in this instance and therefore is initial as well. □

This argument accounts for local existence in sheaf semantics. A local index is allowed to vary with the context. The proof reduces that index to numeral summands, and every summand gives an empty predicate for the fixed counterexample $\bar{g}$. In $\mathcal{S}$ the same phenomenon is visible directly since N^N is the discrete space of all set-theoretic

numerical functions. A recursion principle can introduce a map from specified base and step data even when other maps admitted by the semantics have no program index.

10.4 The effective topos as a boundary example

Take the ordinary effective topos Eff based on Kleene's first partial combinatory algebra. Its full subcategory of assemblies has objects (X, R_X) with a nonempty set $R_X(x) \subseteq \mathbb{N}$ of realizers for each x. A morphism $f : (X, R_X) \to (Y, R_Y)$ is a function equipped, in the sense of existence, with a single partial recursive tracker φ_e. The condition is that for every x and every $r \in R_X(x)$, the computation $\varphi_e(r)$ is defined and belongs to $R_Y(f(x))$. Different trackers for a fixed underlying function give the same morphism. The natural numbers object N_{Eff} is the assembly with $R(n) = \{n\}$ [12, Ch. 3].

Fullness of the assembly inclusion reduces every endomorphism of N_{Eff} to a tracked function. The tracker condition for this assembly is $\varphi_e(n) = f(n)$ on every input. Conversely, any total recursive numerical function supplies its own tracker. Thus the external endomorphisms of N_{Eff} are exactly the total recursive functions. For the internal assertion with quantification over the exponential, Hyland separately proves Church's thesis in Eff [11, Prop. 10.2]. The argument via tracked arrows identifies the external endomorphism set, while the internal theorem governs generalized elements and their witnesses.

There is a direct obstruction to applying Proposition 10.1 to Eff as an ordinary category. If the countable coproduct $\coprod_{n\in\mathbb{N}} 1$ existed, its shift would make it an ordinary natural numbers object by the same recursion proof using sets of arrows. The resulting isomorphism to N_{Eff} would preserve all numeral injections. Choosing their images in N_{Eff} independently would define an endomorphism for every external function $\mathbb{N} \to \mathbb{N}$, contrary to tracking. This recovers the countable-coproduct obstruction of van Oosten [12, Prop. 3.2.3]. In particular Eff cannot be the full subcategory of 0-truncated objects in any Grothendieck ∞-topos, since the countable coproduct of terminal objects constructed above is already 0-truncated.

The argument uses the Grothendieck hypothesis through external coproducts. Rasekh constructs natural numbers objects for a specified elementary higher topos framework by a different argument that avoids assuming these coproducts [13]. A higher analogue of realizability intended to validate absolute Church's thesis must specify a framework outside the hypotheses of Proposition 10.4, or change the numerical predicate or quantifier in the thesis. The comparison in Section 11 addresses the additional choices in passing from ordinary realized predicates to higher families.

11 Higher realizability and the scope of Church's thesis

The ordinary effective topos and the Grothendieck semantics of Section 10 admit different functions on natural numbers. This gives a criterion for comparing a proposed higher realizability model with its ordinary fragment. Suppose the model has set-valued natural numbers and the indexed operations of Theorem 2.2 restrict to its 0-truncated types. Interpreting arithmetic via the same natural numbers object and numerical predicates, the truth of (10.1) is computed by those restricted operations. Each numerical type and context in that sentence is a set, and every existential is an image. Hence the arithmetic sentence has the same truth value in the higher model and in its ordinary fragment. The argument also applies to an elementary higher model with the same restriction, without assuming that model is a Grothendieck ∞-topos.

The distinctions among existing constructions determine which such transfer statements apply. Uemura constructs a cubical-assembly model with a univalent impredicative universe and shows a failure of a form of propositional resizing [14]. Swan and Uemura establish that the ordinary realizability interpretation of Church's thesis and the homotopical interpretation in cubical assemblies can disagree. They construct a left exact modality whose reflective subuniverse validates the homotopical thesis [15]. The existence of the modality and the calculation of the ordinary fragment are independent questions about such a model. The transfer applies when an equivalence of ordinary fragments identifies the arithmetic data.

Awodey and Emmenegger give a candidate effective 2-topos with 0-types equivalent to the effective topos [16]. The comparison with cubical assemblies in their work exhibits a concrete reason to pay attention to exact completion: forming an ordinary exact completion and then introducing a cubical direction can change which ordinary types are recovered after passage to homotopy. Their recovery theorem at the 2-topos level concerns the ordinary fragment and gives a finite-dimensional comparison for the extension to all levels. In particular, a claim about a higher effective model should specify its underlying category, its universe assumptions, and the identification of its 0-types before invoking ordinary realizability theorems in that model.

One can see the arithmetic preservation requirement independently of the geometry of the model. Suppose $f, g : X \to N^N$ are two generalized sections and that a proposition over X asserts the existence of an index computing both. On the support of that proposition the two composites $X \times N \to N$ agree. Pull back along the effective epimorphism supplying the shared index. Over the product of its domain with N, take the fiber product of the two covers supplying halting runs for that input. On this last cover determinism of the machine forces equality of outputs. Equality descends through both covers because it is interpreted by a mono. This argument uses the run

existentials separately in the context of an input, so it requires no choice of a run simultaneously for all inputs. By the exponential adjunction the two original generalized sections agree on that support. The resulting identity is an extensional property of program behavior. An assignment of programs implementing f can carry additional data, such as distinct codes for a single function, which this identity and the truth of the corresponding instance of (10.1) discard. Determining the space of implementations requires the ambient higher model.

12 Scope of the comparison

Theorem 2.2 identifies the set-valued language of an ordinary topos within the higher language while allowing the latter to form identities and dependent products of higher types. The ambient category controls the resulting families: an identity can carry a group of loops, an existential can be true without a global term, and an ordinary context can support a product whose components depend on higher gluing. Each assertion has an example above with the context, family, and operation specified.

There are two distinct extensions of the classical term-model program here. The classifying topoi in Theorem 7.2 give explicit universal models for geometric fragments. Classifying stronger dependent theories with judgmental computation rules requires specifying their syntactic presentations and the morphisms that preserve their operations. Likewise, an effective version of the higher theory needs a framework that keeps the intended computational structure. The Grothendieck assumptions yield the comparison and descent theorems of this article, and Proposition 10.4 gives a precise limit on their use for absolute Church's thesis.

Use of automation. AI tools assisted in drafting, organizing, and checking this manuscript. Arguments are presented as mathematical proofs in the text. No proof assistant verification is claimed. The cited sources and the LaTeX source allow the constructions and calculations to be reviewed directly.

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