Higher Categorical Recursion Theory

Effective trees, admissible computation, and homotopy-coherent witnesses

---

Manuscript prepared for *Higher Categorical Semantics*

November 2026

Preface

A successful computation gives a value and a record. In classical recursion theory the record often fits a finite piece of the oracle, and the value determines the observable behavior. In higher recursion theory the record may use an admissible ordinal bound or a well-founded family of subcomputations. There is then a question of whether the witnesses can be collected at one stage. In a category of spaces there is the further question of how the witnesses are identified as their inputs vary. A family can have a witness in each fiber and have no coherent choice of witnesses.

This book develops the classical material of Sacks's *Higher Recursion Theory* [11] in that setting. From the first chapter we use a slice over the object of inputs. Successful derivations live in the slice. The image of the derivation object is the predicate of convergence, and its relative truncations exhibit how much of the equivalence data the observations retain. We retain explicit recursive or admissible presentations where an effectiveness claim depends on them. The difference between internal existence and a search in one of these presentations recurs in the proofs.

Part A treats effective trees, hyperarithmetic hierarchies, and positive induction. It develops the comparison with indexed polynomial algebras and gives a forcing interpretation of local witnesses. Part B treats metarecursion, admissibility, priority, and degrees. Collections of successful records, bounded supports, and stabilization of proper initial segments have separate roles in these arguments. Part C treats Normann's set recursion, effective selection and bounding, and descent of computation records. It ends with an explicit equivariant lift of the jump and a criterion for effective selection in group families.

We assume elementary recursion theory, set theory with choice in the metatheory, and some familiarity with categories and homotopy types. The exercises vary the sites, recursive presentations, and witness transport. Worked solutions appear together at the end of the book.

Preparation. Generative AI assisted with drafting, mathematical exposition, worked solutions, and typesetting. The original classical theorems are cited where used.

Contents

Preface **ii**

A Effective trees and inductive definitions **1**

I Families of computations **2**
I.1 Presentations and families 2
I.2 Change of parameters 3
I.3 An effective reading of the same diagram 5
I.4 Exercises 6

II Recursive trees and their path spaces **7**
II.1 From arithmetical tests to trees 7
II.2 Path witnesses over a variable base 8
II.3 Accessibility as a dependent polynomial 9
II.4 Effective rank bounds 11
II.5 Basis elements and uniformization 11
II.6 Exercises 12

III Jump hierarchies and oracle families **14**
III.1 Construction on an ordinal notation 14
III.2 The relative halting predicate 15
III.3 A hierarchy over locally constant inputs 17
III.4 Exercises 18

IV Ramification and inductive definitions **19**
IV.1 Ranks of names 19
IV.2 Rules with dependent premises 20
IV.3 Locally inhabited premises 22
IV.4 Exercises 24

V Forcing and sheaves of witnesses **25**
V.1 Three orders on information 25

V.2 What a Boolean truth value retains 26
V.3 A bound from the chain condition 28
V.4 Fusion and effective separation 29
V.5 A bound on the complexity of a separator 30
V.6 Exercises 31

B Admissibility, approximation, and degrees 32

VI Admissible bounds and witness objects 33
VI.1 Set-sized domains of search 33
VI.2 A certificate for coanalytic predicates 34
VI.3 Bounded sets as a regular category 35
VI.4 Bounds in an infinity-topos 36
VI.5 Exercises 38

VII Relative computation and support 39
VII.1 Whole queries 39
VII.2 Regularity, hyperregularity, and compression 40
VII.3 Amalgamation over a parameter space 42
VII.4 Exercises 43

VIII Priority and coherent stabilization 44
VIII.1 Restraints and blocks 44
VIII.2 Stabilization in a slice 45
VIII.3 Observing infinitely many windows 47
VIII.4 Exercises 48

IX Supported reductions and degree structure 49
IX.1 Composition with bounded queries 49
IX.2 A split and the information it preserves 51
IX.3 Bounded records over weaker levels 52
IX.4 Exercises 53

C Set recursion and higher descent 54

X Set recursion and spaces of records 55
X.1 Success as an inductive relation 55
X.2 Indexed signatures and contexts 57
X.3 Divergence certificates 59
X.4 Exercises 59

XI Selection and bounded forcing 61

XI.1 Finding a successful instruction 61
XI.2 Local existence and simultaneous choice 62
XI.3 Extracting a bound from local records 62
XI.4 The effective forcing problem 64
XI.5 Sections and envelopes 65
XI.6 Exercises 65

XII Descent of computation records **67**
XII.1 Algebras over a cover 67
XII.2 Descent through a bounded stage 68
XII.3 An obstruction invisible to components 69
XII.4 Successive truncations 70
XII.5 Exercises 70

XIII Effective oracles with transport **72**
XIII.1 A jump in a family of symmetries 72
XIII.2 Selection in an action groupoid 74
XIII.3 Ranks and relative closure 75
XIII.4 Exercises 76

Worked solutions **77**
I. Families of computations 77
II. Recursive trees and their path spaces 78
III. Jump hierarchies and oracle families 82
IV. Ramification and inductive definitions 84
V. Forcing and sheaves of witnesses 87
VI. Admissible bounds and witness objects 91
VII. Relative computation and support 94
VIII. Priority and coherent stabilization 96
IX. Supported reductions and degree structure 99
X. Set recursion and spaces of records 102
XI. Selection and bounded forcing 105
XII. Descent of computation records 107
XIII. Effective oracles with transport 109

Bibliography **111**

Index **113**

Part A

Effective trees and inductive definitions

Chapter I

Families of computations

I.1 Presentations and families

A partial function on natural numbers can be given by a program, by its graph, or by a family of verified records of its executions. If e is an index and x, y are numbers, a predicate $V(e, x, y, w)$ may state that w is a successful execution of e on x with output y. The projection of the verified tuples onto (e, x, y) gives the graph. The projection onto (e, x) gives the domain of convergence. The projections keep less information than the original relation.

The same description applies when a set of possible subcalls is part of the input. Then a record must justify every required subcall. Even for a deterministic computation, the record may contain choices of a presentation, of a bound, or of an indexing of intermediate data. A family of such records can be studied by retaining its local identifications. We use objects of an ∞-topos to represent these families, and we keep the data certifying effectivity in the chosen recursive or admissible presentation.

An ∞-topos $\mathcal{X}$ is taken in the sense of [7]. Write $\mathcal{X}_{/U}$ for its slice at U. An object there is a map to U, thought of as a family over parameters U. For $v : V \to U$, substitution is the pullback functor

$$\Sigma_v \dashv v^* \dashv \Pi_v.$$

Here Σ_v composes with v, and Π_v takes dependent products. These formulas include the change of the indexing object. For example, over a test map $b : T \to U$, a section of $\Pi_v W$ is equivalent to a section of the pulled-back W over $T \times_U V$. In sets this is a tuple of choices indexed by the elements of each fiber. In spaces the sections must transport coherently over paths of the test object.

Definition I.1. A family of execution records with input object U and output family $Y \to U$ consists of a map $W \to Y$ over U. Its behavior is the subobject

$$\mathrm{im}(W \to Y) \hookrightarrow Y.$$

Its domain is $\mathrm{im}(W \to U) \hookrightarrow U$.

By image we mean the (-1)-truncated part of the effective epimorphism and monomorphism factorization. The record, behavior, and domain are displayed in Figure I.1. In a slice there are also intermediate relative truncations $\tau^{/U}_{\leq n} W$. For $n = 0$ the paths within a fiber are collapsed to equivalence classes. For $n = -1$ all inhabitants of a fiber give the same truth value. The distinctions are fiberwise. Taking connected components of the total space can also identify records from different inputs when those inputs lie in a connected base.

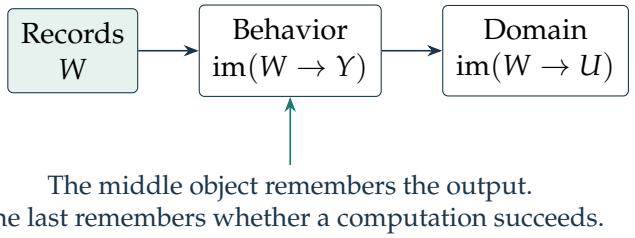


Figure I.1. Two projections of a family of successful computations. All three objects have maps to the input object U.

Proposition I.2 (Ordinary partial maps in a slice)**.** *Suppose the behavior $R \hookrightarrow Y$ has monomorphic projection $R \to U$. It defines a partial section of Y on its domain. If the output family is 0-truncated, a partial section $D \hookrightarrow U$, $D \to Y$ over U, gives such a behavior, with domain D.*

Proof. Factor $R \to U$ through its image D. The map $R \to D$ is both an effective epimorphism and a monomorphism, so it is an equivalence. Thus $R \to Y$ is a section over D. For the converse, the homotopy fiber of $D \to Y$ at an output is a fiber of a map from a subterminal input fiber to a discrete output fiber. Such a fiber is subterminal. Hence $D \to Y$ is itself a mono, and its image has the specified domain. □

Remark I.3. The monomorphism condition on $R \to U$ is natural for discrete outputs. With higher outputs, a chosen section need not be a mono into Y. The graph is then kept as the span $U \leftarrow D \to Y$, or as an object of the slice over Y, before taking any image. For instance $1 \to BG$ classifies an output with automorphisms. For nontrivial G its image is all of BG, whose projection to 1 is not a monomorphism.

I.2 Change of parameters

The image of a map in an ∞-topos pulls back to the image of its pullback. This follows from stability of effective epimorphisms and monomorphisms. For

constructive logic this amounts to the rule that an existential assertion may be pulled back with its witness problem. In the following pullback square p can be a projection from an index family, and q is its change of base.

$$\begin{array}{ccc} E' & \xrightarrow{u} & E \\ {\scriptstyle q}\downarrow & & \downarrow{\scriptstyle p} \\ U' & \xrightarrow[v]{} & U. \end{array}$$

On objects the corresponding comparisons are

$$v^*\Sigma_p \simeq \Sigma_q u^*, \qquad v^*\Pi_p \simeq \Pi_q u^*.$$

The first identifies two ways of writing the same composite pullback. For the second, test the left side and the right side against $T \to U'$. Moving the pullbacks and the dependent product across their adjunctions identifies the two spaces of maps with maps over E from $E \times_U T$. This determines the required equivalence naturally.

Proposition I.4 (Composition and existential behavior)**.** *Let W be a family over $A \times B$ and Z a family over $B \times C$. Form the record of their successive use as $W \times_B Z$ over $A \times C$. Its image is*

$$\exists b \, \big(\mathrm{im}(W)(a,b) \wedge \mathrm{im}(Z)(b,c)\big)$$

in the internal subobject logic. This comparison is compatible with pullback in an additional parameter.

Proof. Write $R = \mathrm{im}(W \to A \times B)$ and $S = \mathrm{im}(Z \to B \times C)$. The arrow $W \times_B Z \to R \times_B S$ is a composite of pullbacks of the two effective epimorphisms onto R, S. It is an effective epimorphism. Thus the two maps to $A \times C$ have the same image. The image of $R \times_B S$ is precisely the displayed predicate. Each construction used here commutes with a change of the common parameter by the pullback formulas. □

For an arbitrary index family $D \to U$, local inhabitation of each W_d does not imply local inhabitation of the whole product over D. The proof of Proposition I.4 uses only a finite matching of witnesses. The finite locally constant case of dependent products is treated in Chapter XI. An argument with larger arities must justify the construction of the family of witnesses. In admissible recursion this is where a collection bound can enter. In a sheaf model this is where covers with progressively smaller neighbourhoods can prevent simultaneous choice.

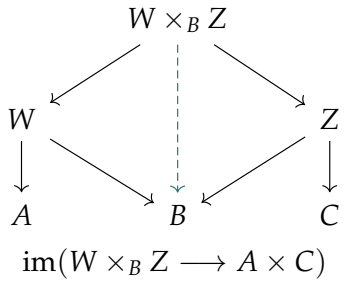


Figure I.2. A composite record contains two records with the same intermediate value. Projection discards that value, and the image asserts the existence of such a pair.

I.3 An effective reading of the same diagram

For a fixed recursive program with oracle $X \subseteq \omega$, a finite successful run on n certifies its output under every oracle agreeing on the finitely many queries in that run. With $[s]$ denoting the cylinder determined by a finite initial oracle segment s, we have

$$\{X : \Phi_e^X(n) \downarrow = m\} = \bigcup \{[s] : \text{the run for } e, n \text{ with } s \text{ returns } m\}.$$

In the right side, a run is suspended if its next query leaves the initial segment. Every successful run over a total oracle is captured by a long enough initial segment. Thus the union is uniformly effectively open. The diagrams of this chapter describe its existential projection. The recursive enumeration of the cylinders proves the effective openness. One needs this enumeration to recover the classical theorem from the semantic calculation.

Another treatment uses assemblies based on a partial combinatory algebra. A map of assemblies has a tracking index acting on the realizers. When the underlying partial application is Kleene application, the tracker gives an ordinary computable operation. Quantification and exact completion supply the effective topos of Hyland [6]. Changing the partial combinatory algebra to represent an additional partial function has a separate universal property [10]. These constructions give places to put the effectiveness data. An univalent family in an ∞-topos classifies a suitable class of maps and records equivalences between fibers [21]. The determination of which presentations are effectively comparable is a further condition on the class and its morphisms.

Example I.5 (A deterministic value with moving records)**.** Let G act on a nonempty set D, and consider the family $EG \times_G D \to BG$ in spaces. Give every record in D the numerical answer zero. The behavior is the constant answer zero, since each fiber is nonempty. Pullback along a loop g of BG transports a record by the permutation g. If D has no fixed point, the record family has no section. This follows because sections correspond to homotopy fixed

points of D, which for a discrete D are ordinary fixed points. The computation is deterministic at the level of answers, while its records carry nontrivial transport.

The subsequent constructions distinguish three uses of an index. A program is indexed by a natural number. A rank bounds when the record of a branched computation appears. A parameter may itself have paths and acts on its record fibers by transport. A numerical index makes a uniform effective assertion possible. A common rank makes a collection argument possible. The parameter pullback makes a comparison of local witnesses possible. None of these constructions requires identifying an ordinal rank with a homotopy dimension.

I.4 Exercises [Worked solutions]

Exercise I.1. Give a recursive family of finite records for the composition of two partial recursive numerical functions. Show by projection that its domain has precisely the elements on which the composite is defined.

Exercise I.2. Let $W \to U$ and $Z \to U$ be families. Prove that the image of $W \times_U Z \to U$ is the intersection of their images, and identify the use of effective epimorphisms.

Exercise I.3. In spaces find a map $Y \to U$ with Y and U both connected for which the relative 0-truncation of Y over U has a fiber with more than one component.

Exercise I.4. Let G act transitively on a two-element set. Describe the relative image and relative 0-truncation of its family over BG, and decide whether the family admits a section.

Exercise I.5. Write down an explicit representation of a finite oracle trace as a cylinder in Cantor space. Determine the condition under which two such cylinders intersect, including when they have different lengths.

Exercise I.6. Use Proposition I.4 to show that three relations compose associatively in the subobject logic. Describe the two parenthesizations before taking the image.

Chapter II

Recursive trees and their path spaces

II.1 From arithmetical tests to trees

A tree on ω is a set of finite strings closed under initial segments. A path is a function $x : \omega \to \omega$ all of whose finite restrictions are in the tree. Use a fixed recursive coding of finite strings. A family T_m is recursive when one program decides whether a string belongs to T_m, with m as an additional numerical input. Write $[T_m]$ for the set of paths. The usual analytical classes below are lightface unless a real parameter is displayed.

Theorem II.1 (Tree normal form)**.** *A predicate $A(m)$ is Σ^1_1 iff there is a recursive family of trees T_m with $A(m)$ equivalent to $[T_m] \neq \varnothing$.*

Proof. Write $A(m)$ as $\exists x\, \theta(m, x)$ with θ arithmetical. Put the number quantifiers in prenex form. A function of the earlier universally quantified numbers can serve as a witness for each existential number quantifier. Combine the finitely many new functions and x into one real z. The result is $\exists z\, \forall k\, R(m, z, k)$, where the test of R halts after finitely many questions to z. Put a string s in T_m when none of the first $|s|$ tests has rejected after $|s|$ steps with partial oracle s. Suspend any test that asks for a value outside s. A rejection persists under extension. A failed total test is eventually seen along the corresponding sequence of prefixes. Thus paths are precisely the witnesses passing all tests. Conversely the condition $x \upharpoonright n \in T_m$ is recursive in m and the prefix, so the existence of a path has the required formula. □

For a real input y, the test can consult finite prefixes of y and of the witness. This gives a recursive tree of pairs (s, t) of equal length. The set of pairs of reals passing the test is effectively closed in Baire space squared, and the desired predicate is its projection on the first coordinate. See [11, Chs. I–II] and [8].

Figure II.1 shows why a test for arbitrarily long finite instances cannot replace the path quantifier on Baire space.

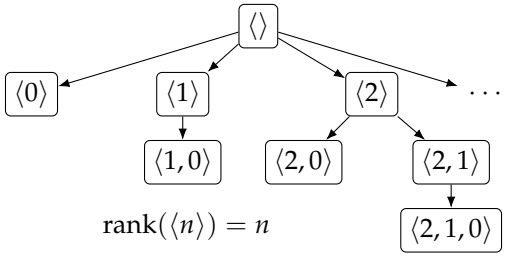


Figure II.1. The tree of strictly decreasing strings. Its root has rank ω, but each choice of first coordinate bounds the length of the remaining string.

II.2 Path witnesses over a variable base

In an ∞-topos $\mathcal{X}$, fix U and a tower

$$T_0 \longleftarrow T_1 \longleftarrow T_2 \longleftarrow \cdots$$

in $\mathcal{X}_{/U}$. A tree coded by subobjects of the finite strings has such a tower. General towers also allow homotopy types as the levels. Set $W(T) = \lim_n T_n$ in the slice. A section $V \to W(T)$ over U is a coherent path parametrized by $V \to U$. Thus the tower expresses compatibility, including the coherence of the restrictions when the levels are higher types. Let $\mathrm{Path}(T) = \tau^U_{\leq -1} W(T)$, regarded as a subobject of U. The expression $\tau^U_{\leq -1}$ means relative propositional truncation, or equivalently the image factor of $W(T) \to U$.

Proposition II.2 (Reindexing paths)**.** *For $f : V \to U$ there are canonical equivalences*

$$f^* W(T) \simeq W(f^* T), \qquad f^* \mathrm{Path}(T) \simeq \mathrm{Path}(f^* T).$$

In spaces, if U and all levels of T are discrete, these constructions give the ordinary family of path sets and its support.

Proof. Pullback in an ∞-topos has a left adjoint, so it preserves the limit. Relative (-1)-truncation is preserved by pullback. One can also check the second claim using the image factorization as an effective epimorphism followed by a monomorphism, both stable under pullback. In spaces, limits of discrete spaces are discrete and the (-1)-truncation of a discrete space is nonempty iff the underlying set is nonempty. Apply this calculation at every point. □

In the case of a subobject $R \hookrightarrow U \times B$, this is the base-change formula for the projection predicate $\exists_{U \times B \to U} R$, illustrated in Figure II.2. For Theorem II.1 the subobject R is the path relation and $B = \omega^\omega$, taken discretely in spaces. The object R can be retained after the existential projection has been computed.

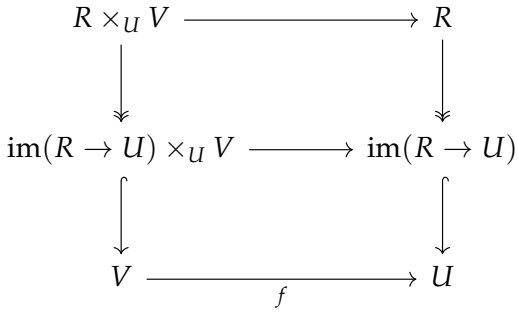


Figure II.2. The two squares are pullbacks. Reindexing carries both the witness object and the proposition that witnesses exist.

Example II.3 (Path transport around a circle)**.** Take $\mathcal{X} = \mathcal{S}$, the ∞-category of spaces, and let $U = S^1$. Put $T_0 = U$ and $T_n = \widetilde{U}$ for $n \geq 1$, where $\widetilde{U} \to U$ is the two-sheeted covering of the circle. Let the first parent map be the covering and every subsequent map be the identity. The fiber tree at any point consists of a root and two rays. Its path object over U is the covering itself. Its image is all of U, whereas a section of the covering would split the homomorphism $\mathbb{Z} \xrightarrow{\times 2} \mathbb{Z}$ induced on fundamental groups. A circuit around the base exchanges the paths.

The same tower construction with the Hopf fibration $S^3 \to S^2$ in place of the two-sheeted covering has connected path fibers of type S^1. Its image is all of S^2, but it has no section because $\pi_2(S^3) = 0$ and $\pi_2(S^2) = \mathbb{Z}$. Here a pointwise path choice fails to extend coherently even though all the pointwise path spaces are connected.

The ordinary normal form singles out recursive codes in addition to forming a limit and a projection. An arbitrary internal tower need not have such a code. A realizability interpretation supplies its own effective presentations, for example via the Kleene application operation in assemblies. Tracking an arrow there is extra data even when its underlying set map has already been specified [6]. The comparison in Proposition II.2 concerns the specified discrete trees and their truth values.

II.3 Accessibility as a dependent polynomial

Let $S \to U$ be an object of vertices and let D be an object of edges. Write $d : D \to S$ for the child and $p : D \to S$ for the parent. On $\mathcal{X}_{/S}$ use the functor $P = \Pi_p d^*$, with Π_p the dependent product. In internal language $P(X)_s$ is the type of assignments of an X-witness to every child of s. This is a dependent polynomial with one constructor for each parent.

Theorem II.4 (Accessibility object)**.** *If P preserves λ-filtered colimits for a regular infinite cardinal λ, its initial algebra is the family obtained from the empty family by λ stages of P, taking colimits at limit stages. The family is subterminal over S and is preserved by change of base in U.*

Proof. The chain of length λ is λ-filtered, and its successor stages are cofinal. Applying P to its colimit therefore gives the same colimit. This is the initial algebra. Indeed, for any algebra, the compatible maps from the chain into its carrier are forced at the empty stage and at successors by the algebra operation, and at limits by the colimit universal property. The chain construction with this compatibility gives a contractible space of algebra maps. The empty family and P preserve the subterminal condition, and filtered colimits preserve it by exactness. Finally, pullback preserves the colimits in the chain and commutes with dependent products by Beck–Chevalley. The pulled-back chain stabilizes at the same stage and gives the initial algebra there. □

In a presentable ∞-topos one can choose a sufficiently large λ, using accessibility of the adjoints between the presentable slices [7]. The formula is the accessibility instance of the construction of dependent well-founded trees. Compare [3]. For sets it says that a vertex is accessible once every child is accessible. With dependent choice, a nonaccessible vertex has an infinite path, since a nonaccessible child can be chosen repeatedly. The converse holds by induction on an accessibility derivation, by passing to the tail of any proposed path. This converse works internally as well.

Example II.5 (No internal path, but no accessibility at the origin)**.** In the topos of sheaves on $\mathbb{R}$, take a global root and let level $n \geq 1$ be the open $U_n = (-1/n, 1/n)$, with inclusions as the parent maps. Regard the same objects as discrete sheaves of spaces. The limit of this tower of subterminal objects is the interior of their intersection, hence empty. At every stage of the accessibility construction, the subobject of accessible vertices in each level excludes the origin. For successors, every neighbourhood of zero at that level contains zero in the next level, where accessibility has failed at the preceding stage. Limits are unions of open subobjects. Thus the root has no accessibility witness near zero even though the path predicate is everywhere false. Away from zero one can use a neighbourhood disjoint from some U_n and prove accessibility by finitely many predecessor steps. The stalk at zero provides a further calculation. Each level has a one-point stalk there, whereas the path object has empty stalk. The stalk functor fails to preserve this countable limit. Reindexing in Proposition II.2 is pullback in a fixed topos and preserves all limits in the slices.

II.4 Effective rank bounds

Adjoin a root to an empty tree whenever necessary. A well-founded tree in sets has a rank defined by the supremum of $\operatorname{rank}(t) + 1$ over its immediate children t. A recursive well-founded tree has rank below ω_1^{CK}. Its Kleene–Brouwer order, with extensions first and larger distinct coordinates later, is a recursive well-order. An infinite descending sequence in that order would yield a path by eventually stabilizing the coordinates one at a time. An order type of this order bounds the tree rank.

Theorem II.6 (A bound for a family)**.** *Let $D \subseteq \omega$ be Σ^1_1, and let T_m be uniformly recursive and well founded for $m \in D$. One recursive ordinal strictly bounds their root ranks for $m \in D$.*

Proof. Choose witness trees S_m for D. Form a recursive tree whose first coordinate chooses m and whose subsequent coordinates give strings of equal length in S_m and T_m. This tree is well founded. A path would witness both $m \in D$ and a path through T_m. If $m \in D$, pairing each string of T_m with the corresponding prefix of one path through S_m embeds T_m above the first coordinate m. The root rank of the combined tree strictly bounds the rank in question. □

For separation, let $A, B \subseteq \omega$ be disjoint Σ^1_1 sets, and choose witness trees T_m for B. By Theorem II.6 choose a recursive well-order of length $\beta + 1$ with β greater than the ranks of the T_m for $m \in A$. At a successor stage prune the vertices with no child in the preceding tree. At a limit take the intersection. A real coding the entire pruning hierarchy of one T_m is given by an arithmetical condition in the recursive well-order. Such a code exists and is unique by transfinite recursion. Thus both assertion and denial of root removal at stage β have a Σ^1_1 description, using the same hierarchy. The root has been removed for $m \in A$, and it survives for $m \in B$, as the vertices of a path survive every stage. Removal at β gives a Δ^1_1 separator. This proof separates the effective contribution, the recursive code for a common bound, from the semantic contribution, transfinite pruning on the indicated presentation.

II.5 Basis elements and uniformization

Gandy's basis theorem gives an element x of any nonempty lightface Σ^1_1 class in ω^ω with $\omega_1^x = \omega_1^{\mathrm{CK}}$. Its relative form uses the join of x with the oracle in the presentation. This conclusion gives an ordinal bound on recursive well-orders in that parameter. The selected point need not be hyperarithmetic. A recursive tree can have a path and contain no hyperarithmetic path. The source of the basis theorem is a refinement of Σ^1_1 conditions controlling each proposed relative well-order, using rank boundedness to control the successful

comparisons [11, Ch. III]. The point satisfying the simultaneous constraints comes from an additional genericity argument on the analytic conditions.

There is a simpler argument in the special case of a unique point. If A is a lightface Σ^1_1 singleton $\{a\}$, the two predicates

$$\exists x\,(x \in A \ \wedge\ x(n) = k), \qquad \exists x\,(x \in A \ \wedge\ x(n) \neq k)$$

are both Σ^1_1 and are disjoint complements. Thus the graph of a is Δ^1_1. Carrying out the argument uniformly for a recursively parameterized family of such singletons gives a single Δ^1_1 graph. A lightface Π^1_1 singleton gives instead a Δ^1_2 graph by the same test with one more existential real quantifier. The effective coanalytic uniformization theorem supplies Π^1_1 singletons in sections of a lightface Π^1_1 relation on recursive Polish spaces, through a uniformizing subrelation [11, Ch. III][8]. Its selection is a statement about pointclasses on the chosen effective spaces.

In the slice of spaces, uniqueness has a stronger homotopical form. If a witness map has contractible homotopy fibers, it is an equivalence, and its section has a contractible space of choices. If each fiber instead has exactly one component, the relative 0-truncation has this uniqueness. The Hopf example above has this latter property without a section of the witness map. Transport therefore distinguishes two ways of interpreting a unique output. One is uniqueness of the equivalence class of the witness. The other is contractibility of the full witness type. The projection formulas carry either assertion under base change, once its appropriate witness object has been specified.

II.6 Exercises [Worked solutions]

Exercise II.1. Make the timed-prefix test in Theorem II.1 primitive recursive by explicitly coding a bounded machine run. Include both a real parameter and a numerical parameter in the test.

Exercise II.2. Replace the two-sheeted covering in Example II.3 by a finite local system over BG for a discrete group G. Characterize existence of a section of $W(T)$ in terms of the action. Compute the path support when the action set is empty.

Exercise II.3. Take two trees over U and put the fiber product of their levels at every depth. Relate its path object to the two path objects. Compare the path support of the product with the intersection of the two supports.

Exercise II.4. Give the accessibility stages for the tree in Figure II.1. Compare the stage of the root with the stage of a root having one child of rank n for every positive even number n.

Exercise II.5. In a metric space, take a nonempty closed nowhere-dense set K and put $U_n = \{x : d(x, K) < 1/n\}$. Use these levels in Example II.5. Determine the root's path support and accessibility support.

Exercise II.6. Relativize Theorem II.6 to a fixed real z. Identify which tree presentations, ranks, and pruning codes now depend on z.

Exercise II.7. Produce a recursive well-founded tree of root rank $\omega + 1$ such that exactly one child of the root has rank at least ω. Construct a second example of rank $\omega \cdot 2$ in which no child has that rank.

Exercise II.8. Given a uniformly Σ^1_1 relation $R(n, x)$ with at most one x in each fiber, determine the complexity of its domain. Under what hypothesis on that domain does the argument for singletons give a Δ^1_1 graph of the numerical values of its partial selector?

Chapter III

Jump hierarchies and oracle families

III.1 Construction on an ordinal notation

Let $X \subseteq \omega$ be an oracle. The Turing jump of X is $X' = \{e : \varphi_e^X(e) \downarrow\}$, for a fixed effective numbering of the oracle programs. If $X \leq_T Y$, an oracle program for X from Y can be substituted into every query in a computation relative to X. Thus $X' \leq_T Y'$. Also $X \leq_T X'$ by asking whether a program halts after a positive answer to one prescribed query. If $X' \leq_T X$, a program with oracle X would halt on its own index exactly when that index does not belong to X', a contradiction. The jump is monotone on Turing degrees, and the jump of a degree is strictly above that degree.

A recursive ordinal is the order type of a recursive well-order on a recursive subset of ω. The recursive ordinals form an initial segment with supremum ω_1^{CK}. The notations $\mathcal{O}$ give presentations of these ordinals with instructions for effective transfinite recursion [11, Ch. I]. Use the convention in which 1 denotes zero, 2^b the successor of $|b|$, and $3 \cdot 5^e$ the limit of an effectively given sequence of notations b_n increasing in the notation order. A valid limit code includes a program computing those notations. It does not include an algorithm recognizing all valid codes.

For $a \in \mathcal{O}$ and initial oracle X, set

$$H_1^X = X, \qquad H_{2^b}^X = (H_b^X)', \qquad H_{3\cdot 5^e}^X = \bigoplus_{n\in\omega} H_{b_n}^X.$$

The join uses one fixed recursive pairing of integers, so a bit of a limit oracle is specified by its column and its row. The notation can also be relative to X. In that case $\mathcal{O}^X$ uses X-recursive sequences, and the possible order types are those below the first non-X-recursive ordinal ω_1^X.

For a fixed valid notation, one real can code the columns at its syntactically reachable subnotations. All equations on this real are arithmetical. At a suc-

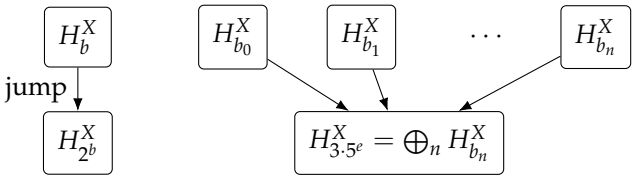


Figure III.1. Two constructors in the jump hierarchy. The successor asks for halting information. The limit preserves all columns and uses the displayed numerical access procedure.

cessor the jump condition uses a finite positive computation for each asserted halting fact and quantifies over stages for a negative one. At a limit the equation is a reindexing of bits. Existence and uniqueness follow by induction on rank among the reachable subnotations. Consequently both bit values of the last column have a $\Sigma^1_1(X)$ definition. The reachable subnotations and equations can be specified arithmetically in the original notation and X, even when their rank cannot be read directly from the notation.

The converse is the Kleene–Spector characterization. Every $\Delta^1_1(X)$ subset of ω is recursive in H^X_a for some $a \in \mathcal{O}^X$ [11, Ch. II][19]. In the separation argument of Chapter II, the common bound is a recursive ordinal relative to the base oracle. Translating the pruning equations into a jump computation along a notation above that bound yields the effective decision procedure. Another theorem of Spector compares H^X_a and H^X_b when the notation ordinals are comparable. In particular $|a| = |b|$ gives equality of Turing degrees. The separate notations still specify separate columns and separate programs for answering queries. Equality of degree alone does not identify these presentations. The hyperjump, represented by $\mathcal{O}^X$ or the well-foundedness test for recursive-in-X trees, gives a complete $\Pi^1_1(X)$ predicate. It tests a proposed valid code by a stronger oracle operation than any single jump in the displayed hierarchy.

III.2 The relative halting predicate

Let $\mathcal{X}$ be an ∞-topos, with natural numbers $\mathbb{N} = \coprod_{n \geq 0} 1$ and $2 = 1 \amalg 1$. Write Ω for its subobject classifier. An oracle over U is an arrow $Q : U \times \mathbb{N} \to 2$. A bounded oracle computation is specified using the ordinary instruction table, finite configurations, and evaluations of Q. The resulting subobject

$$C_Q \hookrightarrow U \times \mathbb{N}_e \times \mathbb{N}_t$$

says that program e halts on input e by stage t. Use $J_Q \hookrightarrow U \times \mathbb{N}_e$ for the image of C_Q under projection. It is classified by an arrow to Ω. Currying also describes this construction as a map $2^{\mathbb{N}} \to \Omega^{\mathbb{N}}$ of oracle and predicate objects. At a fixed

machine input one can retain C_Q as the object of witness times over the halting locus.

Proposition III.1 (Base change and decidability)**.** *For $f : V \to U$, reindexing C_Q and J_Q gives C_{f^*Q} and J_{f^*Q}, respectively. The halting predicate gives an oracle over U precisely when its inclusion is complemented in $U \times \mathbb{N}$.*

Proof. For each numerical e, t, bounded simulation is a finite composition of primitive recursive configuration maps and oracle queries. Reindexing commutes with these operations and with assembling the cases indexed by the two copies of $\mathbb{N}$. This gives the formula for the witness object. Images commute with pullback, giving the formula for J_Q. An arrow to 2 gives the decomposition into its two disjoint pullbacks. Conversely a decomposition into a subobject and its complement gives the map to 2 which is constant on the two summands. These two operations are inverse. □

Example III.2 (An oracle whose jump cannot be read continuously)**.** In sheaves of spaces on Cantor space $K = 2^\omega$, let Q be the generic binary real. Its n-th bit is the continuous function $x \mapsto x(n)$ to 2. A program that queries the bits in order and stops at the first 1 has halting locus

$$U_e = \{x \in K : \exists n\, x(n) = 1\} = K \setminus \{0^\omega\}.$$

A subobject of the terminal sheaf is an open, and complemented subobjects correspond to clopen sets. Each basic neighbourhood of 0^ω meets U_e. Consequently U_e has no open complement and the jump of Q cannot be a binary oracle. Every bounded stage has a clopen halting locus, and their union is precisely U_e.

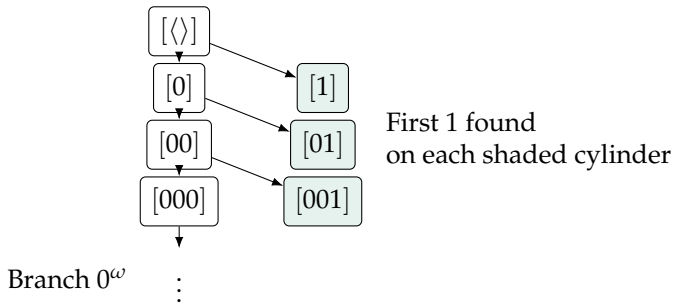


Figure III.2. Halting is the union of the cylinders branching off the all-zero ray. The ray itself is a point of the closure of that open union.

The example shows the difference between closure under truth-valued quantification and closure of a class of total oracles under jumps. At each classical point the oracle and its jump are binary sequences. With the variable base, the same semantic construction gives a predicate that may have no coherent binary decision function. It is already possible for this obstruction to occur in the subobjects of a 1-topos.

III.3 A hierarchy over locally constant inputs

There is a class of observable inputs on which the entire hierarchy can be transported. Denote by $\Delta E = \coprod_{e \in E} 1$ the constant object associated to a set E. Take a fixed ordinary notation $a \in \mathcal{O}$. A locally constant input real here is an arrow $x : U \to \Delta(2^\omega)$, using the set 2^ω of classical reals on the right. Evaluation gives an internal oracle

$$U \times \mathbb{N} \longrightarrow \Delta(2^\omega) \times \Delta\omega \longrightarrow 2.$$

Such a presentation locally specifies the real as a unit. The internal oracle in Example III.2 has each bit locally constant, but the real as a unit has no such presentation.

Theorem III.3 (Locally constant jump families)**.** *For each fixed valid notation a and each such input x, the classical columns reachable from a determine internal oracles over U. These satisfy the successor equations with the halting predicate of Proposition III.1 and the limit equations with the designated column join. They are uniquely determined by these equations and the given base oracle. The hierarchy so constructed commutes with pullback in U.*

Proof. Each external rule $X \mapsto H_b^X$ on the set of reals gives an arrow of constant objects. Compose it with x and evaluate every column to get its oracle. Extensivity and universality of coproducts give a cover of U by the inverse images of the summands indexed by the real X. Over each such piece the initial real is specified. The bounded simulations for each e, t are then either true or false everywhere according to the corresponding classical simulation. Formation of the image over t gives exactly the classical jump, interpreted on that piece. At a limit, each bit is the value read off from the indicated preceding column. These local equations imply the global equations by descent. At a successor the characteristic map of a complemented subobject is determined by its support and at a limit the individual column equations determine all bits. Induction on the fixed notation gives uniqueness. Restriction along a base map preserves the stipulated pieces and the equations, which proves the base-change assertion. □

For a varying notation one can use a map to a constant object of valid pairs (X, a) with $a \in \mathcal{O}^X$. This gives a presentation and a classical validity hypothesis in each summand. It does not give an internal decision procedure for being such a pair. In particular the theorem does not assert that $\Delta(2^\omega) \simeq 2^{\mathbb{N}}$. Infinite products need not be preserved by the constant-object functor.

Example III.4 (Transport of ports)**.** Let $\pi : I \to BG$ be the local system of ports associated to a finite G-set F for a discrete group G. Assign a classical oracle to each orbit, then view this as a locally constant family over I. Apply

Theorem III.3 there. For a test of a bit in one of the resulting columns, take the subobject $I_e \subseteq I$ on which the test succeeds. The dependent sum along π retains the successful ports over BG, and its relative (-1)-truncation says that at least one port succeeds. An action of G which exchanges two successful indices preserves this proposition. It can still obstruct a global choice of a successful index. Base change controls the family of answers and the fibers of the success object even where there is no global enumeration respecting transport.

The classical recursion equations are arithmetical in the hierarchy code and its ordinal presentation. Iteration over arbitrary internal oracles requires a query operation whose results admit the same kinds of successor and limit rules.

III.4 Exercises [Worked solutions]

Exercise III.1. Construct a many-one reduction $X' \leq_m Y'$ from an index witnessing $X \leq_T Y$, uniformly in that index. Analyze the role of diagonal input in the compiler.

Exercise III.2. Give two limit presentations of ω, one using all positive finite ordinals and one using only the positive even ordinals. Construct a Turing reduction of either of the limit joins to the other using the reduction on the finite stages.

Exercise III.3. Replace the search for a first 1 in Example III.2 by a search for two successive 1's. Describe the bounded halting opens, their union, and the interior of its complement.

Exercise III.4. Pull the oracle of Example III.2 back along the point 0^ω and along the open of nonzero sequences. Compare the decidability of the specified jump bit on each restricted base.

Exercise III.5. Take $G = \mathbb{Z}$ and make F the disjoint union of a transitive two-element orbit and a singleton. Give an orbitwise assignment of real oracles and a bit test for which $I_e \to BG$ is surjective and has no section. Give a second test for which there is a section.

Exercise III.6. Use Proposition III.1 to determine which open of Cantor space classifies simultaneous halting of a specified finite collection of searches. Compare the image of the fiber product of their witness-time objects with the intersection of the separate images.

Exercise III.7. An ∞-topos admits internal families of reals whose coordinates are decidable. Formulate a sufficient condition on a chosen subobject of $2^{\mathbb{N}}$ for the successor and limit steps of a fixed notation to stay in that subobject. Explain the difference between the data needed at the two steps.

Chapter IV

Ramification and inductive definitions

IV.1 Ranks of names

Sacks uses ordinal ranks to control which names a quantifier can inspect. Their role is especially clear in the ramified analytic hierarchy [11, III.4]. For a fixed real x, one may start with the number structure and the predicate x, then admit real names defined by formulas using names and set quantifiers of lower rank. A ranked existential quantifier ranges over the names available up to its specified bound. Two names may define the same real. The verification of a bounded truth assertion proceeds by induction on a syntactic rank that decreases when a ranked quantifier is instantiated by an admissible name. This induction leaves open the problem of finding one rank bound for all the instances in a family of assertions.

Here is the semantic form of a ranked existential test. Let I be an object of inputs in an ∞-topos $\mathcal{X}$. Let $N \to I$ be a family of names, let $N_\alpha \hookrightarrow N$ be its increasing filtration by bounds, and let $W \to N$ be the family of witnesses that the named object passes the test. Put

$$W_\alpha = N_\alpha \times_N W, \qquad R_\alpha = \mathrm{im}_I(W_\alpha \longrightarrow I).$$

A point of W_α carries a name and a witness. A point of the subobject R_α asserts local existence of both. In the category of spaces, restricting to discrete names and proof-irrelevant tests recovers the assertion that some bounded name passes the test.

Proposition IV.1. *Suppose $N_\lambda = \bigcup_{\alpha<\lambda} N_\alpha$ at a limit stage. Then*

$$R_\lambda = \bigvee_{\alpha<\lambda} R_\alpha.$$

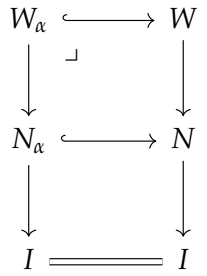


Figure IV.1. The witness type is restricted to bounded names before its image in the input object is taken.

The diagram in Figure IV.1, this equality, and all the individual image tests commute with change of the input base.

Proof. Pullback along $W \to N$ preserves colimits of subobject diagrams, giving $W_\lambda \simeq \operatorname{colim}_{\alpha<\lambda} W_\alpha$. The image functor from $\mathcal{X}_{/I}$ to $\mathrm{Sub}(I)$ is a left adjoint. Applying it carries this colimit to the displayed join. For a new input base, pullback preserves the fiber products and the image factorization, since effective epimorphisms and monomorphisms are stable under pullback. It also preserves the join. □

Equality at a limit identifies which inputs pass some bounded test. A uniform effective bound carries more information. Consider a total oracle routine on a fixed input that asks only finitely many membership questions. If a chain of sets increases to its limit, the answers to the finitely many questions on the actual computation path are correct from some stage onward. Determinism then gives exactly the same output. For functions of a numerical input this is a separate argument at each number. The additional bounding principle in the effective theory controls these individual stages by a recursive ordinal when the evidence is given in the required effective form. An arbitrary external list of recursive ordinal notations can have supremum equal to the Church–Kleene ordinal. A countable cover of bounded tests does not supply the effective evidence in that case.

IV.2 Rules with dependent premises

Fix a diagram

$$I \xleftarrow{s} B \xrightarrow{p} A \xrightarrow{t} I$$

in $\mathcal{X}$. An element of A_i, over $i \in I$, is a rule with conclusion of sort i. For $a \in A_i$ its premise occurrences form the fiber B_a, and $s(b)$ is the sort of premise b. These descriptions are made in generalized elements. They include the variation of

the rule and of its premises along equivalences in the base. Write

$$P(Y) = \Sigma_t \Pi_p s^* Y$$

for the associated polynomial on $\mathcal{X}_{/I}$. We assume this polynomial has an initial algebra $P(D) \simeq D$, for example by the initial-chain construction for accessible polynomials. Its object D gives derivations with their choices of rules and constituent derivations. The internal least predicate of derivability is obtained by a second construction. For $Q \hookrightarrow I$ put

$$\Gamma(Q) = \mathrm{im}_I(P(Q) \longrightarrow I).$$

Then Γ is monotone. Iterate it from the empty subobject, taking joins at limit stages. The stages are increasing by monotonicity, and since $\mathrm{Sub}(I)$ is a set the iteration stabilizes. Write Q_{ind} for the resulting least fixed point. Induction on the stages puts this fixed point below every pre-fixed point.

In sets, $i \in \Gamma(Q)$ means that some rule for i has all its premises in Q. The set-theoretic closure ordinal depends on the presentation of the rules. For example, if all premise sets have cardinality less than an infinite regular cardinal κ, the chain stabilizes by κ. In fact a rule with premises in the union of the first κ stages has those premises in a common earlier stage by regularity. In the recursive setting Spector's result for monotone Π^1_1 inductive definitions gives closure by ω_1^{CK} and a lightface Π^1_1 definition of the fixed point [11, III.8]. The cardinal argument concerns the number of premises. Spector's result concerns the definability of the operator, including uniformity in codes.

Theorem IV.2. *There is a canonical inclusion*

$$Q_{\mathrm{der}} = \mathrm{im}_I(D \to I) \;\leq\; Q_{\mathrm{ind}}.$$

If Π_p preserves the effective epimorphism $s^*D \to s^*Q_{\mathrm{der}}$, this inclusion is an equality. In particular equality holds for a locally finite constant family of premise occurrences over the object A of rules.

Proof. To give $Q \hookrightarrow I$ a P-algebra structure is equivalent to factoring $P(Q) \to I$ through Q. This is possible precisely when $\Gamma(Q) \leq Q$. Initiality therefore maps D over I to any pre-fixed subobject, in particular Q_{ind}. Passing to the image gives the inclusion.

Under the stated hypothesis $P(D) \to P(Q_{\mathrm{der}})$ is an effective epimorphism. Indeed Σ_t composes with t and leaves the map of total objects unchanged. Its two images over I agree. The image of $P(D)$ is Q_{der} by the initial-algebra equivalence. The image of $P(Q_{\mathrm{der}})$ is $\Gamma(Q_{\mathrm{der}})$. Hence Q_{der} is fixed, giving the other inclusion.

For the last assertion, pull back to a cover on which p is a finite coproduct of copies of the base. There Π_p is a finite product. Finite products of effective

epimorphisms in a slice are effective epimorphisms, by stability under pullback and composition. Dependent products commute with this base change, and the effective epimorphism property can be tested on the cover. □

The least predicate is stable under change of a separate parameter object. To see this, treat the signature as a diagram in a slice $\mathcal{X}_{/U}$. Pullback along a map $V \to U$ commutes with the image and dependent-polynomial operations by stability of images and Beck–Chevalley. It commutes with every stage of the predicate chain by induction, using the preservation of joins at limits. Pullback of the stabilized stage is then exactly the stabilized stage for the new base. This stability does not give the extra effective epimorphism in Theorem IV.2. The distinction appears even over a real line.

IV.3 Locally inhabited premises

Example IV.3 (A rule with countably many premises)**.** Work in sheaves of spaces on $\mathbb{R}$, with its usual topology. For each $n \geq 1$ take the subterminal sheaves determined by the opens

$$U_n = (-1/n, 1/n), \qquad V = \mathbb{R} \setminus \{0\}, \qquad E_n = U_n \amalg V.$$

There are countably many premise sorts n and a conclusion sort $*$. The nullary rules for sort n form E_n. The conclusion has one rule with one premise of each sort n. The initial algebra has components $D_n = E_n$ and $D_* = \prod_{n \geq 1} E_n$, since every derivation at $*$ has nullary derivations as children.

Each $E_n \to 1$ is an effective epimorphism because U_n and V cover the line. In the least predicate every premise is therefore true at stage one, and the conclusion is true at stage two. Figure IV.2 depicts the covers. A section of $\prod_n E_n$ near 0 would restrict to a connected interval J containing 0. For each n its projection to E_n makes a locally constant choice of summand on J. This choice is constant. Since $0 \notin V$, it must be the U_n summand. Thus $J \subseteq \bigcap_n U_n = \{0\}$, a contradiction. Away from 0 every premise can be obtained from V, so the image of D_* is exactly V. The conclusion has strictly smaller derivation support than its least predicate.

Example IV.4 (A coherent obstruction to a global derivation)**.** Let $U = BC_2$ in spaces and take the double cover $E \to U$ associated to the free transitive C_2-set. In the slice over U consider one sort, nullary rules parametrized by E, and a unary rule that adds one delay. The associated endofunctor is $Y \mapsto E \amalg Y$, whose initial algebra is $E \times \mathbb{N} \to U$. The algebra follows either from its initial chain or by evaluating one nullary rule followed by a finite number of delays. Its image is all of U. It has no section. A section would give an invariant element of the discrete fiber $C_2 \times \mathbb{N}$, but the nonidentity loop in U interchanges the two

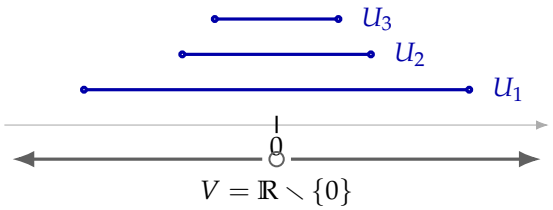


Figure IV.2. Each row supplies a cover by U_n and V. Their product provides no common witness on any interval around zero.

elements with each fixed delay. In this case all arities are finite and the two predicates in Theorem IV.2 agree. The obstruction concerns the global choice of a derivation.

Proposition IV.5 (The arity test in spaces)**.** *In the category of spaces, fix an arity space K. The dependent product $\mathcal{S}_{/K} \to \mathcal{S}$ preserves all effective epimorphisms if and only if K is 0-truncated. Consequently, for a map of spaces $p : B \to A$, preservation of effective epimorphisms by all base changes of Π_p is equivalent to 0-truncation of its homotopy fibers.*

Proof. If K is discrete up to equivalence, the dependent product is a product of spaces indexed by the set $\pi_0 K$. An effective epimorphism is surjective on path components, and a product of such maps is surjective on components by Choice. This uses the identification of the components of a product of spaces with the product of components.

For the converse choose a point in each component and consider the resulting map $e : \pi_0 K \to K$ from a discrete space. The map $e \to \mathrm{id}_K$ in the slice is an effective epimorphism. If its dependent product is an effective epimorphism, then the space of sections of e maps onto the point, so e has a section. A section would exhibit the identity of each connected component of K as homotopic to a constant map, since every map from that component to a discrete space is constant up to homotopy. Each component is then contractible. This proves the first assertion. For the second, necessity follows by base change to a point of A, and sufficiency by testing an effective epimorphism on the homotopy fibers over those points. □

The arity test gives a strict inequality in Theorem IV.2 even in spaces. Take the premise sorts to form the space BC_2 and the nullary rules to form its contractible universal double cover EC_2. Add one rule of a distinct conclusion sort whose premise family is the identity of BC_2. Each premise sort passes the predicate test, so the added conclusion passes it as well. A derivation of that conclusion would be a section of $EC_2 \to BC_2$. This is impossible by the arity test. Ordinary Choice handles arbitrarily large discrete arities in spaces, while the proposed section here must be coherent around loops.

The examples separate failures at different points of an induction. The sheaf example fails at the dependent product before the conclusion is formed. The unary-delay example over BC_2 passes the image test at every stage, but the actual witness type retains its monodromy. A uniform computation over a parameter object must specify which of these two levels of information its witnesses are supposed to provide.

IV.4 Exercises [Worked solutions]

Exercise IV.1. Let $N_\alpha \hookrightarrow N$ be a filtration as in Proposition IV.1. Formulate a hypothesis on an input subobject $K \hookrightarrow I$ under which $K \leq R_\lambda$ implies $K \leq R_\alpha$ for some $\alpha < \lambda$. Test it for compact and noncompact open subobjects in sheaves on a space.

Exercise IV.2. For a monotone operator on a set-sized subobject lattice, verify that the ascending iteration from bottom is below each pre-fixed point. Construct a rule system on $\mathbb{N} \amalg \{*\}$ whose first fixed point occurs at $\omega + 1$, and explain the role of the additional premise arity.

Exercise IV.3. In the category of sets with Choice, establish equality in Theorem IV.2 for arbitrary small premise sets. Identify the choice instance used when the conclusions and rules are indexed by arbitrary sets.

Exercise IV.4. Work in sheaves on a connected space X. Take opens U_n and a point $x \in \bigcap_n U_n$. Set $V = X \smallsetminus \{x\}$ and assume V is open. If X is locally connected, compute the support of $\prod_n (U_n \amalg V) \to 1$ in terms of the interior of $\bigcap_n U_n$.

Exercise IV.5. Let $H \leq G$ be finite groups. Replace the free transitive fiber in the last example by the transitive G-set G/H. Determine precisely which subgroup maps $BK \to BG$ admit a section after base change. Include the effect of conjugating the inclusion of K.

Exercise IV.6. Suppose all rules of a signature have either zero or one premise. Interpret derivations as finite strings of unary rules terminating in a nullary rule. Formulate the appropriate homotopy pullbacks when the objects of sorts and rules are spaces, and show how the finite strings assemble into the initial algebra.

Exercise IV.7. Give two encodings of the same operator on subsets of $\mathbb{N}$ by polynomial signatures whose initial algebras are inequivalent as spaces over $\mathbb{N}$. Fix a sort whose derivation object has different positive-dimensional homotopy in the two encodings.

Chapter V

Forcing and sheaves of witnesses

V.1 Three orders on information

A rank bound makes analytic truth accessible to a stage-by-stage analysis. Chapter IV of Sacks applies this idea to measure, Cohen forcing, and perfect forcing. In each case a smaller condition supplies more information, but the admissible restrictions differ. A Cohen condition is a binary cylinder $[s]$, refined by extending the finite string s. A measure condition is a Borel set of positive measure, ordered by inclusion modulo null sets. A perfect condition is a nonempty perfect closed subspace of Cantor space, refined by taking a perfect subspace. In the effective theory the last conditions carry hyperarithmetic codes. The three choices give different basis and degree theorems. For instance the measure theorem gives a hyperarithmetic member of a positive-measure lightface Π^1_1 set, while the effective perfect forcing theorem gives minimal hyperdegrees [11, IV.2 and IV.5].

For category, a set is meager if it is contained in a countable union of nowhere dense sets. A null set has measure zero for the usual product probability on Cantor space. Even for Borel sets these notions differ. For each m choose, inside every basic cylinder, a smaller cylinder so that the sum of the measures of the chosen cylinders is less than 2^{-m}. Their union O_m is open and dense. The intersection $\bigcap_m O_m$ is dense and null. Its complement is meager and conull. The choice of forcing decides which assertion of existence these sets can witness.

Let P be a nonempty set of conditions, ordered so that $q \leq p$ means q is stronger. The lower sets form a topology on P. Write $B = \mathrm{RO}(P)$ for the complete Boolean algebra of regular opens $U = \operatorname{int} \overline{U}$, and set

$$e(p) = \operatorname{int} \overline{\{q : q \leq p\}}.$$

The image is dense in $B \setminus \{0\}$. Indeed any nonempty regular open contains some p and then contains $e(p)$. Two conditions are compatible precisely when their regular opens have nonempty intersection. For the nontrivial implication,

if r belongs to both regular opens, extend r first below p and then below the second condition. Such refinements exist by the definition of regularization in the lower topology. A downward closed sieve S below p covers in the dense topology precisely when

$$\bigvee_{q \in S} e(q) = e(p).$$

For if S is dense below p, its union of lower opens has the same regularization as the lower open of p. If some refinement $r \leq p$ has no further refinement in S, the nonzero open $e(r)$ is disjoint from the join. This argument also explains how a dense basis of conditions presents the same local truth values as the Boolean algebra.

V.2 What a Boolean truth value retains

Regard a complete Boolean algebra B as a site with covers $b = \bigvee_i b_i$. Let $\mathcal{X}_B = \mathrm{Shv}(B; \mathcal{S})$ be its ∞-topos of sheaves of spaces. Denote by h_b the representable subterminal sheaf for b. Its support is b. In fact a subterminal sheaf is determined by the ideal of objects over which it has a section. The ideal is downward closed and closed under arbitrary joins by descent. It is therefore principal, so $\mathrm{Sub}(1) \cong B$. For a sheaf W let $\|W\| \in B$ denote the support of $W \to 1$. Then

$$\|W\| = \bigvee \{b : W(b) \text{ is nonempty}\}.$$

The formula asserts local existence. The space $W(b)$ may retain the group of automorphisms of a witness and the homotopies between its identifications. Those data do not enter the join.

Lemma V.1. *Every cover of $b \in B$ has a disjoint refinement whose join is b. If $b = \bigvee_i c_i$ is disjoint, evaluation at b satisfies*

$$W(b) \simeq \prod_i W(c_i).$$

Thus $b \leq \|W\|$ implies that $W(b)$ is nonempty.

Proof. Well-order the covering family and replace its i-th member b_i by $c_i = b_i \wedge \neg \bigvee_{j<i} b_j$. Omitting zero terms gives a disjoint family with the same join, by induction on the ordering. In the sheaf topos, $\coprod_i h_{c_i} \to h_b$ is an effective epimorphism by the cover condition. It is a monomorphism because the summands are subterminal and pairwise disjoint, and coproducts are disjoint and universal. It is consequently an equivalence. Mapping this equivalence into W gives the product formula, illustrated in Figure V.1. If $b \leq \|W\|$, intersect a cover by opens carrying sections with b, take a disjoint refinement, and choose the local sections. The product formula amalgamates them. □

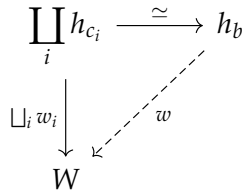


Figure V.1. On a disjoint Boolean partition, a system of local witness maps glues to a witness over the entire condition. The equivalence retains the whole mapping space.

Proposition V.2. *In $\mathcal{X}_B$, all base changes of Π_p preserve effective epimorphisms if and only if p is 0-truncated.*

Proof. First let $Z \to h_b$ be a set-valued sheaf over a representable. Every local section $h_c \to Z$ is a monomorphism. These subterminals cover Z. The lattice $\mathrm{Sub}(Z)$ is Boolean. For a subobject Q, the pullback along each section $z : h_c \to Z$ is $h_d \hookrightarrow h_c$ for some $d \leq c$. Split c into d and $c \wedge \neg d$. On the second part the section has image disjoint from Q, while on the first it factors through Q. The sections jointly cover Z, so Q and its pseudocomplement cover Z. Applying the disjoint-refinement argument of Lemma V.1 in $\mathrm{Sub}(Z)$ writes Z as a coproduct of subterminals.

Suppose $p : C \to A$ is 0-truncated. To test lifting for its dependent product, take a representable $h_b \to A$ and a section of the target dependent product over h_b. By adjunction that section is a map out of $C \times_A h_b$. This arity is a set-valued sheaf over h_b, so it decomposes as above. Each summand lifts through the given effective epimorphism by Lemma V.1 applied to its pullback. Their coproduct is a lift. Representables test effective epimorphisms because they generate sheaves under colimits. The argument applies after arbitrary base change.

Conversely assume preservation after every base change. For $h_b \to A$ take $Z = C \times_A h_b$, and cover Z by a coproduct Y of representables, using its local sections. The dependent product of $Y \to Z$ is then effective epimorphic over h_b, so it has a section over h_b by Lemma V.1. This gives a section of $Y \to Z$. A retract of a 0-truncated object is 0-truncated, so Z is 0-truncated. Testing on the representables covering A yields the claim. □

For example let $b = c \vee d$ be a nontrivial disjoint partition. In $\mathcal{X}_B/h_b$ let a witness be locally an object of the classifying spaces BC_2 over c and BC_3 over d. More precisely take the coproduct of the constant sheaf of value BC_2 in the slice over c and of value BC_3 over d. Its support is b. Under restriction of the witness family to nonzero Boolean pieces of either region, the corresponding finite groups act by automorphisms of the trivial witness. The two actions are distinguishable in the slice. Every nonzero subregion of c still has a nonidentity automorphism of order two on the trivial witness. Over d, a witness is locally isomorphic to the trivial witness with automorphism group C_3. Thus an auto-

morphism of order dividing two is locally the identity and hence the identity by descent. The Boolean truth value discards the variation of the groups.

For truth at a generic, there is an important restriction on the families of joins used. Let M be a transitive ground model with a Boolean algebra which it regards as complete. An M-generic filter G on its nonzero conditions meets every dense set of conditions in M. For a family $(b_i) \in M$,

$$\bigvee_i b_i \in G \quad \Longleftrightarrow \quad b_i \in G \text{ for some } i.$$

Indeed the conditions below some b_i, together with those below $\neg \bigvee_i b_i$, form a dense set in M. The equivalence proves the existential step of the truth lemma for formulas whose ranked name families belong to M. The Boolean and atomic steps then use the ultrafilter property and atomic evaluation. A ground generic filter need not preserve joins from arbitrary families outside the ground model. This truth test is about Boolean values of propositions. It makes no assertion about evaluating higher witness types, or about hyperdescent on arbitrary hypercovers.

V.3 A bound from the chain condition

The following result gives a categorical form of one part of the rank argument. Call B κ-cc if every pairwise disjoint family of nonzero elements has cardinality less than the infinite cardinal κ. Let $(W_\alpha)_{\alpha<\lambda}$ be a chain of sheaves of possible witnesses as the rank bound grows, where λ is a limit ordinal. Its Boolean support values are increasing. This is the situation obtained from a filtered name family as in Proposition IV.1.

Theorem V.3. *Suppose B is κ-cc and $\operatorname{cf}(\lambda) \geq \kappa$. If $\bigvee_{\alpha<\lambda} \|W_\alpha\| \geq b$, there is a single $\beta < \lambda$ for which $W_\beta(b)$ is nonempty.*

Proof. The elements $b \wedge \|W_\alpha\|$ cover b. Refine the cover to a disjoint partition (c_i) by Lemma V.1. Discard zero parts and choose $\alpha_i < \lambda$ such that $c_i \leq \|W_{\alpha_i}\|$. There are fewer than κ such stages. Their supremum is below λ. Take β to be that supremum. By monotonicity every part satisfies $c_i \leq \|W_\beta\|$, so their join b does too. The same lemma supplies the section at stage β. □

The Cohen algebra has a countable dense basis of cylinders. Distinct disjoint nonzero elements must then contain distinct cylinders, so the algebra is $\aleph_1$-cc. In the measure algebra disjoint positive elements with measure at least $1/n$ can occur at most n times, proving the same chain condition. The Church–Kleene ordinal has external cofinality ω. The theorem therefore does not supply the effective bound Sacks obtains for names below that ordinal. For a concrete failure at a countable limit, the cylinders $[0^i 1]$ for $i < \omega$ have join one in the

Cohen algebra. Their union omits just the all-zero sequence. No finite initial portion has join one. The same example works in the measure algebra.

For perfect forcing in its full set-theoretic version, the perfect sets obtained by fixing all even coordinates to x, for $x \in 2^{\mathbb{N}}$, form a disjoint family of conditions. Their corresponding elements of B are disjoint and nonzero. Choose ω_1 of them, denoted d_ξ, and let c be the complement of their join. The elements

$$b_\alpha = c \vee \bigvee_{\xi<\alpha} d_\xi \qquad (\alpha < \omega_1)$$

have join one, and none equals one. Taking $W_\alpha = h_{b_\alpha}$ shows that the stated uniform bound can fail at uncountable cofinality if the chain condition fails. This example concerns full perfect forcing. The countable collection of all hyperarithmetically coded trees is a different forcing presentation.

V.4 Fusion and effective separation

A perfect condition retains several incompatible alternatives. Fusion preserves them at successive depths, which explains why the usual decreasing-sequence construction for Cohen forcing does not suffice. The following version isolates the coding geometry from the required effective choices.

Proposition V.4. *Let $F : P \to 2^{\mathbb{N}}$ be continuous with P a nonempty perfect closed subset of Cantor space. There is a perfect closed $Q \subseteq P$ on which F is constant or a topological embedding.*

Proof. If F is constant on some nonempty relatively clopen subset of P, take that subset. It remains perfect. Otherwise every nonempty relative clopen set has two points with different images. For each node of a full binary tree choose two disjoint nonempty relatively clopen subsets of its current clopen set, of diameters at most 2^{-n} at level n. Choose them near such a pair of points so that their images are contained in distinct cylinders. Continuity supplies these neighborhoods. Figure V.2 shows two levels of the resulting refinement. At a given level the images of any two pieces are in disjoint sets, since this property was imposed where their branches first split. The intersection of the decreasing finite unions of pieces is compact. Along each branch compactness and the diameter bounds give one point, and distinct branches give distinct points. The branch map is continuous by the same bounds, so it is a homeomorphism onto a perfect set Q. Distinct points have different images. The restriction is a continuous injection from a compact space to a Hausdorff space and is therefore an embedding. □

The corresponding lemma in Sacks deals with ranked formulas along hyperarithmetic perfect conditions. In the injective alternative one must also make

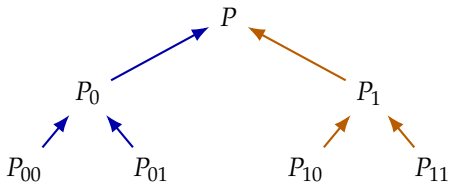


Figure V.2. Fusion keeps both children of each selected piece. Distinct image cylinders at a split remain distinct under every later refinement.

recovery of the branch effective from its output and a fixed auxiliary code. The geometric proof does not control the complexity of the chosen clopen sets. Passing to a Boolean completion discards the rank of a Borel code. For instance a nonempty closed null set has value zero in the measure algebra regardless of how its code was obtained. Forcing can organize proofs of effective separation, but the proof has to work with the coded conditions and their bounds to retain that conclusion.

V.5 A bound on the complexity of a separator

There is a separation result in which the witness to be retained is itself a code. Fix a recursive ordinal $\xi > 0$. If two disjoint lightface analytic sets in Baire space can be separated by a Π^0_ξ set, Louveau's theorem gives a separator in the same Borel class with a hyperarithmetic Borel code [11, Ch. IV, §6]. Taking a set and its complement as the analytic pair gives the corresponding assertion for a hyperarithmetic set of Borel class Π^0_ξ. The theorem keeps the Borel class of the separator, in addition to making the separator effective. For instance a hyperarithmetic G_δ set admits a hyperarithmetic sequence of effective Borel codes for open sets whose intersection is that set, with hyperarithmetic parameters allowed.

The forcing proof uses nonempty analytic sets as conditions and tracks the real witnesses in their tree presentations. The sets used as conditions need not be closed. To pass from a decreasing sequence of such conditions to a point in all of them, the proof must arrange finite approximations to compatible witnesses as well as to the point. The assertion that a point belongs to the intersection is an existential projection of this combined limiting relation. Forming the dependent product of chosen witness presentations before taking its image is the slice-theoretic version of retaining the simultaneous data. Neither the existence of that product nor the stability of its image under pullback proves the effective bound on the Borel code. That bound is the additional conclusion of the classical forcing argument.

V.6 Exercises [Worked solutions]

Exercise V.1. For two conditions p, q in a poset, prove that $e(p) \leq e(q)$ exactly when every extension of p is compatible with q. Use the criterion to describe the quotient that precedes regular-open completion.

Exercise V.2. Let B be complete and atomless. Show that it has no completely prime filter in the full external universe. Explain how a generic filter over a countable ground model can nonetheless preserve every join specified by a family in that model.

Exercise V.3. For the real line with its ordinary topology, exhibit a cover which has no disjoint open refinement. Relate this obstruction to the witness family of Example IV.3.

Exercise V.4. Prove the converse of the chain-condition argument in this form. If B fails κ-cc and κ is regular, construct a chain of subterminal sheaves indexed by κ whose join is terminal and no member is terminal.

Exercise V.5. Let P be obtained by fixing the even coordinates of a binary sequence. Compare the value of the corresponding closed-set assertion in the Cohen, measure, and full perfect forcing algebras. Explain why a positive answer for a perfect condition requires a condition comparison, using the refinement order on perfect conditions.

Exercise V.6. Replace the binary output in Proposition V.4 by a continuous map to a Hausdorff space with a countable separating family of opens. Decide which parts of the construction carry over. State what further effective information would be required to implement branch recovery.

Exercise V.7. Given a disjoint Boolean partition, independently prescribe a connected groupoid as witness type over every part. Determine the space of sections after restricting to a subcondition which meets several parts. Find two examples with the same support and inequivalent section spaces.

Part B

Admissibility, approximation, and degrees

Chapter VI

Admissible bounds and witness objects

VI.1 Set-sized domains of search

Ordinary finite computation has two bounds. A completed run uses finitely many instructions, and a finite family of completed runs can be stored together. The second bound has to be proved anew when finite sets are replaced by sets of an admissible level. It is the bound that permits a family of computations to serve as the input to another computation.

Let $M = L_\alpha$ be a transitive admissible set. The case in which $\omega \in M$ has infinite internal domains. The level L_ω gives the separate case of hereditary finiteness. We use the following fragment of Kripke–Platek set theory. Pairing, Union and bounded Separation construct sets from given data. Bounded Collection says that if

$$M \models (\forall x \in a)\,(\exists w)\,R(x, w, p)$$

for $a, p \in M$ and bounded R, there is a $b \in M$ containing at least one witness for each $x \in a$. Here a formula is bounded when every quantifier ranges over a set displayed in the formula. Transitivity makes its truth on parameters in M absolute. Packing any extra witnesses into w gives the corresponding form of Σ_1-Collection. We allow parameters from M in definitions over M. A set of M is called α-finite. A subset or graph on α is α-recursively enumerable when it is Σ_1 over M. A predicate is α-recursive when it and its complement are both so enumerable. These conventions give Sacks's ordinal recursion theory [11, Ch. VII][1].

The first admissible ordinal above ω is ω_1^{CK}, the supremum of the recursive ordinals. At this level α-finite is called metafinite, and ordinal recursion has the notation presentation called metarecursion. To see why hyperarithmeticity occurs in this presentation, fix a coanalytic system of unique notations for the re-

cursive ordinals. The image by that notation map of an ω_1^{CK}-finite set of ordinals is hyperarithmetic. The equivalence with the constructible presentation uses the coanalytic normal form for $\Sigma_1(L_{\omega_1^{\mathrm{CK}}})$ and the translation of well-founded hyperarithmetic codes to their collapses. In particular a set of integers belongs to $L_{\omega_1^{\mathrm{CK}}}$ exactly when it is hyperarithmetic [11, Chs. III, V and VII]. Thus infinite sets of integers can be metafinite, and a union chosen by an arbitrary external sequence of metafinite sets need not be metafinite.

Proposition VI.1 (Bounded records)**.** *Suppose $a, p \in M$, and a partial function on a is specified over M by*

$$f(x) = y \quad \Longleftrightarrow \quad M \models (\exists z)\, D(x, y, z, p)$$

with D bounded. If f is total on a, its graph belongs to M. Moreover one set in M contains a pair (y, z) witnessing $f(x) = y$ for each $x \in a$.

Proof. Apply Collection to the bounded matrix saying that the two components of a proposed record verify D. Read the first components of the collecting set and form their product with a. Apply bounded Separation to this product, using a bounded existential quantifier over the collected records. Functionality identifies the result with the graph of f. □

A Σ_1 predicate on a may have a partial domain outside M. The proposition uses totality to ensure that Collection applies before the graph is formed. It explains why a uniform enumeration in the theory requires more than an external list of correct local decisions.

VI.2 A certificate for coanalytic predicates

Proposition VI.2 (The numerical Spector–Gandy comparison)**.** *A lightface Π_1^1 predicate on ω is Σ_1 over $L_{\omega_1^{\mathrm{CK}}}$. Conversely, a numerical predicate with a Σ_1 definition over this admissible level, allowing a fixed parameter from the level, is lightface Π_1^1.*

Proof. For the forward direction, take a recursive family T_m whose path predicate is the complementary Σ_1^1 predicate. A certificate of well-foundedness of T_m is an isomorphism between its Kleene–Brouwer order and an ordinal. For a recursive well-founded tree the order, its order type, and its collapse are all in $L_{\omega_1^{\mathrm{CK}}}$. The collapse can be obtained by external induction on the well-order, using Collection in the admissible level to assemble the initial-segment collapses. The test of an isomorphism to a given ordinal is bounded. Existence of this certificate is therefore a Σ_1 predicate over the level. Conversely, code the sets of $L_{\omega_1^{\mathrm{CK}}}$ by well-founded hyperarithmetic relations. The collapse of a valid code belongs to the level. Relative to well-founded codes, the bounded test and its denial have effective inductive evaluations on the coded membership relations.

The Spector–Gandy restricted existential theorem applies to this coding, since the witnesses range over hyperarithmetic reals. It makes the resulting numerical predicate Π^1_1. A code for the fixed parameter is itself hyperarithmetic and can be substituted without increasing that pointclass. See [11, Chs. III and VII] for the restricted existential theorem and the code comparison. □

The certificate in the forward direction is highly rigid. For a fixed presentation the isomorphism to the ordinal, if it exists, is unique. Under transport that preserves the presentation of the order, composition with the transport is the induced map of certificates. It is sometimes preferable to use the ordinal-valued rank function on the tree itself. In the tree consisting of a root with two leaves, the permutation of the leaves preserves the rank assignment, since both leaves have rank zero. It exchanges the labels that determine their relative order in a fixed Kleene–Brouwer presentation. A witness family over BC_2 can use this permutation as transport on the tree. The rank certificate respects that transport. A linearization that assigns a fixed order to the two leaves would instead require a choice of one of two exchanged orderings. Its existence on fibers is compatible with its failure to descend. Thus two equivalent numerical tests of well-foundedness give different requirements on their higher presentations.

VI.3 Bounded sets as a regular category

Write $\mathsf{Set}(M)$ for the category whose objects are the sets in M and whose arrows are the functions with graphs in M. This is an ordinary category viewed from the metatheory. A choice of product, equalizer or union in it comes from the bounded set operations, and its variable a ranges only over M.

Proposition VI.3 (Regular images and substitution)**.** *The category* $\mathsf{Set}(M)$ *is regular. The regular image of* $f : C \to A$ *is the subset* $\{a \in A : (\exists c \in C)\, f(c) = a\}$. *For an arrow* $u : A' \to A$ *the corresponding substitution identity is*

$$u^* \operatorname{im}(f) = \operatorname{im}(u^* f)$$

as subobjects of A'.

Proof. Products, equalizers and the terminal set belong to M, by Collection and bounded Separation. The image formula is bounded. The resulting surjection $e : C \to \operatorname{im}(f)$ is the coequalizer of its kernel pair in this category. Indeed a map $g : C \to D$ equalizing that pair determines a unique map on the image. Its graph is the bounded separation of $\operatorname{im}(f) \times D$ using the existential witness $c \in C$ with $f(c) = a$ and $g(c) = d$. Pullback preserves surjectivity, and a map into A factors through its ordinary set image by the same argument. Thus the surjection–injection factorizations are regular and stable, which gives both assertions. □

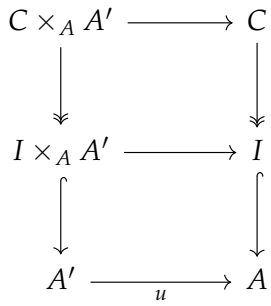


Figure VI.1. Substitution of a bounded witness relation. The middle row is the image before and after substitution.

In a slice the same proof gives left and right adjoints to substitution on subobjects. For $v : Y \to X$ and $S \subseteq Y$ in M, write $\exists_v S$ for the image in X, and let $\forall_v S$ consist of the x such that every member of $v^{-1}(x)$ lies in S. Both predicates are bounded and both satisfy change of base because their witnessing fibers pull back. These operations concern subobjects represented by elements of M. The corresponding class of all external subsets of an $a \in M$ can be larger. In particular these constructions do not require a power object in $\mathsf{Set}(M)$.

Example VI.4 (A classified predicate and an unclassified one)**.** Take $M = L_{\omega_1^{\mathrm{CK}}}$ and $A = \omega$. Take a uniformly recursive family T_e for which the assertion $P(e)$ that T_e is well founded is Π^1_1-complete. The uniform tree normal form gives such a family. Ranking the tree as a well-founded relation is expressible by a Σ_1 witness in M, and the ranking of a recursive well-founded tree belongs to M. The extension of P is therefore not an element of M [11, Chs. I–III]. For a fixed ordinal cutoff $\beta \in M$, the set of indices with a rank less than β is hyperarithmetic. Thus it is a subobject in $\mathsf{Set}(M)$. The full external union of these approximations can fail to lie in M, even though each approximation is represented there. Adding the complement as another semidecidable predicate would give a total two-valued decision on ω and invoke Proposition VI.1.

VI.4 Bounds in an infinity-topos

In an ∞-topos $\mathcal{X}$, fix a base B and a diagram of witnesses $W_\xi \to X$ in $\mathcal{X}_{/B}$, indexed by a nonempty limit ordinal λ. Each transition commutes with the maps to X. Write $W = \operatorname{colim}_{\xi<\lambda} W_\xi$ in the slice and

$$I_\xi = \operatorname{im}(W_\xi \to X), \qquad I = \operatorname{im}(W \to X).$$

Here the image is the relative (-1)-truncation. The map from the witness object to its image is an effective epimorphism. An X-parametrized assertion that witnesses exist locally is expressed by $I = X$. A uniform stage bound is an

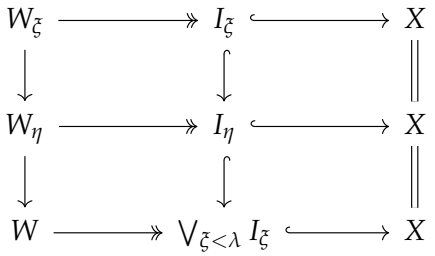


Figure VI.2. Witnesses at stages $\xi \leq \eta < \lambda$, projected to truth values over X. A uniform bound is a stage at which the middle entry equals X. A section would choose witnesses in the left column.

ordinal $\xi < \lambda$ for which $I_\xi = X$. Neither assertion requires a section of the corresponding witness map [7].

Theorem VI.5 (Transfer and detection of a bound)**.** *Let $v : B' \to B$. Pullback along v preserves each I_ξ, their join I, and every uniform stage bound. If v is an effective epimorphism, it also detects a proposed bound at any fixed ξ. If X is κ-compact in $\mathcal{X}_{/B}$ and $\mathrm{cf}(\lambda) \geq \kappa$ for a regular infinite cardinal κ, then $I = X$ implies the existence of a uniform stage bound.*

Proof. Relative (-1)-truncation to subobjects is left adjoint to the inclusion of subobjects in the slice over X. It therefore sends the colimit of the witness diagram to the join of the images. Colimits in an ∞-topos are universal and effective epimorphisms are stable under base change. Uniqueness of the image factorization gives the preservation assertion. If v is an effective epimorphism, so is $X \times_B B' \to X$. If I_ξ pulls back to the top subobject then this effective epimorphism factors through I_ξ, so I_ξ was already top.

For the last assertion the indexing ordinal is κ-filtered by the cofinality assumption. Filtered colimits in an ∞-topos commute with finite limits, so the colimit of the chain of monomorphisms $I_\xi \to X$, in the slice over B, is monic into X and represents their join. Compactness factors the identity of X, up to homotopy, through one I_ξ. A monomorphism with such a section is an equivalence. □

The compactness argument uses the size of the externally given indexing diagram and compactness of the object of inputs. Proposition VI.1 instead uses the definability of the relation and Collection in a particular transitive model. An internal domain can be infinite even when its definable total family has a single bound. Conversely an externally chosen stage assignment on $\omega \in L_{\omega_1^{\mathrm{CK}}}$ can be cofinal in that model, since the model is countable in the ambient universe. Such an assignment cannot have a total Σ_1 graph over the model.

Example VI.6 (Local existence without a uniform bound)**.** In $\mathsf{Sh}_\infty(\mathbb{R})$ let $B = X = 1$ and W_n be the open subobject $(-n, n)$ for positive integers n. The join

of their images is 1, but no one stage has image 1. Over a bounded open, one stage does suffice. This gives local uniform bounds which cannot be promoted to a single global bound.

Example VI.7 (A bound without a section)**.** In spaces, let $X = BC_2$ and take $W_0 \to X$ to be the point equipped with the canonical map. Its fiber is C_2 and the map is an effective epimorphism. Its section would make the identity of BC_2 homotopic to a constant map, which is impossible on π_1. Take a constant stage system with this witness object. Stage zero is a uniform bound and the action on the two witnesses around the generating loop exchanges them. Thus a stage bound does not remove monodromy from the witness family.

VI.5 Exercises [Worked solutions]

Exercise VI.1. Prove directly that a jointly specified Σ_1 relation on a set in L_α that gives both signs of a predicate has its positive part in L_α. State where the two signs are used.

Exercise VI.2. In $\mathsf{Set}(M)$, prove that a bounded equivalence relation on $A \in M$ has a quotient in M. Determine its behavior under pullback along a map into the quotient.

Exercise VI.3. Give an example of a sequence of metafinite sets whose union is an element of M although the sequence itself is not. Explain why this does not give a converse to bounded Collection.

Exercise VI.4. Replace the ordinal diagram in Theorem VI.5 by a small κ-filtered diagram. Give the precise hypotheses needed for compactness to imply that one image is top.

Exercise VI.5. Let K be a compact Hausdorff space. Prove that a directed family of open truth values in $\mathsf{Sh}_\infty(K)$ with join 1 has a stage equal to 1. Compare the conclusion with the example on $\mathbb{R}$.

Exercise VI.6. Pull back $* \to BC_2$ along the basepoint $* \to BC_2$. Draw its homotopy pullback and compute the action on its fiber obtained by transporting around the fundamental loop.

Exercise VI.7. Replace the two-leaf tree by a root with d leaves and transport by a subgroup $G \leq S_d$. Compare global sections of the finite family of linear orders of the leaves with global sections of the ordinal-rank certificate family.

Chapter VII

Relative computation and support

VII.1 Whole queries

Fix $M = L_\alpha$ as in Chapter VI. A signed support on α is a pair $p = (H, J)$ of disjoint α-finite subsets of α. An oracle $A \subseteq \alpha$ satisfies it when $H \subseteq A$ and $J \cap A = \varnothing$. Supports are ordered by extension in both signs. Every support is bounded in α because the union of an α-finite set of ordinals belongs to M.

A weak relative procedure is an α-recursively enumerable relation $R(p, x, y)$ interpreted as a possible certificate for $f(x) = y$, provided A satisfies p. The procedure computes f relative to A if the witnessed graph with that oracle is exactly the graph of f. It need not be functional with other oracles. A strong reduction of A to B gives semidecision procedures, with signed B-supports, for both predicates

$$K \subseteq A, \qquad K \cap A = \varnothing \qquad (K \in M,\ K \subseteq \alpha).$$

These tests concern entire α-finite queries. In particular they jointly provide the two answers for a singleton. These are the weak and strong definitions of Sacks [11, Chs. V and VII].

Proposition VII.1 (Composition of strong reductions)**.** *Strong α-reducibility is transitive.*

Proof. Suppose A is strongly reduced to B, and B to C. An accepting record for a whole query about A uses one support (H, J) on B. Substitute an accepting record for $H \subseteq B$ and one for $J \cap B = \varnothing$ from the second reduction. Each record has a C-support. Take the union of the two sets of positive questions and of the two sets of negative questions. Discard incompatible unions. The test that the three enumerated records have the indicated interfaces is bounded once their enumeration witnesses are given, so existence of the three records is a Σ_1

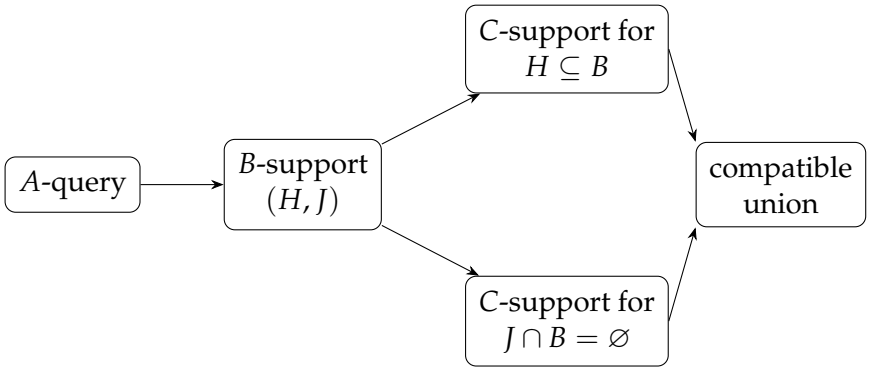


Figure VII.1. Substitution of whole-query records. One intermediate support is discharged by two accepting records.

condition. With oracle C, a successful record implies truth of the original query by the two hypotheses. If the query is true, choose the first reduction's record and the two records verifying its support. Their unions are compatible because they agree with the same oracle. The negative whole-query test is composed by the same construction. □

A weak procedure applied separately to all members of a support would supply a family of new supports. The fact that its index set and every member belong to M does not put the family in M. For instance, if (ξ_n) is an external cofinal sequence in ω_1^{CK}, the supports $(\{\xi_n\}, \varnothing)$ are metafinite. Their aggregate positive part is unbounded, so it is not metafinite. This example isolates the missing bound without claiming that this particular sequence has a given relative computability degree. By contrast, the supports $(\{2n\}, \varnothing)$ have a graph in $L_{\omega_1^{\mathrm{CK}}}$, and their aggregate is the recursive set of even integers.

VII.2 Regularity, hyperregularity, and compression

An oracle A is regular if $A \cap \xi \in M$ for every $\xi < \alpha$. It is hyperregular if every total weakly A-recursive function $\alpha \to \alpha$ has bounded range on each $\xi < \alpha$. These definitions test different aspects of relative computation. Regularity makes any *given* bounded portion of the oracle available as a set. Hyperregularity bounds the resources demanded by an entire family of relative computations. They are related to the expansion of $(M, \in)$ by the predicate A.

Theorem VII.2 (Admissible expansion)**.** *The expansion $(L_\alpha, \in, A)$ satisfies bounded Separation and bounded Collection in its expanded language if and only if A is regular and hyperregular.*

Proof. Expanded Separation gives regularity. Any weak oracle graph has a $\Sigma_1(A)$ definition, because compatibility of a supplied support is a bounded

formula in the expanded language. Expanded Collection therefore places the range of each total weakly A-recursive function on a bounded domain in a set of M. Such a set of ordinals is bounded in α.

For the converse, bounded Separation with a fixed set of parameters requires only a bounded part of A. Indeed, the values reached by bounded quantifiers belong to the transitive closure of the sets being searched and of the parameters. The ordinals there have a single bound in α. The oracle predicate on this domain can be replaced by the set $A \cap \xi$ at that bound.

For Collection, suppose $(\forall x \in a)(\exists y)\, D(x, y, p, A)$ with $a, p \in M$ and D bounded. Use an M-well-order of a with ordinal domain $\gamma < \alpha$, and the usual α-recursive enumeration of M with M-coded bounded prefixes [11, Ch. VII, Secs. 1 and 3]. For each index $i < \gamma$, define $g(i)$ to be the first ordinal code of a witness y for its input x. Set $g = 0$ on the other indices below α. Given a proposed witness index, all earlier candidates lie in one bounded prefix of the enumeration. Take the transitive closure of this prefix together with the parameters and the selected input. It bounds the oracle questions in testing D on all those candidates. A signed table for A on this ordinal bound exists by regularity. Using the table as support, ordinary Σ_1 enumeration certifies that the proposed index is the first success. This describes g as a total weakly A-recursive ordinal function. Hyperregularity bounds $g[\gamma]$. A bounded prefix of the enumeration containing these indices gives the required collecting set. □

Admissibility does not say that every enumerable subset of a given set is available. Let $\rho_1(\alpha)$ be the least ordinal $\rho \leq \alpha$ admitting an injective α-recursive map $\alpha \to \rho$. This is Sacks's first projectum for L_α. It also measures the least ordinal supporting an α-recursively enumerable subset outside M, with the comparison at α understood externally [11, Ch. VII, Sec. 2][2]. Each enumerable $S \subseteq \beta$ has a partial injective α-recursive enumeration with domain an initial segment of α. This enumeration orders first successful certificates and removes duplicates, recording bounded initial histories. If $\beta < \rho_1(\alpha)$, its domain cannot be all of α. On a proper initial segment Proposition VI.1 puts its range in M. Conversely the range of an injection witnessing $\rho_1(\alpha)$ cannot be an M-set. Its inverse would be a total Σ_1 function on that set, and Collection would put its unbounded ordinal range in M.

The projectum measures coding into a short ordinal rather than a cofinal map out of a short ordinal. A total Σ_1 map from $\beta < \alpha$ into α has bounded range, regardless of how small the projectum is. Cofinality from a shorter domain can occur at higher definability complexity, or when the predicate in an expansion breaks Collection. For example $\mathrm{cf}(\omega_1^{\mathrm{CK}}) = \omega$ externally, and $\rho_1(\omega_1^{\mathrm{CK}}) = \omega$, but there is no cofinal Σ_1 map from ω over $L_{\omega_1^{\mathrm{CK}}}$. This distinction constrains the priority schedules that a relative procedure can use.

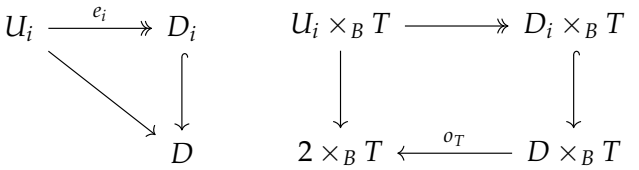


Figure VII.2. Aggregation and its test on a generalized oracle. Agreement on occurrences descends along the effective epimorphism at the top right.

VII.3 Amalgamation over a parameter space

Let $\mathcal{X}$ be an ∞-topos, and work in $\mathcal{X}_{/B}$. To represent a family of queries, take an object D of possible questions and $2 = 1 + 1$ as the object of answers in this slice. An oracle is a point of $O = 2^D$. A signed family of supports indexed by S consists of maps

$$U_1 \longrightarrow S \times_B D, \qquad U_0 \longrightarrow S \times_B D.$$

Its components give the individual occurrence domains and their signs. Repeated occurrences are allowed. Let $D_i \hookrightarrow D$ be the image of $U_i \to D$ for $i = 0, 1$. The cylinders defined by the family and by its aggregate can be compared without a section of either image map.

Theorem VII.3 (Amalgamation of signed families)**.** *In O, the subobject of oracles giving answer i on every U_i is the subobject of those giving answer i on D_i. Consequently the simultaneous cylinder for the family is the cylinder of the two image supports. The identity is stable under arbitrary change of base $B' \to B$. If $D_0 \times_D D_1$ is initial, the aggregate is a disjoint signed support.*

Proof. Test the two subobjects on a map $T \to O$. The corresponding oracle is $D \times_B T \to 2 \times_B T$ over T. The condition that it take value i on $D_i \times_B T$ is the condition that a certain equalizer subobject there be top. Its pullback to $U_i \times_B T$ is top exactly when the original family receives that answer. The map $U_i \times_B T \to D_i \times_B T$ is an effective epimorphism. Thus it detects top subobjects by image factorization, as in Theorem VI.5. Equality with the constant answer has a propositional truth value since the target is 0-truncated. Therefore the tests identify the corresponding subobjects on every T. Images, finite limits, and the exponential constructed in the slice satisfy base change, which proves the stability assertion. □

Example VII.4 (A moving query)**.** In the ∞-topos of spaces with C_2-action, let D be the discrete two-point torsor. The diagonal $D \to D \times D$ gives a family of positive singleton queries parametrized by $S = D$. There is no equivariant global section of $D \to 1$, so a global choice of one of these queries is unavailable. The aggregate has $D_1 = D$, since the second projection of the

diagonal is an equivalence. Its cylinder consists of the oracle that answers 1 at both points. After pullback along the torsor the two labels become available, the two singleton equations become separate, and their intersection gives the same result. The aggregate uses every occurrence and thereby avoids the need to choose a global label.

With a discrete set of questions the last theorem recovers ordinary aggregation. For an α-recursive procedure, the diagram of occurrence domains must be available among the α-finite data. An arbitrary external family can be aggregated as a set or as an object of an ∞-topos. It may fall outside the chosen admissible level. For queries on α with an M-coded occurrence diagram, Collection puts all the queried ordinals in one set of M, so a bounded subset of α suffices for D. The theorem then identifies the aggregate and family cylinders at every base receiving those same bounded data.

VII.4 Exercises [Worked solutions]

Exercise VII.1. Write the composite signed rule in Proposition VII.1 as a Σ_1 formula over L_α. Indicate which parts of the test are bounded.

Exercise VII.2. Suppose a weak oracle procedure $f <_{w\alpha} A$ is total on α, and A is regular and hyperregular. Show that the computation records needed for f on a bounded ordinal can be verified using one bounded oracle table and one bounded enumeration history.

Exercise VII.3. Assume $\beta < \rho_1(\alpha)$ and $H \in L_\alpha$ has an L_α-bijection with β. Prove that every $\Sigma_1(L_\alpha)$ subset of H is in L_α.

Exercise VII.4. Replace the binary object 2 in Theorem VII.3 by any 0-truncated object K, with specified target values for the different signs. Determine when the prescribed answers agree on overlapping supports.

Exercise VII.5. Let D be the discrete C_2-torsor in the moving-query example. In the slice over D, construct a family with one positive and one negative occurrence over the two different labels. Compute its cylinder and determine how the generator acts when it exchanges the chosen labels.

Exercise VII.6. In spaces take a connected space D, two different points as opposite-signed occurrences, and $O = 2^D$. Evaluate the aggregate cylinder. Explain what the images of the two occurrences say about compatibility here.

Chapter VIII

Priority and coherent stabilization

VIII.1 Restraints and blocks

Let α be admissible. A requirement in an α-recursive construction controls a portion of a growing oracle. An action may enumerate an ordinal into the oracle and at the same time protect an α-finite set of negative answers. An injury occurs when an action of higher priority conflicts with a protected answer. The injury discards the affected computation and may create a need for a new witness. The relevant question is whether all the earlier injuries fit in one history in L_α.

For example, to construct incomparable enumerably presented oracles A, B, assign an unbounded supply of distinct witnesses to each proposed reduction. Suppose a stage witnesses that a procedure using B gives answer 0 on a chosen witness x. An action puts x into A and restrains changes in the queried negative part of B. Positive answers to queries about B survive because the enumeration is increasing. If the restraint survives, the reduction fails at x. If the reduction is total and correct, some such action must eventually occur once the higher restraints have settled. An unchanged witness outside A would otherwise get a convergent answer 0 with a genuine use. This is the diagonal argument behind the classical method. The eventual success of a particular priority scheme depends on which histories the ground model can collect.

At an admissible ordinal, the work of Sacks and Simpson proves incomparability even for the weak version of reducibility. The α-finite injury method bounds the uses of predecessor requirements by working below suitable internal cardinals. Other cases require projecting the list of requirements down to a shorter list on which the stabilization argument can be run. Shore's method puts requirements with the same side of the construction into blocks and orders the blocks by a definable cofinal sequence. Its benefit is that requirements in a

block need not injure one another. The proofs use specific connections among projecta, definable cofinalities, and bounded enumerations [11, VIII.1–3][13, 2].

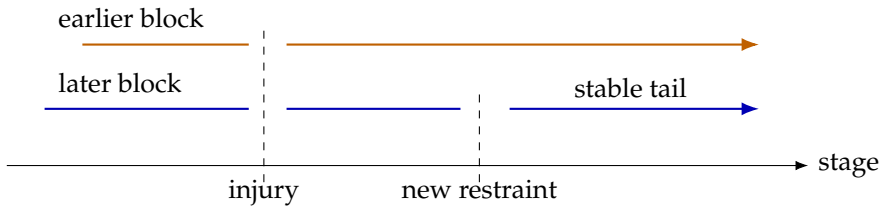


Figure VIII.1. A block history records injuries as well as the restraint that survives. Uniform stabilization of a prefix asks for one tail on which every restraint in that prefix has settled.

Definition VIII.1. An α-recursive approximation $a(s,i)$ to an ordinal function a_∞ on $\gamma \leq \alpha$ is *tame* if for every $\delta < \gamma$ there is $b < \alpha$ such that $a(s,i) = a_\infty(i)$ for all $s \geq b$ and $i < \delta$.

Sacks's tame Σ_2 maps admit approximations of this kind. The Σ_1 projectum is the least $\rho \leq \alpha$ into which α admits a Σ_1-definable injection, with set parameters allowed. A tame Σ_2 projectum instead measures the minimal indexing length of a tame map onto the height. The two constraints are different. A shorter list of codes may have an approximation which keeps changing arbitrarily late on one proper prefix. Admissible collection gives a convenient sufficient condition.

Lemma VIII.2 (Collecting final-action bounds)**.** *Let $d \in L_\alpha$, and suppose $a(s,i)$ is defined for $s < \alpha$, $i \in d$. If a total $\Sigma_1(L_\alpha)$ graph $b : d \to \alpha$ has the property that $a(s,i)$ is constant for $s \geq b(i)$, then there is a common stabilization stage for d.*

Proof. Apply Σ_1 Collection to witnesses for the graph. The range of b is a set in L_α, and its supremum is an ordinal of L_α. Use that supremum as the common stage. □

The Σ_1 hypothesis is on a bound with its witnesses. A proof of eventual constancy for each coordinate does not supply those witnesses automatically. For $\alpha = \omega$, the approximation $a(s,i) = 1$ for $i < s$ and 0 otherwise stabilizes uniformly on every finite prefix. No stage is constant on the entire domain. The same distinction applies at transfinite stages, with the ground notion of boundedness in place of numerical finiteness.

VIII.2 Stabilization in a slice

Let $\mathcal{X}$ be an ∞-topos and $U \in \mathcal{X}$. Let S be a nonempty ordinal of stages and I a small partially ordered set of observation windows. The order $i \leq j$ means that

i is a restriction of j. A *history* in this section is a diagram

$$H : N(I^{\mathrm{op}}) \times N(S) \longrightarrow \mathcal{X}_{/U}.$$

Here N denotes the nerve. The S-arrows advance a record. The I^{op}-arrows discard its observations outside a smaller window. A new stage may discard failed choices even if the underlying oracle grows. For actual construction records these arrows must be supplied with the corresponding coherence data. A sequence of estimates that offers only objects at each stage does not yet define such a history. In particular the following result applies after one has built a coherent record of the operations and cancellations.

Call i eventually stable if every transition on some tail $S_{\geq b_i}$ is an equivalence in the slice. Write $H_\infty(i) = \operatorname{colim}_{s \in S} H(i, s)$. The colimit here is semantic. It need not be formed inside L_α.

Proposition VIII.3 (Coherent eventual objects)**.** *If i is eventually stable, every structural map $H(i, s) \to H_\infty(i)$ with $s \geq b_i$ is an equivalence. If $J \subseteq I$ is a subposet with one bound b for all its windows, then the comparison*

$$H(-, b)|_{J^{\mathrm{op}}} \longrightarrow H_\infty|_{J^{\mathrm{op}}}$$

is an equivalence of diagrams. These assertions are preserved by pullback along any $V \to U$.

Proof. Every tail is cofinal. It has a first object b_i, so the transition maps give a natural transformation from the constant diagram at $H(i, b_i)$ to the tail. This is a pointwise equivalence by the stabilization hypothesis. The indexing nerve is contractible, so the colimit of that constant diagram is $H(i, b_i)$. This proves the assertion at b_i, and the same argument applies starting at a later s. With a common bound b, the structural comparison is natural on J^{op} and equivalences of diagrams are pointwise. Finally, pullback on slices has both adjoints and hence preserves the colimits computing H_∞. □

The square in Figure VIII.2 records a coherence condition in the history. For two stable windows, take a stage later than both bounds to read the restriction map between the eventual objects. Proposition VIII.3 shows that this reading is compatible with all subsequent stages. It also identifies the reading for any subdiagram that has a common bound. This allows the restrictions to retain nontrivial automorphisms of the stable witnesses.

Theorem VIII.4 (Descent of a specified bound)**.** *Let $v : V \to U$ be an effective epimorphism and $b \in S$. If every transition of $v^* H(i, -)$ beyond b is an equivalence for each $i \in J$, then b is a common bound for H on J. For a covering family, separate bounds suffice if they are bounded by a single stage.*

$$\begin{array}{ccccc} H(j,s) & \longrightarrow & H(j,t) & \longrightarrow & H_\infty(j) \\ \Big\downarrow{\scriptstyle \mathrm{res}} & & \Big\downarrow{\scriptstyle \mathrm{res}} & & \Big\downarrow{\scriptstyle \mathrm{res}} \\ H(i,s) & \longrightarrow & H(i,t) & \longrightarrow & H_\infty(i) \end{array}$$

Figure VIII.2. Advancing a record and restricting a window commute with the coherence of the history. The horizontal maps are equivalences once the corresponding row has stabilized.

Proof. For each transition use conservativity of pullback along v. Indeed, slice descent along the Čech nerve of v identifies a morphism over U by its compatible pullbacks. If its pullback to V is an equivalence, then so are its pullbacks in every degree of the Čech nerve. An equivalence in that limit of slices is detected componentwise. For a covering family, pull back first along its coproduct and use the common stage. □

Local existence of a stabilization stage may fail to give a global one. In spaces, take $U = \mathbb{N}$ discrete and $S = \omega$. At stage s let the fiber over n be $B\mathbb{Z}$ when $n \leq s$, and a point otherwise. At the first change use the basepoint inclusion. There is a bound over each component. There is no single bound over U, because the transition where the n-th circle appears changes the fundamental group of that fiber. The eventual family still exists and its pullbacks satisfy Proposition VIII.3.

VIII.3 Observing infinitely many windows

Consider the natural comparison

$$\mathrm{colim}_{s\in S} \lim_{i\in I^{\mathrm{op}}} H(i,s) \longrightarrow \lim_{i\in I^{\mathrm{op}}} \mathrm{colim}_{s\in S}\, H(i,s).$$

If there is a common stabilization stage on I, both sides are computed by the stage diagram at that bound. Stable behavior at individual windows is weaker, even if every proper initial segment of windows has a bound.

Example VIII.5 (Direct sum and product of loop groups)**.** Work in spaces and put $S = I = \omega$. Let $H(i,s) = (B\mathbb{Z})^{\min(i,s)}$. Restriction forgets final coordinates. Advancement appends basepoints. The two operations commute and define a history. At fixed i the eventual value is $(B\mathbb{Z})^i$. At fixed s the inverse limit over windows is $(B\mathbb{Z})^s$. The comparison therefore induces on fundamental groups the inclusion

$$\bigoplus_{n<\omega} \mathbb{Z} \longrightarrow \prod_{n<\omega} \mathbb{Z}.$$

In computing the left group one uses that loops, and homotopies of loops, in a filtered colimit of spaces are represented at finite stages. On the right, loops in the product are products of loops. The all-ones element is absent from the image. These histories are tame on every finite initial window, yet a compatible family of loop classes can require unboundedly many stages.

The distinction explains the role of a bounded family of requirements in an ordinal construction. A finite or adequately bounded collection of windows can be audited from a single stage. The inverse limit of all the eventual windows counts coherent observables regardless of whether a single admissible stage presented them. The effectivity conditions in the classical priority proof address the former operation.

VIII.4 Exercises [Worked solutions]

Exercise VIII.1. Give a priority rule for two conflicting finite signed restraints and analyze what happens if the same requirement has the right to discard its own witness repeatedly. Specify an extra condition on the rule that bounds its own actions when all higher restraints have stabilized.

Exercise VIII.2. For a countable successor-stage construction with finite predecessors, assume each new injury is charged to a distinct higher-priority action. Bound the number of actions at each requirement when every requirement acts at most once in each interval separated by injuries, including the initial and final intervals. A requirement with no injuries may act at most once. State which part of the argument changes if the predecessor list is infinite.

Exercise VIII.3. Replace the ordinal S in Proposition VIII.3 by a small directed poset. Formulate a cofinal-cone hypothesis giving a common representative for the eventual diagram on a given finite family of windows.

Exercise VIII.4. Let G act nontrivially on a finite set E. Regard this action as a family over BG. Construct a history over BG with stable fiber E and describe the distinction between a stable family and a stable section.

Exercise VIII.5. Modify Example VIII.5 to put the obstruction in π_k for each fixed $k \geq 2$. Determine the comparison on all homotopy groups.

Exercise VIII.6. Give a criterion in terms of cofinality of S and cardinality of I that turns separate bounds into a common bound. Prove the corresponding sufficiency for a covering family in Theorem VIII.4.

Chapter IX

Supported reductions and degree structure

IX.1 Composition with bounded queries

A reduction computes an output from an oracle. In the supported version the computation presents its requests and a rule for interpreting their answers. The requests have their own size condition. With a fixed arity budget, a reduction on each individual query might exist even when the complete family of requested reductions exceeds the budget. In ordinal recursion a similar question arises when the input and the supports are required to belong to the ground level.

We first give a parallel model of supported computation. It is useful for showing exactly where the condition on a composite enters. Work in an ∞-topos $\mathcal{X}$, and fix a base U. All the following diagrams are in the slice $\mathcal{X}_{/U}$. An answer family is a map $A \to I$ with request object I. Its fiber at i is the space of permissible answers to that request. Fix a class $\mathcal{A}$ of allowed support maps, containing equivalences and closed under pullback and composition. In spaces one may take the maps with discrete fibers of cardinality less than a fixed infinite regular cardinal κ. Composition then uses regularity of κ to bound a sum of fewer than κ cardinals below κ.

Definition IX.1. A parallel reduction of $A \to I$ to $B \to J$ consists of a span $I \xleftarrow{p} K \xrightarrow{q} J$ with $p \in \mathcal{A}$ and an evaluator

$$\Pi_p q^* B \longrightarrow A$$

over I. An equivalence of reductions is an equivalence of their support spans equipped with compatibility with the evaluator.

In sets, at $i \in I$ one queries $q(k)$ for each $k \in K_i$. The input to the evaluator is an element of $\prod_{k \in K_i} B_{q(k)}$. In a slice, this product is dependent and includes

the appropriate coherences. No choice of a whole family of answers has been assumed. The evaluator acts when that family is supplied. For the composition framework for polynomial functors in locally cartesian closed categories see [20].

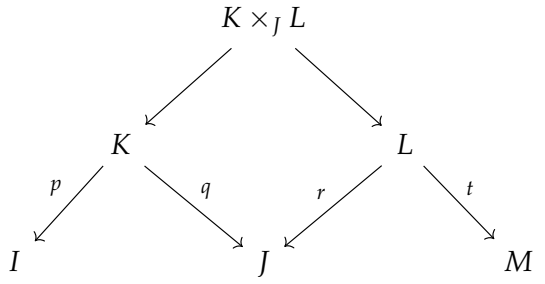


Figure IX.1. Substituting the support L for the intermediate requests in K. The pullback indexes the actual requests in the composite to M.

Theorem IX.2 (Substitution and symmetry)**.** *Parallel reductions compose with identity reductions and associative substitution up to canonical equivalence. In an ordinary locally cartesian closed category they give a bicategory with invertible* 2*-morphisms. In an* ∞*-topos the construction and its comparisons are stable under change of base.*

Proof. For the second reduction, use $J \xleftarrow{r} L \xrightarrow{t} M$ and $\Pi_r t^* C \to B$. In Figure IX.1 write u, v for the two pullback projections to K, L respectively. The composite evaluator is

$$\Pi_{pu}(tv)^* C \simeq \Pi_p q^* \Pi_r t^* C \longrightarrow \Pi_p q^* B \longrightarrow A.$$

The equivalence is the composition law for right adjoints to pullback followed by dependent-product base change. The support pu is in $\mathcal{A}$ by the two closure hypotheses. The identity span and identity evaluator give identity reductions. For three spans the two ways of forming the composite support are limits of the same diagram, with the same projections. The evaluations agree under their canonical comparison. This can be checked by adjunction, since after pulling back to the request objects the counits evaluate the same answers in the same order. The same comparison for four spans identifies the two associativity paths by the universal property of the limit. Reparametrizing by equivalences of spans commutes with these operations. Pullback on slices preserves the support pullbacks and commutes with their dependent products by base change. This proves the last assertion. □

If evaluators must belong to a specified effective calculus, one must also require that the displayed composite evaluator belongs to it. A condition of this kind is necessary for the reflexive and transitive relation $A \leq B$ defined

by existence of an admitted reduction. Quotienting by mutual reduction then gives a poset. That poset does not retain the space of presentations of a given reduction.

Example IX.3 (Symmetries of query indices)**.** Let I, J, A, B each be terminal in sets, and let K be a two-element set requesting the same terminal answer twice. The evaluator is unique. The reduction has the transposition of its queries as a nonidentity automorphism. Its connected component in the groupoid of reductions is equivalent to $B(\mathbb{Z}/2)$. Replacing reductions by their existence relation discards this loop.

Example IX.4 (Local solvability and monodromy)**.** Take $\mathcal{X}$ to be spaces, $U = BG$ for a nontrivial discrete group G, and $I = J = U$ as objects over U. Use $B = U \to J$ as the uninformative answer family, and $A = EG \to BG$ as the result family. Every product of the uninformative family is terminal over its request object. A reduction would thus supply a section of $EG \to BG$. Such a section is impossible by π_1, since EG is contractible. After pullback along $* \to BG$, the result family has the nonempty discrete fiber G, so a reduction exists there. Its descent would require a choice fixed by the monodromy, which acts freely on that fiber.

IX.2 A split and the information it preserves

Let α be admissible. Write $\leq_\alpha$ for strong α-reducibility in the bounded-support sense. In particular one can use an oracle to certify full positive and negative answers on an α-finite support. A set $D \subseteq \alpha$ is regular when $D \cap \eta \in L_\alpha$ for every $\eta < \alpha$. For regular D, the complete answer table on an α-finite request exists in L_α and can be sought via the two signs of the support. Use $B_0 \oplus B_1$ for the oracle whose two tagged columns are B_0, B_1.

Lemma IX.5 (The degree of an enumerated partition)**.** *If C is regular and α-recursively enumerable, and $C = B_0 \sqcup B_1$ with both parts α-recursively enumerable, then both parts are regular and $C \equiv_\alpha B_0 \oplus B_1$.*

Proof. Fix an α-finite set of requests D. Regularity makes $C \cap D$ a set in L_α. Each of its elements appears in the enumeration of exactly one part. Apply Σ_1 Collection in the ground structure to certificates of those appearances. Once both enumerations are run beyond the resulting bound, they give the correct partition on D. This proves regularity of the parts.

To compute the split from C, acquire its answer table on the projections of the requests. Find one stage of the two ground enumerations at which all the positive entries have been assigned their tags. Such a stage exists by the preceding argument. All other entries are negative in both parts. The answer on D thus has a single bounded support in C and a ground certificate for its

labeling. For the reverse reduction, obtain complete support tables from the two columns and take their union on the positive side. The negative answer requires absence from both columns. This is again a computation on the entire support, so the equivalence is strong. □

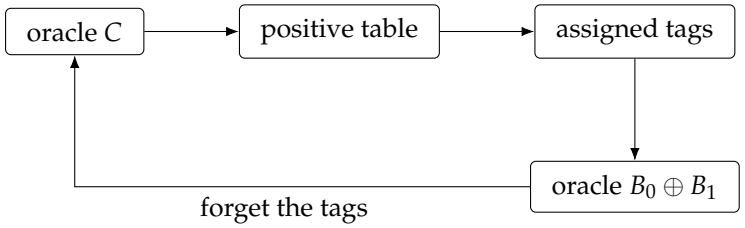


Figure IX.2. Forgetting tags gives the union. Computing the tags requires reaching a stage that accounts for every positive entry of the requested table.

Shore's splitting theorem has further content. For a nonrecursive regular α-recursively enumerable C, it chooses the partition so that $C \not\leq_\alpha B_i$ for $i = 0, 1$ [11, IX.1][14]. The resulting degrees are incomparable. If one part were below the other, their join would have that other degree, contrary to the strict inequality. It is the priority proof that supplies both strict inequalities. The bounded table argument works for any enumerated partition under its hypotheses.

Shore's density theorem begins instead with $D <_\alpha C$ among the α-recursively enumerable sets and produces an α-recursively enumerable B strictly between them [11, IX.5][15]. The lower oracle must be encoded into B, and successful searches relative to it have to be distinguished from provisional searches against a stage approximation. Splitting an upper oracle without these relative conditions does not establish density. Over E-closed levels the need for an explicit hypothesis is even more apparent. Sacks requires the computation trees of a reduction to belong to the ground structure. Direct composition is available when the target oracle preserves that closure, and transitivity on the enumerably presented oracles over an E-closed L_κ requires a separate argument [11, XIII.1].

IX.3 Bounded records over weaker levels

At a limit level L_β, the basic operations of pairing, union and bounded separation are available even when admissible collection fails. A $\Sigma_1(L_\beta)$ description of each certificate separately does not ensure that an entire supported computation can be transcribed there. At $L_{\omega+\omega}$, for example, each finite successor chain starting at ω is a set. The function sending $n \in \omega$ to the last ordinal in the chain of length $n + 1$ has a Σ_1 graph with parameter ω. Its range is cofinal in $\omega + \omega$. A single collecting set in the level cannot contain that range. This isolates the

failure regardless of how a map of the corresponding spaces of certificates is presented.

The weak admissibility condition of β-recursion compares the first projectum to the first definable cofinality. In the convention of [11, IX.6], the cofinality must be at least the projectum. Bounds on proper initial segments can then substitute for particular instances of collection. At the point where a composed reduction needs all its subsidiary trees, a suitable sufficient condition is a ground certificate that all required data have a bounded joint record, together with a ground operation assembling the record. Such hypotheses should be checked for the construction under discussion. Once they hold, the composite is computed by that bounded record and can use the categorical substitution of Theorem IX.2. The theorem itself does not turn a definable class of ground certificates into a set of that ground structure.

IX.4 Exercises [Worked solutions]

Exercise IX.1. Give two consecutive reductions on finite bit strings, each using at most two nonadaptive oracle queries per output bit, whose exhibited composition requires four. Show that a function realized by this example cannot be computed with only two queries.

Exercise IX.2. For three spans as in Theorem IX.2, write the common support as an iterated pullback and compare the two evaluators using the counits. Specify the maps over which the dependent products are taken.

Exercise IX.3. Generalize the two-query symmetry example to n requests. Determine its loop group when all queries are distinct labels and when each label occurs with a specified positive multiplicity.

Exercise IX.4. Replace the universal torsor example by a finite G-set $T \to *$ regarded as a family over BG. Characterize reductions using the trivial answer family as oracle in terms of the fixed points of T, and relate this to descent over the Čech nerve of $* \to BG$.

Exercise IX.5. For an enumerably presented C and an enumerated disjoint partition, specify the local hypothesis on a single α-finite support used in the proof of Lemma IX.5. Determine which direction of the equivalence can be evaluated from already supplied complete tables.

Exercise IX.6. Build a transitive, rudimentarily closed limit level other than $L_{\omega+\omega}$ with a bounded domain supporting a definable cofinal sequence in its height. Describe the resulting obstruction to a supported composite with that domain.

Part C

Set recursion and higher descent

Chapter X

Set recursion and spaces of records

X.1 Success as an inductive relation

An instruction for set recursion has a numerical program e and a finite list $\vec{x}$ of set arguments. Write $\{e\}(\vec{x}) \downarrow = y$ for the judgment that its computation ends with the value y. The Normann schemes are positive rules for this judgment [9, 11]. The first three have no premises. They return a projection, a difference $x_i \setminus x_j$, or an unordered pair $\{x_i, x_j\}$. An enumeration rule invokes the index supplied as an argument. A composition has premises for the inner calls, with values $y_1, \ldots, y_k$, and a premise for the outer call at $(y_1, \ldots, y_k)$. The union rule has one premise for each element of a set. Its form, with d the index of the inner instruction, is

$$\frac{\left(\{d\}(a, \vec{x}) \downarrow = y_a\right)_{a \in D}}{\{e\}(D, \vec{x}) \downarrow = \bigcup_{a \in D} y_a}.$$

The numerator is a family of verified judgments. There is no uniform finite upper bound on its size. With singletons in place of the y_a, the same rule forms the set of attained values.

A successful record lists the instructions and values on a well-founded tree, including the output of each preliminary call needed to determine a later input. Its length is the rank of the root. With length zero at a leaf, the length of a compound call is $\sup_b(\ell_b + 1)$, where b ranges over its immediate premises. A union with no premises has length zero. For fixed program conventions the checks on the data of a record, including a displayed ordinal ranking, can be made with bounded set quantifiers. Equivalently, convergence has a Σ_1 definition by existence of a record [11, Chapter X].

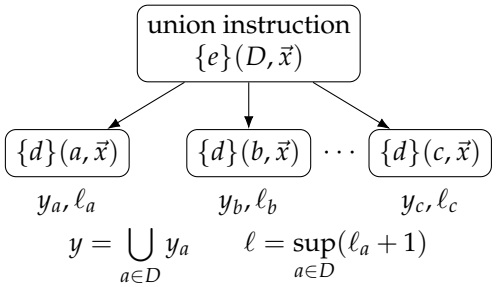


Figure X.1. The arrows point from a union instruction to its premises. Each premise must supply a value and a smaller length, including every premise in an infinite family.

Proposition X.1. *Any two successful records for the same instruction have the same output.*

Proof. Induct on the maximum of the two root lengths. The outer rule is read from the instruction. At a projection, a difference, or a pair the value is fixed. For composition, first compare the records of the inner calls by induction. Their agreed values specify the same outer call, to which induction applies again. Enumeration invokes the same instruction in the two records. In a union the same index set occurs at both roots. The two values at every index agree by induction, and their unions therefore agree. □

Proposition X.2. *Let M be a transitive admissible set. If $e, \vec{x} \in M$ and $\{e\}(\vec{x})$ converges in the universe, then its value and some successful record belong to M.*

Proof. Induct along a successful external record. The primitive operations are available in M. For composition and enumeration one assembles finitely many records, using transitivity and the computed inner values for the later inputs. For a union with parameter $D \in M$, the inductive hypothesis gives an M-record at each $a \in D$. Validity with an ordinal ranking is a bounded test. Collection in M yields a set of such records covering D. Bounded separation and the rudimentary operations extract their values and take the union. Join the certified subrecords under a new root, using an ordinal above all the displayed subrecord ranks. The collecting set and this new record belong to M. Determinacy identifies the resulting value with the external value. □

A transitive set closed under all such values is called E-closed. The least one containing x is denoted by $E(x)$. The relative constructible presentation uses the transitive closure $T_x = \mathrm{TC}(\{x\})$ as its base. In Sacks's convention, $E(x) = L(\kappa^x, T_x)$, where $\kappa^x = \mathrm{Ord} \cap E(x)$ and the zero stage of the relative hierarchy is T_x [11, Chapter X, Proposition 2.10]. The bound κ^x ranges over results with additional arguments from T_x. The ordinal κ_0^x is the supremum of the lengths of computations using x and numerical indices and parameters.

A length is the ordinal of an execution. An ordinal returned by a program is a value, and the two spectra need not agree. Even a projection can return an arbitrarily large input ordinal at length zero. The case $E(\omega) = L_{\omega_1^{\mathrm{CK}}}$ recovers the hyperarithmetic hierarchy at its admissible bound.

X.2 Indexed signatures and contexts

Work in an ∞-topos $\mathcal{X}$ and fix a parameter object U. A small presentation of rules in the slice $\mathcal{X}_{/U}$ has objects I, A, B and maps

$$I \xleftarrow{s} B \xrightarrow{p} A \xrightarrow{t} I.$$

Here I is the object of judgments, A is the object of rule instances, and B describes their premises. A point of A for an ordinary composition rule displays the inner results as local data. A point for a union rule displays the local family of values and its union. The inputs of every premise are then determined, even when the underlying program syntax has dependencies between calls. In the set case one may take a set of allowed arguments and outputs and include exactly the rule instances with data in that set. This gives a fragment of the class theory and recovers every derivation whose data lie in the fragment.

The operation on families of records Z over I is $P(Z) = \Sigma_t \Pi_p s^* Z$. For dependent polynomials in the ordinary setting see [3]. In a slice, the fiberwise product is a dependent product of evidence. Its arguments include one premise record for every branch, and path and higher path data of the indexing objects are respected. A small accessible polynomial gives an initial algebra W by the initial chain, with $W_0 = 0$, $W_{\alpha+1} = P(W_\alpha)$ and a colimit at a limit stage. The maps into W are monomorphisms. Indeed these properties hold initially, polynomial functors preserve monomorphisms, and an increasing directed colimit of subobjects computes their join. Thus the stages give a filtration in the slice. For set signatures W_α consists of the trees of length strictly less than α. The assertion follows by induction, since the strict supremum of the child lengths is at most α exactly when every child has length less than α.

Theorem X.3 (Transport of the stage filtration)**.** *Let P, Q be small accessible polynomials on $\mathcal{X}_{/I}$ with the same judgment object. Suppose the rule map from P to Q preserves conclusions and identifies each family of premises with the corresponding family over the image rule, including its map of premise sorts. The induced map from W^P to W^Q satisfies*

$$W^P_\alpha \simeq W^P \times_{W^Q} W^Q_\alpha$$

for every stage α. In particular the map preserves and reflects membership in each stage.

Proof. The hypotheses give a cartesian transformation $P \to Q$. In fact the rule data of $P(Z)$ over a rule of P are the pullback of those of $Q(Z)$ over its image,

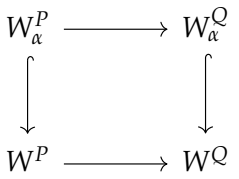


Figure X.2. Preserving the indexed premises makes every stage square a pullback. Instruction labels can change while the bound on a derivation is preserved.

because the premise families are identified. The initial chain induces the map of initial algebras. At stage zero the claim follows from stability of the initial object under pullback. If it holds at α, apply P to the pullback and then use the cartesian transformation for the map $W_\alpha^Q \to W^Q$. Polynomials preserve pullbacks and the initial algebras are fixed points. This gives the required pullback at $\alpha + 1$. At a limit, pullback commutes with the colimit of the earlier stages. Induction proves the result. □

Example X.4 (A homogeneous family of union nodes)**.** Suppose the inputs are unordered configurations of m distinct points of $\mathbb{R}^2$. The ordered configuration space is a finite covering of the unordered space. Let the m premises of a node be the associated family of configuration points. On a simply connected open set one can order the points and write the premises as a product of m named factors. A loop interchanging two points permutes the corresponding factors. The rule that takes the union of their set values descends along the covering, by invariance under the permutation. The ordered rule presentation changes only the labels on the same premise family, so its comparison with the unordered presentation preserves the stage filtration by Theorem X.3. A choice of an ordering for all configurations would be a section of the covering. For $m > 1$ the loop exchanging two entries rules out such a section.

Proposition X.5 (Determinacy descends)**.** *Let $R \to C$ be a space of records over instructions and $v : R \to Y$ a map over U to a* 0*-truncated object over U. Assume that the two outputs agree on $R \times_C R$. Let $D \hookrightarrow C$ be the image of $R \to C$. There is a unique output map $D \to Y$ over U extending v.*

Proof. The map $R \to D$ is an effective epimorphism and its Čech nerve has colimit D. Mapping this colimit into Y over U gives a limit of discrete spaces. Compatibility in degree one implies all higher compatibilities, since both maps from a higher multiple intersection are restrictions of the same agreement on pairs. The resulting unique compatible map is the claimed extension. □

Example X.6 (A stable output with higher evidence)**.** Let $U = K(\mathbb{Z}, 2)$ in spaces and let $R = * \to U$ be the chosen base point. Interpret R as the object of a single

kind of nullary rule over instruction $C = U$. The evidence fiber is equivalent to $K(\mathbb{Z}, 1)$, the loop space of U. The image of $R \to U$ is U, and the output of every record can be the same element of the terminal value object over U. All inhabited record fibers occur at rank zero. A global record would be a section of $* \to U$, making the identity of U homotopic to a constant map. The second homotopy group of U rules this out. The output descends and its evidence retains a higher obstruction.

X.3 Divergence certificates

Once the successful preliminary calls of an instruction have been determined, the operational immediate-subinstruction relation is defined as in the set theory. A Moschovakis witness to divergence is a sequence of immediate subinstructions descending from the given instruction. It can be accompanied by the successful records justifying edges that depend on earlier returns. A sequence of that kind and a successful record for the same instruction cannot coexist, since the successful record supplies a descending sequence of ordinals along the branch. In the classical universe a well-founded operational tree of valid instructions yields a record by recursion, so a divergent valid instruction has an infinite path by dependent choice. Existence of that path in a specified transitive model is a further problem.

An ordinal δ is x-reflecting when the Σ_1 sentences with parameter x true in $L(\delta, T_x)$ are already true in $L(\kappa_0^x, T_x)$. Sacks uses the greatest such ordinal κ_r^x to control the level where divergence witnesses appear. When an E-recursive well-ordering of the transitive base is available, a divergent instruction has a witness definable over $L(\kappa_r^x, T_x)$ [11, Chapter X, Lemma 5.3]. Inside an ∞-topos one can form objects of successful records and of sequences satisfying the witness relation. Their images are disjoint whenever rank decrease along a certified edge has been verified. Their union need not be the entire instruction object. Reflection and dependent choice are the additional ingredients in the classical witness analysis.

X.4 Exercises [Worked solutions]

Exercise X.1. Write a union instruction whose subcalls return singletons. Compute its output and rank in the cases of empty, finite, and infinite input domains, assuming each subcall is an elementary instruction of rank zero.

Exercise X.2. Build a set signature with one judgment for each ordinal below a fixed λ, with one rule at η having premises indexed by $\xi < \eta$. Determine the first stage containing the judgment η.

Exercise X.3. Suppose the map of rule presentations is an effective epimorphism on rules and identifies the premise families. Find sufficient hypotheses on the arities under which it induces an effective epimorphism on records at every stage. Examine the role of simultaneous local choices in a successor step.

Exercise X.4. Give two presentations of the same total operation on a singleton input, one with a longer successful derivation than the other. Explain which hypothesis of Theorem X.3 is absent from a comparison that erases the longer derivation’s extra rule.

Exercise X.5. In the configuration example, fix the ordered presentation locally and give an action of a loop on the space of a two-branch record. Decide when an ordered pair of premise records descends along that loop.

Exercise X.6. Let $R \to D$ be the base-point map $* \to K(A, n)$ for $n \geq 2$ and $A \neq 0$ abelian. Treat it as a nullary rule and compute its evidence fiber and image. Determine for which n that fiber is 1-truncated.

Exercise X.7. For a tree with a specified ordinal rank, show that any alleged infinite operational path produces a descending sequence of ordinals. Formulate the argument in a slice so that it is preserved by pullback.

Chapter XI

Selection and bounded forcing

XI.1 Finding a successful instruction

For fixed x let $S_e(x) = \{n < \omega : \{e\}(n, x) \downarrow\}$. The set is enumerable in the sense of E-recursion. Numerical search by interleaving one step of each computation does not describe its enumeration. A successful subcall can have arbitrary ordinal length. Gandy selection gives a uniform partial E-recursive procedure with arguments e, x which is defined exactly when $S_e(x)$ is nonempty and returns a member in that case [11, Chapter X, Theorem 4.1]. The key comparison examines two calls and obtains the shorter successful length, allowing one call to diverge. The recursion in the comparison decreases a bound supplied by a successful call. With an ordinal bound supplied, testing whether a given computation ends below it is E-recursive.

Consider an index domain D larger than ω. If the least successful length γ can be computed from the input, the bounded-length test recovers the entire nonempty set of minimizers, using the union scheme on singletons. Selecting a member from this set is a separate operation. A given effective well-order of D suffices. The Grilliot and Harrington–MacQueen arguments use access to a power set to compute the minimum for suitable larger domains [5]. A version due to Normann uses a bounded part of a regular cardinal in the relevant E-closure. These theorems specify resources for both the rank comparison and the selection. The existence of a least ordinal uses none of those resources.

For finding a divergent subcall the index class is usually the complement of an enumerable class. The Kechris basis theorem treats just this situation. If D is E-recursive in x, A is E-enumerable in x, and $D \setminus A \neq \varnothing$, there is $b \in D \setminus A$ with $\kappa_r^{x,b} \leq \kappa_r^x$ [11, Chapter X, Theorem 5.1]. A typical application chooses a subcall without raising the reflection bound. For a sequence of such choices one still needs a choice rule and verification that the records for the chosen edges are visible in the given model. The use of an effective well-ordering in the analysis of Moschovakis witnesses supplies this extra uniformity.

XI.2 Local existence and simultaneous choice

Work in an ∞-topos $\mathcal{X}$. Put the possible parameters in U, and let $R \to D \to U$ be a family of candidate records. A parameter in D specifies a candidate instruction. The image of $R \to U$ is the assertion that some candidate has a record. A section of $R \to U$ supplies both a candidate and a record throughout the parameter object. Even existence in every local fiber need not yield such a section. Proposition X.5 shows that a discrete value can descend under a uniqueness hypothesis. The evidence for that value may still have many paths and nontrivial monodromy.

A rule with premises indexed by $p : D \to U$ needs a family of records. Its existence predicate is the image of $\Pi_p R \to U$, where R is now regarded as an object over D. If this image is all of U, then $R \to D$ is an effective epimorphism. Indeed the evaluation map pulls every local section of the dependent product back to witnesses over the corresponding part of D. The converse is valid when p is finite locally constant. To prove it, pass to a cover on which D is a finite disjoint union of copies of the base. Finite products of effective epimorphisms are effective epimorphisms by stability under pullback and composition. The assertion descends along the cover, since equality of subobjects can be tested after such pullback.

Example XI.1 (Individual witnesses on the affine line)**.** Consider sheaves of spaces on the Zariski space $X = \mathbb{A}^1_{\mathbb{Q}}$. For $n \geq 1$ let $U_n = X \setminus \{x = n\}$ and let $V = X \setminus \{x = 0\}$. Regard these opens as subterminals and put $R_n = U_n \amalg V$. Every $R_n \to 1$ is an effective epimorphism because the two opens cover X. A section of R_n over a nonempty open O must lie in a single summand. The open O is irreducible and a choice of summand would otherwise decompose it into two disjoint nonempty opens. If O contains $x = 0$, the V summand is unavailable. A family of all the R_n-sections over O would require O to omit every positive integer point. Every nonempty Zariski open in X contains all but finitely many of those points. Thus the product of the R_n has no local section at $x = 0$. This is a failure to assemble witnesses even if every witness is assigned the same ordinal bound.

XI.3 Extracting a bound from local records

An external increasing filtration $R_{<\gamma} \hookrightarrow R$ of records, indexed by $\gamma < \lambda$ for a limit ordinal λ, can express length restrictions such as those in Chapter X. If the colimit of the filtration is R, then the union of the images of all $R_{<\gamma} \to U$ is the image of $R \to U$. This follows from the adjunction between image and inclusion of subobjects. One asks when the image can be reached at one particular stage.

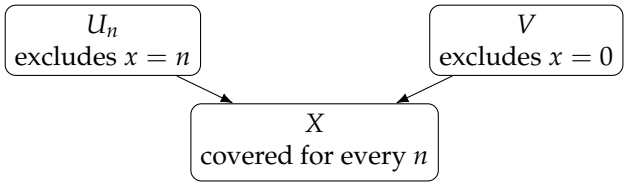


Figure XI.1. The two local domains form a cover for each fixed premise. The countable family has no common neighbourhood of 0 carrying all the chosen witnesses.

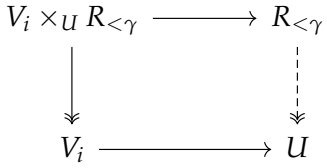


Figure XI.2. At a common bound γ every local pullback covers. Joint coverage by the V_i makes the right map an effective epimorphism.

Theorem XI.2 (Bounding along a small cover)**.** *Let $\{V_i \to U\}_{i \in J}$ be jointly effective epimorphic with $|J| < \mathrm{cf}(\lambda)$, and let $R_{<\gamma} \to U$ be an increasing filtration as above. Suppose for each i there is $\gamma_i < \lambda$ such that $V_i \times_U R_{<\gamma_i} \to V_i$ is an effective epimorphism. Then for some $\gamma < \lambda$ the map $R_{<\gamma} \to U$ is an effective epimorphism. The assertion for the chosen bound is preserved by any pullback in $\mathcal{X}$.*

Proof. Take an ordinal $\gamma < \lambda$ above all the γ_i. The image of $R_{<\gamma}$ pulls back to the whole of every V_i, since images commute with pullback. Therefore every $V_i \to U$ factors through this image. The V_i cover U, so their joint image is U. This proves the first assertion. Effective epimorphisms are stable under base change. □

Example XI.3 (Pointwise bounds in the topos of spaces)**.** When $\mathcal{X}$ is the topos of spaces, the inclusion of one chosen point in each connected component gives a cover of U. If there are fewer than $\mathrm{cf}(\lambda)$ components and every point has a bounded record, the theorem applies. Inside a single component an inhabited fiber at one point implies a local record at every point, allowing the same stage bound. A coproduct of λ points can behave differently. Put one record at the point indexed by ξ and admit it only at stages $\gamma > \xi$. The images cover their colimit. No stage covers all components. The distinction exists already for sets.

The use of $\mathrm{cf}(\lambda)$ in Theorem XI.2 is external. It gives a bound in the metatheory. For finite covers with a supplied finite list of computable ordinal bounds, an effective maximum gives the bound in the same recursion theory. For an

infinite cover this conclusion needs an effective collection of the bounds, or another effective argument. If the local data only assert that some bounded witness exists, there may be no way to extract from them an index computing a bound. Even a computable common bound for all witnesses would leave open the simultaneous choice problem of Figure XI.1.

XI.4 The effective forcing problem

In classical E-recursion let $M = L_\kappa$ be E-closed, let $\mathbb{P} \in M$ be a set forcing, and let $\dot{t}$ be a parameter name in the computation forcing language of [11, Chapter XI]. The relation $p \Vdash |\{e\}(\dot{t})| = \xi$, with ξ specified, can be obtained by E-recursion from the displayed data using the bounded forcing clauses. The clauses for a union instruction first determine the possible preliminary output names and then account for the calls on their members. A fixed bound keeps these operations within the designated sets of codes. Effective bounding says that when p weakly forces convergence, there is a bound γ computed uniformly from the forcing data and the instruction such that p weakly forces the length to be less than γ. The same bound must work throughout the cone below the condition. A separate search of unbounded length at each extension does not provide it.

Here is a sufficient preservation criterion stated in terms of the computational information. Assume that names for arguments in $M[G]$ are available in the forcing language, that convergence of a call at G is witnessed by a condition in G, and that soundness for the forcing clauses holds. Assume further that when $p \in G$ weakly forces success below some $\gamma \in M$, the bounded name construction yields a value in $M[G]$. Under effective bounding, $M[G]$ is closed under those calls. In fact one obtains $p \in G$ by the convergence truth lemma and computes the bound from the input data in M. Closure of M puts the bound in M. Applying the bounded name construction then puts the value in $M[G]$. This isolates the work the forcing theorems need to do beyond stability of images.

There are two different methods for producing the ground bound. In the countably closed case a path through the tree of possibilities is used to choose a decreasing sequence of conditions. When a code for the sequence exists in the ground structure, countable closure supplies a lower bound forcing the corresponding descent of subinstructions. This conflicts with forced convergence. The proof over a Σ_1-inadmissible E-closed L_κ constructs the required ground paths by reflection, with the effective ordinal presentations of conditions used in [12, 16]. The chain-condition method uses antichains and small-domain selection to find sufficiently many possible deciding conditions. The hypotheses and rank comparisons are different from those needed for a lower bound of a sequence [11, Chapter XI, Theorem 4.1]. Theorem XI.2 explains one feature

common to the two approaches. Once local bounds are obtained, the relevant comparison is with a single bound adequate under every allowed refinement.

XI.5 Sections and envelopes

For set types start with $\mathrm{tp}(0) = \omega$, put $\mathrm{tp}(\xi + 1) = \mathcal{P}(\mathrm{tp}(\xi))$, and take unions at limits. Fix an oracle F with its associated E-recursion and base parameters. A δ-section in Sacks's sense records the reachable outputs of type δ. For successor δ, the δ-envelope records the enumerable subsets of $\mathrm{tp}(\delta - 1)$ with the associated inputs and oracle. The type-two existential functional has the hyperarithmetic reals for its 1-section. The compression at higher types depends on what part of the computation is retained. Write $\mathrm{sc}_\delta(F)$ and $\mathrm{en}_\delta(F)$ for the section and envelope in the conventions of [11, Chapter XII]. Sacks's plus-one theorem gives $\mathrm{sc}_\delta(G) = \mathrm{sc}_\delta(F)$ with G of type $\delta + 1$, for F of type $\alpha + 2$, successor $\delta < \alpha + 2$, and successor α if $\delta > 1$. Harrington's plus-two theorem gives $\mathrm{en}_\delta(G) = \mathrm{en}_\delta(F)$ with G of type $\delta + 2$, for successor $\delta < \alpha$ and successor α [4]. The latter requires preservation of the enumerable traces, whose appearance may depend on the reflection point for the computation.

For the numerical distinction in a topos, use record objects $R_0 \to U$ and $R_1 \to U$ for certificates of the two possible answers. If their image subobjects D_0, D_1 are disjoint and jointly cover U, the canonical arrow $D_0 \amalg D_1 \to U$ is both mono and effective epi. It is therefore an equivalence. Sending the two summands to the two points of $1 \amalg 1$ gives a total Boolean answer. The corresponding statement when one has only R_1 is local semidecision by its image. In particular the range of total answers may be studied independently of the individual records supplying the answers. The proof shows what a comparison of enumerable domains must preserve in order to recover a total Boolean behavior. Comparison across higher oracle types also needs the effective ability to construct the witnessing records.

XI.6 Exercises [Worked solutions]

Exercise XI.1. Suppose a common length bound for every convergent call and a finite presentation of a candidate domain are given as part of an E-instruction. Describe the bounded search for the set of successful candidates. Specify the additional datum needed to return its least member.

Exercise XI.2. Prove by a local trivialization that a finite locally constant family $D \to U$ preserves effective epimorphisms under $\Pi_{D \to U}$. Consider also the case of fibers of size zero.

Exercise XI.3. Replace the positive integer points in Figure XI.1 by any infinite sequence of distinct rational points outside 0. Determine the image of the resulting product in the subobject lattice of the terminal sheaf.

Exercise XI.4. Give a version of Theorem XI.2 for a filtered partially ordered set of bounds in place of an ordinal. State the assumption on the family (γ_i) that replaces the cofinality inequality.

Exercise XI.5. Let $R_{<n} \to \mathbb{N}$ in sets have one record over each $k < n$. Compute the least successful bound over each k, the join of all the images, and the image after pulling back along the inclusion of the even integers.

Exercise XI.6. Suppose $D_0, D_1 \hookrightarrow U$ are the convergence images of tests for outputs $0, 1$. Give the criterion, in terms of those subobjects, for these tests to induce a map $U \to 1 \amalg 1$. Analyze the result when their join is a proper subobject.

Exercise XI.7. In the abstract forcing preservation argument, assume the bounds vary over a finite deciding family of conditions and the language supplies names and a finite list of their rank bounds. Give a common effective bound for the family. Explain which step would require a new hypothesis for a ground set of conditions with infinitely many bounds.

Chapter XII

Descent of computation records

XII.1 Algebras over a cover

Fix an ∞-topos $\mathcal{X}$, a base U, and an indexed rule signature in $\mathcal{X}_{/U}$,

$$I \xleftarrow{s} B \xrightarrow{p} A \xrightarrow{t} I.$$

The existence of a rule instance is indexed by A. The corresponding premises are indexed by its fiber in B, with result and argument sorts determined by t and s. Write $P_U = \Sigma_t \Pi_p s^*$ for the polynomial on objects over I. For a map $U' \to U$, write $P_{U'}$ for the change of the whole signature. The comparisons of Chapter I give $v^* P_U \simeq P_{U'} v^*$. They are natural in v. Thus a P_U-algebra can be restricted with its algebra structure.

Let $V \to U$ be an effective epimorphism, and put $V^{[n]} = V \times_U \cdots \times_U V$, with $n+1$ factors. An algebra over V has descent data if its restrictions to $V^{[1]}$ are identified and these identifications extend to the coherent system on the whole Čech nerve. The higher conditions matter when rule instances or their fibers have paths. An equality of the images of the record objects does not include any such identification.

Theorem XII.1 (Descent and local initiality)**.** *The restriction of algebras gives an equivalence*

$$\mathrm{Alg}(P_U) \simeq \lim_{[n] \in \Delta} \mathrm{Alg}(P_{V^{[n]}}).$$

If a compatible algebra on the right is initial at every Čech level, its descended algebra is initial over U.

Proof. Apply descent for slices of an ∞-topos to $I_V = I \times_U V \to I$, an effective epimorphism [7, Ch. 6]. It gives $\mathcal{X}_{/I} \simeq \lim_n \mathcal{X}_{/I_{V^{[n]}}}$. The base change equivalences identify P_U with the functor on this limit given levelwise by the $P_{V^{[n]}}$. The category of algebras for an endofunctor Q of $\mathcal{C}$ is the pullback in Cat_∞ of

$$\mathrm{Fun}(\Delta^1, \mathcal{C}) \xrightarrow{(d_1, d_0)} \mathcal{C} \times \mathcal{C} \xleftarrow{(Q, \mathrm{id})} \mathcal{C},$$

where the pair in the first map is ordered as source and target. Its object is precisely an object W and an arrow $QW \to W$. Arrow categories and the displayed pullbacks commute with limits. This proves the first assertion. A mapping space from the candidate algebra to any algebra is the limit of the corresponding mapping spaces at the Čech levels. These are all contractible for the compatible family in the hypothesis, so their limit is contractible. This proves initiality. □

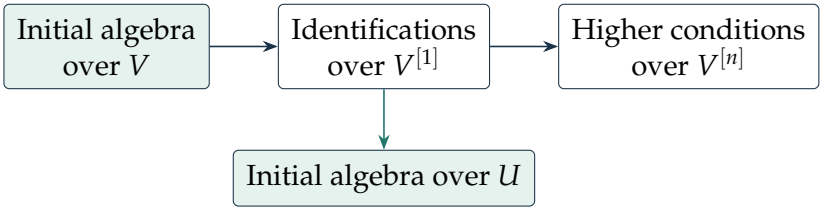


Figure XII.1. The data used to descend an initial algebra. The existence of a local algebra alone supplies only the first object of the diagram.

For small signatures with initial algebras stable under base change, the comparisons between the restrictions of initial algebras have contractible spaces of choices. Thus initiality supplies coherence in this situation. A choice of a particular element of an initial algebra has no such uniqueness. A descent theorem for algebras therefore says little by itself about the existence of a globally chosen execution.

An effective comparison asks for more than the equivalence of Theorem XII.1. The effective structure must be transported too. For example, a local ordinal clock ought to agree with its other presentations under the identifications of records. The necessary bounds are often obtained using the next proposition.

XII.2 Descent through a bounded stage

Let $(W_\xi)_{\xi<\lambda}$ be an increasing family of subobjects of W over U. Assume λ is a limit ordinal. We view the subobject W_ξ as the records available at stage ξ. If a section $w : U \to W$ factors through W_ξ, the factorization is unique in the sense of a contractible space of choices. If the section restricts over a cover and each restriction factors through the same stage, the pullback of $W_\xi \hookrightarrow W$ to U contains the cover and is all of U. This gives a global factorization of the given section.

Proposition XII.2 (Common stage)**.** *Let $\coprod_{j\in J} V_j \to U$ be an effective epimorphism, with $|J| < \mathrm{cf}(\lambda)$ in the metatheory. Suppose the restriction of a section w has a factorization through W_{ξ_j} over each V_j. Then w has a factorization through some W_ξ over U.*

If only the image of W_{ξ_j} covers V_j after pullback, the same bound ξ gives that the image of W_ξ is U.

Proof. Take $\xi = \sup_j \xi_j < \lambda$. In the first case, the pullback of the monomorphism $W_\xi \to W$ along w contains all the cover maps. Its image in U is therefore all of U, so the mono is an equivalence. In the second case, $\mathrm{im}(W_\xi \to U)$ restricts to all of V_j by stability of images and comparison with W_{ξ_j}. The same cover argument proves the assertion about the image. □

The assertion about images is the covering calculation of Theorem XI.2. The first assertion gives the additional factorization when an entire coherent section has been chosen. The rank comparison is a cofinality calculation on a collection already given. In forcing applications the collection may consist of names and records whose ranks must first be found in the ground structure. In admissible computations it may consist of outputs from a total bounded family, with the Σ_1 record predicate used to collect the lengths. The argument here applies after these ranks are available. An unbounded cover can indeed fail to supply one stage. Let $U = \omega$ be discrete, $W = U$ with its identity map, and $W_n = \{0, \dots, n-1\}$. Singletons cover U, every restriction of the identity has a finite stage, and no finite stage covers U.

XII.3 An obstruction invisible to components

A connected cover can exhibit a different failure. Consider the nonsplit exact sequence of finite groups

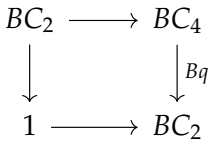
$$1 \longrightarrow C_2 \xrightarrow{i} C_4 \xrightarrow{q} C_2 \longrightarrow 1, \qquad i(1) = 2, \qquad q(k) = k \bmod 2.$$

Let $W = BC_4$ over $U = BC_2$ by Bq. Think of its points over a parameter as successful records, each with the same fixed numerical output. The homotopy fiber of Bq is BC_2. The relative image of $W \to U$ and its relative 0-truncation are both U. Over the base point, the pulled-back family BC_2 has a section. The map $1 \to BC_2$ is an effective epimorphism, so this is a local choice on a cover.

Proposition XII.3. *The record family $BC_4 \to BC_2$ has no section. The obstruction is already visible on the fundamental groups of the total and base spaces.*

Proof. A section would induce a section $C_2 \to C_4$ of q on fundamental groups, up to conjugation. The groups in question are abelian. A possible image of the generator is odd, hence has order four. This does not define a homomorphism from C_2, so no section exists. □

One can see the obstruction directly on the local data. Choose the lift $0 \mapsto 0$, $1 \mapsto 1$ of the elements of C_2. The lift of a product differs from the



The lift of the base generator squares to the nonidentity kernel element.

Figure XII.2. The square is a homotopy pullback. Each record fiber is connected, but the compatibility on loops obstructs a section over the base.

product of the lifts by an element of the kernel. At $(1,1)$ this element is $2 \in C_4$. Replacing the chosen odd lift by the other odd lift leaves its square unchanged. Thus the defect cannot be removed by changing this choice. In the nerve description the failed equality is the composition requirement over the next overlap. Its relative 0-truncated family has a section, since all the fibers there are singletons, and hence cannot see that requirement.

The same family has discrete computational data. At each parameter a verifier may accept the empty trace with output zero. Attaching the group of record identifications does not change this verifier. It changes the question from whether there is a successful result to whether the full record can be chosen compatibly under transport. Thus changing the output test cannot by itself repair the obstruction.

XII.4 Successive truncations

Suppose the natural map $W \to \lim_n \tau^{/U}_{\leq n} W$ in the slice is an equivalence. This is an explicit hypothesis on the family, and it holds in spaces. A compatible system of sections of the relative truncations then gives a section of W. Indeed the mapping-space functor out of the terminal slice object preserves limits. In using this fact it is the compatible system of sections, with higher coherences, which determines the point in the limit. Statements of separate inhabitation at a collection of stages have no such consequence for a general inverse diagram. For example the discrete sets $\{k \in \omega : k \geq n\}$ form a tower with nonempty terms and empty limit. The relevance of truncation is that it permits a sequence of progressively stronger compatibility checks on the same family. It is distinct from the ordinal filtration that bounds the formation of the original records.

XII.5 Exercises [Worked solutions]

Exercise XII.1. Prove the local initiality conclusion of Theorem XII.1 assuming the object is initial over V and all its further restrictions to $V^{[n]}$ preserve initiality.

Exercise XII.2. Let U be discrete and $W_\xi \hookrightarrow W$ be a λ-chain as in Proposition XI-I.2. For a singular limit λ, construct a cover of size $\mathrm{cf}(\lambda)$ on which a section factors locally through stages with no single global stage.

Exercise XII.3. Let $1 \to K \to H \to G \to 1$ be an exact sequence of discrete groups. Determine the relative 0-truncation and the relative 1-truncation of $BH \to BG$. Prove that a section exists iff the group extension splits.

Exercise XII.4. Compute the Čech nerve of the cover $1 \to BG$ in spaces. Express a compatible family of records over its levels in terms of the action of G on a homotopy fiber.

Exercise XII.5. Give a site and a covering family for which nonemptiness of all selected fibers is visible on the cover but a section of their product fails. State exactly which conclusion of Proposition XII.2 still holds for a filtration containing those local witnesses.

Exercise XII.6. Use a family over S^1 with fiber a nonempty set and prescribed permutation monodromy. Determine the components of its section space and compare them with the domain of its convergence predicate.

Chapter XIII

Effective oracles with transport

XIII.1 A jump in a family of symmetries

The degree of an oracle compares which answers can be computed when that oracle is supplied. For a family of oracles there is also a question about the program used to compare them. It may compute the same degree in every fiber and fail to transport its certificates. This chapter gives a class of effective presentations in which both questions can be asked explicitly.

Let G be a discrete group with an effective presentation on a recursive set of natural-number codes, with recursive multiplication and inverse. Assume G acts on the query alphabet ω by a uniformly recursive family of permutations. Its action on oracles $A \subseteq \omega$ is direct image. Let $K(A) = \{e : \Phi_e^A(e)\downarrow\}$, using a fixed effective programming system with parameters. For each $g \in G$ use the same machine numbering on $g^{-1}A$. Put

$$J(A) = \{(g, e) \in G \times \omega : e \in K(g^{-1}A)\}.$$

Give $G \times \omega$ the action $h(g, e) = (hg, e)$. The group coordinate records the presentation of the query alphabet in which the machine is run. The value at the identity coordinate is $K(A)$.

Theorem XIII.1 (Equivariant lift of the jump)**.** *The assignment J is equivariant and $J(A) \equiv_T K(A)$ uniformly in the given recursive group presentation. The successful finite execution records of J also transport equivariantly. If an equivariant transformation F of a G-invariant class of oracles is computed by one oracle program, the induced reduction of $J(F(A))$ to $J(A)$ can be given by a recursive map commuting with the action on $G \times \omega$.*

Proof. For $h \in G$, the membership test for (hg, e) in $J(hA)$ runs the machine with oracle

$$(hg)^{-1}(hA) = g^{-1}A.$$

It runs the very same computation with the same answers to its oracle questions. This proves equivariance of membership and of finite successful records. The identity coordinate reduces $K(A)$ to $J(A)$. In the other direction, a query to $g^{-1}A$ at n is the query to A at gn. Substitute this call in a simulation of e and use the parameter theorem to give the simulator an index uniformly from (g, e). The ambient jump $K(A)$ tests its convergence. This gives a many-one reduction, after the standard encoding of the product query alphabet.

For the last assertion let Ψ^A compute the characteristic function of $F(A)$. There is a recursive transformation $e \mapsto r(e)$ that replaces every query in the machine e by a call to Ψ with the new oracle. Thus $e \in K(F(B))$ iff $r(e) \in K(B)$, for every B in the domain of F. Equivariance gives $F(g^{-1}A) = g^{-1}F(A)$. Apply the same simulator with $B = g^{-1}A$. The map $(g, e) \mapsto (g, r(e))$ gives the claimed reduction and commutes with each left translation on the tag. □

The construction is external and uses a recursive presentation of G. Nevertheless the resulting families are genuinely parametrized by groupoids. An equivariant map on sets induces a map between their action groupoids, preserving the labels of the transport arrows. Applying the nerve and regarding those groupoids as spaces retains every stabilizer and the paths it induces. The map from the action groupoid of successful records to the action groupoid of program-input pairs has image the associated family of convergence predicates. Applying Theorem XIII.1 at the level of those predicates gives the ordinary jump at every untwisted base point. The construction does not require computably quotienting oracles by their orbits.

$$\begin{array}{ccc} (A, g, e) & \xrightarrow{\;h\;} & (hA, hg, e) \\ \Big\downarrow{\scriptstyle \text{oracle and machine}} & & \Big\downarrow{\scriptstyle \text{oracle and machine}} \\ (g^{-1}A, e) & = & ((hg)^{-1}hA, e) \end{array}$$

Figure XIII.1. The transported tag compensates for the change of oracle. The same finite trace certifies both endpoints of the top arrow.

For example, let C_2 exchange $2n$ and $2n + 1$ for all n. A program testing the very first bit of the oracle need not have the same convergence on A and on the translated A. If one translates its group tag together with the oracle, the one-bit record is preserved. For an oracle fixed by the exchange the records over the two tags lie in the same orbit. The underlying numerical jump still has its usual degree. This is a change of presentation allowing transport, rather than an increase in computational power.

XIII.2 Selection in an action groupoid

Let X be a G-set of inputs and let Y be a G-set of proposed witnesses. Let $R \subseteq X \times Y$ be invariant under simultaneous action. Assume that $R_x = \{y : (x, y) \in R\}$ is nonempty for every x. A selector compatible with transport is an equivariant section of $R \to X$. Write $G_x = \{g : gx = x\}$.

Theorem XIII.2 (Selection and stabilizers)**.** *With choice in the metatheory, an equivariant section exists iff $R_x^{G_x}$ is nonempty for one representative x of every orbit. Suppose moreover that the actions and their recursive domains are uniformly computable, R is recursively enumerable, and the presentation effectively gives the following data on an input x.*

1. *A representative $r(x)$ depending only on the orbit of x.*
2. *A transporter $g(x)$ with $g(x)r(x) = x$.*
3. *A finite generating list for $G_{r(x)}$ depending only on the representative.*

Then, under the fixed point condition, an equivariant section is recursive.

Proof. At a representative r, an equivariant section must be fixed by G_r. Conversely choose $y_r \in R_r^{G_r}$ for each representative and prescribe the section at gr to be gy_r. If $gr = g'r$, then $g'^{-1}g$ fixes r and hence fixes y_r. Thus the prescription is well defined. It is equivariant and respects R by its invariance.

For the effective version, enumerate R_r. A candidate y is fixed by G_r iff it is fixed by every member of the finite generating list. Equality of coded witnesses and the action are recursive, so apply this finite test to the enumerated candidates. The search halts by the fixed point hypothesis. Use the first accepted y_r and apply the specified transporter. To ensure the same choice throughout an orbit, run the enumerator with the same representative and list. The proof of well-definedness also proves equivariance of this recursive implementation. □

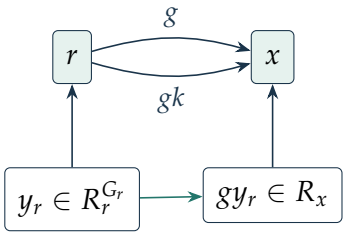


Figure XIII.2. The transporters g and gk give the same transported witness when $k \in G_r$ fixes y_r.

This calculation is the discrete case of selection over a space with transport. Indeed $R//G \to X//G$ is the map of action groupoids, with discrete fibers R_x

over points of $X//G$. A section over a connected component BG_r is equivalent to a homotopy fixed point of R_r. Since R_r is discrete, this is an ordinary fixed point. For instance take $X = \{*\}$, $G = C_2$, and $Y = C_2$ with its regular action. Both witnesses verify the same assertion on the sole input, and an ordinary recursive selector is available. The equivariant selector fails because neither witness is fixed. Changing the witness space to a connected space does not remove all obstructions, as $BC_4 \to BC_2$ in Chapter XII demonstrates.

If the parameterized family supplies recursive finite generators only after choosing the representatives, that choice is part of the computational specification. A statement asserting such generators locally, or a classical axiom of choice supplying them, does not give the effective routine in Theorem XIII.2. This is one of the points at which external computability conditions are needed in the higher treatment.

XIII.3 Ranks and relative closure

The transport calculation respects any rank defined by the call structure of well-founded records. If the transport arrow is an isomorphism of such labeled trees when one changes parameters, an induction from the leaves shows that the rank at each corresponding node is unchanged. At a leaf both ranks are zero. At any other node the correspondence of children identifies the ordinals whose strict supremum gives the rank. Hence the sets of ranks agree on ordinary orbits. This use of transport is entirely extensional. It does not prove that a bound on the ranks of one record in each orbit can be found effectively from a code for the family. The distinction persists even if all ranks have been specified internally as ordinal-valued annotations.

For set recursion the classical test on relative degree comparisons already requires care about which records stay in the ground structure. Sacks's theorem on Post's problem for the E-recursively enumerable degrees uses E-closure, ordinal indexing, and, on inadmissible levels of L, witnesses to divergence [11, Ch. XIII]. The split and density constructions use bounds on proper priority blocks [18, 17]. Equivariant refinements of such comparisons would in addition have to preserve the support of the selected successful or divergent records under transport. A construction on extensional degree classes can erase that requirement, as in the passage from supported reductions to degrees in Chapter IX. The finite-group examples of this chapter show its independence from pointwise enumerability and from the strength of the underlying numerical oracle.

There are accordingly several precise specialization tests for a proposed higher recursion theory. Restricting a record presentation to discrete inputs and outputs should recover the intended partial function or relation. Restricting to its recursive codes should recover the uniform tests and ordinal bounds of the

classical theory. Transporting across a cover should identify compatible records, and projecting to behavior should recover the existential predicates. The first two tests can succeed on fibers in examples where the third gives a genuine obstruction to selection. This supplies a reason to retain the records themselves in the higher construction.

XIII.4 Exercises [Worked solutions]

Exercise XIII.1. For the C_2-action exchanging even and odd queries, compute the two coordinates of $J(A)$ for programs that halt exactly when the answer to their sole oracle query is affirmative.

Exercise XIII.2. Show that the identity-coordinate many-one reduction in Theorem XIII.1 need not be equivariant if the target of the reduction uses the translated tags and the source uses a trivial action. Compare the two actions without changing the ordinary equivalence of degrees.

Exercise XIII.3. A partial computable oracle transformation is given on a G-invariant domain on which every output is a total characteristic function. State conditions under which the simulation part of Theorem XIII.1 still applies there. Verify the induced reduction for a composition of two such transformations.

Exercise XIII.4. Let $G = S_3$ act on the vertices and on the edges of a triangle. Take incidence as a relation between vertices and edges. Determine the existence of an equivariant selector in each direction. Repeat with edges disjoint from a given vertex as witnesses.

Exercise XIII.5. Present two G-invariant families of recursive trees with the same pointwise nonemptiness of branch sets. Arrange that one has an equivariant assignment of branches and that the other fails the fixed point test of Theorem XIII.2.

Exercise XIII.6. Construct a group action on a family of countably branching well-founded trees such that transport is faithful on some fiber, but all trees in the family have the same rank. Explain which data are visible to the classical rank predicate.

Worked solutions

The numbering matches the exercises in each chapter. The number at the start of each solution links to the exercise.

Chapter I Families of computations

I.1. Records of a composite Fix machine indices for f and g. Let $V_f(x, y, u)$ mean that u codes a finite execution of the chosen machine for f, starting on x and halting with output y. Define $V_g(y, z, v)$ in the same way. Bounded checks of the successive configurations and the final halting instruction give recursive predicates V_f, V_g. Use a recursive tuple code and set

$$V(x, z, \langle y, u, v\rangle) \iff V_f(x, y, u) \wedge V_g(y, z, v).$$

If the composite is defined at x, take $y = f(x)$, $z = g(y)$, and the actual finite traces. Conversely a verified triple implies $f(x) = y$ and $g(y) = z$. Thus projecting over z, y, u, v gives exactly the domain of $g \circ f$. The intermediate value belongs to the record even though this final projection forgets it.

I.2. Images of fiber products Factor $W \to U$ as an effective epimorphism onto D_W followed by a monomorphism. Do the same for Z, with image D_Z. The map

$$W \times_U Z \longrightarrow D_W \times_U Z \longrightarrow D_W \times_U D_Z$$

is a composite of pullbacks of the two effective epimorphisms. Hence it is an effective epimorphism. The map $D_W \times_U D_Z \to U$ is the monomorphism of the intersection. The displayed factorization is an image factorization, proving the assertion. Using an ordinary set-theoretic surjection on global points at the first step would miss effective epimorphisms with only local lifts.

I.3. Connected total space and disconnected fibers Use the degree-two cover $S^1 \to S^1$, $z \mapsto z^2$. Both spaces are connected. As a covering map its homotopy fiber over any point is equivalent to its ordinary two-point fiber. A map with discrete fibers is already relatively 0-truncated, so relative truncation leaves these

two components. The loop around the base exchanges them. In comparison, the connected components of the total space form a singleton.

I.4. A transitive two-point fiber Write the G-set as $D = G/H$ for an index-two subgroup H. The associated family has fiber D. Its image is BG because the fiber is inhabited, and its relative 0-truncation is the family itself. A section is a homotopy fixed point of the discrete G-space D. Such a point is exactly one fixed by every $g \in G$. A transitive action on two elements has no fixed element, so the map has no section.

I.5. Cylinders from oracle queries Record the finite set $F \subseteq \omega$ of queried positions and the agreed answers $a : F \to 2$. A purported trace giving inconsistent repeated answers has no realizing oracle. Otherwise the set of realizing oracles is

$$[a] = \{X \in 2^\omega : X \restriction F = a\}.$$

This is a finite-coordinate cylinder. To express it using prefix cylinders, set $N = 1 + \max F$ for nonempty F and $N = 0$ otherwise. Take the union of $[s]$ over all $s \in 2^N$ extending a. Their realizations replay the entire trace, since each successive oracle decision agrees with the recorded answer. Two finite-coordinate cylinders $[a]$, $[b]$ meet iff a and b agree on their common domain. Then their intersection is $[a \cup b]$. For prefix cylinders this condition is that the shorter prefix equals the corresponding initial segment of the longer one.

I.6. Associativity before and after taking images Let $W \to A \times B$, $Z \to B \times C$, $Q \to C \times D$ be witnesses for three relations. Over $A \times D$ the two parenthesizations of the retained records are

$$(W \times_B Z) \times_C Q \qquad \text{and} \qquad W \times_B (Z \times_C Q).$$

The universal property identifies both with the limit of the diagram $W \to B \leftarrow Z \to C \leftarrow Q$, including the homotopies matching the two intermediates. Replacing any inner record object by its image preserves the final image by Proposition I.4. Hence either iterated composition of subobjects is the image of this common limit. Internally both formulas express that some b, c witness the three relations simultaneously.

Chapter II Recursive trees and their path spaces

II.1. Primitive recursion on finite oracle transcripts Fix a primitive recursive coding of finite lists with primitive recursive length, entry, and append operations. For instance, one may keep the length explicitly and use iterated pairing for the entries. Use the same coding for the finite tapes and the states of a fixed

oracle machine testing $R(m, y, x, k)$, where m is the numerical parameter, y is the real parameter, and x codes the witness. Encode a machine configuration by the state and tape data, including the head positions and a flag for suspension. A single step on input m, k, with finite oracle strings p, s in place of y, x, is primitive recursive. At an oracle instruction requesting the i-th value of an oracle, read the specified list entry if it exists. Otherwise set the suspension flag. Acceptance, rejection, and suspension are absorbing states for this bounded simulation. The initial configuration is primitive recursive in m, k. Iterating the step t times by primitive recursion produces a primitive recursive table of the configurations through time t.

For strings p, s of common length n, let $B(m, p, s)$ say that none of the simulations for $k < n$ rejects within n steps. This is a bounded numerical test of the configuration codes. If there was a rejection on shorter prefixes, every step up to that rejection used oracle values already present there. The rejection therefore persists at larger common lengths. Consequently the pairs satisfying B form a recursive tree of pairs, and membership in that tree is primitive recursive as a predicate of the finite pair and m. For the fixed real parameter y, use the tree of all s with $B(m, y \restriction |s|, s)$, uniformly primitive recursive relative to y. If a total test on (y, x) rejects at some k, take n greater than k, the rejection time, and every queried position. Then the corresponding prefix is removed. If all tests accept, no run on the prefixes can reject. The paths are exactly the required witnesses relative to m, y.

II.2. Fixed points of the action Let F be the finite G-set. The positive levels of the tower are all the local system $I = EG \times_G F \to BG$, and its positive-level transition maps are identities. The root level is terminal in the slice over BG. The limit in that slice is consequently I. A section of I is a point of the limit of its straightened functor $BG \to \mathcal{S}$. Since the fibers are discrete, this limit is the ordinary set of vectors of choices compatible with transport. A choice at the single vertex of BG is an element $a \in F$, and compatibility along every $g \in G$ is exactly $g \cdot a = a$. Thus the section space is discrete with underlying set F^G. For $F = \varnothing$ the path object and its support are initial. For $F \neq \varnothing$ the support is all of BG, since the map is surjective on connected components, whether or not F^G is inhabited.

II.3. Intersecting two path supports In the slice, limits in the tower direction commute with the binary limit taking the product. Equivalently, a map from a test object to a coherent cone into the levelwise product is exactly a pair of coherent cones. Thus

$$W(T \times_U T') \simeq W(T) \times_U W(T').$$

Put $S = \mathrm{Path}(T)$, $S' = \mathrm{Path}(T')$, and factor the map on the right by

$$W(T) \times_U W(T') \longrightarrow S \times_U W(T') \longrightarrow S \times_U S' \hookrightarrow U.$$

The first two maps are pullbacks of the respective effective epimorphisms to the path supports. They are therefore effective epimorphisms, as is their composite. The last map is monic. By uniqueness of the image factorization the path support of the product is $S \times_U S' = S \cap S'$. The image records the local existence of a pair of witnesses over the intersection.

II.4. The first stage of accessibility Let A_α be the stages beginning with $A_0 = \varnothing$. On a well-founded tree these stages satisfy

$$s \in A_\alpha \quad \Longleftrightarrow \quad \mathrm{rank}(s) < \alpha.$$

At a successor, the child condition for $A_{\alpha+1}$ is $\mathrm{rank}(t) < \alpha$ for every immediate child t. It is equivalent to $\sup_t(\mathrm{rank}(t) + 1) \leq \alpha$, which is the desired condition at $\alpha + 1$. At a nonzero limit, taking a union implements the strict rank bound for the limit. The assertion at zero follows from the empty stage.

For a nonempty decreasing string with final entry m, its children have final entries precisely the $k < m$. Induction on m gives rank m and first appearance at stage $m + 1$. The root has rank $\sup_{m<\omega}(m + 1) = \omega$. Its children are all present at A_ω, so the root first appears at $A_{\omega+1}$. If the child ranks are only the positive even integers, their successors still have supremum ω. The root again first appears at $A_{\omega+1}$.

II.5. Path and accessibility on shrinking neighbourhoods Write M for the metric space and $U_0 = M$. The intersection of the U_n for $n \geq 1$ is K. The limit of this tower of subterminals is its meet in the frame of opens, namely the interior of that intersection. Since K is closed nowhere dense, the meet is empty. The path object and its support are both initial.

Let V_n^α be the open in U_n supporting accessibility at stage α. The successor clause at level n is the union of the opens $V \subseteq U_n$ satisfying $V \cap U_{n+1} \subseteq V_{n+1}^\alpha$. Indeed a section over V of the dependent product is exactly an accessibility witness on the child locus $V \cap U_{n+1}$. By induction, every V_n^α is disjoint from K. The statement holds at the empty stage and is preserved by unions at limit stages. For a successor, a proposed open containing $x \in K$ has x in its child locus, contrary to the preceding-stage claim there.

Now let $x \in M \setminus K$ and put $\delta = d(x, K) > 0$. Choose N with $1/N < \delta/2$, and take the open ball $V = B(x, \delta/2)$. The triangle inequality implies $V \cap U_N = \varnothing$. Hence $V \cap U_{N-1}$ supports accessibility at stage one. Applying the successor condition back up the levels gives $V \subseteq V_0^N$. Every point outside K has such an open, so the root accessibility support is exactly $M \setminus K$.

II.6. Effective boundedness with an oracle For the fixed z, choose a z-recursive family of witness trees S_m for the $\Sigma_1^1(z)$ set D. Such a presentation follows from the bounded simulation in II.1, using the prefixes of z at the appropriate stages. The given trees T_m are now uniformly z-recursive. Form the tagged tree

of synchronized pairs of prefixes from S_m, T_m, preceded by the tag m. A path would supply $m \in D$ together with a path through T_m, which cannot occur. The tagged tree therefore has a Kleene–Brouwer well-order with a z-recursive presentation. Its order type plus one is a z-recursive ordinal $\beta < \omega_1^z$ strictly bounding the root ranks in question. To see the bound, fix $m \in D$ and take a path through S_m. Pairing every node of T_m with the prefix of that path of the corresponding length gives the tree embedding above the tag m.

The trees and the recursive presentation of the bound may depend on z. The rank of any fixed actual well-founded tree is its ordinary ordinal rank, independent of which oracle presents its membership test. For pruning through β, choose a z-recursive presentation of $\beta + 1$. Membership of stage-labeled nodes in the pruning sequence is specified by the arithmetical recursion equations in that presentation and z. The sequence exists uniquely. Either assertion about a specified bit of its code is expressible by existentially quantifying a real satisfying the same equations. The code is therefore $\Delta^1_1(z)$, and it need not be computable by an ordinary machine with oracle z alone.

II.7. Two recursive trees of prescribed rank For any recursive well-order L, take the tree $D(L)$ of finite strictly L-decreasing strings. These are decidable using the order comparison and membership in the domain. The tree has no path, since a path would be a descending sequence in L. A nonempty string ending in a has rank $\mathrm{otp}(L_{<a})$. To prove this, induct on a and use the rank equation

$$\mathrm{rank}(s) = \sup_{b <_L a} (\mathrm{otp}(L_{<b}) + 1) = \mathrm{otp}(L_{<a}).$$

The supremum over the one-element strings gives root rank $\mathrm{otp}(L)$.

For rank $\omega + 1$, use positive integers in their ordinary order, and place the code 0 last. Then $\langle 0 \rangle$ is the only child of rank at least ω, and its rank is exactly ω. For rank $\omega \cdot 2$, use $\{0, 1\} \times \omega$ with the first coordinate compared first in lexicographic order. The child $\langle (0, n) \rangle$ has rank n, while $\langle (1, n) \rangle$ has rank $\omega + n$. Their ranks are all strictly below the root rank, and the successors of their ranks are cofinal in $\omega \cdot 2$.

II.8. The domain and the bit graph of a partial selector Write $D = \{n : \exists x\, R(n, x)\}$. Existential real quantification preserves Σ^1_1, so D is Σ^1_1. This upper bound is exact. For any Σ^1_1 set $A \subseteq \omega$, put $R(n, x)$ iff $n \in A$ and x is the constant zero real. This is a uniformly Σ^1_1 relation with unique inhabited fibers and domain A. In particular the domain can be Σ^1_1-complete.

The numerical graph is

$$G(n, k, m) \iff \exists x\, [R(n, x) \wedge x(k) = m].$$

It is Σ^1_1. Uniqueness of the real in each inhabited fiber gives

$$\neg G(n, k, m) \iff n \notin D \vee \exists x\, [R(n, x) \wedge x(k) \neq m].$$

If D is Δ^1_1, the right-hand side is Σ^1_1, so G is Δ^1_1. This includes the total case $D = \omega$. Conversely, $D(n)$ holds iff $\exists m\, G(n, 0, m)$. The lightface class Δ^1_1 is closed under number quantification. For the nontrivial form of this closure, countably many real witnesses to a universal family of Σ^1_1 assertions can be coded as one real with countably many columns. Thus G can be Δ^1_1 only if D is Δ^1_1.

Chapter III Jump hierarchies and oracle families

III.1. Compiling a reduction between jumps Let i index a total oracle computation $\varphi_i^Y = \chi_X$. Given i, e, build a Y-oracle program which ignores its own input and simulates $\varphi_e^X(e)$. On a query of the form $k \in X$, it runs $\varphi_i^Y(k)$, answers the query with that bit, and resumes the simulation. It halts exactly when the simulated program halts. All query simulations halt and supply the right bits by the hypothesis on i. The resulting program has an index $q(i, e)$, computed uniformly from i, e by the parameter theorem. Because it ignores its input, it has the same halting behavior at its own index $q(i, e)$ as at every other input. Thus

$$e \in X' \iff \varphi_e^X(e) \downarrow \iff \varphi_{q(i,e)}^Y(q(i,e)) \downarrow \iff q(i,e) \in Y'.$$

The simulated diagonal input e must be built into the target code. The index used to query Y' is the target index, so substituting that index as input to the original machine would in general test a different event.

III.2. Changing the columns of an ω-jump Use $a_0 = 1$ and $a_{n+1} = 2^{a_n}$ for finite ordinal notations. One limit code uses the sequence $n \mapsto a_{n+1}$ and the other uses $j \mapsto a_{2j+2}$. Both sequences are recursive and strictly increasing in the notation order. Each has supremum ω. Denote their joins by L_{all} and L_{even} respectively, using the same initial oracle X. A query $\langle j, k\rangle$ to L_{even} is the query $\langle 2j+1, k\rangle$ to L_{all}.

Conversely a query $\langle n, k\rangle$ to L_{all} asks about finite stage $n+1$. Put $j = \lfloor n/2 \rfloor$. Then $2j+2$ is either $n+1$ or $n+2$. In the first case query row k of the j-th even column. In the second case use a fixed recursive function p such that for every oracle Z, $k \in Z$ iff $p(k) \in Z'$. It sends k to a program which halts on every input precisely when its oracle answers yes on k. Query row $p(k)$ in column j. These procedures are uniform in X. They give many-one reductions for the displayed joins, in particular the requested Turing reductions.

III.3. A search for a pair of consecutive ones Count the clock by inspected bits. The locus for stopping after inspecting at most t bits is

$$U_t = \bigcup_{i+1<t} \{x \in 2^\omega : x(i) = x(i+1) = 1\}.$$

Each summand is clopen since it imposes two coordinate constraints. The union is finite, and for $t \leq 1$ it is empty. Measuring time in individual machine steps can change the bounded opens. Each occurrence of 11 is still detected eventually, so the increasing union is unchanged. The full halting locus $U = \bigcup_{t<\omega} U_t$ is the open of sequences with an occurrence of 11. Every basic cylinder meets U because one may extend its defining prefix by 11. The complement consists of sequences with no adjacent ones and is a closed set with empty interior. In particular the open complement in the sheaf truth-value lattice, which is the interior of the set complement, is empty.

III.4. Restricting to the zero point and its complement At the point 0^ω every coordinate of the pulled-back oracle is zero. Every bounded check for the program searching for a first 1 is false there. Accordingly its jump bit is false and has the constant binary decision 0 at that point. On the open $U = 2^\omega \setminus \{0^\omega\}$, the halting locus restricts to all of the open. Every point belongs to a cylinder where some inspected coordinate is 1, and those cylinders cover U. The jump bit thus has the constant binary decision 1 over the open. These computations can be made by restricting the bounded halting loci first, because inverse image at either restriction preserves finite limits and the colimit expressing the union of the bounded loci. The zero singleton has no open neighbourhood disjoint from U, so the two restrictions do not supply a clopen partition of the original base.

III.5. Success loci in a local system with two orbits Let the generator of $\mathbb{Z}$ exchange ports p, q and fix port r. Assign $X = \{0\}$ at p, q and $Y = \{1\}$ at r. For the first test use the jump bit given by a program that ignores its input and halts exactly if its oracle contains 0. Its success locus on the fiber is $\{p, q\}$. The associated map $I_e \to B\mathbb{Z}$ is surjective on components with nonempty two-point fibers. A section would pick a point invariant under the generator, and neither port in this orbit is invariant.

For the second test use the jump bit for the same program with query 1 in place of query 0. Its success locus is the singleton $\{r\}$. The action there is trivial. Transport preserves r, so its choice defines a section of the success object over $B\mathbb{Z}$.

III.6. Simultaneous bounded halting Index the searches by $j = 1, \ldots, m$, and let $U_{j,t}$ be the locus where search j succeeds by time t. A bounded execution of an oracle program with binary responses has only finitely many possible transcripts on the time interval in question. Each successful transcript restricts only finitely many coordinates. Thus $U_{j,t}$ is a union of cylinders, and each

$U_j = \bigcup_t U_{j,t}$ is open. Simultaneous success has locus

$$\bigcap_{j=1}^{m} U_j = \bigcup_{(t_1,\ldots,t_m)\in\omega^m} \bigcap_{j=1}^{m} U_{j,t_j} = \bigcup_{t<\omega} \bigcap_{j=1}^{m} U_{j,t}.$$

The last equality follows from monotonicity of the bounds by taking $t = \max_j t_j$. For the empty family the intersection is the whole base. Since the family is finite, these are identities of opens as well as identities of sets. At the sheaf level the product of the witness-time objects consists locally of tuples of successful time bounds. Its image is the union over the corresponding tuple loci in the middle expression. Equivalently, the product of the effective epimorphisms from witness objects to their separate images is an effective epimorphism onto the intersection of those images, by pullback stability and composition.

III.7. Closure rules for iteration on a class of oracles Let $Q = 2^{\mathbb{N}}$, and choose a subobject $C \hookrightarrow Q$. Assume the initial oracle family factors through C. A sufficient successor condition is that the relative halting predicate for the universal oracle over C be complemented in $C \times \mathbb{N}$, and that its characteristic function, curried to $C \to Q$, factor through C. This is a single uniform condition. By base change, a family classified by $U \to C$ has a complemented relative halting locus and the resulting jump is again classified by $U \to C$.

For a sufficient limit condition, suppose the universal join $C^{\mathbb{N}} \to Q$ factors through C. This map uses the chosen pairing of numbers and sends row $\langle n, k\rangle$ to the k-th bit of the n-th input column. At a limit notation, the previous columns give a map $U \to C^{\mathbb{N}}$ by the product universal property. Composing with the factored join produces the limit column in C. One could weaken this condition to closure for the particular column lists appearing in the chosen notation. Well-founded induction on subnotations proves closure under the entire iteration. The successor uses a binary decision of an existential halting predicate, whereas the limit uses an assembled sequence of preceding oracles and a closure property for its reindexing by pairing.

Chapter IV Ramification and inductive definitions

IV.1. Compactness and a bound on names A sufficient condition is that K be a compact element of the lattice $\mathrm{Sub}(I)$. In other words, whenever K lies below the join of a directed family, it lies below some member of that family. Proposition IV.1 identifies R_λ with the directed join of the R_α, so this condition supplies the bound.

For sheaves on a topological space X, take $I = 1$, so the subobjects are open sets ordered by inclusion. If the open K is quasi-compact as a space, a directed open cover of K admits a finite subcover. Directedness puts those finitely many

members below a common member of the family. Conversely let K have an open cover admitting no finite subcover, and choose from that cover a subcover $(U_i)_{i<\kappa}$ of least cardinality. The infinite cardinal κ is taken as an initial ordinal. Put $R_\alpha = \bigcup_{i<\alpha} U_i$ for $\alpha < \kappa$. The union of the R_α contains K, but each R_α uses fewer than κ members of the given cover and therefore misses some point of K. To make these open sets a ranked test in the stated form, use $N = W = I = 1$, identity witness map $W \to N$, and let N_α be the subterminal associated to R_α. Its images are the given R_α. This failure uses a limit length adapted to the cover. At length ω a concrete example is $K = \mathbb{R}$ with increasing opens $(-n, n)$.

IV.2. A fixed point reached at a successor of a limit Write $Q_0 = \bot$, $Q_{\alpha+1} = \Gamma(Q_\alpha)$, and $Q_\lambda = \bigvee_{\alpha<\lambda} Q_\alpha$. For any pre-fixed point S, the base inequality $Q_0 \leq S$ follows from bottom, and the successor inequality follows from

$$Q_{\alpha+1} = \Gamma(Q_\alpha) \leq \Gamma(S) \leq S.$$

At a limit the join of elements below S is below S. These three cases give the result by transfinite induction.

For the explicit system, give sort 0 a nullary rule and give sort $n+1$ a unary rule with premise n. Give $*$ a single rule with one premise of sort n for each $n \in \mathbb{N}$. Then $Q_k = \{0, \ldots, k-1\}$ at every finite stage k, $Q_\omega = \mathbb{N}$, and $Q_{\omega+1} = \mathbb{N} \amalg \{*\}$. The last is fixed and all earlier successive inclusions are strict. The rule for $*$ becomes available when the stages are joined, since its whole premise set is never contained in one finite stage. With only finite premise sets, any rule applicable to the increasing union at ω already has all its premises in one finite stage. Then Q_ω is fixed.

IV.3. Lifting a set of premises Let A be the set of rules, B_a the premise occurrences of a specific rule a, and D_i the derivations of a conclusion of sort i. The map from derivations to their support is the map $D_i \to 1$ for each inhabited D_i, and the unique map between empty sets otherwise. The fiber at a of the relevant dependent product map is

$$\prod_{b \in B_a} D_{s(b)} \longrightarrow \prod_{b \in B_a} Q_{\mathrm{der}, s(b)}.$$

If the target is empty the map is surjective. If the target is inhabited then every $D_{s(b)}$ is inhabited. Apply Choice to the set-indexed family $(D_{s(b)})_{b \in B_a}$ to obtain the required lift. The empty premise set gives the identity of a singleton. This proves pointwise surjectivity over the whole rule set and hence the hypothesis of Theorem IV.2, so the two predicates agree.

For the support assertion, the choices are made separately for any proposed rule application. If one asks for a simultaneous section of all the product maps with nonempty target, Choice can instead be applied to the one family of derivation sets indexed by pairs (a, b), where a is applicable and $b \in B_a$.

This also handles repeated occurrences of the same sort. No choice of such a simultaneous section is required for equality of supports.

IV.4. Support of the product near the omitted point Put $J = \mathrm{int}_X \bigcap_n U_n$. The desired support is the open

$$(X \setminus \{x\}) \cup J.$$

There is a section over $X \setminus \{x\}$ obtained by taking the V summand for every n. There is also a section over J, obtained by taking the U_n summand for the n-th coordinate. Thus the displayed open lies in the support. To see that there is no extra support at x, suppose some open neighborhood of x has a section. Local connectedness gives a connected open neighborhood W of x inside it. A section of $U_n \amalg V$ over W splits W into two disjoint open parts carrying the two summands. Connectedness makes the choice constant on W, and the point $x \notin V$ rules out the V summand. Therefore $W \subseteq U_n$ for every n, and $x \in J$. The support contains x only if there is such a local section. When J contains x, the support is all of X. Otherwise it is precisely $X \setminus \{x\}$.

IV.5. Conjugacy and sections of transitive fibers Let $\iota : K \to G$ be the given subgroup map. For the cover with fiber G/H, pullback along $B\iota$ gives the locally constant discrete family of fiber G/H with K acting through ι. A section is a fixed element of that action. Indeed a section has to transport its chosen element to itself around every loop of BK. Conversely such a fixed element defines a section, and the compatibility conditions have no higher ambiguity because the fiber is discrete. The fixed-coset test is

$$\iota(k)gH = gH \text{ for every } k \in K \quad \Longleftrightarrow \quad g^{-1}\iota(K)g \leq H.$$

Thus the pullback admits a section exactly when the image of K is contained in a conjugate of H. If the unary delay from the example is included, its parameter $n \in \mathbb{N}$ is fixed by the action, so the set of sections is $(G/H)^K \times \mathbb{N}$, with the same criterion for nonemptiness.

Replacing ι by $k \mapsto a\iota(k)a^{-1}$ sends each fixed coset gH to agH. This is a bijection of the sets of sections for the two maps, with inverse left multiplication by a^{-1}. Equivalently the two base-change maps are homotopic using the conjugating element as a natural isomorphism between the corresponding groupoid functors.

IV.6. The finite strings of unary constructors Let $L \to I$ be the object of nullary rules and let R be the space of unary rules, with input sort $s : R \to I$ and output sort $t : R \to I$. On $\mathcal{S}_{/I}$ the polynomial is

$$P(Y) = L \amalg F(Y), \qquad F(Y) = \Sigma_t s^* Y.$$

Put $D_0 = L$, and define

$$D_{n+1} = R \times^h_{I, s, q_n} D_n, \qquad q_{n+1} = t \circ \mathrm{pr}_R : D_{n+1} \longrightarrow I,$$

where q_n is the output sort of D_n. Thus an element of D_{n+1} is a unary rule, a shorter derivation, and an identification of its required input sort with the shorter derivation's output sort. These are homotopy pullbacks and so retain the identifications even for non-discrete I. Take $D = \coprod_{n \geq 0} D_n$. Pullback in spaces preserves coproducts, as does composition with t, so $F(D) \simeq \coprod_{n \geq 0} D_{n+1}$. This gives the algebra equivalence $L \amalg F(D) \simeq D$.

For initiality, let $L \amalg F(Y) \to Y$ have components $a_0 : L \to Y$ and $a_1 : F(Y) \to Y$. A map of algebras from D amounts to maps $f_n : D_n \to Y$ over I with homotopies

$$f_0 \simeq a_0, qquad f_{n+1} \simeq a_1 \circ F(f_n).$$

Here the description follows by splitting the map and its algebra-compatibility homotopy along the coproduct summands. At each finite step a new map and a path from it to a prescribed map form a contractible space, since this is the space of paths with one endpoint fixed. The space of all maps of algebras is the limit of the resulting tower of spaces of finite choices. Each of those spaces is contractible and hence so is the limit. This proves initiality with the higher identifications included.

IV.7. Equal predicates and different homotopy Take sorts $\mathbb{N}$ and use only nullary rules at sort 0. In one encoding the space of such rules is a point, and in the other it is BC_2 for the cyclic group of order two. In terms of the polynomial signature, the premise-occurrence object is empty, the rule object maps constantly to 0, and the polynomial is constant at that rule object over $\mathbb{N}$. Both encodings therefore induce the constant operator $Q \mapsto \{0\}$ on subsets, since both rule spaces are inhabited.

The initial algebra of a constant endofunctor at A is A with its identity structure map. An algebra is a map $A \to Y$, and the pair of an algebra map from A and its compatibility homotopy is the space of maps to Y with a specified path to that structure map. This space is contractible. At sort 0 the first derivation object has trivial fundamental group, while the second has fundamental group C_2. Thus the initial algebras are inequivalent even over a discrete object of sorts.

Chapter V Forcing and sheaves of witnesses

V.1. The separative quotient In the topology of lower sets, the smallest open neighborhood of r is $\downarrow r$. Thus

$$r \in \operatorname{int} \overline{\downarrow q} \quad \Longleftrightarrow \quad (\forall t \leq r)(\exists u \leq t)\, u \leq q.$$

The right side says exactly that every extension of r is compatible with q. Since $p \in e(p)$, the inequality $e(p) \leq e(q)$ implies this test at p. Conversely, if the test

holds, $p \in e(q)$ and hence $\downarrow p \subseteq e(q)$. Regularizing both sides gives $e(p) \leq e(q)$, since $e(q)$ is regular open.

Define $p \preceq q$ by this compatibility test, and identify p and q if each precedes the other. By the characterization just proved, they are identified exactly when $e(p) = e(q)$. The classes, ordered by $\preceq$, embed as the dense suborder $e(P)$ of the nonzero regular opens. The quotient is separative. If $[p] \not\leq [q]$, the failed test yields an extension $r \leq p$ incompatible with q, and $[r]$ is below $[p]$ and incompatible with $[q]$.

V.2. Complete primality and ground-model genericity Suppose F is a proper completely prime filter of a complete Boolean algebra B. Ordinary primality makes it an ultrafilter. Set $a = \bigwedge_{b \in F} b$. If $\neg a$ were in F, then complete primality applied to $\neg a = \bigvee_{b \in F} \neg b$ would put some $\neg b$ in F together with b, a contradiction. Hence $a \in F$, so $a \neq 0$. For any $0 < c \leq a$, the ultrafilter contains either c or $\neg c$. The second possibility would give $a \leq \neg c$, contrary to $0 < c \leq a$. The first gives $a \leq c$, so $c = a$. Thus a is an atom. The filter is exactly the principal filter above that atom, which cannot exist when B is atomless.

In the ground-model statement, let (b_i) belong to the countable model M and let b be its join as computed there. The set of nonzero conditions below $\neg b$ or below some b_i is dense and belongs to M. Indeed a nonzero condition meeting b has nonzero meet with at least one b_i, by distributivity of meet over that join. A ground-generic filter meets this dense set. If it contains b, it cannot meet the part below $\neg b$, so it contains some b_i by upward closure. The converse also follows from upward closure. One obtains such a filter externally by enumerating the ground-model dense sets and taking the filter generated by a sequence of successive refinements meeting them. This test applies to ground-model families. The family $(b)_{b \in F}$ used in the external atom argument need not belong to M, so genericity supplies no preservation test for its meet or the complementary join.

V.3. A cover without disjoint open refinements The two proper opens $(-\infty, 1)$ and $(0, \infty)$ cover $\mathbb{R}$. Any partition of a connected space into disjoint nonempty open sets has one part, since each part has open complement given by the union of the others. A disjoint open refinement covering $\mathbb{R}$ would therefore have $\mathbb{R}$ as its single nonempty member, and that member would have to lie in one of the two proper opens. This is impossible.

The same reasoning applies to each pair U_n, V in Example IV.3, since both opens are proper and cover the line. A section of $U_n \amalg V$ on an open set gives precisely a partition into open pieces selecting one summand at each point. On a connected neighborhood of zero there can be no switch between summands, so the only permissible summand is U_n. For a section of the product on one such neighborhood, the argument applies at once to every coordinate. The neighborhood would lie in $\bigcap_n U_n = \{0\}$. Each individual support is therefore

all of the line, but the support of the product omits zero. In a Boolean site the disjoint-refinement lemma would allow local lifts to be glued whenever they cover a condition.

V.4. Failure of the bound at a regular cardinal Failure of κ-cc supplies κ pairwise disjoint nonzero elements $d_\xi \in B$, indexed by $\xi < \kappa$. Let

$$d = \bigvee_{\xi<\kappa} d_\xi, \qquad c_\alpha = \neg d \vee \bigvee_{\xi<\alpha} d_\xi \quad (\alpha < \kappa).$$

The sequence is increasing, and its join is $\neg d \vee d = 1$. For each $\alpha < \kappa$, the positive element d_α is disjoint from c_α, so $c_\alpha \neq 1$. The chain of subterminal sheaves h_{c_α} has join terminal under the identification $\mathrm{Sub}(1) \cong B$, but none of its members is terminal. Taking the index ordinal to be the regular infinite cardinal κ gives a counterexample with the cofinality hypothesis of Theorem V.3 still satisfied.

V.5. The closed-set event in three forcing algebras Write $P = \{y : (\forall n)\, y(2n) = x(n)\}$ for the prescribed binary sequence x. This is closed and nowhere dense. Every basic cylinder has a refinement whose string disagrees with x at an even coordinate beyond the end of the original string. Thus the conditions forcing $\dot{y} \notin P$ are dense in Cohen forcing and $[\![\dot{y} \in P]\!] = 0$. Its measure is zero as well. The event of agreement at the first m even coordinates has measure 2^{-m}, and these events decrease to P. Continuity from above gives $\mu(P) = 0$. Its class in the measure algebra is therefore zero.

For full perfect forcing, P is itself a condition. The unrestricted odd coordinates give a homeomorphism from Cantor space onto P. A perfect condition $Q \subseteq P$ forces membership, since a refinement deciding any finite initial segment of the generic point can decide it only consistently with the fixed coordinates. If $Q \not\subseteq P$, choose a point in $Q \setminus P$ and a cylinder containing it and disjoint from P. Intersecting Q with that cylinder gives a nonempty clopen subspace of Q, hence a perfect extension forcing nonmembership. Consequently the exact Boolean value in full perfect forcing is $e(P)$. To see the upper bound explicitly, a Boolean value larger than $e(P)$ would have a nonzero part disjoint from $e(P)$, hence contain $e(Q)$ for a condition forcing membership but incompatible with P. The refinement just described rules out that situation. The lower bound comes from P itself. A cylinder contradicting one fixed coordinate is also a perfect condition disjoint from P, so $0 < e(P) < 1$. The decisive comparison in this algebra is refinement by inclusion of perfect sets. The fact that P has empty interior in Cantor space leaves its status as a perfect condition intact.

V.6. Fusion for Hausdorff-valued outputs Let $F : P \to Y$ be continuous with Y Hausdorff and P a nonempty perfect closed subspace of Cantor space. If there is a nonempty relatively clopen part on which F is constant, use it. In the

remaining case, from any nonempty relatively clopen part of P take two points with distinct images. The Hausdorff property gives disjoint open neighborhoods of the two images. Their preimages are relatively open in P, so the two points have relatively clopen neighborhoods in those preimages, inside the parent part and with diameters as small as desired. These neighborhoods are the children at the next stage. This is sufficient to run the binary fusion of Proposition V.4. Nesting, compactness, and the diameter bounds identify branches with points of a perfect closed subspace. The distinct images at every split give injectivity, and compactness of that subspace together with the Hausdorff property gives a topological embedding. The argument did not require the countable separating family. The open neighborhoods used at a split need not be chosen from that family.

For an effective version, a sufficient set of data includes effective codes for the successive child pieces, with effective information determining longer and longer binary prefixes along a branch. It also includes two coded disjoint target neighborhoods at each split, known to contain the images of the corresponding two children. Membership in a coded neighborhood should be semidecidable from an output name, uniformly in the code. Given an output name in the image of the fusion set, run the two membership tests in parallel at the current split. The output is in exactly one of the two neighborhoods, so one test succeeds and determines the next child. Iterate to recover the branch, and use the child-piece data to compute every finite input prefix. A bare countable separating family provides neither a uniform method for these assignments nor a computable interpretation of target names.

V.7. Section spaces over a Boolean partition Let $b = \bigvee_i b_i$ be a disjoint partition of the support and let $r \leq b$. Write $r_i = r \wedge b_i$. Then $(r_i)_i$ is a disjoint partition of r. Suppose the witness sheaf on the i-th part is constant at a connected groupoid $\mathcal{G}_i$. By Lemma V.1 the space of sections over r is the product of the section spaces of those constant groupoids over their respective nonzero parts r_i.

One can describe each factor explicitly. Choose an object of $\mathcal{G}_i$ with automorphism group H_i, so its nerve is equivalent to BH_i. In the slice over h_{r_i}, the constant sheaf at BH_i classifies torsors for the constant discrete group sheaf with fiber H_i. Indeed the constant-object functor preserves finite products and geometric realizations, so it carries the usual bar construction to the required classifying object. Any such torsor has an effective epimorphism to h_{r_i}. It has a section by the disjoint-refinement lemma and is therefore trivial. The groupoid of sections is consequently connected, and the automorphisms of its trivial torsor form the group $\mathrm{LC}(r_i, H_i)$ of locally constant sections of the group sheaf. A member of this group can be described by disjoint elements $(c_h)_{h \in H_i}$ with join r_i, labelling the part c_h by h. Group multiplication takes common refinements

and multiplies the two labels. It follows that the section space is

$$\prod_{\{i : r_i \neq 0\}} B\,\mathrm{LC}(r_i, H_i).$$

For an example take the Boolean algebra of subsets of $\{0,1,2\}$ with partition $\{0\}, \{1,2\}$, and restrict to $r = \{0,1\}$. Taking trivial groups on both parts gives a point as section space over r. Taking C_2 on the first part and the trivial group on the second gives BC_2 there. Both witness sheaves have full support, while the two section spaces have different fundamental groups.

Chapter VI Admissible bounds and witness objects

VI.1. Separation by a pair of signed tests Let $a \in L_\alpha$ and put $M = L_\alpha$. Write the two tests as

$$P_i(x) \iff M \models \exists w\, R_i(x, w, p) \qquad (i = 0, 1),$$

where the matrices are bounded, and suppose exactly one sign holds at every $x \in a$. Pair the sign and the witness to form one record $v = \langle i, w \rangle$. The assertion that v is a valid record for x is bounded and total on a. Bounded Collection gives a set $C \in M$ containing at least one valid record for every x. Bounded Separation now forms

$$a_1 = \{x \in a : (\exists v \in C)\, v \text{ is a valid record of sign 1 for } x\}.$$

If sign 1 is true at x, the collecting set must contain a record of that sign, because any valid record for x has it. The reverse inclusion follows from validity. The negative test supplies a record at every negative input, making the combined relation total. The exclusivity of the signs lets us use any valid record in the collecting set. A positive semidecision alone can have a domain on which Collection does not apply.

VI.2. Quotients of bounded equivalence relations For $x \in A$, bounded Separation gives $C_x = \{y \in A : xEy\} \in M$. There is a bounded test that $z = C_x$, namely $z \subseteq A$ and $(\forall y \in A)(y \in z \leftrightarrow xEy)$. Collection provides a set containing C_x for all $x \in A$. Discard its extra elements by bounded Separation to obtain $Q = \{C_x : x \in A\} \in M$. The same class-description test cuts out the graph of $q : A \to Q$, $x \mapsto C_x$, from $A \times Q$. The kernel pair of q is E. If $g : A \to Z$ equalizes E, its unique factorization has the graph cut out from $Q \times Z$ by $(\exists x \in A)(q(x) = c \wedge g(x) = z)$. Hence q coequalizes E universally in $\mathsf{Set}(M)$.

For $u : Q' \to Q$ in $\mathsf{Set}(M)$, let $A' = A \times_Q Q'$. The map $q' : A' \to Q'$ is again surjective. Its kernel pair relates $(x, b), (y, c)$ precisely when $b = c$, which

entails $q(x) = q(y)$ and thus xEy. It is the relation obtained by pulling back $E = A \times_Q A$ as an object over Q along u. The same bounded graph formula, with A', q' in place of A, q, proves that q' coequalizes this relation universally. This proves pullback stability, including when Q' is empty.

VI.3. A metafinite union with an external presentation Work at $M = L_{\omega_1^{\mathrm{CK}}}$. Choose a nonhyperarithmetic set $P \subseteq \omega$ containing 0, for instance a shifted copy of Kleene's $\mathcal{O}$ with 0 adjoined. Define

$$S_n = \begin{cases} \{0\} & n \in P, \\ \varnothing & n \notin P. \end{cases}$$

Each S_n is metafinite and $\bigcup_n S_n = \{0\} \in M$. If the sequence graph G were in M, the bounded formula $(\exists z \in \{\varnothing, \{0\}\})(\langle n, z\rangle \in G \wedge 0 \in z)$ and Separation on $\omega \in M$ would give $P \in M$. This contradicts the characterization of its reals as hyperarithmetic. Collection bounds witnesses to a relation of the stated definability that is total over an internal domain. The single union here gives no such relation describing which set is assigned to which integer, and the assignment itself cannot have a total functional $\Sigma_1(M)$ graph by the bounded-record proposition in Chapter VI.

VI.4. A filtered family of witness stages Let κ be infinite regular, let J be small and κ-filtered, and take a diagram $j \mapsto (W_j \to X)$ in the slice $\mathcal{X}_{/X}$, where $X \to B$ belongs to an ∞-topos. Suppose $X \to B$ is κ-compact in $\mathcal{X}_{/B}$. Let $I_j \hookrightarrow X$ be the image of $W_j \to X$. Image factorization is the reflector from objects over X to subobjects of X, so the image of the colimit of the W_j is the join of the I_j. Since J is filtered, the colimit of the monomorphisms $I_j \to X$ in $\mathcal{X}_{/X}$ is still monic. Indeed its diagonal is the colimit of the diagonals, since filtered colimits in this slice commute with finite limits. This monomorphism therefore represents the join. If the join is top, the map $\operatorname{colim}_J I_j \to X$ is an equivalence.

The underlying colimit over B is computed by forgetting the map to X, since slice colimits are computed on the underlying objects. Compactness now gives

$$\operatorname{Map}_{/B}\bigl(X, \operatorname{colim}_J I_j\bigr) \simeq \operatorname{colim}_J \operatorname{Map}_{/B}(X, I_j).$$

Take the component corresponding to id_X. It is represented by some $X \to I_j$, because the functor π_0 on spaces preserves colimits. Its composite into X is homotopic to the identity, and $I_j \hookrightarrow X$ is monic. It follows that $I_j = X$. Base change and detection along an effective epimorphism carry over as in the bound transfer theorem in Chapter VI, because those arguments apply to each image and any join. For the stage assertion it also suffices to assume directly that the top of $\mathrm{Sub}(X)$ is compact for the given class of joins. This order-theoretic hypothesis describes the part of compactness actually used for images.

VI.5. Uniform stages over a compact space Subobjects of 1 in $\mathrm{Sh}_\infty(K)$ are the opens of K, with joins given by union. Let (U_i) be the directed family. Its union K is an open cover, so some finite subfamily covers K by compactness. Directedness gives one U_j containing each member of this finite subfamily, and hence $U_j = K$. Here directedness includes nonemptiness, which also handles the case $K = \varnothing$. The argument uses compactness alone among the topological hypotheses. On $\mathbb{R}$, the opens $(-n, n)$ for $n \geq 1$ have union $\mathbb{R}$ and are directed, but every finite subunion remains bounded and omits points.

VI.6. The monodromy of the universal two-point family The pullback square in spaces is

$$\begin{array}{ccc} \Omega BC_2 & \longrightarrow & * \\ \downarrow & & \downarrow \\ * & \longrightarrow & BC_2. \end{array}$$

The upper-left object represents paths in BC_2 from the basepoint to itself. It is therefore the loop space and is equivalent to the discrete group C_2. For transport in the original family $* \to BC_2$, represent it by the endpoint fibration of paths starting at the basepoint. Transport along a loop γ changes a fiber element represented by p to the path p followed by γ. With multiplication defined by that concatenation convention, this acts on $\Omega BC_2 \simeq C_2$ by right multiplication by $[\gamma]$. The nonidentity loop exchanges the two fiber elements. The pullback itself has base a point. This action describes how its fiber is transported in the original family over BC_2.

VI.7. Orders and rank certificates under a permutation group Write D for the set of d leaves, and equip the rooted tree $\{r\} \amalg D$ with the action of G fixing r. The family of linear orders has fiber $\mathrm{Lin}(D)$ over BG with G acting by relabelling. Its sections are the G-fixed orders, since this is a local system of discrete sets. Every bijection preserving a finite linear order is the identity. Thus a fixed order is possible exactly when every element of G acts trivially on D, which here is equivalent to $G = 1$. For $G = 1$ there are $d!$ sections.

The canonical ordinal rank assigns 0 to each leaf and 1 to r when $d > 0$. For $d = 0$ it assigns 0 to the only vertex. The assignment is G-invariant and therefore gives a section for any G. If the family is defined by the exact rank recursion, this assignment is unique. If the family allows any ordinal-valued function that strictly decreases from a vertex to its children, the canonical rank remains a section. The other sections are precisely those assignments that are constant on each leaf orbit and have root value strictly larger than each leaf value, subject to any imposed ordinal bound.

Chapter VII Relative computation and support

VII.1. A formula for the composite support Let $R_\varepsilon(H, J, K)$ enumerate the supports for sign ε in the reduction of A to B. The two signs refer to $K \subseteq A$ and $K \cap A = \varnothing$, respectively. Let $S_+(U, V, H)$ and $S_-(U, V, J)$ enumerate supports for the positive and negative queries of the reduction of B to C. Write their Σ_1 definitions using bounded matrices $\theta_\varepsilon, \sigma_+, \sigma_-$ and witness variables w, w_+, w_-. For proposed C-support (X, Y) in the ε-query about K, use

$$\begin{aligned}\exists H, J, U_+, V_+, U_-, V_-, w, w_+, w_- \ \big[&\theta_\varepsilon(H, J, K, w)\\ &\wedge \sigma_+(U_+, V_+, H, w_+)\\ &\wedge \sigma_-(U_-, V_-, J, w_-)\\ &\wedge X = U_+ \cup U_- \ \wedge \ Y = V_+ \cup V_-\\ &\wedge X \cap Y = \varnothing\big].\end{aligned}$$

The finite tuple of witnesses can be packed as a single existential variable. In addition to the bounded matrices, the union tests are bounded by the supplied operands and result. For example one tests $X \subseteq U_+ \cup U_-$ by quantifying over X, and tests the two reverse inclusions by quantifying over U_+, U_-. Disjointness uses a bounded quantifier over X. Truth with oracle C means this new support is compatible with C. Any such record certifies both halves of the intermediate support and then the original query. Conversely a true query supplies a compatible B-support and hence two compatible verifications in C. Their respective unions are compatible because both were satisfied by C.

VII.2. One enumeration history and oracle table By the admissible-expansion theorem in Chapter VII, $(M, \in, A)$ satisfies bounded Collection in the expanded language. Let $\gamma < \alpha$ bound the inputs under discussion, and write the enumeration of the weak rule as

$$R(H, J, x, y) \iff \exists z\, D(H, J, x, y, z, p)$$

with D bounded. Totality on γ says that for every $x \in \gamma$ there is a record $\langle H, J, y, z\rangle$ such that D holds, $H \subseteq A$, and $J \cap A = \varnothing$. All these tests are bounded in the expanded language. Apply expanded Collection to obtain one set of records containing at least one valid record for every input. Choose $\eta < \alpha$ such that L_η contains that collecting set, γ, and the parameters of D, using that L_α is a limit of the levels below it. By transitivity L_η contains the components of all collected records, and all ordinals in their supports are below η.

For an enumeration history, keep only record components in L_η and use $(\exists z \in L_\eta) D(H, J, x, y, z, p)$ as the enumeration test. This entire history on the bounded candidates is available by bounded Separation from a finite power of

L_η, which belongs to M. Regularity gives the single oracle table $A \cap \eta \in M$. It decides all the support checks for candidates in the history, since such supports belong to L_η. Each $x \in \gamma$ has a verified record among the collected ones. Any verified record in the history is also valid with the full oracle, by transitivity and bounded absoluteness. Functionality therefore gives exactly the prescribed graph of f on γ.

VII.3. Transfer along an internal bijection Let $j : \beta \to H$ be the given bijection in M. For a subset $P \subseteq H$ given by a $\Sigma_1(M)$ formula, $S = \{b < \beta : j(b) \in P\}$ is also $\Sigma_1(M)$, since evaluation of the graph of j on the bounded set H uses only bounded quantifiers. As in the first-projectum argument of Chapter VII, take an injective α-recursive enumeration of S on an initial segment of α. Its domain could be all of α only if it gave an α-recursive injection into $\beta < \rho_1(\alpha)$. Thus the domain is a proper initial segment. the bounded-record proposition in Chapter VI puts its graph and hence S in M. Finally $P = j[S]$ is the image under an M-function of an M-set and is formed by bounded Separation on H.

VII.4. Overlap compatibility for arbitrary discrete answers Work over B and let $K \to B$ be 0-truncated, with chosen sections $k_i : B \to K$ for the signs $i = 0, 1$. For any generalized oracle $T \to K^D$, its equality with k_i on $D_i \times_B T$ holds exactly when its pullback agrees with k_i on $U_i \times_B T$. In fact the equalizer is monic because $K \to B$ is 0-truncated, and topness of that equalizer is detected on the effective epimorphism $U_i \times_B T \to D_i \times_B T$. This proves amalgamation with K as the answer object.

Let $Q = D_0 \times_D D_1$. The prescribed values agree on the overlap exactly when the two maps $Q \to K$ obtained from k_0, k_1 are equal over B. Equivalently the structure map $Q \to B$ factors through the equalizer of k_0, k_1 in B. This condition is necessary and sufficient to glue the answers to a prescription on the union $D_0 \vee D_1$. To see sufficiency, use descent along $D_0 \amalg D_1 \to D_0 \vee D_1$, which is an effective epimorphism. Its kernel pair consists of the two same-support diagonals and the two copies of Q. Equality holds automatically on the diagonal terms and by hypothesis on Q. Since the target is 0-truncated, these equalities give the requisite compatible maps on the entire Čech nerve. A global extension from the union to all of D is an additional condition. For example, prescribe different values on the negative and positive open half-lines in $\mathsf{Sh}_\infty(\mathbb{R})$, with $D = B = 1$ and $K = 2$. The overlap is empty and the union admits the prescription. A map $1 \to 2$ is a locally constant function on $\mathbb{R}$, so it cannot extend that prescription.

VII.5. The complementary oracles on a torsor Write τ for the involution of D. In the slice over D the object of query labels is $D \times D \to D$ by the first projection. Give the positive occurrence by the graph of the identity and the

negative occurrence by the graph of τ. Explicitly the sections are

$$s_+(u) = (u, u), \qquad s_-(u) = (u, \tau u).$$

They are equivariant and disjoint. At basepoint u of the torsor, the cylinder consists of exactly one oracle o_u, with $o_u(u) = 1$ and $o_u(\tau u) = 0$. The cylinder is thus a section of the pulled-back oracle object over D. Transport by the generator sends (u, o_u) to $(\tau u, o_u \circ \tau^{-1}) = (\tau u, o_{\tau u})$. In fixed coordinates on the two-point set, $o_{\tau u} = 1 - o_u$. Consequently the two complementary assignments form one free orbit in the oracle object over 1. The displayed slice section uses the selected frame u. An asymmetric assignment in that orbit has no equivariant global section over 1.

VII.6. Connected labels with opposite answers A map from a connected space D to 2 factors through the single component of D and is constant, since 2 is discrete. Thus no such map takes distinct prescribed values on the two points, and the aggregate cylinder is initial. More explicitly, each map from a point into D is an effective epimorphism in spaces, since its map on π_0 is surjective. Its image subobject is therefore D. The two opposing signs each have all of D as their aggregate support. Their support overlap is all of D, on which the two constant answer maps disagree. Distinct points of a connected space can consequently prescribe incompatible answers even though their point names are different.

Chapter VIII Priority and coherent stabilization

VIII.1. Conflicting restraints and self-injury Represent a restraint by a finite partial map into $\{0, 1\}$. For an upper restraint h and a lower restraint ℓ, the two are compatible precisely when their values agree on the intersection of their domains. When they are compatible, preserve the union. On a conflict preserve h and discard the lower witness together with its restraint. A new lower witness is eligible only if its computation presents a restraint compatible with h. For an enumeration in which positive oracle answers are permanent, also require a new restraint to agree with the positive answers already enumerated.

This rule by itself permits unboundedly many lower actions. For example, let the upper restraint protect $h(0) = 1$ forever. After discarding an incompatible restraint prescribing $\ell(0) = 0$, the lower requirement can take the witness n with restraint $\{(n, 0)\}$ for each $n = 1, 2, \ldots$. These restraints are all compatible with h. A rule allowing the lower requirement to discard each successful witness and take the next one gives infinitely many actions even with no further upper action.

Add a rule that a lower requirement installs at most one witness until an actual higher-priority action invalidates it. A failed search for an eligible witness

does not count as an installation. After an installation, keep the witness and its restraint unless a higher action conflicts with a recorded answer. After the upper restraints cease changing, the lower requirement can install at most once. This bound is on actions, while eventual satisfaction also requires an opportunity to find an eligible witness when the lower requirement is still pending.

VIII.2. Charging injuries to higher actions Number requirements by increasing priority index, with R_0 highest. Let a_n be the total number of actions of R_n. Use the one-action rule on each interval separated by injuries, including the interval before the first injury and the interval after the last injury. Thus a requirement with no injuries can act at most once.

Each injury of R_n is charged to a distinct action of one of $R_0, \dots, R_{n-1}$. If their action counts are finite, there are at most $\sum_{j<n} a_j$ injuries. Consequently

$$a_0 \leq 1, \qquad a_n \leq 1 + \sum_{j<n} a_j \leq 2^n.$$

The final inequality follows by induction using $\sum_{j<n} 2^j = 2^n - 1$. The same estimate applies using the number of strict predecessors when the finite predecessor set is listed in a compatible linear order.

With infinitely many predecessors, finiteness of every summand no longer gives a finite bound on the number of charges. For example, place Q below requirements R_n for $n < \omega$. At stage $2n$ let Q act once, and at $2n + 1$ let R_n act for its only time and injure Q. This respects distinct charges and the interval rule. It supplies infinitely many actions of Q, so a separate bound on the collected higher actions is needed.

VIII.3. A common cofinal cone Use as the common-cone hypothesis the existence of $b \in S$ such that for every i in the chosen family J, the transitions of $H(i, -)$ on $S_{\geq b}$ are equivalences. The subposet of I on J keeps all its restriction arrows. If separate principal cones $S_{\geq b_i}$ have this property, directedness gives a single upper bound b on the finite list of b_i, and restriction to $S_{\geq b}$ gives the common hypothesis.

For every $s \in S$ the poset of common upper bounds of s and b is nonempty and directed. Its nerve is contractible, since any finite simplicial diagram of its vertices admits a cone at an upper bound. The finality criterion therefore makes $S_{\geq b} \hookrightarrow S$ cofinal for colimits. Within that cone the canonical map from the constant diagram at $H(i, b)$ is an objectwise equivalence, and the cone has an initial object b. Thus $H(i, b) \to \operatorname{colim}_{s \in S} H(i, s)$ is an equivalence. The comparison uses the colimit cocones, so it is natural in all the restriction arrows on J^{op}. This gives the representative $H(-, b)|_{J^{\mathrm{op}}}$ for the entire eventual diagram on these windows.

VIII.4. A stable family with monodromy Take I to have one window and use stages $s < \omega$. At stage zero use the initial object $\varnothing \to BG$. At every later stage use the associated family $EG \times_G E \to BG$, with identity transition maps. Use the unique map out of the initial object for the first transition. The bound is 1. The eventual fiber is E, transported along a loop $g \in G$ by the action of g.

A section of the stable family corresponds to a G-equivariant map $EG \to E$, where EG is contractible and E is discrete. Every such map is constant with value e, and equivariance says $ge = e$ for all g. Conversely, each such point gives a section over BG. The space of sections is thus the discrete set E^G. A nontrivial action can have fixed points, as when the cyclic group of order two exchanges two points and fixes a third. If it acts by exchanging just two points, E^G is empty. In both cases the family is stable. Its stage bound certifies equivalences on the tail independently of whether the family admits a section.

VIII.5. An obstruction in a chosen homotopy group Fix $k \geq 2$, choose a pointed Eilenberg–Mac Lane space $X = K(\mathbb{Z}, k)$, and put $H_k(i, s) = X^{\min(i,s)}$. Drop final factors when restricting a window and append basepoints when advancing a stage. These operations commute. At fixed i the eventual value is X^i. At fixed s the inverse limit over windows is X^s, since the window diagram is constant for $i \geq s$ and that tail computes its inverse limit. The comparison is the usual map from the filtered colimit of finite products under insertion of basepoints to the full countable product.

All the spaces involved are connected. In spaces, filtered colimits commute with the computation of homotopy groups of pointed spaces. Indeed a pointed sphere map and a homotopy between sphere maps can be represented at a common stage by compactness of finite simplicial models for the sphere and its cylinder. Products commute with based loop spaces and compute products of homotopy groups. It follows that for $j \geq 1$ the comparison on π_j is

$$\begin{cases} \bigoplus_{n<\omega} \mathbb{Z} \hookrightarrow \prod_{n<\omega} \mathbb{Z}, & j = k, \\ 0 \longrightarrow 0, & j \neq k. \end{cases}$$

The π_k map takes a tuple to its infinite extension by zeros. It misses the sequence with every coordinate 1. Both component sets are singletons.

VIII.6. Cofinality and bounds on a cover Let S be a nonempty limit ordinal. Any set of fewer than $\mathrm{cf}(S)$ stages is bounded in S, by the definition of cofinality. If $|I| < \mathrm{cf}(S)$, the set of bounds b_i therefore has a common upper bound $b \in S$. A successor ordinal has a greatest stage, which gives a bound with no cardinality hypothesis. The strict inequality cannot be replaced by an unrestricted weak inequality at a limit. Take I discrete of size $\mathrm{cf}(S)$ and assign a cofinal family of stages b_i. Let $H(i, s)$ be empty before b_i and a point afterward. Each window stabilizes, but no stage is a bound for them all.

Suppose $V_j \to U$ is a covering family and each V_j has a bound b_j common for the desired windows. If the index set of the cover has size less than $\mathrm{cf}(S)$, choose $b \in S$ above all the b_j. The coproduct $\coprod_j V_j \to U$ is an effective epimorphism. After pullback to it, each transition on the tail from b is an equivalence on every summand, hence an equivalence there. Theorem VIII.4 makes it an equivalence over U. If a bound is given separately for each pair (j, i), the same proof works when the number of pairs is less than $\mathrm{cf}(S)$.

Chapter IX Supported reductions and degree structure

IX.1. Query bounds under composition From four input bits x_0, x_1, x_2, x_3, form the two-bit string

$$y_0 = x_0 \oplus x_1, \qquad y_1 = x_2 \oplus x_3,$$

where $\oplus$ denotes addition modulo two. Each output bit queries two fixed positions of the input. Now query y_0, y_1 and output $y_0 \oplus y_1$. Substitution queries all four distinct input positions and computes parity of the original string.

In fact no deterministic algorithm for this function uses fewer than four queries on any input. If an algorithm stops before it has inspected some position i, consider the input obtained by flipping the bit at i. On the new input all inspected answers are the same. Induction on the query trace shows that it asks the same questions and stops with the same output. Parity changes, a contradiction. This proves the required lower bound for nonadaptive queries and also for adaptive ones. Thus the two-query policy fails closure under substitution.

IX.2. One support and two ways to substitute Write $A_r \to I_r$ for the answer families for $0 \le r \le 3$. Write

$$I_{r-1} \xleftarrow{p_r} K_r \xrightarrow{q_r} I_r, \qquad e_r : \Pi_{p_r} q_r^{*} A_r \longrightarrow A_{r-1} \quad (1 \le r \le 3)$$

for the supports and evaluators. Use the common support

$$T = K_1 \times_{I_1} K_2 \times_{I_2} K_3,$$

where each matching pair of arrows is q_r, p_{r+1}. Both parenthesizations exhibit this as a limit of the same diagram. Put $T_2 = K_1 \times_{I_1} K_2$, write $u_2 : T_2 \to K_1$ and $u_3 : T \to T_2$ for the projections dropping the last factors, and set

$$P = p_1 u_2 u_3 : T \longrightarrow I_0, \qquad Q = q_3 \,\mathrm{pr}_3 : T \longrightarrow I_3.$$

The product on the composite is $\Pi_P Q^{*} A_3$. First use the composition of right adjoints for p_1, u_2, u_3. The square defining u_3 pulls p_3 back along $q_2 \,\mathrm{pr}_2 : T_2 \to$

I_2, and the square defining u_2 pulls p_2 back along q_1. The two base-change isomorphisms give

$$\Pi_P Q^* A_3 \simeq \Pi_{p_1} q_1^* \Pi_{p_2} q_2^* \Pi_{p_3} q_3^* A_3.$$

Apply e_3 under the two outer products and pullbacks, then e_2 under the remaining outer product and pullback, and finally e_1.

To compare the evaluations intrinsically, one can start with a generalized tuple of answers, a map $D \to \Pi_P Q^* A_3$ over I_0. Its transpose is a map $P^* D \to Q^* A_3$ over T. Currying this map along u_3 gives the k_3-family over T_2. The dependent-product base-change comparison is characterized by pulling back the counit of $p_3^* \dashv \Pi_{p_3}$, so currying presents this family as an input to e_3 pulled back along $q_2 \operatorname{pr}_2$. Applying that evaluator gives the A_2-answer over T_2. Now curry along u_2 and apply the pullback of e_2, using the corresponding pulled-back counit for p_2. Curry the A_1-answers along p_1, then apply e_1. In either parenthesization the transposed tuple is evaluated in this order, with the same projections and counits. The naturality and triangle identities for those adjunctions identify the maps, giving equality in an ordinary category and the canonical homotopy in the higher setting.

IX.3. Automorphisms of repeated requests Take the output request space to be a point and let $q : K \to J$ label the n support positions. Use the terminal answer family over each request object, as in the two-query example. Every permutation σ of K fixes the left leg. It fixes the right leg exactly when $q\sigma = q$, which is equivalent to preserving each fiber $K_j = q^{-1}(j)$ setwise. With terminal answer families the evaluator is unique, so there is no further restriction. Thus the automorphism group is

$$\prod_{j \in q[K]} \operatorname{Sym}(K_j).$$

The loop group of the corresponding component in the groupoid of reductions is this automorphism group. Write $\operatorname{Sym}(m)$ for the permutation group of an m-element set. Distinct labels give singleton fibers and a trivial group. A label of multiplicity m_j gives the factor $\operatorname{Sym}(m_j)$, and repeated requests of just one label give $\operatorname{Sym}(n)$.

With a general evaluator e, take the subgroup that also fixes e under the induced coordinate permutations of the dependent product. For example, with two equal labels whose answer set is $\{0, 1\}$, an evaluator selecting the first coordinate excludes the transposition.

IX.4. Fixed points and the descent condition Let $A = EG \times_G T \to BG$ be the desired answer family and use the trivial oracle $B = BG \to BG$. There is one request over each base point. A pullback of B is terminal in its slice and its

dependent product remains terminal. Thus a parallel evaluator yields a section of A. Conversely, a section gives a reduction by using the identity support and evaluating its trivial answer with that section. This works provided identity supports are admitted, as in the definition.

The section space is the space of G-equivariant maps $EG \to T$. Since EG is connected and T is discrete, each such map is constant. A constant value descends exactly when $gt = t$ for every g. The section space is consequently T^G as a discrete space. This calculation can also be made by descent. Pulling back to the point of BG gives the local set T. The double intersection is equivalent to the discrete group G, and compatibility over its component g says that the transported gt equals t. The higher Čech conditions are then satisfied by the identity paths because the action obeys the group law and the fibers are discrete. They cannot rectify a failed condition on the double intersection. In particular a fixed-point-free, nonempty action gives local reductions that do not descend.

IX.5. The local information in the partition argument Let D be the single α-finite support being queried, or the projection to α of a tagged support. The local regularity input is the existence of the full untagged positive table $P = C \cap D$ in L_α. The negative part $D \setminus P$ then belongs to L_α by bounded Separation. A bound on the successful certificates assigning a tag to each $x \in P$ is the other ingredient of the computation. Under the lemma's hypotheses the ground enumerations give a positive $\Sigma_1(L_\alpha)$ certificate of a color for every $x \in P$. The disjoint partition ensures that color is unique. Collection over P supplies one bound on all those occurrences. If such a bound is supplied directly for this D, one may use it in place of the collection argument.

Given both tagged tables, deciding the untagged answer is immediate. Take the union of their positive entries on D, and complement it inside D for the negative entries. This direction requires no decisions about when entries first appear. Given the complete untagged table instead, run both enumerations until every positive entry of P has appeared in a color. Read these colors and mark the points of $D \setminus P$ negative in both columns. This terminates by the bound above. The untagged table alone does not specify the colors. With $D = C = \{x\}$, assigning x to either part gives the same untagged table and different tagged answers.

IX.6. Cofinality at a different limit level Take $\beta = \omega \cdot 3$, $M = L_\beta$, and $a = \omega \cdot 2$. The level is transitive. It is closed under rudimentary functions because any one such function takes arguments from L_ξ into some finite successor extension $L_{\xi+n}$. Here pairing, difference, products, union, and bounded Separation take only finitely many definability steps. Close under finite compositions and the bounded unions used in building rudimentary functions. For a bounded union, apply the uniform finite definability bound to all elements of the bounding argument and take their union at the next definability step. For every $\xi < \beta$

and finite n, $\xi + n < \beta$. In particular $a, \omega \in M$, and all the individual finite sequences below have codes in M.

Define $f(n) = a + n$ for $n \in \omega$. Give a $\Sigma_1(M)$ definition by asking for a sequence w with domain $n+1$, first value a, last value the proposed output, and $w(i+1) = w(i) \cup \{w(i)\}$ at each coordinate $i < n$. The sequence properties and successor checks can be written with bounded quantifiers on the displayed data. Each w exists by finitely many rudimentary operations with the appropriate ordinals as inputs. The recurrence determines the values uniquely by induction. The domain is the set $\omega \in M$, and the outputs are cofinal in β.

A ground record collecting all the outputs would have to contain every $a + n$. Such a record cannot be an element of M, because the ordinal members of any set in L_β are bounded in β. The same rank argument applies if a record contains those ordinals in its transitive closure as decoded witnesses. Consequently, a support for ω many subsidiary requests can lie in M, and every request can admit a certificate in M, without the substituted computation having a single ground record containing all the certificates. This is the obstruction for a computation calculus that requires the joint record to be a ground set.

Chapter X Set recursion and spaces of records

X.1. Union of singleton outputs Use the elementary pairing instruction d with both selected arguments equal to its first argument. Its value at a is $\{a, a\} = \{a\}$, and its length is zero. Apply the union scheme with index d on the domain D. The resulting value and length are

$$\bigcup_{a \in D} \{a\} = D, \qquad \sup_{a \in D} (0 + 1) = \begin{cases} 0 & D = \varnothing, \\ 1 & D \neq \varnothing. \end{cases}$$

For a nonempty finite domain this is the union of finitely many singleton values. For an infinite domain the union has infinitely many premises, still of length zero. The rank bounds branch lengths, so their number does not change the second calculation.

X.2. An ordinal signature Take the judgment and rule sets to be $I = A = \lambda$. Set $t(\eta) = \eta$, let $B = \{(\eta, \xi) : \xi < \eta < \lambda\}$, and use the maps $p(\eta, \xi) = \eta$ and $s(\eta, \xi) = \xi$. The induced polynomial on I-sorted sets sends a family (Z_ξ) to $(\prod_{\xi < \eta} Z_\xi)_{\eta < \lambda}$. There is exactly one tree w_η of judgment η. Inductively its child at $\xi < \eta$ is the unique tree w_ξ. For $\eta = 0$ the product is empty and the tree is a leaf. The root rank satisfies

$$\ell(w_\eta) = \sup_{\xi < \eta} \big(\ell(w_\xi) + 1\big) = \eta$$

by induction, including at limit η. The first stage containing w_η is $W_{\eta+1}$, because W_α contains exactly the trees whose lengths are less than α.

X.3. Lifting records along a cover of rules Suppose that the arity map $p_Q : B_Q \to A_Q$ in the target signature is finite locally constant. Let $a : A_P \to A_Q$ be the effective epimorphism of rule objects. The premise condition gives

$$P(Z) \simeq A_P \times_{A_Q} Q(Z).$$

Thus $P(Z) \to Q(Z)$ is an effective epimorphism by pullback. The functor Q preserves effective epimorphisms. Premise pullback preserves them, and the dependent product over p_Q preserves them by local trivialization and finite products. Indeed a finite product of effective epimorphisms is a composite of their pullbacks. Postcomposition along the conclusion map preserves the underlying effective epimorphism.

Now $W_0^P \to W_0^Q$ is the identity of the initial object. If $W_\alpha^P \to W_\alpha^Q$ is an effective epimorphism, the successor factorization

$$P(W_\alpha^P) \longrightarrow Q(W_\alpha^P) \longrightarrow Q(W_\alpha^Q)$$

is a composite of effective epimorphisms. At a limit stage, the image of the map to the target colimit contains the image of every target stage, since each target stage has an effective epimorphic source stage. Those images jointly cover the target colimit. The stage map is therefore an effective epimorphism at the limit as well.

For an infinite arity, coverage of each individual premise may fail to give any simultaneous local lift. Use the $R_n \to 1$ of Figure XI.1. Give the target one nullary rule at each judgment j_n, and one rule at i with a premise j_n for each $n \geq 1$. In the source, the nullary labels at j_n are R_n, and the rule at i stays fixed. This comparison covers rule objects and identifies all arities. At stage two the map of records for i is $\prod_{n\geq 1} R_n \to 1$, which fails to cover at $x = 0$. The finite arity proof uses simultaneous local choice precisely at the product over premises.

X.4. Erasing a unary step Take two judgments i, j, with i the result judgment and j an auxiliary one. Give both presentations a nullary rule of judgment j. In the first presentation the only rule at i is unary with premise j. In the second it is nullary. Let all rules return the unique element of the value set. Both presentations compute the same total operation on the singleton input. The unique successful record at i has rank 1 in the first and rank 0 in the second.

An erasure sends the true one-branch derivation to the nullary derivation at i. The premise fiber of the unary rule has one element, while the premise fiber of its image is empty. They cannot be identified. This is the premise-family hypothesis in Theorem X.3. The comparison fails to reflect membership in W_1 exactly at this derivation.

X.5. Interchanging premises Let γ be a loop exchanging the two points of a configuration. Fix a local ordering at its base point, with premise record spaces A and B. Transport of records along the two point paths gives equivalences $u : A \to B$ and $v : B \to A$. In the original order at the base point the transport on the product is

$$T(a, b) = (v(b), u(a)).$$

The pullback record family over the loop is obtained by identifying the ends of a family over an interval using T. A section with prescribed pair $z = (a, b)$ over the base point is therefore a path in the product from $T(z)$ to z, using the transport to identify the endpoint fibers. Equivalently, z must be equipped with a homotopy fixing it under one turn of the loop. The generator of the circle supplies the only gluing condition in this description.

For discrete record spaces this requires $a = v(b)$ and $b = u(a)$. If the two discrete premise record spaces are identified with one fixed set and the only action is exchange of the factors, a pair descends exactly when its entries coincide. For spaces of records, different paths witnessing the homotopy may give different sections.

X.6. The evidence of a nullary rule At the base point the homotopy fiber of $* \to K(A, n)$ is $\Omega K(A, n) \simeq K(A, n-1)$. Every other fiber is equivalent to this one by transport along a path from the base point, since $K(A, n)$ is connected. In particular the map is surjective on connected components and its image is the whole base. As a nullary rule it supplies records at length zero in every fiber. The only nonzero homotopy group of the evidence fiber lies in degree $n - 1$. Since $A \neq 0$ and $n \geq 2$, the fiber is 1-truncated exactly when $n = 2$.

X.7. Rank excludes infinite descent Let $\rho(t)$ be the given ordinal rank of a tree node. If $(t_n)_{n<\omega}$ is an operational path, the node t_{n+1} is an immediate subinstruction of t_n, hence

$$\rho(t_{n+1}) < \rho(t_n)$$

for every n. In sets the nonempty set of ranks of the nodes on the path has a least element. The next node would have a smaller rank, a contradiction.

For a slice statement let U be the parameter object and let ranks take values in the relative discrete ordinal $\coprod_{\xi<\vartheta} U$ for some external ordinal bound ϑ. Use the strict order induced by ordinary ordinal comparison. An infinite path in the slice consists of a sequence of nodes whose consecutive pairs lie in the immediate-subinstruction relation. Applying the rank to each node gives a map from its path object to the object of strictly descending sequences of ranks. Over first term ξ, the latter object is the coproduct, over $\zeta < \xi$, of the objects of strictly descending sequences with first term ζ. The equivalence takes the tail of the sequence and remembers its first term. The fiber at $\xi = 0$ is initial, and induction on ξ shows every fiber is initial. The total sequence object is initial,

so the path object is initial by strictness of the initial object in a topos. Pullback between slices preserves the relative ordinal, the sequential limit defining paths, and the initial object. The same argument applies over every new base.

Chapter XI Selection and bounded forcing

XI.1. Searching a finite list below a rank bound Let e be the candidate program, x its fixed parameter, and D the presented finite set of candidates. For each $d \in D$ test the bounded condition $|\{e\}(d,x)| < \gamma$. Bounded-length testing is E-recursive. Output the singleton $\{d\}$ when the test succeeds and the empty set otherwise. The union of the resulting sets is

$$D_{<\gamma} = \{d \in D : \{e\}(d,x) \downarrow \text{ with length less than } \gamma\}.$$

An inclusive bound may be replaced by its successor. If γ bounds every convergent call on D, this produces the full success set. If γ only bounds the length of one known or promised success, the answer is the bounded part $D_{<\gamma}$, which is nonempty in that case.

To return the least member one needs a specified effective order on D and a nonempty result. When presentation order is meant, a given finite enumeration already supplies the order by first occurrence. Scan that enumeration and stop at the first positive test. A different order requires its own comparison data.

XI.2. Choice over a finite locally constant family Let $q : R \to S$ be an effective epimorphism over D. Take a cover $V_i \to U$ trivializing $p : D \to U$, so over V_i the family is a coproduct of m_i copies of V_i. The components $R_{ij} \to S_{ij}$ are effective epimorphisms by pullback. Base change for Π_p identifies the local map $\Pi_p(q)|_{V_i}$ with

$$\prod_{j<m_i} R_{ij} \longrightarrow \prod_{j<m_i} S_{ij} \quad \text{in } \mathcal{X}_{/V_i}.$$

This map is a composite that replaces the factors one at a time. Every step is a pullback of $R_{ij} \to S_{ij}$, so all the steps and their composite are effective epimorphisms. When $m_i = 0$ the map is the identity of the terminal object V_i of the slice. The image of $\Pi_p(q)$ pulls back to its entire codomain on every piece of the cover, and the pullbacks of the V_i cover the codomain. The image is therefore the whole codomain and $\Pi_p(q)$ is an effective epimorphism.

XI.3. The precise open of simultaneous witnesses Let $a_1, a_2, \ldots$ be the distinct nonzero rational points and set $U_n = X \setminus \{a_n\}$, $V = X \setminus \{0\}$. The image of $\prod_n (U_n \amalg V) \to 1$ is V as a subterminal. Over V, choose the second summand in every factor to obtain one simultaneous section. This proves that V is contained in the image.

Conversely, let O be a nonempty open containing 0. Every nonempty open $W \subseteq O$ containing 0 is irreducible. A section of $U_n \amalg V$ over W chooses a single summand, since it would otherwise decompose W into two disjoint nonempty opens. At 0 the only available summand is U_n. A simultaneous section over W would thus require that W omit every a_n. Its complement in $\mathbb{A}^1_{\mathbb{Q}}$ is a proper closed subset and contains only finitely many rational points. No such section exists on any open neighbourhood of 0, so 0 lies outside the image.

XI.4. Bounds in a filtered poset Let P be a filtered poset and $R_b \to U$ an increasing family for $b \in P$. Let $(V_i \to U)_{i \in J}$ be a jointly effective epimorphic family, and suppose $V_i \times_U R_{b_i} \to V_i$ is effective epimorphic. The replacement for $|J| < \mathrm{cf}(\lambda)$ is that the set of chosen b_i have an upper bound b in P. For example one may assume P is μ-directed and $|J| < \mu$, where μ-directed means every set of fewer than μ elements has an upper bound.

Since $b_i \leq b$, the image of R_b contains that of R_{b_i}. Its restriction to each V_i is therefore all of V_i. Joint coverage shows that the image of R_b equals U. This entire comparison is preserved by pullback. Ordinary filteredness alone gives upper bounds for finite subsets. To see the limitation for an infinite cover, take $P = \mathbb{N}$, $U = \mathbb{N}$ in sets, and $R_n = [0, n)$. Every singleton of U is covered at some stage, while no finite stage covers all of U.

XI.5. Initial segments and even inputs The stage n has a record at k exactly when $k < n$. Its least successful bound at k is $n = k + 1$. The image is $[0, n)$, and the join over all n is $\mathbb{N}$. Let $i : 2\mathbb{N} \hookrightarrow \mathbb{N}$ be the inclusion of the even integers. The image of $i^* R_{<n}$ in $2\mathbb{N}$ is

$$2\mathbb{N} \cap [0, n) = \{2m : 2m < n\}.$$

The join of these images is $2\mathbb{N}$. The least successful bound at the pulled-back point $2m$ remains $2m + 1$.

XI.6. When the two tests define a Boolean classifier A total map $f : U \to 1 \amalg 1$ whose fibers are exactly D_0, D_1 exists if and only if $D_0 \times_U D_1 = 0$ and $D_0 \vee D_1 = U$. Necessity follows by pulling back the two disjoint coproduct inclusions along f, using universality of coproducts. Conversely, disjointness makes $D_0 \amalg D_1 \to U$ a monomorphism and the join assumption makes it an effective epimorphism. The map is therefore an equivalence. The Boolean classifier sends the two summands to 0 and 1, and these prescribed restrictions determine it uniquely.

If the join is proper but the images are disjoint, the same construction defines a canonical partial classifier with domain $D_0 \vee D_1$. A total map merely extending that partial map exists exactly when D_0, D_1 are contained respectively in two complementary subobjects of U. Such an extension need not exist. For the terminal object in sheaves on $\mathbb{R}$, use the subterminal opens $(-\infty, 0)$ and $(0, \infty)$.

A total classifier would partition the connected space into two disjoint nonempty clopen sets, which is impossible. If the two images overlap, even a partial classifier on their join cannot have the two stipulated values there.

XI.7. A finite common bound and the collection needed for infinite families Let the finite deciding family be $(p_i)_{i<m}$, and suppose the given data assign γ_i with

$$p_i \Vdash^* |\{e\}(\dot{t})| < \gamma_i$$

on the convergence cases under consideration. Use the maximum of the γ_i as a common bound, or the successor of that maximum if the supplied tests are inclusive. Finite maximum and ordinal successor are E-recursive from the displayed list. If there are no convergence cases, any bound suffices. On each case the new condition on length follows by monotonicity in the bound. A generic satisfying a deciding case therefore supplies the same rank bound for the bounded name construction, independently of which of the cases it satisfies.

For an infinite ground set of conditions one needs a way to collect the individual rank bounds into one ordinal of the ground structure. For instance, if the domain lies in an E-closed ground structure and the map $i \mapsto \gamma_i$ is uniformly E-recursive and total on that domain, apply the union scheme to take their supremum. The supremum stays in the ground structure by E-closure. When the only information is that each i has a bound somewhere in the ground structure, the individual bounds may fail to be collectable and may be cofinal in its ordinal height. A finite maximum proof gives neither an effective choice of the bounds nor a bound for such a cofinal family.

Chapter XII Descent of computation records

XII.1. Testing initiality on the Čech nerve The given algebra and its restrictions constitute the compatible family in Theorem XII.1. For any target algebra Z over U, its mapping space out of the global candidate is the limit of the mapping spaces to $Z|_{V^{[n]}}$ in the local algebra categories. The assumed initiality of each restriction makes each of these spaces contractible. A diagram of contractible spaces is equivalent to the constant diagram on the terminal space, so the limit is contractible. This is the defining mapping-space property of an initial algebra. In particular, a proof that restriction preserves initiality is needed for the higher Čech levels when the only supplied initiality is over V.

XII.2. A cover at the cofinality threshold Set $\mu = \mathrm{cf}(\lambda)$ and choose a strictly increasing cofinal sequence $(\xi_i)_{i<\mu}$ in λ. Let $U = W = \mu$ as discrete spaces, with the identity structure map. Define the increasing filtration by $W_\xi = \{i < \mu : \xi_i \le \xi\}$, and take $w = \mathrm{id}_U$. The singleton cover indexed by μ is jointly

effectively epimorphic. The restriction of w to the i-th singleton factors through W_{ξ_i}, whereas for every $\xi < \lambda$ there is i with $\xi_i > \xi$. Thus there is no global stage. The construction works for every limit ordinal admitting this cofinal sequence, including one of singular cofinality.

XII.3. Truncation of a group extension The homomorphism $H \to G$ is onto. The homotopy fiber of $BH \to BG$ is therefore connected with fundamental group equal to its kernel K and all higher homotopy groups zero. Equivalently the fiber is BK. The relative 0-truncation is $BG \to BG$, and the relative 1-truncation is $BH \to BG$. A splitting homomorphism gives a section on taking classifying spaces. Conversely a section yields a homomorphism $\pi_1(BG) \to \pi_1(BH)$ whose composite with the quotient is the identity up to inner automorphism of G, after choices of base points. Precomposing by the inverse of this inner automorphism gives an actual splitting. In the based case the correction is unnecessary.

XII.4. The descent nerve of a point The homotopy fiber product of two points over BG is $\Omega BG \simeq G$ with its discrete topology. The $(n+1)$-fold fiber product specifies n successive paths and is equivalent to G^n. The outer faces forget the first or last path, the inner faces compose successive paths, and degeneracies insert the identity. These are the nerve maps of the one-object groupoid. A family of records has a fiber F over its object and transport along each $g \in G$, with the multiplication and unit coherences. Thus it is an action $BG \to \mathcal{S}$. A section is a compatible point in this diagram, that is, a point of F^{hG}. When F is discrete this reduces to a fixed element of F.

XII.5. A covered product with no global section Use sheaves of spaces on $\mathbb{R}/\mathbb{Z}$. Take as the first family the sheaf of sections of the double cover $\mathbb{R}/2\mathbb{Z} \to \mathbb{R}/\mathbb{Z}$, and take every other member of the countable family to be terminal. Their product is equivalent to the first sheaf. Two proper open arcs whose union is the circle give a finite cover. On each arc the double cover is trivial, so one can choose a section of the product there. A global section would give a section of the double cover and hence a right inverse to multiplication by two on $\pi_1(S^1) \cong \mathbb{Z}$. This is impossible.

Place these local records in stages of a filtration satisfying the common-bound hypothesis of Proposition XII.2. The second conclusion applies to the product record object, since on each arc a bounded stage is locally inhabited. A single stage then has image equal to the base. The first conclusion would require a section already given over the whole circle. The arc sections supply no such section, since their chosen sheets cannot agree around the loop.

XII.6. Sections of a permutation family Let F be the nonempty discrete fiber and let $\pi : F \to F$ be the monodromy. The section space is the homotopy

equalizer of the identity and π on F. Indeed a section is a choice of value at the base point and an identification with its transport along the generator of the loop. Since F is discrete, that identification exists exactly for $x = \pi(x)$, and then its space of choices is a point. The components, and in fact the whole section space, are the fixed-point set F^{π}. The domain of convergence is all of S^1 because each fiber is inhabited. A permutation without fixed points gives a total convergence predicate with empty section space.

Chapter XIII Effective oracles with transport

XIII.1. The two tags for a one-query machine Write $C_2 = \{1, s\}$, and let $s(2n) = 2n + 1$ and $s(2n + 1) = 2n$. Let e_m query its oracle at m and halt iff that answer is affirmative. Then

$$(1, e_m) \in J(A) \iff m \in A, \qquad (s, e_m) \in J(A) \iff s(m) \in A.$$

The second equality follows from $m \in s^{-1}A$ iff $s(m) \in A$. In particular at the two positions $2n, 2n + 1$ the first tag reads the two bits in their original order and the second reads them in the opposite order.

XIII.2. The identity tag and equivariance Consider the underlying reduction $j : e \mapsto (1, e)$. With trivial source action we have $j(he) = j(e)$ for every $h \in G$. But the translated tag gives $hj(e) = (h, e)$, which differs from $j(e)$ when $h \neq 1$. More generally the free translation on the tag alphabets gives no equivariant map from a nonempty trivial G-set to $G \times \omega$. None of these equalities changes the ordinary many-one reduction, since the identity tag still answers the original jump query. The uniform simulation of transformed queries in the other direction gives the same ordinary degree as in Theorem XIII.1.

XIII.3. Simulation on a partial domain Suppose a single oracle machine $\Psi^A(n)$ defines $F(A)$ on a G-invariant domain $\mathcal{D}$, with total binary output as a function of n for each $A \in \mathcal{D}$. Suppose also $F(gA) = gF(A)$ there. No decidability of membership in $\mathcal{D}$ is needed for the index transformation. Simulate the machine with index e by replacing its oracle queries with calls to Ψ, and use the parameter theorem to make a recursive index transformation $e \mapsto r_F(e)$. Under the domain and totality assumptions every simulated query finishes, so

$$(g, e) \in J(F(A)) \quad \iff \quad (g, r_F(e)) \in J(A).$$

This reduction commutes with left translation on g.

Now let H be another such transformation, defined on a G-invariant domain containing $F(\mathcal{D})$. Simulate the H-oracle for e in $F(A)$, then the F-oracle in A. This gives $(g, e) \mapsto (g, r_F(r_H(e)))$ on $\mathcal{D}$. A separately compiled simulator for

$H \circ F$ makes the same halting decisions there. The verification concerns their computations on that domain and allows different numerical indices for the simulators.

XIII.4. Selectors on the triangle The stabilizer of a vertex v in S_3 interchanges the two other vertices. It exchanges the two incident edges at v, so the fiber of incident edges has no fixed element. There is no equivariant selector of an incident edge at a vertex. In the reverse direction the stabilizer of an edge interchanges its endpoints, so there is no equivariant selector of an endpoint either. For disjointness there is exactly one edge disjoint from each vertex, namely the opposite edge, and each edge misses exactly one vertex. The resulting bijection and its inverse both commute with the S_3-action, giving selectors in both directions.

XIII.5. Branch sets with different transport Let $G = C_2$ act on the alphabet $\{0, 1, 2, 3\}$ by exchanging $0, 1$ and fixing $2, 3$. It acts coordinatewise on strings and on infinite branches. Take the parameter set to be a singleton with trivial action. One recursive tree consists of all prefixes of the constant 2 and constant 3 sequences. The second tree consists of prefixes of the constant 0 and constant 1 sequences. Both trees are invariant and have two branches, so their branch existence predicates are identical. Either fixed branch of the first tree is an equivariant choice. The two branches of the second tree are exchanged, and the stabilizer is all of G, so no equivariant assignment exists.

XIII.6. Faithful transport with constant rank Let G be any nontrivial countable group. Form a tree with a root and one child (g, n) for each $g \in G$ and $n < \omega$. Continue below (g, n) by a single chain of n edges. Left translation acts by sending the chain at (g, n) to the chain at (hg, n), preserving distance along it. This action is faithful, since a nonidentity group element moves the child with group coordinate 1. Each branch is finite, the rank of (g, n) is n, and the root rank is $\sup_n(n + 1) = \omega$. Use this as a family over BG, or take disjoint copies over several components with the same action. Every fiber has rank ω. A classical rank test therefore sees the same rank and the absence of infinite branches at each such parameter, independently of the labels of the transport arrows. The transport action remains visible in the family.

Bibliography

[1] Jon Barwise. *Admissible Sets and Structures*. Springer, 1975. Cambridge University Press reissue, 2017. doi: 10.1017/9781316717196.

[2] Chi Tat Chong and Sy-David Friedman. Ordinal recursion theory. In *Handbook of Computability Theory*, pp. 277–317. North-Holland, 1999. Author's copy.

[3] Nicola Gambino and Martin Hyland. Wellfounded trees and dependent polynomial functors. In *Types for Proofs and Programs*, LNCS 3085, pp. 210–225. Springer, 2004. Springer.

[4] Leo Anthony Harrington. *Contributions to Recursion Theory on Higher Types*. PhD thesis, Massachusetts Institute of Technology, 1973. MIT Libraries.

[5] Leo A. Harrington and David B. MacQueen. Selection in abstract recursion theory. *Journal of Symbolic Logic* 41, 153–158, 1976. Publisher.

[6] J. Martin E. Hyland. The effective topos. In *The L. E. J. Brouwer Centenary Symposium*, pp. 165–216. North-Holland, 1982. doi: 10.1016/S0049-237X(09)70129-6.

[7] Jacob Lurie. *Higher Topos Theory*. Princeton University Press, 2009. arXiv manuscript.

[8] Yiannis N. Moschovakis. *Descriptive Set Theory*. Second edition. American Mathematical Society, 2009. doi: 10.1090/surv/155.

[9] Dag Normann. Set recursion. In *Generalized Recursion Theory II*, pp. 303–320. North-Holland, 1978. doi: 10.1016/S0049-237X(08)70938-8.

[10] Jaap van Oosten. A general form of relative recursion. *Notre Dame Journal of Formal Logic* 47, 311–318, 2006. doi: 10.1305/ndjfl/1163775438.

[11] Gerald E. Sacks. *Higher Recursion Theory*. Springer, 1990. Cambridge University Press reissue, 2017. doi: 10.1017/9781316717301.

[12] Gerald E. Sacks and Theodore A. Slaman. Inadmissible forcing. *Advances in Mathematics* 66, 1–30, 1987. doi: 10.1016/0001-8708(87)90028-4.

[13] Richard A. Shore. Σ_n sets which are Δ_n-incomparable (uniformly). *Journal of Symbolic Logic* 39, 295–304, 1974. Publisher.

[14] Richard A. Shore. Splitting an α-recursively enumerable set. *Transactions of the American Mathematical Society* 204, 65–77, 1975. doi: 10.1090/S0002-9947-1975-0379154-1.

[15] Richard A. Shore. The recursively enumerable α-degrees are dense. *Annals of Mathematical Logic* 9, 123–155, 1976. Publisher.

[16] Theodore A. Slaman. Reflection and forcing in E-recursion theory. *Annals of Pure and Applied Logic* 29, 79–106, 1985. doi: 10.1016/0168-0072(85)90036-3.

[17] Theodore A. Slaman. Reflection and the priority method in E-recursion theory. In *Recursion Theory Week*, LNM 1141, pp. 372–404. Springer, 1985. doi: 10.1007/BFb0076231.

[18] Theodore A. Slaman. The E-recursively enumerable degrees are dense. In *Recursion Theory*, Proc. Sympos. Pure Math. 42, pp. 195–213. American Mathematical Society, 1985. Author's bibliography, item 127.

[19] Clifford Spector. Recursive well-orderings. *Journal of Symbolic Logic* 20, 151–163, 1955. doi: 10.2307/2266902.

[20] Nicola Gambino and Joachim Kock. Polynomial functors and polynomial monads. *Mathematical Proceedings of the Cambridge Philosophical Society* 154, 153–192, 2013. arXiv manuscript.

[21] David Gepner and Joachim Kock. Univalence in locally cartesian closed infinity-categories. *Forum Mathematicum* 29, 617–652, 2017. arXiv manuscript.

Index

accessibility, 9
analytical hierarchy, 7

change of base, 6
coanalytic uniformization, 12

dependent polynomial, 9
descent, 70

effective boundedness, 11
effective epimorphism, 70

Gandy basis theorem, 12

halting predicate, 15
hyperarithmetic hierarchy, 14

jump
    equivariant lift, 76

Kleene–Brouwer order, 11

locally constant oracle, 17
Louveau separation theorem, 30

ordinal notation, 14

path object, 8
propositional truncation, 8

rank certificate, 35
record
    coherent choice, 70
    rank, 76
record of computation, 6
recursive ordinal, 14
recursive tree, 7
relative truncation, 6

selection
    stabilizers, 76
separation theorem, 11
Spector–Gandy theorem, 35

Turing jump, 14