Generators and Classifiers in Effective Toposes: Image Semantics, Semidecision, and Higher Fibers

Anonymous submission draft

29 October 2026 Version 1.0

Abstract

We give a precise account of the sense in which generating objects and classifying their membership are equivalent. In any elementary topos, a generating presentation $g : A \to X$ determines the predicate $\exists a\, g(a) = x$, and every predicate arises in this way. The equivalence identifies presentations with the same image; it does not identify their domains, programs, witnesses, or execution costs. Categorically, image factorization is a reflector from the slice to its subobjects and exhibits a localization at image-preserving maps. We establish compatibility with substitution, relational composition, and geometric inverse image. For computational semantics, we distinguish arbitrary truth values, semidecidable truth values, and Booleans. Under explicit dominance, countable-existential, equality, and enumeration hypotheses, partial natural-number generators correspond extensionally to semidecidable classifiers. We supply uniform program translations for Kleene's first model and counterexamples to Boolean decision, universal enumerability, and reconstruction of witnesses. In an infinity-topos the support classifier is the propositional truncation of the generating fiber; a size-bounded object classifier instead retains the whole family up to equivalence. The paper is a self-contained synthesis of standard image, realizability, and truncation constructions, with an explicit statement of their computational limits.

Keywords: effective topos; realizability; subobject classifier; dominance; computable enumeration; image factorization; propositional truncation.
MSC 2020: 18B25, 03D25, 03F50, 18N60.

1 The question and its correct level of generality

The statement "generators and classifiers are equivalent" has several inequivalent readings. A generator can mean a program that enumerates values, a map presenting a family of witnesses, or a categorical generating object. A classifier can mean a Boolean decision procedure, a semidecision procedure, a subobject classifier, or an object classifying families. A formal treatment must fix both meanings before asserting an equivalence.

Our basic generator is a morphism $g : A \to X$. Its domain is an object of witnesses; its generated subobject is its image. Our basic classifier is a characteristic map $p : X \to \Omega$. Thus the broad theorem applies to *every elementary topos*, not merely one informally described as computable. The phrase "arbitrary computable/effective topos" is not used as an additional axiom: the effective topos $\mathbf{Eff} = \mathbf{RT}(\mathcal{K}_1)$ is a particular topos, while a realizability topos $\mathbf{RT}(\mathcal{A})$ depends on a chosen partial combinatory algebra and, in relative versions, a chosen class of realizers.

The distinctions can be summarized as follows.

Level	Generator	Classifier and equivalence
Elementary topos	Arbitrary $A \to X$	$X \to \Omega$, after identifying equal images
Effective fragment	Partial $\mathbb{N} \rightharpoonup X$ with admissible domain	$X \to \Sigma$, under the hypotheses of Theorem 6.2
Decision	A generated subobject with a complement	$X \to \mathbf{2}$; enumeration alone is insufficient
Higher support	$A \to X$ in an infinity-topos	Proposition-valued truncation of each fiber
Higher family	A suitably small family over X	A map to an object classifier, retaining fiber homotopy types

Image factorization and characteristic maps are standard topos theory [1]. Hyland's construction supplies the effective setting [2]; van Oosten gives the broader realizability framework [3]. Our contribution is an organized comparison of these meanings, with proofs of the exact quotient and hypotheses involved. We make no priority claim for the component theorems and no claim that an equivalence of denotations is an invertible compiler or a complexity equivalence.

2 Conventions and semantic presentations

Let $\mathcal{E}$ be a locally small elementary topos. We use its finite limits, stable regular-epimorphism/monomorphism factorizations, and subobject classifier $\mathsf{true} : 1 \hookrightarrow \Omega$. No natural numbers object, excluded middle, axiom of choice, well-pointedness, or countable coproducts is assumed until specified. Size statements about collections of presentations can be interpreted in a larger metatheoretic universe; the subobject poset of any fixed object is a set.

Write $\mathrm{Sub}_{\mathcal{E}}(X)$ for the poset of subobjects of X, up to isomorphism over X, ordered by factorization. Characteristic maps carry the corresponding order:

$$p \le q \quad \Longleftrightarrow \quad \mathcal{E} \models \forall x : X \; (p(x) \Rightarrow q(x)).$$

In this formula $p(x)$ abbreviates the proposition classified by p, not an assertion that $p(x)$ is a machine-readable Boolean. This makes $\mathrm{Hom}_{\mathcal{E}}(X, \Omega)$ a poset rather than merely a set.

Definition 2.1. A *generating presentation over* X is an object $g : A \to X$ of $\mathcal{E}/X$. A morphism from g to $h : B \to X$ is a map $u : A \to B$ with $hu = g$. Define

$$g \sim_X h \iff \mathrm{im}(g) = \mathrm{im}(h), \qquad g \preceq_X h \iff \mathrm{im}(g) \le \mathrm{im}(h).$$

Equality here means equality of subobjects, not equality of selected global elements or of external underlying supports.

The relation $\preceq_X$ is deliberately broader than the existence of a map $A \to B$ over X: inclusion of images does not generally supply a uniform choice of a witness in B for each witness in A. The distinction becomes important in realizability and in toposes without choice.

A *categorical generator*, in the usual distinct sense, is an object G for which $\mathrm{Hom}(G, -)$ is faithful. For example, 1 is a generator of **Set**, whereas the subobject classifier is $\mathbf{2}$. These are not isomorphic. In the topos of actions of a nontrivial group, the regular action is a generator and Ω is the trivial two-element action. No theorem below asserts that generating objects and classifying objects coincide.

3 The image–classifier equivalence

Theorem 3.1 (Extensional equivalence)**.** *For every $X \in \mathcal{E}$ there is an order isomorphism*

$$\mathrm{Gen}_{\mathcal{E}}(X)/{\sim_X} \;\cong\; \mathrm{Sub}_{\mathcal{E}}(X) \;\cong\; \mathrm{Hom}_{\mathcal{E}}(X, \Omega).$$

Here the quotient is the poset of generating presentations ordered by image inclusion. The forward map sends g to $\chi_{\mathrm{im}(g)}$. Its internal expression is

$$P_g(x) \;\equiv\; \exists a : A \; (g(a) = x). \tag{1}$$

The inverse sends p to the comprehension inclusion $m_p : P_p \hookrightarrow X$, obtained by pulling back true *along p.*

Proof. Factor g as $A \twoheadrightarrow I_g \hookrightarrow X$. Images are unique as subobjects, so $g \mapsto I_g$ is well defined on the stated quotient. Every subobject $m : P \hookrightarrow X$ is the image of itself, proving surjectivity. Equal outputs are exactly the relation $\sim_X$, proving injectivity and compatibility with the defined orders. The second isomorphism is the defining universal property of Ω. In the internal language, regular existential quantification is interpreted by images, yielding (1). Pulling back true along the characteristic map of I_g reconstructs I_g, and the characteristic map of m_p is p by uniqueness. These are the two inverse identities. □

Proposition 3.2 (The image reflector)**.** *Let* $\mathrm{Mono}(\mathcal{E}/X)$ *be the full subcategory of the slice on monomorphisms. Image factorization defines an adjunction*

$$\mathrm{im}_X : \mathcal{E}/X \rightleftarrows \mathrm{Mono}(\mathcal{E}/X) : J_X, \qquad \mathrm{im}_X \dashv J_X.$$

The right adjoint is fully faithful; its counit is an isomorphism. The unit at g is the regular epimorphism $A \twoheadrightarrow I_g$ over X.

Proof. For a mono $m : P \hookrightarrow X$, a map $g \to m$ in the slice is a factorization of g through m, and is unique if it exists. Such a factorization exists exactly when $I_g \leq P$. One direction follows from minimality of the image. For the other, compose the regular epimorphism $A \twoheadrightarrow I_g$ with the inclusion $I_g \hookrightarrow P$. This gives the required natural bijection of hom-sets. The image of a mono is itself, proving the counit assertion and the description of the unit. □

Corollary 3.3 (Localization)**.** *Let W_X consist of the morphisms $u : g \to h$ in the slice for which the induced map $I_g \to I_h$ is an isomorphism. Then image reflection exhibits*

$$(\mathcal{E}/X)[W_X^{-1}] \simeq \mathrm{Sub}_{\mathcal{E}}(X).$$

In particular, this is a localization of presentations, not an equivalence of the original slice with a poset.

Proof. The reflector inverts W_X, and every component of its unit lies in W_X. If $F : \mathcal{E}/X \to \mathcal{C}$ inverts W_X, then F applied to the unit gives a natural isomorphism $F \cong F J_X \,\mathrm{im}_X$. Thus F factors through the reflective subcategory up to natural isomorphism. The fully faithful inclusion and invertible counit give the uniqueness of this factorization and of its natural transformations. This is the universal property of localization; replacing the mono category by its equivalent subobject poset gives the displayed form. □

Remark 3.4. A map in W_X need not itself be an epimorphism: the inclusion $1 \to \mathbf{2}$ over the terminal object of **Set** preserves the image. Also, equal images do not imply equal kernel pairs, equal numbers of witnesses, or isomorphic domains. The maps $1 \to 1$ and $\mathbb{N} \to 1$ already demonstrate this.

4 Substitution, composition, and change of topos

Proposition 4.1 (Substitution)**.** *For $f : Y \to X$ and $g : A \to X$, let $f^*g : A \times_X Y \to Y$ be the pullback presentation. Then*

$$\mathrm{im}(f^*g) = f^* \,\mathrm{im}(g), \qquad P_{f^*g} = P_g \circ f.$$

Consequently the equivalences of Theorem 3.1 are compatible with substitution and assemble into an isomorphism of the associated indexed posets on $\mathcal{E}^{\mathrm{op}}$.

Proof. Pull back the regular-epi/mono factorization of g. Regular epimorphisms are stable under pullback and the second factor remains monic, so the resulting factorization is the image factorization of f^*g. Characteristic maps of pulled-back subobjects are obtained by composition with f. The identities respect compositions of substitutions by uniqueness of subobjects, independently of the choices used to form pullbacks. □

For $f : X \to Y$, direct image defines $\exists_f : \mathrm{Sub}(X) \to \mathrm{Sub}(Y)$ by $\exists_f(U) = \mathrm{im}(U \hookrightarrow X \xrightarrow{f} Y)$. It is left adjoint to inverse image:

$$\exists_f(U) \le V \quad \Longleftrightarrow \quad U \le f^*V. \tag{2}$$

Indeed, both sides say that $f|_U$ factors through V. If $g : A \to X$, then

$$P_{fg}(y) \equiv \exists x : X \; (P_g(x) \wedge f(x) = y).$$

The same image calculation proves Beck–Chevalley for a pullback square and the Frobenius identity

$$\exists_f(U \wedge f^*V) = \exists_f(U) \wedge V.$$

For the latter, both sides are the image of $U \times_Y V \to Y$. These are semantic laws for combining generators, not assumptions that an implementation can decide the resulting predicates.

Proposition 4.2 (Witness relations compose extensionally)**.** *Let $r : A \to X \times Y$ and $s : B \to Y \times Z$ present relations R and S. Pair compatible witnesses using $C = A \times_Y B$ and map $C \to X \times Z$ by the remaining coordinates. Its image has predicate*

$$T(x, z) \equiv \exists y : Y \; (R(x, y) \wedge S(y, z)).$$

This construction depends only on R, S, and is associative at the level of images, with diagonal relations as identities.

Proof. Membership in the image means that there exist $a : A, b : B$ with the specified endpoints and equal middle coordinates. Substituting the image definitions of R and S gives exactly the displayed predicate. This manipulation uses only conjunction, equality, and existential quantification. For three relations both parenthesizations assert the existence of the same two intermediate coordinates and three witnesses. A diagonal imposes equality of adjacent coordinates and hence changes no predicate. □

This is the regular relational semantics of witness-producing computations. Recovering a particular witness from the composite predicate is additional data; the existential quantifier supplies no general inverse to witness erasure.

Proposition 4.3 (Geometric inverse image)**.** *Let $F : \mathcal{E} \to \mathcal{F}$ be the inverse-image functor of a geometric morphism. Then*

$$F(\mathrm{im}_{\mathcal{E}}\, g) \cong \mathrm{im}_{\mathcal{F}}(Fg)$$

as subobjects of FX. If $\theta : F\Omega_{\mathcal{E}} \to \Omega_{\mathcal{F}}$ classifies $F\mathsf{true}$, then

$$\chi_{\mathrm{im}(Fg)} = \theta \circ F(\chi_{\mathrm{im}(g)}).$$

Proof. The functor F preserves finite limits and is a left adjoint. Thus it preserves monomorphisms, kernel pairs, and their coequalizers, hence regular epimorphisms. Apply it to the image factorization and to the pullback square defining the original characteristic map. The two assertions follow from the uniqueness of image and characteristic factorizations. □

The comparison θ need not be an isomorphism. An inverse-image functor need not preserve power objects, universal quantification, or an independently chosen computational dominance. No uniform invariance of algorithms across arbitrary geometric morphisms is asserted.

5 Realizability: truth is not a Boolean output

In the realizability tripos over a partial combinatory algebra $\mathcal{A}$, a predicate on an external index set I is represented by a family $\varphi(i) \subseteq \mathcal{A}$ of realizers. Entailment $\varphi \vdash_I \psi$ requires a single allowed realizer e satisfying

$$\forall i \in I\ \forall a \in \varphi(i), \quad ea \downarrow \text{ and } ea \in \psi(i).$$

For relative realizability, "allowed" refers to the chosen subalgebra or filter. This *uniform* entailment is stronger than pointwise nonemptiness implication. After tripos-to-topos construction it supplies the internal logic of the resulting topos. A description of predicates is not, by itself, a procedure for enumerating the external set I.

For $\mathcal{K}_1$, application is $e \cdot n = \varphi_e(n)$, using a standard acceptable numbering of the partial recursive functions. An assembly consists of a set $|A|$ and nonempty sets $E_A(a) \subseteq \mathbb{N}$ of realizers. A map is tracked by one partial recursive function transforming every input realizer into an appropriate output realizer. The standard natural numbers assembly has $E_{\mathbb{N}}(n) = \{n\}$. These conventions follow the standard realizability presentations [2, 3, 4].

Example 5.1 (A subobject with non-enumerable support)**.** Let $S \subseteq \mathbb{N}$ be any external subset, with no computability assumption. The assembly $(S, n \mapsto \{n\})$ includes monomorphically into the natural numbers object by the identity tracker. The full embedding of assemblies into **Eff** makes this a subobject and therefore supplies an Ω-valued classifier. Taking S to be the complement of the diagonal halting set gives a subobject whose external membership is not computably enumerable. Thus the existence of its characteristic map does not yield a semidecider.

Example 5.2 (Equal external support is insufficient)**.** Let $b : \mathbb{N} \to \mathbf{2}$ be an external noncomputable function. Define an assembly A_b with underlying set $\mathbb{N}$ and realizers

$$E_{A_b}(n) = \{\langle n, b(n)\rangle\}.$$

First projection tracks the underlying identity $m : A_b \to \mathbb{N}$. The map is monic. If it were an isomorphism in **Eff**, fullness of the embedding of assemblies would give a tracked inverse. Applying its tracker to n and taking second projection would compute $b(n)$, a contradiction. Hence m is a proper subobject although its underlying set of elements is all of $\mathbb{N}$. Equality of generated subobjects must retain realizability evidence; checking external supports alone would falsely identify m with $\mathrm{id}_{\mathbb{N}}$.

These examples do not contradict the principle that total maps between standard natural numbers objects are computable. The classification map has codomain Ω, and an arbitrary subobject domain can carry information that no uniform numerical program can supply from its underlying element alone.

6 Dominances and a conditional effective equivalence

We now state exactly what is needed to interpret the correspondence as enumeration versus semidecision. The assumptions in this section are extra structure, not consequences of the word "topos".

Definition 6.1. A *dominance* for this paper is a mono $j : \Sigma \hookrightarrow \Omega$ through which true factors, such that the monos classified through j contain identities and are closed under composition. Call these monos Σ-*admissible*; they are automatically stable under pullback. We also assume that false factors through j and that every complemented mono is Σ-admissible. Thus Booleans are included among the allowed truth values.

The associated partial maps $X \rightharpoonup_\Sigma Y$ are spans $X \xleftarrow{i} D \xrightarrow{v} Y$ with i an admissible mono, modulo isomorphism of spans. When described by the associated lifting object, the standard partial-map-classifier correspondence is

$$\mathrm{Par}_\Sigma(X, Y) \cong \mathrm{Hom}_{\mathcal{E}}(X, L_\Sigma Y), \qquad L_\Sigma Y = \sum_{p:\Sigma} Y^{[p]}.$$

Here $[p]$ is the subterminal truth fiber of p; the sum is a dependent sum in the internal language. In particular $L_\Sigma 1 \cong \Sigma$. This classifies *partial maps*; it should not be confused with classifying the range of a partial map. Dominances and lifting are developed in synthetic domain theory [5, 6].

Assume a natural numbers object $\mathbb{N}$ and the following countable-existential closure property:

$$R \hookrightarrow \mathbb{N} \times Y \text{ admissible} \quad \Longrightarrow \quad \exists_{\pi_Y} R \hookrightarrow Y \text{ admissible}, \quad \text{for every } Y. \tag{3}$$

This is an explicit hypothesis. The mere internal assertion that each proposition individually has some semidecision structure need not produce a uniformly given family of such structures. In constructive settings, converting this property into a dominance can require choice principles; Knapp analyzes precisely this obstruction [6]. We do not import countable choice into an arbitrary topos.

Theorem 6.2 (Countable generators and admissible classifiers)**.** *Suppose that $\mathcal{E}$ has $\mathbb{N}$, a dominance as in Definition 6.1, and property (3). Let X have an admissible diagonal $\Delta_X \hookrightarrow X \times X$, and a specified epimorphism $q : \mathbb{N} \twoheadrightarrow X$. Then*

$$\mathrm{Par}_\Sigma(\mathbb{N}, X)/{\sim_X} \;\cong\; \mathrm{Sub}_\Sigma(X) \;\cong\; \mathrm{Hom}_{\mathcal{E}}(X, \Sigma),$$

where a partial generator is identified with another exactly when their value maps have the same image. The quotient is ordered by image inclusion. Decidable equality of X is sufficient for the diagonal hypothesis.

Proof. Let g be $\mathbb{N} \xleftarrow{i} D \xrightarrow{v} X$. Its graph is the mono $\Gamma_g = (i, v) : D \hookrightarrow \mathbb{N} \times X$. The graph $D \hookrightarrow D \times X$ of v is a pullback of Δ_X, hence is admissible. The map $i \times \mathrm{id}_X : D \times X \hookrightarrow \mathbb{N} \times X$ is also admissible. Closure under composition makes Γ_g admissible. Property (3) makes its projection image admissible; that image is exactly $\mathrm{im}(v)$. This defines the forward map to $\mathrm{Sub}_\Sigma(X)$ and hence its unique classifier $p_g : X \to \Sigma$.

Conversely, let $m : P \hookrightarrow X$ be classified by $p : X \to \Sigma$. Form $D = \mathbb{N} \times_X P$. Its inclusion $i : D \hookrightarrow \mathbb{N}$ is admissible by pullback stability. Define the partial generator's value map by $v = qi$. The projection $D \to P$ is the pullback of the epimorphism q, hence is an epimorphism. Therefore $\mathrm{im}(v) = P$. This proves surjectivity. Injectivity on the stated quotient and preservation of its order follow immediately from equality and inclusion of images. The second isomorphism is the restricted classifier property of j. □

Remark 6.3 (Semantic closure versus effective code)**.** The theorem constructs maps inside $\mathcal{E}$. Extracting a program additionally requires an effective presentation of q, tracking data for equality, and realizers for the closure operations. In **Eff** the usual semidecidable truth-value object supplies this interpretation on $\mathbb{N}$. In an arbitrary realizability topos, the same symbols refer to its chosen model of computation, which can differ from ordinary Turing computability.

Corollary 6.4 (Boolean classifiers are the complemented fragment)**.** *For any elementary topos and any $P \hookrightarrow X$, its characteristic map factors through the canonical $\mathbf{2} = 1 + 1 \hookrightarrow \Omega$ if and only if P is complemented. In particular, the hypotheses of Theorem 6.2 do not make its classifiers Boolean.*

Proof. A map $X \to 1+1$ pulls back the two coproduct injections to complementary subobjects. Conversely, if P, Q are disjoint subobjects whose join is X, the induced map $P + Q \to X$ is both mono and epi and hence an isomorphism. Assign the true summand to P and the false summand to Q. Its composite with $\mathbf{2} \hookrightarrow \Omega$ is the characteristic map of P. □

7 Uniform program translations in the effective topos

The preceding theorem specializes to $X = \mathbb{N}$, $q = \mathrm{id}_{\mathbb{N}}$, and the usual semidecidable dominance. Operationally, a truth value can be represented by a computable bit sequence which contains a 1 exactly when the proposition holds. A tracker for $\mathbb{N} \to \Sigma$ produces such a sequence uniformly in the input; searching it gives a semidecider. Conversely, bounded simulation of a semidecider produces the sequence. This is the standard Sierpinski/Rosolini presentation [4]; the distinction between computability as supplied structure and as a mere property is essential [6].

Theorem 7.1 (Uniform enumeration–semidecision equivalence)**.** *For $S \subseteq \mathbb{N}$, the following are equivalent:*

(i) $S = \mathrm{rng}(g)$ *for a partial recursive* $g : \mathbb{N} \rightharpoonup \mathbb{N}$*;*

(ii) $S = \mathrm{dom}(c)$ *for a partial recursive* $c : \mathbb{N} \rightharpoonup 1$*;*

(iii) $n \in S \iff \exists t\, R(n, t)$ *for a total recursive Boolean relation* R*;*

(iv) S *is the set of numerical values output by a total recursive* $a : \mathbb{N} \to \mathbb{N} + 1$*, where the extra value is "no output."*

There are uniform effective translations on program indices around these equivalences. The round trips preserve the generated set, not the original code or its runtime.

Proof. For (i)⇒(ii), on input x perform stages $s = 0, 1, \ldots$. At stage s simulate each $g(n)$ for s steps for all $n \leq s$. Halt with the unique unit value when a simulation returns x. Every stage is finite. If $g(n)$ returns x in t steps, some stage $s \geq \max(n, t)$ discovers it; no other acceptance occurs. This is fair dovetailing, not sequential waiting for each $g(n)$ to terminate.

For (ii)⇒(i), define $g_c(n)$ to run $c(n)$ and return n if it terminates. For (ii)⇒(iii), take $R(n, t)$ to mean that $c(n)$ halts within t steps, a decidable bounded simulation. For (iii)⇒(ii), search $t = 0, 1, \ldots$ and halt at the first success.

For (i)⇒(iv), decode an input as a pair (n, t) using a computable pairing bijection. Run $g(n)$ for t steps; return its value if it has halted and return "no output" otherwise. For (iv)⇒(i), define a partial program that returns $a(k)$ when it is numerical and diverges when it is "no output." These procedures give algorithms transforming source code with its index as a parameter; acceptable numbering, equivalently parameter specialization, makes the index translations total recursive. The proofs explicitly identify the same output set in every direction. □

The empty set is covered by an everywhere-divergent partial generator or an always-blank total optional-output generator. It cannot be the range of a total map $\mathbb{N} \to \mathbb{N}$. Every nonempty computably enumerable set has a total recursive enumeration with repetition: dovetail until a first output is found, use it as a default, and on input k use bounded simulation to report any further output encoded by k. Nonemptiness is needed for termination of the initial search. No injectivity, increasing order, or duplicate elimination with a total output schedule is implied.

For a computably enumerated data type $q : \mathbb{N} \twoheadrightarrow X$ with computable equality, the reverse translation runs a recognizer on $q(n)$ and outputs $q(n)$ when accepted. The forward translation compares candidate outputs using equality and dovetails. This is the operational content of Theorem 6.2; it is not an algorithm for every object of **Eff**.

8 Obstructions to stronger equivalences

Proposition 8.1 (Generation does not imply decision)**.** *The diagonal halting set $K = \{e \mid \varphi_e(e) \downarrow\}$ has a computable generator and a semidecider, but no total recursive Boolean classifier.*

Proof. Simulating $\varphi_e(e)$ semidecides membership, and Theorem 7.1 supplies a generator. Suppose a total recursive $d : \mathbb{N} \to \mathbf{2}$ decided K. Define a partial program $r(e)$ to halt when $d(e) = 0$ and to loop when $d(e) = 1$, and choose an index k of r. Then

$$k \in K \iff r(k) \downarrow \iff d(k) = 0 \iff k \notin K,$$

a contradiction. Thus the classifier $\mathbb{N} \to \Omega$ of this subobject cannot be replaced by a total Boolean computation. Its semidecidable classifier is a genuinely weaker computational interface. □

If both a set and its complement have semideciders, running them in parallel decides membership: exactly one eventually accepts. Hence $\mathbb{N} \setminus K$ is not computably enumerable, justifying the choice in Example 5.1.

Proposition 8.2 (No decidable test for extensional equality)**.** *There is no total algorithm deciding whether two partial recursive generators have equal ranges.*

Proof. Given e, uniformly define $g_e(n)$ to simulate $\varphi_e(e)$ and return 0 if the simulation halts. Let $z(n) = 0$. Then $\mathrm{rng}(g_e) = \mathrm{rng}(z)$ exactly when $e \in K$. A range-equality decider would decide K, contradicting Proposition 8.1. □

Thus the quotient in Theorem 7.1 is a mathematical quotient, not a computable canonicalization of programs. Complexity is not preserved either: arbitrary delays and repeated outputs can change computation costs while leaving the image fixed.

Proposition 8.3 (Not every object is countably presented)**.** *In a nondegenerate elementary topos with a natural numbers object, there is no epimorphism $\mathbb{N} \twoheadrightarrow \Omega^{\mathbb{N}}$.*

Proof. Suppose q were such an epimorphism. Define internally $d : \mathbb{N} \to \Omega$ by $d(n) \equiv \neg q(n)(n)$. Internal surjectivity of q gives $\exists k\ (q(k) = d)$. For such a k, the proposition $p = q(k)(k)$ satisfies $p \leftrightarrow \neg p$. Intuitionistically this implies contradiction: $p \Rightarrow \neg p$ yields $\neg p$, and $\neg p \Rightarrow p$ then yields p. Existential elimination therefore gives false globally, contrary to nondegeneracy. No global selection of k was used. □

The enumeration hypothesis in Theorem 6.2 is therefore a real restriction even in a computational topos. A more elementary restriction already holds in **Set**: a countable stream cannot enumerate an uncountable set, although every subset has a characteristic function in the set-theoretic sense.

Proposition 8.4 (Image data do not select witnesses)**.** *Recovering, for every presentation $g : A \to X$, a section of its image epimorphism $A \twoheadrightarrow I_g$ would imply that every epimorphism in $\mathcal{E}$ splits.*

Proof. Apply the proposed recovery to any epimorphism $e : A \twoheadrightarrow B$. Its image is B, so the recovered section is a section of e. □

This splitting assertion fails in general toposes. For instance, in the topos of actions of a nontrivial group G, the equivariant map $G \to 1$ from the regular action is an epimorphism with no section, since the regular action has no fixed point. The existence of a classifier does not overcome this obstruction. Even where a witness-selection operation is available for a particular computable presentation by search, it need not be canonical or reconstruct the original witnesses.

If "generator" instead denotes a probabilistic sampler, the support predicate loses still more information. On $\{0, 1\}$, Bernoulli distributions of parameters $1/3$ and $2/3$ have the same support classifier and different laws. The equivalence proved here concerns logical membership; it does not equate generative probability models with discriminative statistical models.

9 Higher fibers, propositional truncation, and object classifiers

Let $\mathcal{X}$ now be an infinity-topos and $g : A \to X$ a morphism. Its fiber is the dependent type

$$F_g(x) = \sum_{a:A} (g(a) = x).$$

This retains witnesses and paths among witnesses. Its support is the proposition

$$P_g(x) = \|F_g(x)\|_{-1}\,, \tag{4}$$

where $\|-\|_{-1}$ is propositional truncation. We use truncation in the slice over X, not a claim that truncation preserves every finite limit. The relevant general constructions are developed in higher topos theory [7] and in the type-theoretic account of truncation and images [8].

Theorem 9.1 (Higher support reflection)**.** *In every infinity-topos, the inclusion of (-1)-truncated objects of $\mathcal{X}/X$ has a left adjoint $\tau^{/X}_{\leq -1}$. Applied to g, it gives its image mono $I_g \hookrightarrow X$, with fiber (4). This reflection commutes with base change and localizes the slice at morphisms inducing equivalences on images.*

Proof. For a proposition-valued family Q over X, the universal property of propositional truncation gives a natural equivalence

$$\mathrm{Map}_{/X}(I_g, Q) \simeq \mathrm{Map}_{/X}(A, Q).$$

Thus I_g is the reflection of A into the (-1)-truncated objects of the slice. Equivalently, use the effective-epimorphism/monomorphism factorization of g. Both classes are stable under base change, so pulling back that factorization computes the image of the pulled-back family. The unit of the reflection induces an equivalence of images. Any functor inverting all such morphisms therefore inverts the unit and factors, by the adjunction, through the reflective subcategory. This proves the localization assertion in its infinity-categorical universal sense. □

The full subcategory of monos over X is equivalent to a poset: its mapping spaces are empty or contractible. Its maximal groupoid of objects is classified by $\mathrm{Map}(X, \Omega)$, where Ω is the higher topos's subobject classifier. The order of inclusion on subobjects is additional

structure; the mapping space $\mathrm{Map}(X, \Omega)$ alone should not be confused with that ordered category.

In spaces, $S^1 \to 1$ and $1 \to 1$ have identical support classifiers, yet their fibers are not equivalent, since their fundamental groups differ. Even $\mathbf{2} \to 1$ and $1 \to 1$ have the same support and different sets of witnesses. Propositional truncation is exactly the operation responsible for this loss, rather than an unspecified failure of computational precision.

There is a second, stronger reading of "classifier." Fix a class of suitably small maps for which an object classifier $\widetilde{U}_\kappa \to U_\kappa$ exists; in an infinity-topos, sufficiently large regular cardinal bounds provide such classes [7, Section 6.1.6]. Its defining universal property is

$$(\mathcal{X}_{/X}^{\kappa\text{-small}})^{\simeq} \simeq \mathrm{Map}_{\mathcal{X}}(X, U_\kappa). \tag{5}$$

The left side is the *maximal infinity-groupoid*, containing only equivalences between small families, not all slice morphisms. Under (5), a generator understood as a witness family really is recovered, up to equivalence, by pulling back the universal family. Its support classifier is obtained afterward by (-1)-truncation. Thus we have a precise hierarchy:

$$\text{family with witnesses} \longrightarrow \text{proposition-valued support} \longleftarrow \text{image subobject.}$$

The first arrow forgets higher and multiplicity information; the second identification is exact. No object classifying all maps of all sizes is asserted. Nor do these higher statements construct an "effective infinity-topos" from an arbitrary partial combinatory algebra, establish univalence for an ordinary realizability topos, or supply effective algorithms for arbitrary higher fibers.

10 Scope, verification, and formalization boundary

The general semantic conclusion is

$$\boxed{\text{generating presentations modulo image equality} \;\cong\; \text{subobject predicates.}}$$

Under the explicit computational hypotheses, it refines to

$$\boxed{\text{partial computable generators modulo range equality} \;\cong\; \text{semidecidable classifiers.}}$$

Boolean classifiers require complementation, and full higher-family reconstruction requires an appropriate object classifier. These are different levels of information, not interchangeable terminology.

The arguments in this paper are mathematical proofs checked by inspection, not kernel-verified derivations. The accompanying Python artifact performs exhaustive finite-set checks of the image universal property, substitution, Frobenius, and relational composition on its stated finite bounds. It also checks the bounded-stage behavior of the two operational translations on finite synthetic machines, including empty ranges, duplicates, and delayed termination. These checks can detect specification or transcription mistakes, but do not establish the theorems for all toposes, prove nontermination, or verify an arbitrary program's range.

A machine formalization should keep three developments separate: (a) regular image reflection and its subobject classifier; (b) a specified partial-computation model with a proved dovetailing scheduler and sound realizability interpretation; and (c) slice truncation and a size-bounded universe. Linking them requires an explicit soundness theorem, not a renaming of types. We have not supplied a Lean, Coq, or Agda formalization, and no such verification is claimed.

Authorship and automation. This manuscript was prepared with OpenAI ChatGPT/Codex assistance for an anonymous human submitter. Automated assistance was used for drafting, mathematical cross-checking, reference retrieval, typesetting, and finite executable checks. The submitting human retains responsibility for review, attribution, and any decision to submit. This version has not been independently refereed, and its existence does not imply editorial acceptance or publication.

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