

# Marginal Descent and Wright Manifolds in Evolutionary Games: Toward Higher Topos Semantics

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## 1. Introduction and scope

An evolutionary population can be described by a distribution over complete contingent plans or by separate distributions over their local coordinates. Marginalization passes from the first description to the second and forgets correlations. Whether this loss is harmless depends on the dynamics as well as the payoffs. Cressman's treatment of extensive-form evolutionary games explains why the normal form alone need not preserve the relevant dynamical structure [1].

This note gives a precise marginal-descent construction, self-contained proofs relating product distributions to additive selection, and an explicit separation of state descent from dynamical closure. The probability and dynamical calculations are elementary; no priority claim is made for them. The connection to higher topos semantics is a research direction supported by these calculations.

Two neighboring programs motivate the formulation. Open games give a compositional account of strategic interaction [2]. Sheaf-theoretic contextuality studies compatibility and global distributions over overlapping contexts [3]. Neither identification alone supplies evolutionary dynamics on a higher topos. We use the language of sheaves of spaces [4], while specifying which constructions here are only set-valued.

A role coordinate below is a component of a contingent plan. It need not be a player, an information set, or a proper subgame. Product marginals are not asserted to be Kuhn's behavioral representation of an arbitrary extensive-form game. Reach probabilities, information constraints, and perfect recall need additional treatment. Our site is a finite context site, not a reconstruction of an extensive-form tree.

## 2. A finite context site

Let  $U = \{1, \dots, N\}$ ,  $N \geq 1$ , and let  $S_k$  be finite with  $|S_k| \geq 2$ . For  $I \subseteq U$ , put  $S_I = \prod_{k \in I} S_k$  and  $S_\emptyset = \{*\}$ . Write  $\Delta(T)$  for the probability simplex on a finite set  $T$ , of dimension  $|T| - 1$ .

Let  $\mathcal{C}$  be the poset of all subsets of  $U$ , including the empty subset, with inclusions as arrows. A family  $\{I_\alpha \hookrightarrow I\}$  covers  $I$  when  $\bigcup_\alpha I_\alpha = I$ . The empty family covers  $\emptyset$ . These covers give the topology  $J$  of the finite discrete space  $U$ : intersections supply pullbacks, union covers remain covers after intersection, and composites of union covers are covers.

Define presheaves of sets

$$F(I) = \Delta(S_I), \quad B(I) = \prod_{k \in I} \Delta(S_k).$$

Restrictions in  $F$  are marginalizations; restrictions in  $B$  forget coordinates. Marginalization is functorial because finite sums commute. Both presheaves take  $\emptyset$  to a singleton. There is a natural transformation

$$m_I : F(I) \longrightarrow B(I), \quad m_I(p) = (p^k)_{k \in I}, \quad p_r^k = \sum_{i \in S_I : i_k = r} p_i.$$

Values in this section are sets: a simplex is not being replaced by its contractible homotopy type.

### Proposition 2.1. Marginal sheafification

The presheaf  $B$  is a sheaf, and  $m : F \rightarrow B$  exhibits  $B$  as the set-valued sheafification  $a_J F$ .

*Proof.* A matching family in  $B$  assigns the same marginal to each coordinate wherever that coordinate occurs. These marginals determine a unique element of  $B(I)$ .

For any sheaf  $Q$ , the disjoint singleton cover gives a canonical bijection  $Q(I) \cong \prod_{k \in I} Q(\{k\})$ , since  $Q(\emptyset)$  is terminal. Given a natural transformation  $\eta : F \rightarrow Q$ , define  $\bar{\eta}_I : B(I) \rightarrow Q(I)$  by gluing  $\eta_{\{k\}}(p^k)$ . This is natural. The restrictions of  $\bar{\eta}_I(m_I(p))$  and  $\eta_I(p)$  to each singleton agree, so the sheaf axiom makes them equal. Any factorization must have those singleton restrictions and is therefore unique. This proves the universal property.  $\square$

This site permits singleton covers even when payoffs couple coordinates. Proposition 2.1 concerns distributions, not decomposition of payoffs or trajectories. For a topology that forbids some singleton covers to respect interactions, its conclusion must be reconsidered.

### 3. The Wright locus and lost information

The natural product map

$$\sigma_I : B(I) \longrightarrow F(I), \quad \sigma_I(x)_i = \prod_{k \in I} x_{i_k}^k$$

satisfies  $m_I \sigma_I = \text{id}$ . Its image at  $U$  is the linkage-equilibrium Wright locus

$$W = \left\{ p \in \Delta(S_U) : p_i = \prod_{k=1}^N p_{i_k}^k \right\}.$$

Thus  $W$  is a selected family of joint distributions representing marginal data. Sheafification identifies all distributions with the same marginals; it does not assert that every joint distribution is independent.

**Proposition 3.1. Product locus and splicing equations**

With their Euclidean topologies,  $W$  and  $B(U)$  are homeomorphic under  $m_U$  and  $\sigma_U$ . Moreover,  $W$  consists precisely of the distributions satisfying

$$p_i p_j = p_{i_I j_{I^c}} p_{j_I i_{I^c}} \quad (i, j \in S_U, \quad I \subseteq U),$$

where  $i_I j_{I^c}$  takes coordinates from  $i$  on  $I$  and from  $j$  on its complement.

*Proof.* The maps are continuous and inverse on these subspaces. Product distributions satisfy the splicing equations by multiplication. Conversely, summing an equation over  $j$  gives  $p_i = p_I(i_I) p_{I^c}(i_{I^c})$ , where the factors are joint marginals. Successively splitting off individual coordinates yields  $p_i = \prod_k p_{i_k}^k$ . The argument also applies at the boundary, without division by zero.  $\square$

**Proposition 3.2. A static strong deformation retraction**

The formula

$$H(p, t) = (1 - t)p + t\sigma_U(m_U(p)), \quad 0 \leq t \leq 1,$$

defines a strong deformation retraction of  $\Delta(S_U)$  onto  $W$ , preserving all one-coordinate marginals.

*Proof.* The segment lies in the simplex. Affinity of marginalization gives  $m_U(H(p, t)) = m_U(p)$ . Its endpoints are  $p$  and  $\sigma_U(m_U(p)) \in W$ . If  $p \in W$ , both terms equal  $p$  for every  $t$ . Continuity is immediate.  $\square$

This is a static homotopy; its curves are not claimed to solve an evolutionary equation. Both the simplex and  $W$  are contractible. Their ordinary homotopy types therefore do not record linkage.

For example, with  $S_1 = S_2 = \{0, 1\}$ , the family

$$p^\lambda = \begin{pmatrix} \frac{1}{4} + \lambda & \frac{1}{4} - \lambda \\ \frac{1}{4} - \lambda & \frac{1}{4} + \lambda \end{pmatrix}, \quad |\lambda| \leq \frac{1}{4},$$

has constant marginals  $x = y = (1/2, 1/2)$ . Only  $\lambda = 0$  lies in  $W$ . The whole interval is collapsed by  $m_U$ .

**4. Additive selection and dynamical closure**

Suppose the payoff to plan  $i$  in population  $p$  is additive:

$$f_i(p) = \sum_{k=1}^N a_k(i_k, m_U(p)), \quad \bar{f}(p) = \sum_i p_i f_i(p).$$

Assume  $a_k$  are Lipschitz on the product of marginal simplices. The replicator equation is

$$\dot{p}_i = p_i(f_i(p) - \bar{f}(p)).$$

For instance, a role model may have  $a_k(r, x) = \sum_{\ell} \rho_{k\ell} (A_{k\ell} x^{\ell})_r$  for fixed weights and payoff matrices. Additivity is a hypothesis; arbitrary extensive-form payoffs need not satisfy it.

**Proposition 4.1. The covariance correction**

Put  $x = m_U(p)$  and  $\bar{a}_k = \sum_r x_r^k a_k(r, x)$ . Then

$$\dot{x}_r^k = x_r^k (a_k(r, x) - \bar{a}_k) + \sum_{\ell \neq k} \text{Cov}_p(\mathbf{1}_{\{i_k=r\}}, a_{\ell}(i_{\ell}, x)).$$

Covariance is over the joint distribution  $i \sim p$ , with  $x$  held fixed when evaluating the random variables. On  $W$  these covariance terms vanish,  $W$  is invariant, and  $m_U$  conjugates the restricted flow to

$$\dot{x}_r^k = x_r^k (a_k(r, x) - \bar{a}_k).$$

*Proof.* Sum  $\dot{p}_i$  over  $i_k = r$  to obtain  $\mathbb{E}_p[\mathbf{1}_{\{i_k=r\}} f_i] - x_r^k \mathbb{E}_p[f_i]$ . The  $k$ -th payoff term gives the first term displayed; the others give the covariances. Under a product distribution distinct coordinates are independent. Differentiating  $p_i = \prod_k x_{i_k}^k$  along the coordinate equations gives the global equation, also when some factors vanish. Lipschitz uniqueness proves invariance and conjugacy.  $\square$

**Example 4.2. Equal marginals, different velocities**

Take two binary coordinates with constant local payoffs  $a_1(0) = 1$ ,  $a_1(1) = 0$ , and  $a_2 = 0$ . The correlated state  $p_{00} = p_{11} = 1/2$ ,  $p_{01} = p_{10} = 0$ , and the independent state  $p_{ij} = 1/4$  have identical marginals. At the correlated state,  $\dot{x}_0^2 = 1/4$ ; at the independent state,  $\dot{x}_0^2 = 0$ . Selection on the first coordinate changes the second marginal through correlation.

Marginalization therefore need not push the global vector field forward to a well-defined vector field on  $B(U)$ . State sheafification does not by itself make dynamics descend. Proposition 4.1 supplies closed marginal dynamics after restriction to  $W$ .

**Proposition 4.3. Conserved odds ratios**

For two roles and a strictly positive state, define

$$K_{ij, k\ell}(p) = \frac{p_{ij} p_{k\ell}}{p_{i\ell} p_{kj}}.$$

Under additive selection each odds ratio is constant along a trajectory. Strictly positive distributions  $p, q$  have identical odds ratios precisely when

$$p_{ij} = \frac{q_{ij} u_i v_j}{Z}, \quad u_i, v_j > 0, \quad Z = \sum_{i,j} q_{ij} u_i v_j.$$

*Proof.* Its logarithmic derivative is  $f_{ij} + f_{k\ell} - f_{i\ell} - f_{kj} = 0$  by additivity; mean fitness cancels. For the second statement, put  $h_{ij} = \log(p_{ij}/q_{ij})$ . Equality of odds ratios gives  $h_{ij} + h_{k\ell} - h_{i\ell} - h_{kj} = 0$ . A reference row and column express  $h_{ij} = b_i + c_j$ . Exponentiation and normalization give the formula; conversely the factors cancel.  $\square$

These level sets describe positive generalized Wright loci in this two-role model. All odds ratios equal one precisely on the positive part of  $W$ . These are algebraic invariants. No coefficient sheaf, transition maps, or torsor structure has been supplied to identify them with Čech cohomology classes. Vanishing of an unspecified cohomology class cannot serve as a criterion for independence.

## 5. Role replacement: attraction and retraction

For two roles, let  $x_i = \sum_j p_{ij}$  and  $y_j = \sum_i p_{ij}$ . Let  $q^1(t)$  and  $q^2(t)$  be probability distributions used for replacement at rates  $c_1, c_2 > 0$ . Along any absolutely continuous solution of

$$\dot{p}_{ij} = c_1(q_i^1 y_j - p_{ij}) + c_2(x_i q_j^2 - p_{ij}),$$

the choices may, for example, be measurable best-response selections to the current marginals. The argument is conditional on a solution; it makes no uniqueness claim for discontinuous best responses.

### Proposition 5.1. Exact linkage decay

For  $D_{ij} = p_{ij} - x_i y_j$ ,

$$\dot{D}_{ij} = -(c_1 + c_2)D_{ij}, \quad D_{ij}(t) = e^{-(c_1+c_2)t} D_{ij}(0).$$

*Proof.* Summation gives  $\dot{x}_i = c_1(q_i^1 - x_i)$  and  $\dot{y}_j = c_2(q_j^2 - y_j)$ . Hence  $\frac{d}{dt}(x_i y_j) = c_1 q_i^1 y_j + c_2 x_i q_j^2 - (c_1 + c_2)x_i y_j$ . Subtraction proves the differential identity almost everywhere and thus its integrated form.  $\square$

Linkage decays exponentially here even when the marginals keep moving. This does not make the solution flow a strong deformation retraction onto  $W$ : a strong retraction fixes every point of  $W$ , whereas replacement generally moves such points. A limit at infinite time also needs convergence of the marginals. Proposition 3.2 supplies a static retraction independently.

## 6. What a higher semantics must retain

The standard infinity-topos of sheaves of spaces on this site is

$$\mathcal{E} = \mathrm{Sh}_\infty(\mathcal{C}, J) \simeq \prod_{k \in U} \mathcal{S},$$

where  $\mathcal{S}$  denotes the infinity-category of spaces [4]. Disjoint singleton covers determine a sheaf from its singleton values, with its value on  $I$  their product. The set-valued sheaf  $B$  embeds as a 0-truncated object. This gives an ambient higher topos; it does not yet encode a nontrivial higher theory of evolution.

There are two distinct coefficient choices. Discrete sets of distributions retain their identities but introduce no higher paths. Singular complexes of the Euclidean simplices instead make  $F(I)$  and  $B(I)$  contractible for every  $I$ . These presheaves of spaces are then objectwise equivalent to the terminal object. Ordinary homotopy types of bare simplices forget the probabilistic information being studied. Richer semantics must retain structure such as convex operations, payoffs, distinguished subspaces, or directed dynamics.

For a continuous semiflow  $\phi$  on  $X$ , a time-labelled trajectory category has states as objects and arrows  $(p, t) : p \rightarrow \phi_t(p)$  for  $t \geq 0$ . Composition adds elapsed time; identity arrows have time zero. Its ordinary nerve satisfies the category axioms by construction. A homotopical treatment of its topology requires a further specified construction.

Positive-time arrows are not invertible even on a periodic orbit: composing arrows of positive total time cannot give a zero-time identity. Thus the maximal groupoid of this category does not automatically detect periodic orbits. A chosen periodic orbit has the ordinary homotopy type of a circle; the entire state simplex remains contractible. Computing the fundamental group of one cannot substitute for computing that of the other.

Coupled dynamics also need not restrict to smaller contexts. Proposition 4.1 shows how marginal velocities can depend on omitted correlations. A sheaf of trajectories needs restriction maps compatible with the dynamics; restriction of states alone does not provide them.

## 7. Local observation and open problems

For pure plans,  $S(I) = \prod_{k \in I} S_k$  is a sheaf on the context site. Given a prior plan  $e \in S(U)$  and an observation  $w \in S(K)$ , playwise pasting is the unique plan

$$e[K \leftarrow w]_k = \begin{cases} w_k, & k \in K, \\ e_k, & k \notin K. \end{cases}$$

This is gluing along the disjoint cover  $\{K, U \setminus K\}$ . It specifies which coordinates change after an observation. An interpretation as a Kan extension would require a functor, its domain and codomain, and a universal property; none is asserted here. A stochastic imitation rule also needs switching probabilities, observation frequencies, and payoff assumptions before a replicator equation follows.

Three extensions would move this construction toward extensive-form evolutionary semantics. First, build contexts from an actual information structure and justify covers by reachability and observation. Second, retain probability and payoff structure in semantic objects and prove when their dynamics permit restriction and gluing. Third, define any cohomological obstruction with explicit coefficients and comparison maps, distinguishing incompatible local data from compatible marginals with many correlated realizations.

Equilibrium sets pose a separate problem. A topology on contexts does not make evolutionarily stable sets a presheaf without a definition of restriction for games and stability conditions. Likewise, asymptotic attraction alone gives neither a continuous endpoint map nor a neighborhood deformation retraction. Such statements need their own hypotheses and proofs. The results here establish a comparison between marginal descent, product structure, and dynamical closure; general trees and higher obstruction theory remain open within this program.

## References

1. Ross Cressman. *Evolutionary Dynamics and Extensive Form Games*. MIT Press, 2003. ISBN 9780262033053. [Publisher record](#).
2. Neil Ghani, Jules Hedges, Viktor Winschel, and Philipp Zahn. “Compositional Game Theory.” *Proceedings of LICS 2018*, pp. 472-481. DOI: [10.1145/3209108.3209165](https://doi.org/10.1145/3209108.3209165). [Author preprint](#).

3. Samson Abramsky and Adam Brandenburger. “The Sheaf-Theoretic Structure of Non-Locality and Contextuality.” *New Journal of Physics* **13**, 113036, 2011. DOI: [10.1088/1367-2630/13/11/113036](https://doi.org/10.1088/1367-2630/13/11/113036). [Author preprint](#).
4. Jacob Lurie. *Higher Topos Theory*. Annals of Mathematics Studies 170, Princeton University Press, 2009. [Author preprint](#).

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