

# Abstract Memetics: Capability-Relative Replication and the Evolution of Semantic Universes

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## Part 1: Capability-Relative Memetic Replication

### Abstract

We formulate memetic replication as a capability-relative notion of semantic persistence inside an  $\infty$ -topos. The central structure is a hierarchy of left-exact localizations

$$L_c : E \longrightarrow E_c$$

indexed by cognitive capability levels  $c$ . A memetic presentation  $x \in M$  is perceived by an agent of capability  $c$  only through its localized image

$$\eta_c(x) \in L_c M.$$

Two presentations are therefore semantically identical relative to  $c$  precisely when their localized images are equivalent. A transmission from an agent  $a$  to an agent  $b$  counts as replication of a meme  $m$  when the transmitted realization and the recipient-localized realization of  $m$  are connected by an identity path in  $L_{\kappa(b)} M$ .

We prove that capability-relative semantic equivalence is an equivalence relation, that equivalence at a finer cognitive resolution implies equivalence at every coarser resolution, that the converse fails exactly when the comparison map between capability levels identifies distinct semantic states, and that replication witnesses compose along chains of transmission. We then define realization spaces, cognitive thresholds, mutation relative to capability, and the internal replication correspondence.

The resulting framework makes precise the proposition that memetic identity is not identity of presentation but equivalence after passage through a capability-dependent semantic modality.

### Notation

The symbols of the note, in the order they are used.

- $E$  — Ambient  $\infty$ -topos of states, agents, observations, and semantic structure.
- $c \leq d$  — Capability  $d$  distinguishes at least the semantic structure visible at  $c$ .
- $L_c$  — Accessible left-exact localization: the semantic modality at capability  $c$ .
- $\eta_c$  — Unit of the localization, carrying a presentation to its  $c$ -semantic state.
- $\rho_{dc}$  — Comparison from the finer modality  $L_d$  to the coarser modality  $L_c$ .
- $[m]_c$  — The  $c$ -semantic realization  $\eta_c(m)$  in  $L_c M$ .
- $x \approx_c y$  — Semantic equivalence: the images of  $x$  and  $y$  are identical in  $L_c M$ .
- $T_{ab}$  — Transmission of a semantic state from agent  $a$  into the space of agent  $b$ .

- $\text{Rep}_{a \rightarrow b}(m)$  — Space of paths witnessing that the transmission replicates  $m$ .
- $\text{Real}_c(m)$  — Ways an agent at capability  $c$  can internally realize the structure of  $m$ .
- $\chi(m)$  — Cognitive threshold: the least capability at which  $m$  is realizable.

## 1 Ambient categorical setting

We work in an  $\infty$ -topos  $E$ . Objects of  $E$  may be interpreted as spaces of possible states, representations, agents, observations, or semantic structures. Mapping objects and identity types are understood homotopy-coherently.

Let  $(C, \leq)$  be a small partially ordered set of cognitive capability levels. The intended interpretation is that  $c \leq d$  when capability  $d$  can distinguish at least as much semantic structure as capability  $c$ . Thus increasing capability corresponds to increasing semantic resolution.

### Definition 1.1 Capability modality

A **capability modality** at level  $c \in C$  is an accessible left-exact localization

$$L_c : E \longrightarrow E_c \subseteq E$$

with unit

$$\eta_c : \text{id}_E \Longrightarrow L_c.$$

Left exactness means that  $L_c$  preserves finite limits. Equivalently,  $E_c$  is a reflective sub- $\infty$ -category of  $E$ , and the reflector preserves finite limits.

### Definition 1.2 Capability hierarchy

A family  $\{L_c\}_{c \in C}$  is a **capability hierarchy** if, whenever  $c \leq d$ , there exists a natural comparison transformation

$$\rho_{dc} : L_d \Longrightarrow L_c$$

such that

$$\rho_{dc} \circ \eta_d \simeq \eta_c$$

and

$$L_c L_d \simeq L_c.$$

For  $c \leq d \leq e$ , we require the coherence

$$\rho_{ec} \simeq \rho_{dc} \circ \rho_{ed}.$$

Thus

$$L_e X \longrightarrow L_d X \longrightarrow L_c X$$

successively forgets semantic distinctions.

### Example 1.3 Truncation hierarchy

The fundamental example is  $C = N$  with  $L_n = \tau_{\leq n}$ , where  $\tau_{\leq n}$  denotes  $n$ -truncation. For  $n \leq m$ ,

$$\tau_{\leq n} \tau_{\leq m} X \simeq \tau_{\leq n} X.$$

Hence

$$\tau_{\leq m} X \longrightarrow \tau_{\leq n} X$$

forgets homotopical information above degree  $n$ . This is the canonical model of increasing semantic resolution.

## 2 Agents and cognitive capability

### Definition 2.1 Capability-stratified population

A population of agents is a family of objects  $\{A_c\}_{c \in C}$  of  $E$ . Its total object is

$$A := \coprod_{c \in C} A_c.$$

For an agent  $a \in A_c$ , we write  $\kappa(a) = c$ . The function

$$\kappa: A \longrightarrow C$$

is called the **capability assignment**. Strictly internally one may replace the external poset  $C$  by a poset object of  $E$ , but the externally indexed version suffices for the present results.

## 3 Memetic presentations and semantic identity

Let  $M \in E$  be an object whose points represent possible memetic presentations. A point

$$m: 1 \longrightarrow M$$

may represent, for example, an utterance, diagram, procedure, program, behavioral pattern, mathematical statement, narrative, policy, or other transmissible structure. No assumption is made that  $M$  is 0-truncated. Consequently, semantic identity may itself contain nontrivial higher homotopy.

### Definition 3.1 Capability-relative semantic state

For  $m \in M$  and capability  $c$ , define the  $c$ -semantic realization of  $m$  to be

$$[m]_c := \eta_c(m) \in L_c M.$$

**Definition 3.2 Capability-relative semantic equivalence**

For  $x, y \in M$ , define  $x \simeq_c y$  to mean that the identity type

$$\text{Id}_{L_c M}(\eta_c(x), \eta_c(y))$$

is inhabited. Equivalently,

$$x \simeq_c y \iff \eta_c(x) \simeq \eta_c(y) \text{ in } L_c M.$$

We call this \*semantic equivalence at capability  $c$ .\*

**Proposition 3.3 Equivalence is an equivalence relation**

For every fixed  $c$ , the relation  $\simeq_c$  on points of  $M$  is reflexive, symmetric, and transitive.

**Proof.**

Reflexivity follows from the identity path  $\text{id}_{\eta_c(x)}: \eta_c(x) = \eta_c(x)$ .

Given  $p: \eta_c(x) = \eta_c(y)$ , path inversion gives  $p^{-1}: \eta_c(y) = \eta_c(x)$ , so the relation is symmetric.

Given  $p: \eta_c(x) = \eta_c(y)$  and  $q: \eta_c(y) = \eta_c(z)$ , path composition gives  $q \circ p: \eta_c(x) = \eta_c(z)$ . Therefore  $\simeq_c$  is an equivalence relation.

**4 Monotonicity with respect to capability**

The basic structural result is that finer semantic equivalence implies coarser semantic equivalence.

**Theorem 4.1 Downward persistence of semantic equivalence**

Let  $c \leq d$ . If  $x \simeq_d y$ , then  $x \simeq_c y$ .

**Proof.**

Suppose  $p: \eta_d(x) = \eta_d(y)$  is a path in  $L_d M$ . Apply the comparison map

$$\rho_{dc, M}: L_d M \longrightarrow L_c M.$$

Functoriality of identity types produces

$$\rho_{dc, M}(p): \rho_{dc, M}(\eta_d(x)) = \rho_{dc, M}(\eta_d(y)).$$

By compatibility of the comparison transformation with the localization units,

$$\rho_{dc, M} \circ \eta_d \simeq \eta_c.$$

Hence  $\eta_c(x) = \eta_c(y)$ , and therefore  $x \simeq_c y$ .

#### Corollary 4.2 Nested semantic equivalence classes

For  $c \leq d$ , every  $d$ -equivalence class is contained in a  $c$ -equivalence class. Symbolically,

$$[x]_d \subseteq [x]_c.$$

Thus increasing capability can refine semantic equivalence classes but cannot merge classes that were already distinct at lower capability.

#### Corollary 4.3 Truncation monotonicity

If  $n \leq m$  and  $\tau_{\leq m}(x) \approx \tau_{\leq m}(y)$ , then  $\tau_{\leq n}(x) \approx \tau_{\leq n}(y)$ .

**Proof.**

Take  $L_n = \tau_{\leq n}$  and  $L_m = \tau_{\leq m}$  in Theorem 4.1.

### 5 Failure of the converse

The converse to Theorem 4.1 need not hold. A low-capability observer may identify two structures that remain distinguishable at higher capability.

#### Definition 5.1 Semantic collapse

For  $c \leq d$ , a pair  $x, y \in M$  exhibits \*semantic collapse from  $d$  to  $c$ \* if

$$x \not\approx_d y$$

but

$$x \approx_c y.$$

#### Theorem 5.2 Characterization of semantic collapse

Let  $c \leq d$ . Semantic collapse occurs in  $M$  if and only if the comparison map

$$\rho_{d,c,M}: L_d M \longrightarrow L_c M$$

has a homotopy fiber containing at least two inequivalent points belonging to the image of  $\eta_d: M \longrightarrow L_d M$ .

**Proof.**

Suppose first that semantic collapse occurs. Then there exist  $x, y \in M$  such that  $\eta_d(x) \neq \eta_d(y)$  but  $\eta_c(x) \approx \eta_c(y)$ . Since

$$\rho_{d,c,M} \eta_d(x) \approx \eta_c(x)$$

and

$$\rho_{dc,M} \eta_d(y) \approx \eta_c(y),$$

their images under  $\rho_{dc,M}$  are equivalent. Thus  $\eta_d(x)$  and  $\eta_d(y)$  determine inequivalent points in a common homotopy fiber of  $\rho_{dc,M}$ .

Conversely, suppose some homotopy fiber contains inequivalent points  $u \approx \eta_d(x)$  and  $v \approx \eta_d(y)$  coming from  $M$ . Since  $u$  and  $v$  lie in the same homotopy fiber,

$$\rho_{dc,M}(u) \approx \rho_{dc,M}(v).$$

Hence  $\eta_c(x) \approx \eta_c(y)$ , while  $\eta_d(x) \not\approx \eta_d(y)$ . Therefore semantic collapse occurs.

### Corollary 5.3 Criterion for strict refinement

The capability increase  $c < d$  strictly refines the semantics of  $M$  precisely when  $\rho_{dc,M}$  is non-injective on connected components of the localized image of  $M$ .

### Example 5.4 Homotopical collapse

Suppose  $M = S^1$ . At capability level  $L_0 = \tau_{\leq 0}$ , one has

$$\tau_{\leq 0} S^1 \approx *,$$

so every point becomes semantically indistinguishable. At capability level  $L_1 = \tau_{\leq 1}$ , the fundamental loop structure remains. Thus information encoded in path structure can be invisible to a 0-truncated semantic observer while remaining meaningful to a 1-truncated observer.

## 6 Transmission

Semantic equivalence alone does not specify replication. We must also specify a transmission operation.

### Definition 6.1 Transmission map

Let  $a \in A_c$  and  $b \in A_d$ . A **transmission channel**  $a \rightarrow b$  is a morphism

$$T_{ab}: L_c M \longrightarrow L_d M.$$

The map  $T_{ab}$  describes how a memetic state available to  $a$  is transformed into a semantic state accessible to  $b$ . Transmission need not preserve presentations.

### Definition 6.2 Replication witness space

Let  $m \in M$ . The **replication witness space** for transmission  $a \rightarrow b$  is

$$\text{Rep}_{a \rightarrow b}(m) := \text{Id}_{L_d M}(T_{ab}(\eta_c(m)), \eta_d(m)),$$

where  $c = \kappa(a)$  and  $d = \kappa(b)$ .

### Definition 6.3 Successful replication

The meme  $m$  replicates successfully from  $a$  to  $b$  if  $\text{Rep}_{a \rightarrow b}(m)$  is inhabited. Thus replication means

$$T_{ab}(\eta_c(m)) \simeq \eta_d(m).$$

### Remark 6.4 Presentation is not semantic identity

Successful replication does not require equality of underlying presentations in  $M$ . It requires only equality in the recipient's capability-localized semantic space. Thus  $m' \neq m$  may nevertheless count as successful replication if  $\eta_d(m') \simeq \eta_d(m)$ . This formalizes the distinction between **presentation identity** and **semantic identity**.

## 7 Replication under composition

Suppose  $a \in A_c$ ,  $b \in A_d$ , and  $e \in A_f$ . Let

$$T_{ab}: L_c M \rightarrow L_d M$$

and

$$T_{be}: L_d M \rightarrow L_f M.$$

Their composite is

$$T_{ae} := T_{be} \circ T_{ab}.$$

### Theorem 7.1 Composition of replication witnesses

If  $\text{Rep}_{a \rightarrow b}(m)$  and  $\text{Rep}_{b \rightarrow e}(m)$  are inhabited, then  $\text{Rep}_{a \rightarrow e}(m)$  is inhabited.

**Proof.**

Let  $p: T_{ab}(\eta_c(m)) = \eta_d(m)$  be a replication witness from  $a$  to  $b$ . Applying  $T_{be}$  gives

$$T_{be}(p): T_{be} T_{ab}(\eta_c(m)) = T_{be}(\eta_d(m)).$$

Let  $q: T_{be}(\eta_d(m)) = \eta_f(m)$  be a replication witness from  $b$  to  $e$ . Path composition yields

$$q \circ T_{be}(p): T_{be} T_{ab}(\eta_c(m)) = \eta_f(m).$$

But  $T_{be} T_{ab} = T_{ae}$ . Therefore  $T_{ae}(\eta_c(m)) = \eta_f(m)$ , so  $\text{Rep}_{a \rightarrow e}(m)$  is inhabited.

### Corollary 7.2 Iterated replication

For a chain of agents  $a_0 \rightarrow a_1 \rightarrow \dots \rightarrow a_n$ , if  $m$  replicates successfully across every edge, then  $m$  replicates successfully under the composite transmission  $a_0 \rightarrow a_n$ .

## 8 Mutation

Replication and mutation are not mutually exclusive notions. Mutation depends on which semantic resolution is being used.

### Definition 8.1 Capability-relative mutation

Suppose transmission produces a presentation  $m' \in M$ . Relative to capability  $c$ , the transmission is:

- **presentation-preserving** if  $m' \approx m$  in  $M$ ;
- **\*semantically faithful at  $c$ \*** if  $m' \approx_c m$ ;
- **a  $c$ -detectable mutation\*** if  $m' \not\approx_c m$ .

### Proposition 8.2 Mutation can disappear under coarse-graining

Let  $c \leq d$ . It is possible that  $m' \not\approx_d m$  while  $m' \approx_c m$ . In that case the same transmission is classified as mutation by capability  $d$  and faithful replication by capability  $c$ .

**Proof.**

This is precisely semantic collapse as defined in Definition 5.1.

### Corollary 8.3 Capability-relative fidelity

There is no capability-independent binary distinction between faithful copying and mutation unless the family of modalities is jointly conservative on  $M$ .

### Definition 8.4 Joint conservativity

The capability hierarchy is **\*jointly conservative on  $M$ \*** if

$$(\forall c \in C, \eta_c(x) \approx \eta_c(y)) \implies x \approx y.$$

When joint conservativity fails, two genuinely different ambient structures can be indistinguishable to every capability level represented in the population.

## 9 Cognitive realization

So far we have described semantic discrimination. We now introduce the ability of an agent actually to instantiate a memetic structure. Let  $P_m \in E$  denote a presentation object for meme  $m$ . For every capability level  $c$ , let  $C_c \in E_c$  be a cognitive state object describing realizable structures for agents at capability  $c$ .

### Definition 9.1 Realization space

The **realization space** of  $m$  at capability  $c$  is

$$\text{Real}_c(m) := \text{Map}_{E_c}(L_c P_m, C_c).$$

The meme is **\*realizable at capability  $c$ \*** if  $\text{Real}_c(m) \neq \emptyset$ .

**Definition 9.2 Capability-monotone realization system**

A realization system is **capability-monotone** if, for every  $c \leq d$ , there is a map

$$\Phi_{cd,m} : \text{Real}_c(m) \longrightarrow \text{Real}_d(m)$$

natural in  $m$ . This expresses the assumption that a more capable cognitive architecture can realize every structure realizable by a less capable one.

**Theorem 9.3 Upward closure of realizability**

In a capability-monotone realization system,  $\text{Real}_c(m) \neq \emptyset$  and  $c \leq d$  imply  $\text{Real}_d(m) \neq \emptyset$ .

**Proof.**

Choose  $r \in \text{Real}_c(m)$ . Applying  $\Phi_{cd,m}$  gives

$$\Phi_{cd,m}(r) \in \text{Real}_d(m).$$

Thus the latter realization space is inhabited.

**Definition 9.4 Cognitive support**

Define the **cognitive support** of  $m$  by

$$\Theta(m) := \{ c \in C \mid \text{Real}_c(m) \neq \emptyset \}.$$

**Corollary 9.5 Upward closure of support**

If the realization system is capability-monotone, then  $\Theta(m)$  is upward closed:

$$c \in \Theta(m), \quad c \leq d \implies d \in \Theta(m).$$

**Definition 9.6 Cognitive threshold**

If  $\Theta(m)$  possesses a least element, define

$$\chi(m) := \min \Theta(m).$$

We call  $\chi(m)$  the **cognitive threshold** of  $m$ . It is the least capability at which the memetic structure can be internally realized.

**10 Comprehension versus replication**

Realization and replication are logically distinct.

**Definition 10.1 Comprehension condition**

An agent  $b \in A_d$  can **comprehend**  $m$  if  $\text{Real}_d(m)$  is inhabited.

**Definition 10.2 Replication condition**

A transmission from  $a$  to  $b$  successfully replicates  $m$  if  $\text{Rep}_{a \rightarrow b}(m)$  is inhabited.

**Proposition 10.3 Logical independence**

Without additional assumptions,

$$\text{Real}_d(m) \neq \emptyset$$

does not imply

$$\text{Rep}_{a \rightarrow b}(m) \neq \emptyset,$$

and

$$\text{Rep}_{a \rightarrow b}(m) \neq \emptyset$$

does not imply any richer realization property beyond that explicitly encoded in  $L_d M$ .

**Proof.**

The first statement follows because realizability concerns the mapping space  $\text{Map}(L_d P_m, C_d)$ , whereas replication concerns the identity type  $T_{ab}(\eta_c(m)) = \eta_d(m)$ . No morphism relating these two spaces has been assumed.

For the converse, replication identifies localized semantic states but does not imply that the recipient possesses any additional map  $L_d P_m \rightarrow C_d$  unless such a connection is separately specified. Thus the two notions are independent in the bare theory.

**11 Internal replication correspondence**

The local definitions can be assembled into a global object. Assume transmission maps vary coherently with pairs of agents. Define an object

$$R \longrightarrow A \times A \times M$$

whose homotopy fiber over  $(a, b, m)$  is  $\text{Rep}_{a \rightarrow b}(m)$ .

**Definition 11.1 Memetic replication correspondence**

The span

$$A \times M \xleftarrow{s} R \xrightarrow{t} A \times M$$

is the **memetic replication correspondence**. The source map records  $(a, m)$ , while the target map records the corresponding recipient state  $(b, m)$ . More general versions may allow the target presentation to be  $m'$ , producing  $(a, m) \rightsquigarrow (b, m')$ .

**Definition 11.2 Generalized replication correspondence**

Define

$$\mathcal{R} \longrightarrow A \times A \times M \times M$$

with fiber

$$\mathcal{R}_{a,b,m,m'} : = \text{Id}_{L_{\kappa(b)}M}(T_{ab}(\eta_{\kappa(a)}m), \eta_{\kappa(b)}m').$$

Then  $(a, m) \rightsquigarrow (b, m')$  is a valid transmission exactly when this identity type is inhabited. Faithful replication of  $m$  corresponds to the diagonal case  $m' = m$ . Capability-relative mutation corresponds to  $m' \neq m$  while  $m' \simeq_{\kappa(b)} m$ .

## 12 Replication as a homotopy-coherent dynamical system

A sequence

$$(a_0, m_0) \rightsquigarrow (a_1, m_1) \rightsquigarrow \cdots \rightsquigarrow (a_n, m_n)$$

defines a path in the correspondence-generated dynamics. At each stage we may have  $m_i \neq m_{i+1}$  in the ambient object  $M$ , even though  $m_i \simeq_{\kappa(a_{i+1})} m_{i+1}$ . Thus memetic persistence is naturally represented not by a constant point of  $M$ , but by a coherently related sequence of points whose images agree after appropriate localization.

### Definition 12.1 Capability-relative memetic lineage

A **memetic lineage** is a sequence  $(m_i, a_i)_{i \in I}$  together with transmission witnesses

$$p_i : T_{a_i a_{i+1}}(\eta_{\kappa(a_i)}m_i) = \eta_{\kappa(a_{i+1})}m_{i+1}.$$

### Definition 12.2 Stable memetic lineage

Such a lineage is **semantically stable relative to the recipients** if

$$m_{i+1} \simeq_{\kappa(a_{i+1})} m_i$$

for every  $i$ . It need not be stable in the ambient space  $M$ .

## 13 The central representation theorem

We may now state the basic conceptual result of the framework.

### Theorem 13.1 Capability-relative criterion for memetic identity

Let  $m, m' \in M$ . For an agent  $a \in A_c$ , the two presentations  $m, m'$  represent the same memetic state accessible to  $a$  if and only if

$$\eta_c(m) \simeq \eta_c(m')$$

in  $L_c M$ . Equivalently,  $m \simeq_c m'$ .

**Proof.**

By construction, all semantic states available at capability  $c$  factor through the localization  $\eta_c : M \rightarrow L_c M$ . Hence an agent at capability  $c$  has access only to the images  $\eta_c(m)$  and  $\eta_c(m')$ . If these images are equivalent, they determine the same state of  $L_c M$  up to its intrinsic homotopy notion of identity.

Conversely, if they are inequivalent in  $L_c M$ , then the localized semantic system contains a distinction between them. Therefore semantic identity relative to capability  $c$  is exactly equality in  $L_c M$ .

### Theorem 13.2 Replication preserves semantic identity

A transmission  $T_{ab} : L_{K(a)} M \rightarrow L_{K(b)} M$  successfully replicates  $m$  if and only if

$$T_{ab}([m]_{K(a)}) \simeq [m]_{K(b)}.$$

**Proof.**

By Definition 6.2,

$$\text{Rep}_{a \rightarrow b}(m) = \text{Id}_{L_{K(b)} M}(T_{ab}([m]_{K(a)}), [m]_{K(b)}).$$

This space is inhabited precisely when the two displayed points are equivalent.

## 14 Apparent fidelity and cognitive coarseness

The theory has an immediate consequence relevant to populations with heterogeneous cognitive capabilities.

### Theorem 14.1 Coarse observers weakly increase apparent fidelity

Let  $c \leq d$ . If a transmission is semantically faithful at capability  $d$ , then it is semantically faithful at capability  $c$ . The converse need not hold.

**Proof.**

Faithfulness at  $d$  means  $m' \simeq_d m$ . By Theorem 4.1,  $m' \simeq_c m$ . The converse fails whenever semantic collapse occurs between  $d$  and  $c$ , by Theorem 5.2.

### Corollary 14.2 Measured fidelity can rise as resolution falls

A decrease in cognitive resolution can increase the measured rate of faithful replication without increasing presentation-level fidelity. This follows because the equivalence classes  $[-]_c$  become coarser as capability decreases.

### Remark 14.3 Sloganization and semantic loss

This provides a formal mechanism for phenomena in which many distinct presentations are classified as instances of one simple semantic pattern. Examples may include simplification, sloganization, compression, ritualized repetition, or semantic loss under repeated transmission. The mathematical claim does not depend on any particular empirical theory of human cognition; it follows from the localization structure once such a structure is adopted as a model.

## 15 Memetic complexity

Capability thresholds induce a natural notion of semantic complexity.

### Definition 15.1 Capability complexity

Suppose  $C$  is totally ordered and every realizable meme possesses a cognitive threshold. Define

$$\text{Comp}(m) := \chi(m).$$

Thus  $\text{Comp}(m)$  is the least capability required to realize  $m$ .

### Proposition 15.2 Complexity monotonicity under realization

If  $\text{Comp}(m) = c$ , then every  $d \geq c$  can realize  $m$ , assuming capability-monotone realization.

**Proof.**

Immediate from Theorem 9.3.

## 16 Semantic compression

The localizations themselves can be interpreted as semantic compression operators. For  $c \leq d$ , the comparison

$$\rho_{dc, M} : L_d M \rightarrow L_c M$$

forgets distinctions inaccessible at level  $c$ .

### Definition 16.1 Capability-relative compression

The map  $\rho_{dc, M}$  is the \*semantic compression map from  $d$  to  $c$ \*. Its homotopy fibers measure how many finer semantic states collapse into one coarser state.

### Proposition 16.2 Fibers detect collapsed distinctions

If a homotopy fiber of  $\rho_{dc, M}$  contains multiple connected components, then capability  $c$  identifies semantic states distinguishable at capability  $d$ .

**Proof.**

This is the fiberwise form of Theorem 5.2.

## 17 Information loss as failure of conservativity

A transmission may destroy distinctions.

### Definition 17.1 Semantically conservative transmission

A transmission  $T : L_c M \rightarrow L_d M$  is **semantically conservative** if it reflects equivalences:

$$T(x) \simeq T(y) \implies x \simeq y.$$

### Definition 17.2 Semantic information loss

Transmission exhibits **semantic information loss** if there exist

$$x \neq y$$

such that

$$T(x) \approx T(y).$$

### Proposition 17.3 Conservativity is absence of loss

A transmission has no semantic information loss if and only if it is conservative on the relevant full sub- $\infty$ -groupoid of  $L_c M$ .

**Proof.**

This is the definition of conservativity specialized to the semantic states under consideration.

## 18 Replication, compression, and selection

The foregoing structures distinguish three independent components.

They correspond respectively to the realization space, the existence and properties of the transmission maps, and the identity type witnessing replication. Any quantitative evolutionary or probabilistic theory built on top of the present framework should distinguish at least these three factors. One may schematically write

$$\begin{aligned} & \text{effective replication rate} \\ \sim & \text{exposure} \times \text{realizability} \times \text{transmission probability} \times \text{capability-relative fidelity}, \end{aligned}$$

but such a numerical formula requires additional probabilistic structure not assumed here.

## 19 Formal specification

The complete deterministic specification may be summarized by the tuple

$$\begin{aligned} M = & (E, C, \{L_c\}_{c \in C}, \{A_c\}_{c \in C}, \\ & M, \{C_c\}_{c \in C}, \{T_{ab}\}). \end{aligned}$$

- $E$ : Ambient  $\infty$ -topos.
- $C$ : Partially ordered capability levels.
- $L_c$ : Capability-dependent semantic modality.
- $A_c$ : Agents of capability  $c$ .
- $M$ : Space of memetic presentations.
- $C_c$ : Cognitive realization object at level  $c$ .
- $T_{ab}$ : Transmission morphisms.

The fundamental identity criterion is

$$m \simeq_c m' \iff \text{Id}_{L_c M}(\eta_c(m), \eta_c(m')) \text{ is inhabited.}$$

The fundamental replication criterion is

$$\text{Rep}_{a \rightarrow b}(m) = \text{Id}_{L_{\kappa(b)}M}(T_{ab}(\eta_{\kappa(a)}m), \eta_{\kappa(b)}m).$$

## 20 Main structural consequences

The preceding results may be summarized as follows.

- Theorem A — Equivalence. For each capability  $c$ , the relation  $\approx_c$  is an equivalence relation.
- Theorem B — Capability monotonicity. If  $c \leq d$ , then  $x \approx_d y$  implies  $x \approx_c y$ .
- Theorem C — Strict semantic refinement. The converse fails precisely when  $L_d M \rightarrow L_c M$  collapses distinct semantic states.
- Theorem D — Composition. Replication witnesses compose along transmission chains.
- Theorem E — Capability-relative mutation. A transformation may be mutation at capability  $d$  and faithful replication at capability  $c < d$ .
- Theorem F — Cognitive threshold. Under capability-monotone realizability, the set of capabilities able to realize a meme is upward closed.

## 21 Interpretation

The principal conceptual claim can now be stated precisely.

A meme is not intrinsically identified with a unique physical, syntactic, or representational presentation. Instead, relative to cognitive capability  $c$ , its operative identity is its image

$$[m]_c = \eta_c(m) \in L_c M.$$

Hence memetic identity is indexed by capability:

$$m \mapsto \{[m]_c\}_{c \in C}.$$

Replication likewise cannot be defined solely by duplication of presentations. It is instead preservation of localized semantic identity:

$$T_{ab}([m]_{\kappa(a)}) \approx [m]_{\kappa(b)}.$$

Memetic replication is capability-relative semantic invariance under transmission.

A lower-capability system corresponds to a coarser localization and therefore possesses larger semantic equivalence classes. A higher-capability system corresponds to a finer localization and may distinguish structures previously collapsed together. Thus  $c \leq d$  induces the refinement

$$[m]_d \mapsto [m]_c,$$

and consequently

$$m \simeq_d m' \implies m \simeq_c m',$$

$$m \simeq_c m' \text{ does not imply } m \simeq_d m'.$$

This provides the formal sense in which memetic replication is relative to cognitive capability.

## 22 Conclusion

The  $\infty$ -topos formulation replaces a naive conception of a meme as an invariant token by a hierarchy of capability-relative semantic equivalence classes.

Given an ambient  $\infty$ -topos  $E$ , a hierarchy of left-exact localizations  $L_c: E \rightarrow E_c$ , and a memetic state object  $M$ , the identity of a meme for an observer of capability  $c$  is represented by its localized state  $\eta_c(m) \in L_c M$ .

Transmission from  $a$  to  $b$  constitutes replication exactly when

$$T_{ab}(\eta_{\kappa(a)} m) \simeq \eta_{\kappa(b)} m.$$

This produces a homotopy-coherent notion of replication in which copying, mutation, semantic compression, comprehension, and capability are formally separated.

Identity of memes is not identity of presentations.

\*\*

Identity is equivalence in the semantic localization determined by cognitive capability.

Consequently, memetic replication is not an absolute relation on presentations. It is a capability-indexed relation internal to the appropriate localized  $\infty$ -topos.

## Part 2: Operadic Memetic Replication

### Abstract

We extend the theory of observer-relative memetic replication from unary transmission operators to genuinely multi-input recombination. Let

$$H = \text{Sh}_\infty(C, J)$$

be the  $\infty$ -topos associated to a cognitive  $\infty$ -site. Rather than assuming that a memetic state evolves only by an endomorphism

$$R: M \rightarrow M,$$

we equip the memetic field  $M \in H$  with an algebra structure over an  $\infty$ -operad  $O$ . An  $n$ -ary operation  $o \in O(n)$  then induces a recombination map

$$\mu_o^n: M^n \longrightarrow M.$$

This permits memes to be copied, composed, abstracted, translated, fused, specialized, compressed, or otherwise transformed by operations of arbitrary finite arity.

Replication histories are consequently no longer linear chains but rooted  $O$ -decorated trees. Operadic composition assigns every such tree a coherent composite operation, and the algebra structure evaluates the tree to a memetic state. We prove that semantic equivalence is preserved by all operadic operations, that tree evaluation is invariant under operadic reassociation, that restriction between cognitive architectures commutes with recombination, and that spaces of recombination witnesses satisfy descent.

The free  $O$ -algebra provides a canonical space of formal memetic traces. Its evaluation map

$$\text{ev}_M: F_O(UM) \longrightarrow M$$

identifies syntactically or causally different histories that realize the same semantic state. This yields a precise distinction between operational history and semantic identity.

The unary theory of replication, developed in the preceding note, is recovered as a special case.

## Notation

The symbols of this note, in the order they become active.

- $(C, J)$  — Cognitive  $\infty$ -site. Architectures are objects of  $C$ ;  $J$  is the Grothendieck topology.
- $H$  — The  $\infty$ -topos of sheaves  $\text{Sh}_\infty(C, J)$ , with cartesian product.
- $M(c)$  — Memetic states distinguishable at architecture  $c$ .
- $u^*$  — Restriction along a morphism  $u: c \rightarrow d$ , from states at  $d$  to states at  $c$ .
- $O(n)$  — The space of  $n$ -ary memetic operations.
- $\mu_n$  — The algebra action  $O(n) \times M^{\wedge n} \rightarrow M$ .
- $x \simeq_c y$  — Semantic identity: a path in  $M(c)$ .
- $\text{Rec}_c$  — Witness that an operation on given inputs yields a stated output.
- $\text{Rep}^{(j)}$  — Witness that the  $j$ -th parent is reproduced.
- $F_O$  — Free  $O$ -algebra: formal finite operadic expressions.
- $\epsilon_M$  — Trace evaluation, from formal histories to semantic states.

## 1 Ambient cognitive semantics

Let  $(C, J)$  be a cognitive  $\infty$ -site and put

$$H := \text{Sh}_\infty(C, J).$$

We use the cartesian symmetric monoidal structure

$$(H, \times, 1).$$

Let  $M \in H$  be a memetic field. For every cognitive architecture  $c \in C$ , the space  $M(c)$  is interpreted as the space of memetic states distinguishable at  $c$ .

For every architecture morphism  $u: c \rightarrow d$  there is a restriction map

$$u^*: M(d) \longrightarrow M(c).$$

## 2 Internal $\infty$ -operads of memetic transformations

We require a structure encoding operations of arbitrary finite arity.

### Definition 2.1 Internal memetic operad

An **internal memetic  $\infty$ -operad**  $O$  in  $H$  consists, schematically, of objects

$$O(n) \in H, \quad n \geq 0,$$

equipped with symmetric-group actions, units, and homotopy-coherent substitution maps

$$\gamma: O(k) \times O(n_1) \times \cdots \times O(n_k) \longrightarrow O(n_1 + \cdots + n_k)$$

satisfying the usual operadic associativity, equivariance, and unit axioms up to coherent higher homotopy.

A section  $o \in O(n)(c)$  is interpreted as an  $n$ -ary transformation available at architecture  $c$ .

Examples include:

- copy
- concatenate
- summarize
- translate
- generalize
- specialize
- compare
- combine
- infer

The mathematics does not privilege any particular interpretation.

## 3 Memetic algebras

### Definition 3.1 O-algebra

An  $O$ -memetic algebra is a memetic field  $M \in H$  together with morphisms

$$\mu_n: O(n) \times M^n \longrightarrow M$$

compatible with the operadic composition, symmetry, and unit structure.

For  $o \in O(n)(c)$  and  $x_1, \dots, x_n \in M(c)$ , we write

$$\mu_c(o; x_1, \dots, x_n) \in M(c).$$

The basic dynamics is therefore

$$(x_1, \dots, x_n) \mapsto \mu(o; x_1, \dots, x_n).$$

Unary transmission is merely the case  $n = 1$ . Binary recombination is

$$\mu: O(2) \times M \times M \rightarrow M.$$

#### 4 Semantic identity

For  $x, y \in M(c)$ , define  $x \simeq_c y$  when

$$\text{Id}_{M(c)}(x, y)$$

is inhabited. As before, semantic identity is therefore architecture-relative and homotopical.

#### 5 Operadic operations respect semantic identity

The first structural theorem follows simply from functoriality.

##### Theorem 5.1 Homotopy invariance of recombination

Let  $o, o' \in O(n)(c)$  and  $x_i, x'_i \in M(c)$ . Suppose  $o \simeq o'$  and, for every  $i$ ,  $x_i \simeq_c x'_i$ . Then

$$\mu_c(o; x_1, \dots, x_n) \simeq_c \mu_c(o'; x'_1, \dots, x'_n).$$

**Proof.**

The algebra structure is a map of spaces  $\mu_{n,c}: O(n)(c) \times M(c)^n \rightarrow M(c)$ . The assumed equivalences give a path

$$(o, x_1, \dots, x_n) \simeq (o', x'_1, \dots, x'_n)$$

in its domain. Applying  $\mu_{n,c}$  yields a path

$$\mu_c(o; x_1, \dots, x_n) \simeq \mu_c(o'; x'_1, \dots, x'_n)$$

in  $M(c)$ .

## Corollary 5.2 Semantic congruence

Capability-relative semantic equivalence is a congruence with respect to every available memetic operation. Thus

$$x_i \simeq_c x'_i \quad \forall i$$

implies

$$o(x_1, \dots, x_n) \simeq_c o(x'_1, \dots, x'_n).$$

This is the operadic analogue of substitutivity of equality.

## 6 Recombination witnesses

### Definition 6.1 Recombination witness space

For  $o \in O(n)(c)$  and  $x_1, \dots, x_n, y \in M(c)$ , define

$$\text{Rec}_c(o; x_1, \dots, x_n; y)$$

by

$$\text{Rec}_c(o; x_1, \dots, x_n; y) := \text{Id}_{\{M(c)\}} \bigl( \mu_c(o; x_1, \dots, x_n), y \bigr).$$

The recombination succeeds with output  $y$  when this space is inhabited. The definition separates inputs, operation, output, and the witness of semantic agreement.

## 7 Replication as a special kind of recombination

A recombination need not preserve any input. Replication is obtained by requiring the output to remain in the semantic class of some designated parent or target.

### Definition 7.1 Parent-relative replication

Let  $1 \leq j \leq n$ . Define  $\text{Rep}_c^{(j)}(o; x_1, \dots, x_n)$  by

$$\text{Id}_{M(c)}(\mu_c(o; x_1, \dots, x_n), x_j).$$

If this space is inhabited, the  $j$ -th input has reproduced its semantic type through the operation  $o$ .

### Definition 7.2 Target-relative replication

For a distinguished semantic target  $m \in M(c)$ , define

$$\text{Id}_{M(c)}(\mu_c(o; \vec{x}), m).$$

This is useful when the output need not be identical to any particular input.

## 8 Unary replication is recovered

Let  $r \in O(1)$ . The corresponding operation is

$$R = \mu_r : M \rightarrow M.$$

Then parent-relative replication is the identity type

$$\text{Id}_{M(c)}(R_c(x), x).$$

Thus the previous fixed-point theory  $\text{Eq}(R, \text{id}_M)$  is exactly the unary subtheory of the operadic construction.

## 9 Operadic substitution

Suppose  $o \in O(k)$  and  $o_i \in O(n_i)$  for  $1 \leq i \leq k$ . Their operadic composite is

$$o \circ (o_1, \dots, o_k) \in O(n_1 + \dots + n_k).$$

The algebra axioms provide a coherent equivalence

$$\begin{aligned} & \mu(o; \mu(o_1; \vec{x}_1), \dots, \mu(o_k; \vec{x}_k)) \\ & \simeq \mu(o \circ (o_1, \dots, o_k); \vec{x}_1, \dots, \vec{x}_k). \end{aligned}$$

This equation is the formal mechanism by which several generations of recombination collapse to one composite operation.

## 10 Tree-structured memetic histories

Operadic composition is naturally encoded by rooted trees.

### Definition 10.1 O-decorated memetic tree

An  $O$ -decorated memetic tree  $T$  consists of a finite rooted tree; an operation  $o_v \in O(n_v)$  assigned to every internal vertex  $v$  of arity  $n_v$ ; and memetic states assigned to the leaves. The root represents the final output.

### Definition 10.2 Evaluation of a memetic tree

The evaluation  $\text{ev}_T$  is obtained recursively by applying the operation at every vertex to the evaluations of its incoming branches. Thus

$$\text{ev}_T(\vec{x}) \in M.$$

## 11 Tree contraction theorem

### Theorem 11.1 Operadic contraction of replication histories

For every finite  $O$ -decorated rooted tree  $T$ , there exists a composite operation  $o_T \in O(N)$ , where  $N$  is the number of leaves, such that

$$\text{ev}_T(x_1, \dots, x_N) \approx \mu(o_T; x_1, \dots, x_N).$$

The composite  $o_T$  is well defined up to the coherent equivalence supplied by the  $\infty$ -operad structure.

**Proof.**

Proceed by induction on the number of internal vertices.

If  $T$  has one internal vertex, the result is immediate.

Suppose the root is decorated by  $o \in O(k)$  and its  $k$  input subtrees are  $T_1, \dots, T_k$ . By induction, each subtree determines  $o_{T_i} \in O(n_i)$  such that  $\text{ev}_{T_i} \approx \mu(o_{T_i}; -)$ . Therefore

$$\text{ev}_T \approx \mu(o; \mu(o_{T_1}; -), \dots, \mu(o_{T_k}; -)).$$

By the algebra compatibility with operadic substitution,

$$\text{ev}_T \approx \mu(o \circ (o_{T_1}, \dots, o_{T_k}); -).$$

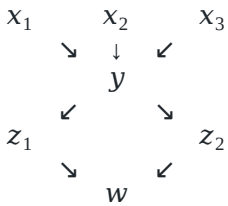
Define  $o_T := o \circ (o_{T_1}, \dots, o_{T_k})$ . Higher associativity coherence of the  $\infty$ -operad guarantees independence, up to canonical coherent equivalence, from choices of recursive decomposition.

## 12 Lineages become trees rather than chains

In the unary theory a history had the form

$$x_0 \rightarrow x_1 \rightarrow x_2 \rightarrow \dots.$$

In the operadic theory one instead has branching and merging:



The appropriate causal object is therefore a rooted tree or, after allowing sharing and feedback, a directed acyclic graph or higher categorical network.

## 13 Semantic identity of histories

Two distinct causal histories can produce the same semantic result.

### Definition 13.1 Semantic equivalence of histories

Let  $T$  and  $T'$  be two operadic histories with outputs  $\text{ev}_T(X)$  and  $\text{ev}_{T'}(Y)$  at architecture  $c$ . Define  $T \approx_c T'$  if

$$\text{ev}_T(\tilde{X}) \simeq_c \text{ev}_{T'}(\tilde{Y}).$$

Thus history identity is strictly finer than semantic identity.

## 14 The free operadic algebra

Assume the forgetful functor

$$U: \text{Alg}_O(H) \longrightarrow H$$

admits a left adjoint

$$F_O: H \longrightarrow \text{Alg}_O(H).$$

This holds in the standard presentable setting for suitably small operads.

The object  $F_O(X)$  is the free  $O$ -algebra generated by  $X$ . Its elements are formal finite operadic expressions built from points of  $X$ . Schematically,

$$x, \quad o(x_1, \dots, x_n), \quad o(o_1(\dots), \dots, o_k(\dots)),$$

and so forth.

## 15 Formal memetic traces

### Definition 15.1 Trace object

For an  $O$ -algebra  $M$ , define its formal trace algebra

$$\text{Tr}_O(M) := F_O(UM).$$

A point of this object represents a finite formal history of operadic transformations whose leaves are memetic states. The counit of the free–forgetful adjunction gives an  $O$ -algebra morphism

$$\epsilon_M: F_O(UM) \longrightarrow M.$$

We call this the trace evaluation map:

$$\text{ev}_M: \text{Tr}_O(M) \longrightarrow M.$$

## 16 Trace-to-semantic-class map

### Definition 16.1 Semantic equivalence of traces

For traces  $t, t' \in \text{Tr}_O(M)(c)$ , define  $t \equiv_c t'$  if

$$\text{ev}_M(t) \approx_c \text{ev}_M(t').$$

Equivalently,

$$\text{Id}_{M(c)}(\text{ev}_M(t), \text{ev}_M(t'))$$

is inhabited. This gives the fundamental map

formal transformation history maps to a semantic operation class.

Many distinct traces can map to the same semantic class. Hence

$$t \neq t'$$

does not imply

$$\text{ev}(t) \neq \text{ev}(t').$$

## 17 Universal property of the trace algebra

### Theorem 17.1 Universal evaluation theorem

Let  $M$  be an  $O$ -algebra. Every map of underlying objects  $f: X \rightarrow UM$  extends, uniquely up to contractible choice, to an  $O$ -algebra morphism

$$\bar{f}: F_O(X) \rightarrow M.$$

**Proof.**

This is the defining adjunction  $F_O \dashv U$ . Explicitly,

$$\text{Map}_{\text{Alg}_O}(F_O(X), M) \simeq \text{Map}_H(X, UM).$$

The homotopy fiber over  $f$  is therefore contractible.

### Corollary 17.2 Canonical trace evaluation

Taking  $X = UM$  and  $f = \text{id}_{UM}$  produces the canonical trace evaluation map

$$\epsilon_M: F_O(UM) \rightarrow M.$$

## 18 Operational nonuniqueness

The trace formalism gives an immediate nonuniqueness theorem.

### Theorem 18.1 Semantic equality does not imply historical equality

Suppose the evaluation map  $\epsilon_M : F_O(UM) \rightarrow M$  has a homotopy fiber with more than one connected component. Then there exist formally inequivalent histories  $t \neq t'$  such that  $\epsilon_M(t) \simeq \epsilon_M(t')$ .

**Proof.**

Choose  $t, t'$  in distinct connected components of one homotopy fiber of  $\epsilon_M$ . By definition of the fiber,  $\epsilon_M(t) \simeq \epsilon_M(t')$ . But  $t \neq t'$  in the free algebra.

Thus uniqueness of semantic output never implies uniqueness of causal or computational realization unless the evaluation map is appropriately conservative. See also Corollary 17.2.

### 19 Architecture restriction commutes with recombination

Because  $\mu_n : O(n) \times M^n \rightarrow M$  is a morphism of sheaves, it is natural in the architecture.

#### Theorem 19.1 Recombination commutes with cognitive restriction

Let  $u : c \rightarrow d$ . Then

$$u^* \mu_d(o; x_1, \dots, x_n) \simeq \mu_c(u^* o; u^* x_1, \dots, u^* x_n).$$

**Proof.**

Naturality of the sheaf morphism  $\mu_n : O(n) \times M^n \rightarrow M$  gives the commutative square

$$\begin{array}{ccc} O(n)(d) \times M(d)^n & \xrightarrow{\mu_d} & M(d) \\ \downarrow & & \downarrow \\ O(n)(c) \times M(c)^n & \xrightarrow{\mu_c} & M(c). \end{array}$$

The stated equivalence follows.

### 20 Observer-relative recombination

The preceding theorem implies that performing a transformation and then restricting is equivalent to restricting the inputs and the operation, and then performing the transformation. Thus the operadic dynamics is compatible with observer relativity.

#### Corollary 20.1 Restriction of a successful recombination

If  $\mu_d(o; \vec{x}) \simeq_d y$ , then

$$\mu_c(u^* o; u^* \vec{x}) \simeq_c u^* y.$$

**Proof.**

Apply  $u^*$  to the original equivalence and use Theorem 19.1.

## 21 Apparent fidelity under semantic collapse

Suppose  $u: c \rightarrow d$  is nonconservative. It may happen that  $\mu_d(o; \vec{x}) \simeq_d y$  while  $u^* \mu_d(o; \vec{x}) \simeq_c u^* y$ . Using Theorem 19.1,

$$\mu_c(u^* o; u^* \vec{x}) \simeq_c u^* y.$$

Thus an operation classified as mutation at  $d$  may be classified as faithful replication at  $c$ . The same capability-relative fidelity phenomenon therefore persists for arbitrary arity.

## 22 The internal recombination object

We can package recombination witnesses into an object of the  $\infty$ -topos. Let

$$E_n := O(n) \times M^n \times M.$$

There are two maps  $E_n \rightrightarrows M$ . The first sends  $(o, \vec{x}, y)$  to  $\mu(o; \vec{x})$ , and the second sends it to  $y$ .

### Definition 22.1 Recombination object

Define

$$R_n := \text{Eq}(E_n \rightrightarrows M)$$

as a homotopy equalizer. A local point of  $R_n$  is precisely a tuple

$$(o, x_1, \dots, x_n, y, p)$$

with

$$p: \mu(o; x_1, \dots, x_n) \simeq y.$$

## 23 Descent of recombination

### Theorem 23.1 Operadic recombination satisfies descent

For every  $n$ ,  $R_n$  is an object of  $\text{Sh}_\infty(C, J)$ . Consequently, for every cover  $U. \rightarrow c$ ,

$$R_n(c) \simeq \lim_{[k] \in \Delta} R_n(U_k).$$

**Proof.**

The object  $E_n = O(n) \times M^n \times M$  is formed by finite products in  $H$ . The recombination object is then a finite limit,  $\text{Eq}(E_n \rightrightarrows M)$ . Since an  $\infty$ -topos is closed under finite limits,  $R_n \in H$ . Every object of  $H$  satisfies descent.

## 24 Distributed recombination theorem

### Corollary 24.1 Descent of local recombination events

Suppose  $\{C_i \rightarrow C\}$  is a cognitive cover. A homotopy-coherent family of local recombination events

$$(o_i, \vec{x}_i, y_i, p_i) \in R_n(C_i)$$

which agrees on all overlaps determines a global recombination event  $(o, \vec{x}, y, p) \in R_n(C)$ .

Thus recombination can itself be distributed across cognitive subsystems. No single subsystem need contain the full operation, all inputs, and the complete semantic witness independently. What must exist is a coherent descent datum.

## 25 Operadic replication closure

Given a family of initial states  $S \rightarrow M$ , one can ask for all states constructible by repeated application of operations in  $O$ . The free algebra provides the formal version.

### Definition 25.1 Formal operadic closure

The formal operadic closure of  $S$  is  $F_O(S)$ . A map  $S \rightarrow UM$  induces an evaluation  $F_O(S) \rightarrow M$ . The semantic states produced by finite recombination histories are therefore represented by the image of this evaluation:

$$\text{im}(F_O(S) \rightarrow M).$$

This is the memetic analogue of closure under an algebraic signature.

## 26 Memetic genotype and phenotype

The trace formalism suggests a useful formal distinction. A formal history is a point

$$t \in F_O(UM),$$

while its semantic phenotype is

$$\epsilon_M(t) \in M.$$

The evaluation map  $\epsilon_M: F_O(UM) \rightarrow M$  then plays the role of a genotype-to-phenotype map. Different formal histories may have the same semantic phenotype:  $t \neq t'$  but  $\epsilon_M(t) \simeq \epsilon_M(t')$ . Conversely, small changes in a trace may produce semantically distinguishable outputs. This analogy is structural rather than biological.

## 27 Mutation under an operadic action

### Definition 27.1 Parent-relative mutation

For a designated parent  $x_j$ , the operation  $o(x_1, \dots, x_n)$  is a detectable mutation at architecture  $C$  when

$$\text{Id}_{M(C)}(\mu_C(o; \vec{x}), x_j)$$

is empty. It is faithful replication relative to that parent when the same identity space is inhabited.

### Definition 27.2 Novel recombination

An output  $y = \mu(o; \bar{x})$  is novel relative to the input family at  $c$  if  $y \not\approx_c x_i$  for every  $i$ . Thus recombination can generate a semantic class not represented by any individual input.

## 28 Higher-order replication

An operation itself may be represented as memetic data. Suppose there exists an operation object

$$\text{Op}(M)$$

together with an evaluation map

$$\text{ev}: \text{Op}(M) \times M^n \rightarrow M.$$

If operations are themselves points of a memetic universe, then the same operadic machinery may act upon them. One can then distinguish replication of states, replication of transformations, and replication of transformation-generating procedures. This produces an iterated hierarchy

$$M, \text{Op}(M), \text{Op}(\text{Op}(M)), \dots$$

A reflective or self-interpreting system requires structure relating these levels.

## 29 Recursive memetic transformation

Suppose there is a map  $q: M \rightarrow \text{Op}(M)$  assigning to some memetic states executable transformation semantics. Then a state  $m \in M$  can act on memetic states through  $q(m)$ . One obtains an evaluation composite

$$M \times M^n \xrightarrow{q \times \text{id}} \text{Op}(M) \times M^n \xrightarrow{\text{ev}} M.$$

Thus a meme may encode an operation which transforms memes. This is the natural operadic setting for self-modifying semantic systems.

## 30 Closure under recursive improvement

If a state  $m$  encodes an operation that produces a new state  $m'$  which in turn encodes another operation, we obtain

$$m_0 \mapsto m_1 \mapsto m_2 \mapsto \dots$$

inside a larger operadic algebra. The crucial distinction remains: identity of traces is not identity of semantic outputs. The iteration can therefore change implementation while preserving some chosen semantic invariant.

## 31 Operadic invariant

### Definition 31.1 Replication invariant

Let  $Q: M \rightarrow X$  be a semantic observable. We call  $Q$  an invariant of an operation  $o \in O(n)$  relative to designated input  $j$  if

$$Q \circ \mu_o \approx Q \circ \pi_j,$$

where  $\pi_j: M^n \rightarrow M$  is projection onto the  $j$ -th input. Then

$$Q(\mu_o(x_1, \dots, x_n)) \approx Q(x_j).$$

This generalizes literal replication. Rather than demanding preservation of the entire state, one can specify exactly which semantic observable must persist.

### 32 Relative memetic identity as preservation of invariants

Let  $Q_c: M(c) \rightarrow X_c$  encode the distinctions available to architecture  $c$ . Define  $x \sim_{Q_c} y$  when  $Q_c(x) \approx Q_c(y)$ . If  $Q_c$  is a localization map, this recovers the previous capability-relative theory. Replication relative to  $Q_c$  is then

$$Q_c(\mu(o; \vec{x})) \approx Q_c(x_j).$$

Thus the entire framework can be summarized as preservation of selected observables under an operadic dynamics.

### 33 Operadic semantic quotient

Suppose  $L: H \rightarrow H_L$  is a left-exact localization compatible with the  $O$ -algebra structure. Then  $LM$  inherits an  $O$ -algebra structure.

#### Proposition 33.1 Localization preserves congruence

If  $x_i \approx_L x'_i$  for every  $i$ , then

$$\mu(o; x_1, \dots, x_n) \approx_L \mu(o; x'_1, \dots, x'_n).$$

**Proof.**

Apply  $L$  to the operation and then use Theorem 5.1 in the localized algebra.

Hence semantic localization is compatible with arbitrary finite recombination whenever the localization preserves the relevant operadic algebraic structure.

### 34 Operadic heredity

We can now isolate the abstract notion of hereditary persistence.

#### Definition 34.1 Hereditary semantic class

A semantic class  $[m]_c$  is hereditary under an operation family  $P \subseteq O$  if, whenever a designated parent belongs to  $[m]_c$ , so that  $x_j \approx_c m$ , every permitted replication operation satisfies  $\mu(o; \vec{x}) \approx_c m$ . Equivalently, the class  $[m]_c$  is closed under the designated action of  $P$ .

### 35 Replicator subobject

One may package self-preserving configurations as an object. For fixed  $n$  and designated parent  $j$ , define  $\text{Rep}_n^{(j)}$  by the homotopy equalizer of

$$O(n) \times M^n \rightrightarrows M,$$

where the two maps are

$$(o, \vec{x}) \mapsto \mu(o; \vec{x})$$

and

$$(o, \vec{x}) \mapsto x_j.$$

Thus

$$\text{Rep}_n^{(j)} = \text{Eq}(\mu_n, \pi_j).$$

Its points are precisely parent-relative replication events.

### 36 Replicator descent theorem

#### Theorem 36.1 Replication witnesses are sheaves

For every  $n$  and  $j$ ,  $\text{Rep}_n^{(j)}$  is a sheaf and therefore satisfies descent.

**Proof.**

It is a finite limit of objects in the  $\infty$ -topos, namely  $\text{Eq}(\mu_n, \pi_j)$ . Hence it is an object of  $H$ , so it satisfies descent.

This is the multi-input extension of the unary replication descent theorem.

### 37 The operadic lineage theorem

We can combine tree contraction with semantic replication.

#### Theorem 37.1 Treewise heredity

Let  $T$  be a finite  $O$ -decorated tree with a distinguished leaf  $x$ . Suppose that at every vertex along the unique path from  $x$  to the root, the corresponding operation preserves the semantic class of its distinguished incoming branch. Then the root output is semantically equivalent to  $x$ .

**Proof.**

Enumerate the vertices on the path from the distinguished leaf to the root as  $v_1, \dots, v_k$ . By assumption, after the operation at  $v_1$ , its output  $y_1$  satisfies  $y_1 \approx x$ . At  $v_2$ , the distinguished input is  $y_1$ , and the preservation assumption gives  $y_2 \approx y_1$ . Thus, by transitivity,  $y_2 \approx x$ . Continuing inductively gives  $y_k \approx x$ . The state  $y_k$  is the root evaluation of  $T$ . Therefore  $\text{ev}_T \approx x$ .

This theorem formalizes semantic heredity through an arbitrarily complex recombination history.

### 38 Recombination versus copying

The resulting theory distinguishes four notions. Neither replication nor mutation is primitive. Both are predicates imposed on the more general operadic transformation dynamics.

- Notion — Meaning —
- Copying — Preservation of the complete state. —
- Replication — Preservation of a chosen semantic class. —
- Recombination — Multi-input production of a semantic output. —
- Mutation — Failure of a specified semantic invariant. —

### 39 Full formal specification

The operadic theory is specified by

$(C, J, H, O, M, \mu, A, T).$

- $C$ : Cognitive architecture  $\infty$ -category.
- $J$ : Grothendieck topology.
- $H$ : The  $\infty$ -topos of sheaves on  $(C, J)$ .
- $O$ : Operad of memetic transformations.
- $M$ :  $O$ -algebra of memetic states.
- $\mu$ : Operadic action.
- $A$ : Agent field.
- $T$ : Agent-indexed transmission structure.

The primitive transformation is

$$\mu_n: O(n) \times M^n \rightarrow M.$$

The primitive semantic witness is

$$\text{Id}(\mu(o; \vec{x}), y).$$

Parent-relative replication is

$$\text{Id}(\mu(o; \vec{x}), x_j).$$

Formal histories are represented by

$$F_O(UM),$$

and their semantic evaluation is

$$\epsilon_M: F_O(UM) \rightarrow M.$$

## 40 Main theorem

### Theorem 40.1 Operadic memetic replication theorem

Let  $H = \text{Sh}_\infty(C, J)$  be a cognitive  $\infty$ -topos, let  $O$  be an internal  $\infty$ -operad, and let  $M$  be an  $O$ -algebra. Then:

- **(i)** Every  $n$ -ary memetic transformation is represented by  $\mu_n: O(n) \times M^n \rightarrow M$ . See Definition 3.1.
- **(ii)** Semantic equivalence is a congruence for every operadic operation. See Theorem 5.1.
- **(iii)** Finite transformation histories are represented by  $O$ -decorated rooted trees. See Definition 10.1.
- **(iv)** Every such tree contracts coherently to a composite operadic operation. See Theorem 11.1.
- **(v)** Restriction between cognitive architectures commutes with operadic evaluation. See Theorem 19.1.
- **(vi)** Recombination witnesses and replication witnesses form objects of the  $\infty$ -topos and therefore satisfy descent. See Theorem 23.1 and Theorem 36.1.
- **(vii)** Formal histories are represented by the free algebra  $F_O(UM)$ . See Definition 15.1.
- **(viii)** The counit  $F_O(UM) \rightarrow M$  is the canonical trace-to-semantic-state map. See Corollary 17.2.
- **(ix)** Distinct operational histories may have equivalent semantic outputs whenever this evaluation map is nonconservative. See Theorem 18.1.
- **(x)** The unary replication theory is recovered by restricting to arity-one operations. See Note I, and Section 8 above.

## 41 Conceptual conclusion

The unary model treats replication as iteration:

$$x \mapsto R(x) \mapsto R^2(x) \mapsto \dots$$

The operadic model replaces this by finite compositional networks:

$$(x_1, \dots, x_n) \mapsto_o y.$$

Repeated transformation therefore produces trees  $T$  rather than merely sequences. The correct algebraic object of formal transformation histories is the free operadic algebra  $F_O(UM)$ , while the actual semantic realization is obtained through

$$\epsilon_M: F_O(UM) \rightarrow M.$$

Hence the fundamental distinction becomes

A transformation history is not a semantic operation class.

Two histories may be computationally, physically, causally, or syntactically different while satisfying

$$\epsilon_M(t) \approx \epsilon_M(t').$$

Replication is therefore not fundamentally literal copying. It is preservation of a specified semantic invariant through an operad of possible transformations. The general specification is an operadic action on a semantic object in an  $\infty$ -topos.

Observer relativity is implemented by sheaf restriction  $u^* : M(d) \rightarrow M(c)$ , distributed realization by descent  $M(c) \approx \text{Tot} M(U.)$ , and multi-stage recombination by operadic composition  $o \circ (o_1, \dots, o_k)$ . The three structures therefore fit together as

Operadic composition, topos-theoretic descent, and capability-relative identity.

This yields a single framework in which copying, recombination, mutation, semantic compression, distributed cognition, lineage, and recursive transformation are all special cases of one homotopy-coherent algebraic semantics.

## Part 3: Polynomial and Traced Network Semantics

### Abstract

We extend the operadic theory of memetic replication from rooted-tree histories to general computational and communicative networks with typed interfaces, fan-out, shared subcomputations, and feedback.

Let

$$H = \text{Sh}_\infty(C, J)$$

be a cognitive  $\infty$ -topos. A typed family of memetic states is represented by an object over a type object  $X \rightarrow I$ . A polynomial

$$I \xleftarrow{s} E \xrightarrow{p} B \xrightarrow{t} I$$

determines an endofunctor

$$P = \Sigma_t \Pi_p \Delta_s : H_{/I} \longrightarrow H_{/I},$$

whose points encode typed operations with input slots and outputs. Polynomial monads generalize operads and provide the syntax of finite typed recombination.

Iteration of polynomial operations still produces trees. To represent sharing of subcomputations one must additionally allow copying and discarding, equivalently a suitable commutative comonoid or cartesian structure. To represent cyclic dependence one adds a trace operator

$$\text{Tr}_{X,Y}^U : \text{Map}(X \otimes U, Y \otimes U) \longrightarrow \text{Map}(X, Y).$$

Finally, organizing networks, transformations between networks, and higher equivalences naturally leads to a symmetric monoidal  $(\infty, 2)$ -category.

The resulting theory separates four layers:

Polynomial syntax, sharing, feedback, and higher semantics.

This allows memetic replication to be modeled as preservation of semantic invariants by arbitrary resource-sensitive network computations, rather than merely by unary copying or tree-shaped recombination. The unary theory is Note I; the operadic theory is Note II.

## Notation

The symbols of this note, in the order they become active.

- $X \rightarrow I$  — A typed family of memetic states. The fiber over  $i$  is the space of states of type  $i$ .
- $I \xleftarrow{s} E \xrightarrow{p} B \xrightarrow{t} I$  — A polynomial: operations  $B$ , slots  $E$ , slot assignment  $p$ , input types  $s$ , output types  $t$ .
- $P = \Sigma_t \Pi_p \Delta_s$  — The polynomial endofunctor on the slice over  $I$ .
- $\Delta_X$  — Copying, when a comonoid or cartesian structure supplies it.
- $\text{Tr}^U$  — Feedback: close a wire of type  $U$ .
- $\llbracket N \rrbracket$  — The semantic process interpreted from a formal network  $N$ .
- $\approx_{\text{op}}$ ,  $\approx_{\text{sem}}$  — Operational equivalence of diagrams, versus equivalence of their meanings.
- $\rho$  — A resource valuation: energy, latency, memory, bandwidth, or operation count.
- $M$  — A symmetric monoidal  $(\infty, 2)$ -category of networks and transformations between them.
- $Q$  — A semantic observable. Replication relative to  $Q$  is preservation of a  $Q$ -class.

## 1 Ambient semantic setting

Let  $(C, J)$  be a cognitive  $\infty$ -site and let

$$H = \text{Sh}_{\infty}(C, J)$$

be its  $\infty$ -topos. Let  $I \in H$  be an object of semantic types. A typed memetic state is an object

$$q: X \rightarrow I.$$

The fiber  $X_i$  over  $i \in I$  is the space of memetic states of semantic type  $i$ . The natural ambient category is therefore the slice  $H_{/I}$ .

## 2 Internal polynomial functors

### Definition 2.1 Polynomial

A **polynomial** over  $I$  is a diagram

$$I \xleftarrow{s} E \xrightarrow{p} B \xrightarrow{t} I$$

in  $H$ . Its parts are interpreted as follows.

- $B$  — Object of operations.
- $E$  — Object of input positions.
- $p: E \rightarrow B$  — Assignment of input slots to operations.

- $s: E \rightarrow I$  — Input typing.
- $t: B \rightarrow I$  — Output typing.

### Definition 2.2 Polynomial endofunctor

The polynomial determines

$$P: H_{/I} \rightarrow H_{/I}$$

by

$$P = \Sigma_t \Pi_p \Delta_s.$$

Here  $\Delta_s$  is pullback,  $\Pi_p$  is dependent product, and  $\Sigma_t$  is dependent sum. In internal type-theoretic notation,

$$P(X)_i \simeq \sum_{b: t(b)=i} \prod_{e \in E_b} X_{s(e)}.$$

A point of  $P(X)_i$  consists of an operation  $b$  whose output has type  $i$ , together with one input state of type  $s(e)$  for every input slot  $e \in E_b$ .

## 3 Polynomial memetic dynamics

### Definition 3.1 Polynomial memetic algebra

A **polynomial memetic algebra** is an object  $X \rightarrow I$  together with a morphism

$$\alpha: P(X) \rightarrow X$$

in  $H_{/I}$ . The map  $\alpha$  evaluates one layer of typed memetic recombination. Internally,

$$\alpha_i: \sum_{b: t(b)=i} \prod_{e \in E_b} X_{s(e)} \longrightarrow X_i.$$

## 4 Operads as a special case

An operad records operations together with substitution. A polynomial functor records only the one-step shape of operations. To obtain coherent substitution we impose monadic structure.

### Definition 4.1 Polynomial monad

A **polynomial monad**  $(P, \eta, \mu)$  is a monad on  $H_{/I}$  whose underlying endofunctor  $P$  is polynomial and whose unit and multiplication are represented by compatible polynomial morphisms. The monad multiplication

$$\mu: P \circ P \rightarrow P$$

encodes substitution of operations into input slots.

### Proposition 4.2 Tree-shaped composition

Every algebra for a polynomial monad possesses coherent finite tree-shaped composition of its primitive operations.

**Proof.**

An algebra is a map  $\alpha: P(X) \rightarrow X$  satisfying  $\alpha \circ \eta_X \simeq \text{id}_X$  and  $\alpha \circ P(\alpha) \simeq \alpha \circ \mu_X$ . The second equation states that evaluating an outer operation after evaluating its immediate suboperations agrees with first substituting all operations into one composite polynomial operation and then evaluating. Induction over finite rooted trees therefore gives coherent evaluation of every polynomial tree.

## 5 Why polynomial syntax alone still yields trees

The free algebra of a polynomial monad is generated by finite substitution. Consequently, its formal expressions have a rooted-tree shape. If a value is used twice, the tree contains two occurrences of that value. This distinguishes duplication of syntax from sharing of one computed value. These are not the same operational structure.

### Example 5.1 Recomputation versus fan-out

Consider

$$f(g(x), g(x)).$$

As an ordinary term tree, the subterm  $g(x)$  appears twice. A shared computational graph instead has one node computing  $y = g(x)$ , followed by a fan-out  $y \mapsto (y, y)$  feeding both inputs of  $f$ . Hence term trees cannot by themselves distinguish recomputation from sharing.

## 6 Sharing via comonoid structure

Let  $(V, \otimes, 1)$  be a symmetric monoidal  $\infty$ -category of semantic processes.

### Definition 6.1 Copying structure

A copying operation on  $X \in V$  is a morphism

$$\Delta_X: X \rightarrow X \otimes X.$$

A discarding operation is

$$\varepsilon_X: X \rightarrow 1.$$

If these form a coherently commutative comonoid structure, then outputs may be copied and discarded. If  $V$  is cartesian monoidal, so that  $\otimes = \times$ , then every object possesses canonical maps

$$\Delta_X: X \rightarrow X \times X, \quad !_X: X \rightarrow 1.$$

Thus cartesian structure provides canonical fan-out and deletion.

## 7 Shared memetic computations

Suppose  $g: X \rightarrow Y$  and  $f: Y \otimes Y \rightarrow Z$ . The shared network

$$X \xrightarrow{g} Y \xrightarrow{\Delta_Y} Y \otimes Y \xrightarrow{f} Z$$

has composite

$$f \circ \Delta_Y \circ g.$$

This represents one execution of  $g$  followed by reuse of its result. By contrast, the tree-like map is

$$X \otimes X \xrightarrow{g \otimes g} Y \otimes Y \xrightarrow{f} Z,$$

possibly preceded by  $\Delta_X: X \rightarrow X \otimes X$ . The distinction is operational even when the resulting semantic maps happen to agree.

## 8 Term graphs

### Definition 8.1 Memetic term graph

A **memetic term graph** is a directed acyclic network whose vertices are labeled by primitive polynomial operations and whose edges carry typed memetic states, with fan-out permitted. Unlike an operadic tree, one output edge may feed several later operations. A term graph therefore represents a computation with shared intermediate results.

There is an unfolding map

$$U: \text{Graph}_P \longrightarrow \text{Tree}_P$$

obtained by duplicating every shared subgraph once for each use.

### Proposition 8.2 Semantic invariance under unfolding

Assume the semantic category is cartesian monoidal and all graph nodes are interpreted functorially. Then a finite acyclic term graph and its complete tree unfolding have equivalent semantic evaluations.

**Proof.**

Every shared node  $y$  with  $k$  outgoing uses is interpreted using the  $k$ -fold diagonal  $\Delta_Y^{(k)}: Y \rightarrow Y^k$ . Naturality and the universal property of finite products imply that composing once with  $g: X \rightarrow Y$  and then diagonally copying its output is equivalent to the map

$$x \mapsto (g(x), \dots, g(x)).$$

The unfolded tree computes exactly these  $k$  components separately. Induction on the finite acyclic graph therefore yields equality of the resulting morphisms.

### Remark 8.3 Cost is not preserved

Operational cost need not be preserved. The unfolded tree may recompute a subexpression many times, while the graph computes it once. Thus semantic equivalence does not imply resource equivalence.

## 9 Resource-sensitive semantics

To distinguish such implementations, the semantic category should not automatically identify copying as free. Let  $(V, \otimes, 1)$  be symmetric monoidal but not necessarily cartesian. Then a state  $x$  cannot in general be duplicated unless a specific map  $\Delta_X: X \rightarrow X \otimes X$  is supplied. This permits explicit accounting of resources.

### Definition 9.1 Resource-sensitive memetic process

A resource-sensitive memetic process is a morphism

$$f: X_1 \otimes \cdots \otimes X_n \longrightarrow Y_1 \otimes \cdots \otimes Y_m$$

in a symmetric monoidal semantic category. Inputs are consumed and outputs produced according to the monoidal structure.

## 10 Resource observables

Let  $R$  be a commutative ordered monoid, for example  $(R_{\geq 0}, +, 0, \leq)$ .

### Definition 10.1 Resource valuation

A resource valuation is a function

$$\rho: \text{Mor}(V) \rightarrow R$$

satisfying, at minimum,

$$\rho(g \circ f) \leq \rho(f) + \rho(g)$$

and

$$\rho(f \otimes g) \leq \rho(f) + \rho(g).$$

Examples include energy, latency, memory, bandwidth, and the number of primitive operations. Two networks may therefore satisfy

$$\llbracket G \rrbracket \simeq \llbracket H \rrbracket$$

while  $\rho(G) \neq \rho(H)$ . This is precisely the distinction between semantic and operational equivalence.

## 11 Feedback

Trees and directed acyclic graphs remain acyclic. Recursive self-reference and recurrent processing require cycles. The appropriate categorical structure is a trace.

### Definition 11.1 Traced symmetric monoidal structure

A **traced symmetric monoidal  $\infty$ -category** is a symmetric monoidal  $\infty$ -category together with operations

$$\mathrm{Tr}_{X,Y}^U : \mathrm{Map}(X \otimes U, Y \otimes U) \longrightarrow \mathrm{Map}(X, Y)$$

satisfying homotopy-coherent versions of naturality, dinaturality, vanishing, superposing, and yanking.

A morphism  $f : X \otimes U \rightarrow Y \otimes U$  can be pictured as a process from  $X$  to  $Y$  with an auxiliary wire of type  $U$ .

The trace closes that wire:

$$\mathrm{Tr}^U(f) : X \rightarrow Y.$$

### 12 Feedback memetic process

Suppose  $F : X \otimes S \rightarrow Y \otimes S$ , where  $S$  is an internal state. Then

$$\mathrm{Tr}^S(F) : X \rightarrow Y$$

represents a process in which the state output is fed back as the next state input. This models recurrence, memory, self-conditioning, iterative refinement, and feedback control.

### 13 Feedback replication

Let  $m \in M$  be a memetic signal and suppose a recurrent transformation has semantic map

$$R = \mathrm{Tr}^S(F) : M \rightarrow M.$$

Replication is again the identity type

$$\mathrm{Id}_M(R(m), m).$$

Thus feedback changes the causal architecture but not the fundamental semantic identity criterion.

### 14 Fixed points and traces

A feedback loop should not automatically be identified with an ordinary fixed-point equation. A trace provides a way to close a wire:

$$F : X \otimes U \rightarrow Y \otimes U$$

becomes

$$\mathrm{Tr}^U(F) : X \rightarrow Y.$$

A fixed point instead concerns a point  $x$  satisfying  $f(x) \simeq x$ . The two notions coincide only under additional hypotheses, such as an appropriate Conway or iteration structure. This distinction is important for recursive systems.

## 15 Memetic networks

### Definition 15.1 Memetic network

A memetic network is a finite typed string diagram generated by:

- polynomial operations;
- sequential composition;
- parallel tensor product;
- copying and discarding operations, when available;
- symmetry morphisms;
- feedback via trace.

Its semantic interpretation is a morphism

$$\llbracket N \rrbracket: X \rightarrow Y$$

in the chosen semantic category.

## 16 Network equivalence

### Definition 16.1 Operational equivalence

Two networks are **operationally equivalent** if they are equivalent as network presentations under the chosen graphical calculus. Write  $N \simeq_{\text{op}} N'$ .

### Definition 16.2 Semantic equivalence

They are **semantically equivalent** if

$$\llbracket N \rrbracket \simeq \llbracket N' \rrbracket.$$

Write  $N \simeq_{\text{sem}} N'$ . In general

$$N \simeq_{\text{op}} N' \implies N \simeq_{\text{sem}} N',$$

but not conversely. Thus the semantic quotient is typically many-to-one.

## 17 Network-to-semantic-class map

Let  $\text{Net}(X, Y)$  denote the space of formal networks from  $X$  to  $Y$ . Evaluation gives

$$\text{ev}_{X,Y}: \text{Net}(X,Y) \longrightarrow \text{Map}_V(X,Y).$$

This is the network analogue of the earlier trace evaluation map  $F_O(UM) \rightarrow M$  from Note II. The homotopy fiber

$$\text{hofib}_f(\text{ev}_{X,Y})$$

is the space of operational realizations of a semantic map  $f: X \rightarrow Y$ .

## 18 Nonuniqueness theorem

### Theorem 18.1 Semantic uniqueness does not imply implementation uniqueness

Suppose  $\text{hofib}_f(\text{ev}_{X,Y})$  has more than one connected component. Then there exist operationally inequivalent networks  $N \simeq_{\text{op}} N'$  such that

$$\llbracket N \rrbracket \simeq f \simeq \llbracket N' \rrbracket.$$

**Proof.**

Choose  $N, N'$  from different connected components of the same homotopy fiber. By membership in that fiber,  $\llbracket N \rrbracket \simeq f$  and  $\llbracket N' \rrbracket \simeq f$ . Hence  $\llbracket N \rrbracket \simeq \llbracket N' \rrbracket$ . But the two points are not connected in the implementation space. Therefore the networks are operationally inequivalent.

## 19 Resource nonuniqueness

### Corollary 19.1 Cost is not a semantic invariant

Semantically equivalent replication networks may have different resource costs. That is,

$$\llbracket N \rrbracket \simeq \llbracket N' \rrbracket$$

does not imply  $\rho(N) = \rho(N')$ . Hence semantic identity does not determine energy, latency, memory, or physical implementation.

## 20 From an $(\infty,1)$ -category to an $(\infty,2)$ -category

So far networks and their semantic maps live in mapping spaces. To explicitly retain transformations between processes, introduce an  $(\infty,2)$ -category  $C$ . Its structure may be interpreted as semantic interfaces, memetic networks, and transformations between networks, with higher invertible coherences above dimension 2.

## 21 Memetic $(\infty,2)$ -category

### Definition 21.1 Memetic process $(\infty,2)$ -category

A **memetic process  $(\infty,2)$ -category**  $M$  is a symmetric monoidal  $(\infty,2)$ -category in which objects are typed semantic interfaces, 1-morphisms are computational or communicative processes, and 2-morphisms are process transformations.

For  $f, g: X \rightarrow Y$ , a 2-morphism  $\alpha: f \Rightarrow g$  may represent:

- optimization
- compilation
- refactoring
- learning
- simulation
- implementation replacement

## 22 Semantic invariants of process transformations

Let  $Q: M \rightarrow S$  be a semantic functor into another  $(\infty, 2)$ -category.

### Definition 22.1 Semantics-preserving transformation

A 2-morphism  $\alpha: f \Rightarrow g$  is  $Q$ -semantics preserving if  $Q(\alpha)$  is an equivalence between  $Q(f)$  and  $Q(g)$ . Thus an implementation can change while its chosen semantics remains invariant.

## 23 Replication as a 2-categorical invariant

Suppose  $f: X \rightarrow X$  is a replication process. Let  $Q(f)$  be its observable semantic action. A transformed implementation  $g: X \rightarrow X$  is the same replicator relative to  $Q$  whenever there exists  $\alpha: f \Rightarrow g$  such that  $Q(\alpha)$  is invertible.

Hence the replicator is more naturally an equivalence class of 1-morphisms under semantics-preserving 2-morphisms.

## 24 Compilation as a 2-morphism

Suppose  $f$  is a high-level network and  $g$  a lower-level implementation. A compiler can be represented abstractly by  $\alpha: f \Rightarrow g$ . Correctness means preservation of semantics:

$$\llbracket f \rrbracket \approx \llbracket g \rrbracket.$$

Resource optimization can simultaneously require  $\rho(g) < \rho(f)$ . Thus optimization is naturally a transformation inside a semantic equivalence class.

## 25 Recursive improvement

A recursive improvement step is then not necessarily an endomorphism of a state object. It may instead be a 2-morphism

$$\alpha_n: F_n \Rightarrow F_{n+1}$$

between successive system implementations. Semantic preservation requires

$$Q(F_n) \approx Q(F_{n+1}),$$

while improvement may be expressed by a monotone resource functional

$$\rho(F_{n+1}) < \rho(F_n)$$

or by a performance functional

$$P(F_{n+1}) > P(F_n).$$

Thus recursive improvement takes the form

$$F_0 \Rightarrow F_1 \Rightarrow F_2 \Rightarrow \dots$$

inside a 2-category of implementations.

## 26 Resource-relative improvement

### Definition 26.1 Improvement preorder

Let  $\rho: \text{Mor}_1(M) \rightarrow R$  be a resource cost. For semantically equivalent processes  $f, g$ , define

$$g \leq_\rho f$$

when  $\rho(g) \leq \rho(f)$ . If  $\rho(g) < \rho(f)$ , write  $g <_\rho f$ .

### Definition 26.2 Strict semantic improvement

A transformation  $\alpha: f \Rightarrow g$  is a strict  $(Q, \rho)$ -improvement if  $Q(f) \approx Q(g)$  and  $\rho(g) < \rho(f)$ .

## 27 Replication under optimization

### Theorem 27.1 Optimization preserves replication

Let  $f, g: X \rightarrow X$  be processes and suppose  $f \Rightarrow g$  is  $Q$ -semantics preserving. If  $f$  replicates a semantic invariant  $q$ , then  $g$  also replicates  $q$ .

**Proof.**

Suppose replication by  $f$  means  $Q(f)(q) \approx q$ . Since  $Q(f) \approx Q(g)$ , we obtain

$$Q(g)(q) \approx Q(f)(q) \approx q.$$

Therefore  $g$  preserves the same semantic invariant.

This formalizes the idea that a replicating operation may change implementation without ceasing to be the same semantic replicator.

## 28 Observer-relative network semantics

Return now to the cognitive site  $(C, J)$ . Suppose every architecture  $c$  possesses a semantic process category  $M_c$ . An architecture morphism  $u: c \rightarrow d$  induces reindexing

$$u^*: M_d \rightarrow M_c.$$

Then a network  $N$  at  $d$  has observer-relative image  $u^* N$ . Two networks may satisfy  $N \simeq_d N'$  but  $u^* N \simeq_c u^* N'$ . Thus semantic collapse extends from states to entire causal networks.

## 29 Network descent

Assume the assignment  $c \mapsto M_c$  forms a stack of symmetric monoidal  $(\infty, 2)$ -categories. Then for a cover  $U. \rightarrow c$  one obtains

$$M_c \simeq \lim_{[n] \in \Delta} M_{U_n}.$$

Hence networks themselves can be assembled by descent from local process descriptions.

## 30 Distributed causal structure

This means that a globally defined memetic process can be represented by networks

$$N_i$$

on each subsystem, together with equivalences

$$N_i \mid_{ij} \simeq N_j \mid_{ij},$$

higher equivalences on triple overlaps, and all required coherence data. Thus the causal architecture of a meme can itself be distributed.

## 31 Replication networks as invariant-preserving morphisms

Let  $Q: M \rightarrow S$  be an observable semantic quotient. A network

$$N: M^{\otimes n} \rightarrow M$$

replicates the  $j$ -th input relative to  $Q$  if

$$Q \circ N \simeq Q \circ \pi_j.$$

This generalizes every earlier definition.

## 32 Network replication theorem

### Theorem 32.1 Semantics-preserving transformations preserve replication

Suppose  $N: M^{\otimes n} \rightarrow M$  satisfies  $Q \circ N \simeq Q \circ \pi_j$ . If  $N'$  is related to  $N$  by a  $Q$ -semantics-preserving 2-morphism, then  $N'$  also replicates the  $j$ -th semantic input relative to  $Q$ .

**Proof.**

A  $Q$ -semantics-preserving transformation implies  $Q \circ N' \simeq Q \circ N$ . By hypothesis  $Q \circ N \simeq Q \circ \pi_j$ . Therefore  $Q \circ N' \simeq Q \circ \pi_j$ .

### 33 Composition theorem

Suppose  $N: M^{\otimes n} \rightarrow M$  replicates input  $j$  relative to  $Q$ , and  $K: M^{\otimes m} \rightarrow M$  replicates one input carrying the output of  $N$ . Then the composite network preserves the original  $Q$ -class.

#### Theorem 33.1 Network heredity

If every node along a causal path preserves a semantic invariant  $Q$ , then the entire composite network preserves that invariant along that path.

#### Proof.

Each local preservation gives an equivalence  $Q(x_{k+1}) \simeq Q(x_k)$ . Composition of these equivalences yields  $Q(x_{\text{out}}) \simeq Q(x_{\text{in}})$ . The conclusion is independent of whether the intervening network contains branching, sharing, or feedback, provided the global composite is defined and each relevant preservation equation holds.

### 34 Polynomial syntax and traced execution

The complete picture is now five layers.

polynomial monad

specifies the primitive typed operations and their substitution.

symmetric monoidal structure

specifies parallel composition and resource combination.

comonoid or cartesian structure

specifies sharing and deletion.

trace

specifies feedback.

$(\infty, 2)$ -structure

specifies transformations between networks.

### 35 Formal total specification

A sufficiently rich memetic network theory is a tuple

$(C, J, H, I, P, V, \Delta, \text{Tr}, M, Q, \rho)$ .

- $C$ : Cognitive architectures.
- $J$ : Cognitive coverage.
- $H$ : Semantic  $\infty$ -topos.
- $I$ : Semantic type object.
- $P$ : Polynomial monad of primitive transformations.
- $V$ : Symmetric monoidal semantic category.
- $\Delta$ : Sharing structure.
- $\text{Tr}$ : Feedback operator.
- $M$ : Symmetric monoidal  $(\infty, 2)$ -category of networks.
- $Q$ : Semantic observable or localization.
- $\rho$ : Resource valuation.

### 36 Fundamental equations

Primitive recombination:

$$P(X)_i \simeq \sum_{b: t(b)=i} \prod_{e \in E_b} X_{s(e)}.$$

Evaluation:

$$\alpha: P(X) \rightarrow X.$$

Sharing:

$$\Delta_X: X \rightarrow X \otimes X.$$

Feedback:

$$\text{Tr}_{X,Y}^U: \text{Map}(X \otimes U, Y \otimes U) \rightarrow \text{Map}(X, Y).$$

Semantic network evaluation:

$$\text{ev}_{X,Y}: \text{Net}(X, Y) \rightarrow \text{Map}(X, Y).$$

Replication relative to  $Q$ :

$$Q \circ N \simeq Q \circ \Pi_j.$$

Semantics-preserving optimization:

$$Q(N) \simeq Q(N'), \quad \rho(N') < \rho(N).$$

### 37 Central structural theorem

#### Theorem 37.1 General network memetic replication theorem

Let  $R$  be a system as above. Then:

- **(i)** Polynomial monads generate coherent typed tree-shaped transformations. See Proposition 4.2.
- **(ii)** Commutative comonoid structure extends tree syntax to networks with fan-out and shared subcomputations. See Sections 6–8.
- **(iii)** Traced symmetric monoidal structure extends acyclic networks to networks with feedback. See Sections 11–14.
- **(iv)** A symmetric monoidal  $(\infty, 2)$ -category organizes networks and transformations between them. See Sections 20–24.
- **(v)** Evaluation sends every formal network to a semantic process. See Definition 15.1.
- **(vi)** Distinct network realizations may evaluate to the same semantic process. See Theorem 18.1.
- **(vii)** Semantic equivalence does not imply resource equivalence. See Corollary 19.1.
- **(viii)** Replication relative to an observable  $Q$  is exactly preservation of the corresponding  $Q$ -class. See Section 31.
- **(ix)** Semantics-preserving optimization preserves replication. See Theorem 27.1.
- **(x)** The unary, operadic, and capability-localized theories arise as special cases. See Note I and Note II.

### 38 Relation among the successive theories

The entire development can now be organized as a chain of strict generalizations:

Unary dynamics, then operadic trees, then polynomial typed trees, then shared DAGs, then feedback networks.

At the semantic level:

Sets, then spaces, then  $\infty$ -topoi, then indexed  $\infty$ -topoi.

At the process level:

Functions, then monoidal categories, then traced monoidal categories, then  $(\infty, 2)$ -categories.

These are independent axes of generalization.

### 39 Consequence for the concept of a meme

A meme should no longer be identified with a string, a token, or even merely a point  $m \in M$ . The full theory distinguishes six levels, and these levels need not determine one another uniquely.

- $m \in M$ : Semantic state.
- $t \in F_P(M)$ : Formal construction.
- $N \in Net$ : Causal network.
- $\llbracket N \rrbracket$ : Semantic action.
- $Q(\llbracket N \rrbracket)$ : Observer-relative image.
- $\rho(N)$ : Resource realization.

## 40 Final principle

The strongest formulation is therefore:

Memetic replication is preservation of a semantic invariant under a typed network transformation.

The network may contain composition, parallelism, sharing, recombination, feedback, and optimization. What makes it replication is not its graphical form. It is the existence of an invariant  $Q$  such that

$$Q \circ N \simeq Q \circ \Pi_j.$$

Different implementations may realize the same invariant:

$$N / \approx_{\text{op}} N' \quad \text{while} \quad Q(N) \simeq Q(N').$$

They may also differ radically in resource usage:

$$\rho(N) \neq \rho(N').$$

Hence the semantic object, operational trace, causal network, and physical realization must be regarded as distinct levels. The complete conceptual hierarchy is

Physical realization, then causal network, then semantic process, then observer-relative invariant.

Memetic identity lives at the final level selected by the observer's semantic modality, whereas replication is the persistence of that invariant through transformations at all preceding levels.

## Part 4: Stochastic Memetic Replication

### Abstract

We extend the  $\infty$ -topos and network-theoretic formulation of memetic replication from deterministic transformations to stochastic population dynamics.

Let

$$H = \text{Sh}_{\infty}(C, J)$$

be a cognitive semantic  $\infty$ -topos, and let  $S \in H$  be an object of semantic memetic classes. Since probability is not canonically determined by an arbitrary  $\infty$ -topos, we assume explicitly that  $H$  is equipped with a commutative probability monad

$$P: H \rightarrow H$$

or, equivalently for the present purposes, with a compatible Markov-category structure.

A population is a probability state

$$\mu: 1 \rightarrow P(S).$$

Selection is specified by a positive fitness observable

$$w: S \rightarrow R_{>0},$$

while mutation and imperfect transmission are represented by a Markov kernel

$$K: S \rightarrow P(S).$$

The resulting discrete-time evolutionary operator is

$$R_{w,K}(\mu) = K_* \left( \frac{w\mu}{\int_S w d\mu} \right).$$

Equivalently, introducing the positive transfer operator  $L_{w,K} \mu = K_*(w\mu)$ , one has

$$R_{w,K}(\mu) = \frac{L_{w,K} \mu}{L_{w,K} \mu(S)}.$$

For a finite semantic state space this becomes the classical replicator–mutator equation

$$x'_j = \frac{\sum_i x_i w_i Q_{ij}}{\sum_i x_i w_i}.$$

We prove normalization, characterize stationary populations as positive eigenmeasures of the transfer operator, derive both continuous-time replicator and mutation–selection equations, establish a coarse-graining theorem showing precisely when observer-relative semantic quotients inherit closed Markovian dynamics, and derive the Perron–Frobenius asymptotics of finite primitive systems.

The resulting framework identifies stochastic memetic evolution with normalized positive dynamics on capability-relative semantic classes. The deterministic theories are Note I, Note II, and Note III.

## Notation

The symbols of this note, in the order they become active.

- $S$  — Semantic state object: the distinctions an observer retains.
- $P$  — A commutative probability monad on the cognitive  $\infty$ -topos.
- $\mu: 1 \rightarrow P(S)$  — A population: a probability state on semantic classes.
- $w$  — Positive fitness. Mean fitness is the integral of  $w$  against  $\mu$ .
- $K$  — A Markov kernel. Mutation, noise, and imperfect transmission.
- $\text{Sel}_w$  — Selection: reweight  $\mu$  by fitness and renormalize.
- $R_{w,K}$  — The replicator–mutator operator, selection followed by mutation.
- $L_{w,K}$  — The positive transfer operator. Evolution is its projectivization.
- $Q_{ij}$  — Finite mutation matrix. Rows are probability distributions.

- $q$  — A coarse-graining. Dynamics on the quotient closes only when it is lumpable.

## 1 Semantic state space

Let  $(C, J)$  be a cognitive  $\infty$ -site and let

$$H = \text{Sh}_\infty(C, J).$$

Let  $M \in H$  be an underlying object of memetic realizations, and let  $Q: M \rightarrow S$  be a semantic observable, localization, or quotient. The object  $S \in H$  is the semantic state object.

Two realizations  $m, m' \in M$  are equivalent relative to  $Q$  when  $Q(m) \approx Q(m')$ . Thus  $S$  represents the memetic distinctions retained by the observer or cognitive architecture under consideration.

## 2 Probability structure

Probability is additional structure. An arbitrary  $\infty$ -topos does not by itself determine a canonical notion of probability measure. We therefore assume the following.

### Remark 2.1 Probability monad

The  $\infty$ -topos  $H$  is equipped with a commutative probability monad

$$P: H \rightarrow H$$

with unit

$$\eta_X: X \rightarrow P(X)$$

and multiplication

$$m_X: P(P(X)) \rightarrow P(X).$$

We assume sufficient integration structure to make expressions of the form

$$\int_X f \, d\mu$$

well defined for suitable internal nonnegative real-valued functions  $f: X \rightarrow R_{\geq 0}$ . In ordinary measurable spaces one may take the Giry probability monad. For finite objects,  $P(X)$  is the probability simplex on  $X$ .

## 3 Markov kernels

### Definition 3.1 Stochastic memetic transformation

A stochastic transformation from  $X$  to  $Y$  is a Kleisli morphism

$$K: X \longrightarrow P(Y).$$

We call such a morphism a Markov kernel. For  $x \in X$ , the probability state  $K(x) \in P(Y)$  describes the distribution of possible outputs conditional on input  $x$ .

### Definition 3.2 Composition

If  $K: X \rightarrow P(Y)$  and  $L: Y \rightarrow P(Z)$ , their Kleisli composite

$$L \odot K: X \rightarrow P(Z)$$

is defined by

$$m_Z \circ P(L) \circ K.$$

In ordinary integral notation,

$$(L \odot K)(x) = \int_Y L(y) K(x, dy).$$

Thus stochastic transmission is closed under composition.

## 4 Population states

### Definition 4.1 Population distribution

A memetic population state on  $S$  is a probability state

$$\mu: 1 \rightarrow P(S).$$

The space of populations is therefore  $P(S)$ . For finite  $S = \{1, \dots, n\}$ , we identify  $\mu$  with a vector  $x = (x_1, \dots, x_n)$  satisfying

$$x_i \geq 0, \quad \sum_i x_i = 1.$$

## 5 Fitness

### Definition 5.1 Fitness observable

A fitness function is a positive semantic observable

$$w: S \rightarrow R_{>0}.$$

For a state  $s$ , the value  $w(s)$  is its expected reproductive weight. The population mean fitness is

$$w_\mu = \int_S w(s) d\mu(s).$$

We assume  $0 < w_\mu < \infty$  for every population under consideration.

## 6 Selection

### Definition 6.1 Selection operator

The fitness-weighted population  $\text{Sel}_w(\mu)$  is the probability distribution defined by

$$d\text{Sel}_w(\mu)(s) = \frac{w(s)}{\bar{w}_\mu} d\mu(s).$$

Equivalently, for an event  $A \subseteq S$ ,

$$\text{Sel}_w(\mu)(A) = \frac{\int_A w d\mu}{\int_S w d\mu}.$$

### Proposition 6.2 Selection preserves normalization

$$\text{Sel}_w(\mu)(S) = 1.$$

**Proof.**

By definition the numerator and the denominator are the same integral:

$$\text{Sel}_w(\mu)(S) = \frac{\int_S w d\mu}{\int_S w d\mu} = 1.$$

## 7 Mutation and imperfect transmission

### Definition 7.1 Mutation kernel

A mutation kernel is a Markov kernel

$$K: S \rightarrow P(S).$$

For  $s \in S$ , the distribution  $K(s)$  describes the possible semantic type of an offspring or transmitted realization originating from  $s$ . Mutation here includes any stochastic failure of exact semantic inheritance:

- copying errors
- reinterpretation
- translation
- recombination noise

or any other probabilistic change of semantic class.

## 8 Pushforward through a Markov kernel

A kernel induces a map on populations

$$K_* : P(S) \rightarrow P(S),$$

given by

$$(K_* \mu)(A) = \int_S K(s, A) d\mu(s).$$

### Proposition 8.1 Mutation preserves probability mass

If  $\mu$  is normalized, then  $K_* \mu$  is normalized.

**Proof.**

Since each  $K(s)$  is a probability measure,  $K(s, S) = 1$ . Hence

$$(K_* \mu)(S) = \int_S 1 d\mu(s) = 1.$$

## 9 The categorical replicator–mutator operator

We now compose selection and mutation.

### Definition 9.1 Replicator–mutator operator

Define

$$R_{w,K} = K_* \circ \text{Sel}_w.$$

Thus

$$R_{w,K}(\mu) = K_* \left( \frac{w\mu}{\int_S w d\mu} \right).$$

Explicitly, for an event  $A$ ,

$$R_{w,K}(\mu)(A) = \frac{\int_S w(s) K(s, A) d\mu(s)}{\int_S w(s) d\mu(s)}.$$

## 10 Normalization theorem

### Theorem 10.1 The operator preserves probability

For every probability distribution  $\mu$ ,  $R_{w,K}(\mu)$  is again a probability distribution.

**Proof.**

Selection produces a probability state by Proposition 6.2. The Markov pushforward  $K_*$  preserves probability mass by Proposition 8.1. Therefore their composite does also.

Thus

$$R_{w,K}: P(S) \rightarrow P(S)$$

defines a dynamical system on the population simplex.

## 11 The positive transfer operator

Normalization hides a useful linear structure. Let  $M_+(S)$  denote the cone of finite nonnegative measures on  $S$ .

### Definition 11.1 Mutation–selection transfer operator

Define

$$L_{w,K}: M_+(S) \rightarrow M_+(S)$$

by

$$(L_{w,K} \nu)(A) = \int_S w(s) K(s, A) d\nu(s).$$

Equivalently,  $L_{w,K} = K_* \circ M_w$ , where  $M_w(\nu) = w\nu$ . The operator is positive and linear on finite measures.

### Proposition 11.2 Total mass is mean fitness

For a probability measure  $\mu$ ,  $L_{w,K} \mu(S) = w_\mu$ .

**Proof.**

$$\begin{aligned} L_{w,K} \mu(S) &= \int_S w(s) K(s, S) d\mu(s) \\ &= \int_S w(s) d\mu(s) \\ &= w_\mu. \end{aligned}$$

### Corollary 11.3 Projectivization

The replicator–mutator operator is the projectivization of the positive transfer operator:

$$R_{w,K}(\mu) = \frac{L_{w,K} \mu}{L_{w,K} \mu(S)}.$$

This is the basic categorical structure behind the theory. Evolution takes place in the projectivization of a positive cone.

## 12 Finite-state form

Let  $S = \{1, \dots, n\}$ . Write  $x_i = \mu(\{i\})$ , let  $w_i > 0$ , and define  $Q_{ij} = K(i, \{j\})$ . The matrix  $Q$  is row-stochastic:

$$Q_{ij} \geq 0, \quad \sum_j Q_{ij} = 1.$$

### Theorem 12.1 Finite replicator–mutator equation

The categorical operator reduces to

$$x'_j = \frac{\sum_i x_i w_i Q_{ij}}{\sum_i x_i w_i}.$$

**Proof.**

By Definition 9.1, the mass of  $\{j\}$  is

$$R_{w,K}(\mu)(\{j\}) = \frac{\sum_i w_i K(i, \{j\}) x_i}{\sum_i w_i x_i}.$$

Substituting  $K(i, \{j\}) = Q_{ij}$  gives the result.

In row-vector notation,

$$x' = \frac{xWQ}{xW1},$$

where  $W = \text{diag}(w_1, \dots, w_n)$ . Since  $Q1 = 1$ , this becomes

$$x' = \frac{xA}{xA1}, \quad A = WQ.$$

### 13 Pure replication

Set  $K = \text{id}$ . Then  $Q = I$ , and the update becomes

$$x'_i = \frac{x_i w_i}{w},$$

where  $w = \sum_i x_i w_i$ . This is the discrete replicator equation.

### 14 Pure mutation

If fitness is constant,  $w(s) = c > 0$ , then selection is trivial:  $\text{Sel}_w(\mu) = \mu$ . Hence

$$\mu' = K_* \mu.$$

For a finite state space,  $x' = xQ$ . Ordinary Markov dynamics is the zero-selection limit of the memetic evolutionary theory.

### 15 Stationary populations

#### Definition 15.1 Evolutionary equilibrium

A population  $\mu_* \in P(S)$  is stationary if

$$R_{w,K}(\mu_*) = \mu_*.$$

## 16 Eigenmeasure theorem

### Theorem 16.1 Equilibria are normalized eigenmeasures

A probability distribution  $\mu_*$  is stationary if and only if

$$L_{w,K} \mu_* = \lambda \mu_*$$

for  $\lambda = \int_S w d\mu_*$ .

**Proof.**

Suppose first that  $R_{w,K}(\mu_*) = \mu_*$ . By Corollary 11.3,

$$\mu_* = \frac{L_{w,K} \mu_*}{L_{w,K} \mu_*(S)}.$$

Hence  $L_{w,K} \mu_* = L_{w,K} \mu_*(S) \mu_*$ . By Proposition 11.2,  $L_{w,K} \mu_*(S) = \int_S w d\mu_*$ . Therefore  $L_{w,K} \mu_* = \lambda \mu_*$ .

Conversely, suppose  $L_{w,K} \mu_* = \lambda \mu_*$ . Since  $\mu_*$  has total mass 1,  $L_{w,K} \mu_*(S) = \lambda$ . Therefore

$$R_{w,K}(\mu_*) = \frac{\lambda \mu_*}{\lambda} = \mu_*.$$

Thus evolutionary equilibrium is a Perron–Frobenius-type eigenproblem.

## 17 Iteration

### Proposition 17.1 Linearization of the orbit

Let  $\mu_{n+1} = R_{w,K}(\mu_n)$ . When  $w$  and  $K$  are time-independent,

$$\mu_n = \frac{L_{w,K}^n \mu_0}{L_{w,K}^n \mu_0(S)}.$$

**Proof.**

Normalization factors are scalars and therefore cancel under successive projectivization. Induction gives the formula.

The nonlinear normalized dynamics is induced by an underlying positive linear dynamics.

## 18 Perron–Frobenius theorem in the finite case

Let  $A = WQ$ .

### Theorem 18.1 Primitive finite mutation–selection dynamics

Suppose  $A$  is primitive: there exists  $k$  such that every entry of  $A^k$  is strictly positive. Then:

- $A$  has a unique positive dominant eigenvalue  $\lambda_{\text{PF}} > 0$ .
- There is a unique probability row vector  $x_*$  with strictly positive components satisfying  $x_* A = \lambda_{\text{PF}} x_*$ .
- $x_*$  is the unique interior fixed point of the replicator–mutator map.
- For every nonzero initial vector  $x_0 \geq 0, *$

$$\frac{x_0 A^n}{x_0 A^n 1} \longrightarrow x_*.$$

**Proof.**

This is the Perron–Frobenius theorem for primitive nonnegative matrices applied to  $A$ . The fixed-point characterization follows from Theorem 16.1. The convergence statement follows from domination of all other eigenvalues in modulus by  $\lambda_{\text{PF}}$ .

Thus mutation plus selection can produce a unique asymptotic semantic population even though individual replication events remain stochastic.

## 19 Continuous-time pure selection

Let the fitness over a short interval  $h$  be

$$e^{hf_i} = 1 + hf_i + O(h^2).$$

With no mutation,

$$x_i(t+h) = \frac{x_i(t) e^{hf_i}}{\sum_j x_j(t) e^{hf_j}}.$$

Expanding,

$$x_i(t+h) = x_i + h x_i(f_i - \bar{f}) + O(h^2),$$

where  $\bar{f} = \sum_j x_j f_j$ . Hence

$$\dot{x}_i = x_i(f_i - \bar{f}).$$

This is the continuous replicator equation.

## 20 Mutation generated continuously

Let  $L = (L_{ij})$  be a Markov generator:  $L_{ij} \geq 0$  for  $i \neq j$ , and  $\sum_j L_{ij} = 0$ . Set

$$Q(h) = I + hL + O(h^2), \quad W(h) = I + hF + O(h^2),$$

where  $F = \text{diag}(f_i)$ . The finite-step update is

$$x(t+h) = \frac{x(t) W(h) Q(h)}{x(t) W(h) 1}.$$

### Theorem 20.1 Continuous selection–mutation equation

The infinitesimal evolution is

$$\dot{x}_j = x_j(f_j - \bar{f}) + \sum_i x_i L_{ij}.$$

**Proof.**

Expand

$$(I + hF)(I + hL) + O(h^2) = I + h(F + L) + O(h^2).$$

The numerator is therefore

$$x + h(xF + xL) + O(h^2).$$

Since  $L1 = 0$ , the denominator is  $1 + h\bar{f} + O(h^2)$ . Using  $(1 + h\bar{f})^{-1} = 1 - h\bar{f} + O(h^2)$  gives

$$x + h(xF - \bar{f}x + xL) + O(h^2).$$

The  $j$ -th component yields the stated equation.

## 21 Mutation at reproduction

There is another standard continuous-time model. Suppose type  $i$  reproduces at rate  $f_i$ , and each offspring has type  $j$  with probability  $Q_{ij}$ . Then the production rate of type  $j$  is  $\sum_i x_i f_i Q_{ij}$ . Maintaining constant total population produces

$$\dot{x}_j = \sum_i x_i f_i Q_{ij} - \bar{f} x_j,$$

where  $\bar{f} = \sum_i x_i f_i$ .

### Proposition 21.1 Conservation of total probability

$$\sum_j \dot{x}_j = 0.$$

**Proof.**

$$\begin{aligned}
\sum_j \dot{x}_j &= \sum_i x_i f_i(\sum_j Q_{ij}) - \bar{f} \sum_j x_j \\
&= \sum_i x_i f_i - \bar{f} \\
&= 0.
\end{aligned}$$

## 22 Semantic coarse-graining

Return now to observer-relative semantics. Suppose

$$q: X \rightarrow S$$

maps fine-grained memetic states to coarse semantic classes. Let  $\tilde{w}: X \rightarrow R_{>0}$  be microscopic fitness and

$$K: X \rightarrow P(X)$$

the microscopic mutation kernel. We ask when there is an induced closed evolutionary dynamics on  $S$ . This is a categorical lumpability problem.

## 23 Fitness compatibility

### Definition 23.1 Fitness descends through $q$

Fitness is  $q$ -observable if there exists  $w: S \rightarrow R_{>0}$  such that

$$\tilde{w} = w \circ q.$$

Thus all fine states in one semantic equivalence class possess the same fitness.

## 24 Mutation compatibility

### Definition 24.1 Kernel lumpability

The fine mutation kernel  $K$  is  $q$ -lumpable if there exists  $K: S \rightarrow P(S)$  such that

$$P(q) \circ K \simeq K \circ q.$$

Diagrammatically,

$$\begin{array}{ccc}
X & \xrightarrow{K} & P(X) \\
\downarrow q & & \downarrow P(q) \\
S & \xrightarrow{K} & P(S).
\end{array}$$

The probability distribution of the coarse offspring type therefore depends only on the coarse parent type.

## 25 Coarse-graining theorem

Let  $q_*: P(X) \rightarrow P(S)$  denote pushforward.

### Theorem 25.1 Exact semantic coarse-graining commutes with evolution

Suppose

$$\tilde{w} = w \circ q \quad \text{and} \quad P(q) \circ \tilde{K} \simeq K \circ q.$$

Then

$$q_* \circ R_{\tilde{w}, \tilde{K}} \simeq R_{w, K} \circ q_*.$$

**Proof.**

Let  $\mu \in P(X)$ . First consider selection. For a coarse observable  $f: S \rightarrow R$ ,

$$\int_X f \, d(q_* \text{Sel}_{\tilde{w}}(\mu)) = \frac{\int_X f(q(x)) \tilde{w}(x) \, d\mu(x)}{\int_X \tilde{w} \, d\mu}.$$

Since  $\tilde{w} = w \circ q$ , the right-hand side is

$$\frac{\int_S f(s) w(s) \, d(q_* \mu)(s)}{\int_S w(s) \, d(q_* \mu)(s)}.$$

Hence

$$q_* \text{Sel}_{\tilde{w}}(\mu) = \text{Sel}_w(q_* \mu).$$

Lumpability gives

$$q_* \circ K_* = K_* \circ q_*.$$

Therefore

$$\begin{aligned} q_* R_{\tilde{w}, \tilde{K}}(\mu) &= q_* K_* \text{Sel}_{\tilde{w}}(\mu) \\ &= K_* q_* \text{Sel}_{\tilde{w}}(\mu) \\ &= K_* \text{Sel}_w(q_* \mu) \\ &= R_{w, K}(q_* \mu). \end{aligned}$$

## 26 Consequence for cognitive capability

Suppose  $q_c: M \rightarrow S_c$  is the semantic localization associated with cognitive architecture  $c$ . Then the population dynamics perceived by  $c$  is autonomous precisely when both fitness and transition probabilities depend only on  $q_c(m)$ . Otherwise the coarse semantic dynamics need not be Markovian.

### Corollary 26.1 Hidden semantic structure can induce apparent memory

If mutation or fitness varies inside the fibers of  $q: X \rightarrow S$ , then the coarse population process on  $S$  generally fails to close as a first-order Markov process.

Two fine populations can have the same coarse distribution  $q_* \mu = q_* \nu$  while having different distributions inside each fiber. If those hidden differences influence reproduction or mutation, then

$$q_* R(\mu) \neq q_* R(\nu).$$

The next coarse state therefore cannot be determined from the current coarse state alone. Thus insufficient cognitive resolution can convert microscopically Markovian dynamics into apparently non-Markovian dynamics.

### 27 Observer-relative fitness

Fitness itself may depend on the observer or environment. For each architecture  $c$ , let  $w_c: S_c \rightarrow R_{>0}$ . Then the same underlying memetic population may possess different induced evolutionary dynamics at different semantic resolutions.

For  $u: c \rightarrow d$ , compatibility requires

$$w_c \circ u^* \simeq w_d$$

in the appropriate contravariant sense. Without this compatibility, there is no reason for selection to commute with change of observer.

### 28 Replication probability

Let  $[m]_c$  be a semantic equivalence class. A kernel  $K$  defines a probability that state  $m$  remains in its class after transmission.

#### Definition 28.1 Semantic fidelity

Define

$$F_c(m) = K(m, [m]_c).$$

Thus  $0 \leq F_c(m) \leq 1$ . If  $c$  is a coarser observer than  $d$ , then typically  $[m]_d \subseteq [m]_c$ . Consequently

$$F_c(m) \geq F_d(m),$$

provided both classes are measured under the same underlying transmission kernel.

#### Proposition 28.2 Coarser semantics weakly increase measured fidelity

If  $[m]_d \subseteq [m]_c$ , then  $K(m, [m]_d) \leq K(m, [m]_c)$ .

**Proof.**

Probability measures are monotone under inclusion of measurable events.

This is the stochastic analogue of the earlier theorem that coarse observers weakly increase apparent deterministic fidelity.

### 29 Effective reproductive success

For a state  $m$ , take  $w(m)$  as exposure or reproductive intensity and  $F_c(m)$  as semantic fidelity. A simple capability-relative effective self-reproduction factor is then

$$w(m) F_c(m).$$

More generally, one should retain the full offspring kernel rather than compressing it to a scalar. Nevertheless, the factorization of reproductive intensity by semantic fidelity is useful conceptually.

### 30 Selection of semantic classes

Let  $A \subseteq S$  be a class of memes. The expected post-reproduction mass entering  $A$ , before normalization, is

$$\int_S w(s) K(s, A) d\mu(s).$$

This decomposes evolutionary success into current prevalence, reproductive weight, and transition probability into the target class. A class can increase either because it reproduces rapidly or because other classes mutate into it.

### 31 Mutation–selection balance

At equilibrium,

$$L_{w,K} \mu_* = \lambda \mu_*.$$

Hence for every measurable  $A$ ,

$$\lambda \mu_*(A) = \int_S w(s) K(s, A) d\mu_*(s).$$

This is the abstract mutation–selection balance equation. The quantity  $\lambda$  is the asymptotic population growth factor before normalization.

### 32 Multi-parent stochastic recombination

The Markov-kernel formalism also extends the operadic theory of Note II. Let  $O(n)$  be an object of  $n$ -ary operations. A stochastic operadic transformation is a kernel

$$K_n: O(n) \times S^n \longrightarrow P(S).$$

Given parents  $(s_1, \dots, s_n)$  and an operation  $o$ , the output is distributed according to  $K_n(o; s_1, \dots, s_n)$ .

#### Definition 32.1 Parent-relative stochastic fidelity

Relative to parent  $j$ ,

$$K_n(o; \mathfrak{S}, [s_j]_c)$$

is the probability that the recombination event preserves the semantic class of the  $j$ -th parent. Deterministic operadic replication is recovered when the kernel is a Dirac distribution.

### 33 Stochastic network semantics

More generally, replace deterministic network morphisms  $X \rightarrow Y$  by stochastic morphisms  $X \rightarrow P(Y)$ . Sequential network composition becomes Kleisli composition. Parallel composition is supplied by commutativity of the probability monad. Hence a probabilistic memetic network defines

$$N: X \rightarrow P(Y).$$

Its semantics is a conditional probability law rather than a deterministic function.

### 34 Replication relative to an observable

Let  $Q: Y \rightarrow Z$  be a semantic observable. For an input  $x$ , the output semantic distribution is  $P(Q)(N(x))$ . If the target semantic class associated with  $x$  is represented by  $q(x) \in Z$ , then exact stochastic replication means

$$P(Q)(N(x)) = \delta_{q(x)}.$$

Approximate replication permits a nonzero error distribution around this Dirac state.

### 35 Error kernel

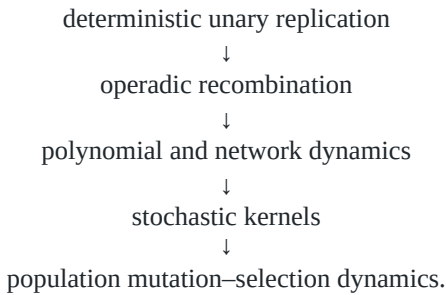
Define the semantic error kernel by

$$E_Q = P(Q) \circ N.$$

The deterministic theory corresponds to the special case in which  $E_Q(x)$  is always a Dirac measure. The stochastic theory instead retains the entire distribution  $E_Q(x) \in P(Z)$ . Thus mutation is not a Boolean event but a probability distribution over semantic deviations.

### 36 Categorical evolutionary hierarchy

The successive theories now form the hierarchy



At each stage, the earlier theory is recovered by restriction. A deterministic morphism  $f: X \rightarrow Y$  embeds into stochastic dynamics through the Dirac kernel  $x \mapsto \delta_{f(x)}$ .

### 37 Full stochastic specification

The resulting model is the tuple

$(C, J, H, P, M, Q, S, w, K)$ .

- $C$ : Cognitive architectures.
- $J$ : Coverage.
- $H$ : Semantic  $\infty$ -topos.
- $P$ : Probability monad.
- $M$ : Fine memetic realization object.
- $Q: M \rightarrow S$ : Semantic quotient.
- $S$ : Semantic class object.
- $w$ : Positive fitness.
- $K$ : Mutation or transmission kernel.

The evolutionary operator is

$$R_{w,K}(\mu) = K_* \left( \frac{w\mu}{\int_S w d\mu} \right).$$

### 38 Main theorem

#### Theorem 38.1 Categorical replicator–mutator theorem

Let  $H$  be a cognitive semantic  $\infty$ -topos equipped with a compatible probability monad  $P$ . Let  $S \in H$ , let  $w: S \rightarrow R_{>0}$ , and let  $K: S \rightarrow P(S)$ . Then:

- **(i)** Selection defines a normalization-preserving map  $\text{Sel}_w: P(S) \rightarrow P(S)$ . See Proposition 6.2.
- **(ii)** The Markov kernel induces a probability-preserving pushforward  $K_*$ . See Proposition 8.1.
- **(iii)** Their composite  $K_* \circ \text{Sel}_w$  defines a discrete-time population dynamical system. See Definition 9.1.
- **(iv)** This nonlinear system is the projectivization of the positive linear operator  $L_{w,K}$ . See Corollary 11.3.
- **(v)** Its stationary populations are precisely normalized positive eigenmeasures of  $L_{w,K}$ . See Theorem 16.1.
- **(vi)** For finite semantic state spaces it reduces to

$$x'_j = \frac{\sum_i x_i w_i Q_{ij}}{\sum_i x_i w_i}.$$

See Theorem 12.1.

- **(vii)** Its infinitesimal pure-selection limit is  $\dot{x}_i = x_i(f_i - \bar{f})$ . See Section 19.

- **(viii)** With a continuous mutation generator  $L$ , its infinitesimal limit is  $\dot{x}_j = x_j(f_j - \bar{f}) + \sum_i x_i L_{ij}$ . See Theorem 20.1.
- **(ix)** A semantic quotient  $q: X \rightarrow S$  inherits closed replicator–mutator dynamics whenever fitness is constant on its fibers and the microscopic mutation kernel is  $q$ -lumpable. See Theorem 25.1.
- **(x)** Failure of these conditions can make a coarse observer perceive non-Markovian dynamics even when the underlying dynamics is Markovian. See Corollary 26.1.

### 39 Conceptual consequence

The deterministic theory characterized memetic replication as preservation of a semantic invariant:

$$Q \circ N \simeq Q \circ \Pi_j.$$

The stochastic theory replaces this by a conditional distribution over semantic outcomes. Exact inheritance is Dirac-valued. Imperfect inheritance is a Markov kernel on semantic classes.

Population evolution is then obtained by weighting those states by reproductive success and integrating over all transition possibilities. The fundamental operator is

$$\mu \mapsto \frac{K_*(w\mu)}{\int_S w d\mu}.$$

### 40 Final synthesis

The entire theory developed so far can now be compressed into the sequence

Realizations, then semantic classes, then selection, then mutation, then normalization, then population evolution.

At the individual level,

$$m \rightsquigarrow K(Q(m))$$

gives a distribution of possible semantic descendants. At the population level,

$$\mu \rightsquigarrow R_{w,K}(\mu).$$

At equilibrium,

$$L_{w,K} \mu_* = \lambda \mu_*.$$

At finite dimension this is a Perron–Frobenius eigenproblem.

At different cognitive resolutions, exact coarse-graining is governed by the commuting square  $P(q) \circ \mathcal{K} \simeq K \circ q$  together with  $\mathcal{W} = w \circ q$ . Consequently, cognitive capability does not merely affect which individual transmissions count as faithful. It determines whether the population dynamics itself closes at the observer’s semantic resolution.

The strongest form of the theory is therefore

Normalized positive stochastic dynamics on observer-relative semantic classes.

Equivalently,

$$\text{Projectivize}(K_* \circ M_w).$$

This connects the  $\infty$ -topos formulation of observer-relative identity directly to Markov kernels, mutation–selection systems, transfer operators, and the classical replicator–mutator equation.

## Part 5: Endogenous Fitness and Evolutionary Games

### Abstract

We extend stochastic memetic replication from exogenously specified fitness to endogenous, frequency-dependent evolutionary dynamics.

Let

$$H = \text{Sh}_\infty(C, J)$$

be a cognitive semantic  $\infty$ -topos equipped with a commutative probability monad

$$P: H \rightarrow H.$$

Let  $S \in H$  denote an observer-relative semantic state object, so that a population is a probability state  $\mu \in P(S)$ .

Rather than assuming a fixed fitness map  $w: S \rightarrow R_{>0}$ , we introduce an endogenous fitness field

$$w: S \times P(S) \times E \longrightarrow R_{>0},$$

where  $E$  is an environment object. Mutation and transmission may themselves depend on the population and environment through kernels

$$K: S \times P(S) \times E \longrightarrow P(S).$$

The induced nonlinear evolutionary operator is

$$R(\mu, e) = (K_{\mu, e})_* \left( \frac{w(-, \mu, e) \mu}{\int_S w(s, \mu, e) d\mu(s)} \right).$$

Its equilibria satisfy a nonlinear eigenmeasure equation

$$L_{\mu_*, e} \mu_* = \lambda \mu_*.$$

For finite state spaces, pairwise payoff interactions recover the classical frequency-dependent replicator equation

$$\dot{x}_i = x_i((Ax)_i - x^\top Ax),$$

and mutation gives the coupled mutation–selection equation.

We derive a generalized Price identity, characterize interior equilibria, prove a Lyapunov theorem for symmetric interaction matrices, extend the construction to multiple interacting populations, and formulate open evolutionary components as stochastic lenses. Memetic evolution is then not a fixed transfer operator but a nonlinear feedback system in which the population changes the fitness landscape that subsequently changes the population.

Semantic population, interaction, fitness field, replication, and then a new semantic population.

The loop is naturally expressible using stochastic lenses and open-game semantics. The exogenous theory is Note IV.

## Notation

- $w(s, \mu, e)$  — Endogenous fitness: the value of a semantic state can depend on the whole population and the environment.
- $K_{\mu, e}$  — A mutation kernel that may itself depend on the population and the environment.
- $R(\mu, e)$  — The nonlinear replicator–mutator step at population  $\mu$  and environment  $e$ .
- $L_{\mu, e}$  — The transfer operator frozen at  $(\mu, e)$ . Equilibria are its eigenmeasures.
- $A$  — A payoff matrix. Frequency-dependent replicator dynamics is the weak-selection limit.
- $P$  — A tuple of populations. Each population’s fitness can depend on all of them.
- $(L^+, L^-)$  — A stochastic lens: a forward exposure and a backward probability kernel.
- BR — Best-response correspondence. It is not the same map as the evolutionary update.

## 1 Ambient semantic probability theory

Let  $(C, J)$  be a cognitive  $\infty$ -site and let

$$H = \text{Sh}_\infty(C, J).$$

Assume  $H$  is equipped with a commutative probability monad  $P: H \rightarrow H$ . Let  $S \in H$  be a semantic state object. A population state is  $\mu: 1 \rightarrow P(S)$ , and  $E \in H$  is an environment object. Expressions

$$\int_S f(s) d\mu(s)$$

are used under the standing assumption that the required internal integration theory exists.

## 2 Endogenous fitness

### Definition 2.1 Endogenous fitness field

An endogenous fitness field is a morphism

$$w: S \times P(S) \times E \longrightarrow R_{>0}.$$

For a fixed population  $\mu$  and environment  $e$ , define  $w_{\mu, e}(s) = w(s, \mu, e)$ . The reproductive value of a semantic state  $s$  may depend on  $s$  itself, on the complete current population  $\mu$ , and on the environment  $e$ .

## Definition 2.2 Mean endogenous fitness

Define

$$w(\mu, e) = \int_S w(s, \mu, e) d\mu(s).$$

We assume  $0 < w(\mu, e) < \infty$ .

## 3 Population-dependent mutation

The transmission law itself may depend on the population state.

### Definition 3.1 Endogenous mutation kernel

An endogenous mutation kernel is a morphism

$$K: S \times P(S) \times E \longrightarrow P(S).$$

For fixed  $(\mu, e)$ , write  $K_{\mu, e}: S \rightarrow P(S)$ . Then  $K_{\mu, e}(s)$  is the conditional distribution of semantic descendants of  $s$  under population state  $\mu$  and environment  $e$ .

## 4 Nonlinear replicator–mutator operator

### Definition 4.1 Endogenous selection

Define  $\text{Sel}_w(\mu, e)$  by

$$d\text{Sel}_w(\mu, e)(s) = \frac{w(s, \mu, e)}{w(\mu, e)} d\mu(s).$$

### Definition 4.2 Endogenous evolutionary operator

Define

$$R(\mu, e) = (K_{\mu, e})_* \text{Sel}_w(\mu, e).$$

Equivalently, for an event  $A$ ,

$$R(\mu, e)(A) = \frac{\int_S w(s, \mu, e) K_{\mu, e}(s, A) d\mu(s)}{w(\mu, e)}.$$

Unlike the exogenous case,  $R: P(S) \times E \rightarrow P(S)$  is generally nonlinear in  $\mu$ .

## 5 Population-indexed transfer operators

For each  $(\mu, e)$ , define  $L_{\mu, e}: M_+(S) \rightarrow M_+(S)$  by

$$(L_{\mu, e} \nu)(A) = \int_S w(s, \mu, e) K_{\mu, e}(s, A) d\nu(s).$$

For fixed  $(\mu, e)$  the operator is linear in  $\nu$ . The assignment  $\mu \mapsto L_{\mu, e}$  is generally nonlinear.

### Proposition 5.1 Projectivization at a frozen operator

The evolutionary operator satisfies

$$R(\mu, e) = \frac{L_{\mu, e} \mu}{L_{\mu, e} \mu(S)}.$$

**Proof.**

Since  $K_{\mu, e}(S, S) = 1$ ,

$$L_{\mu, e} \mu(S) = \int_S w(s, \mu, e) d\mu(s) = w(\mu, e).$$

Substitution gives the result.

Endogenous evolution remains a projectivized positive operator, but the operator itself now depends on the point being evolved.

## 6 Nonlinear eigenmeasure equilibrium

### Definition 6.1 Endogenous equilibrium

For a fixed environment  $e$ , a population  $\mu_*$  is an equilibrium if  $R(\mu_*, e) = \mu_*$ .

### Theorem 6.2 Nonlinear eigenmeasure characterization

A population  $\mu_*$  is an equilibrium if and only if

$$L_{\mu_*, e} \mu_* = \lambda \mu_*,$$

where  $\lambda = w(\mu_*, e)$ .

**Proof.**

By Proposition 5.1,  $R(\mu_*, e) = \mu_*$  if and only if  $L_{\mu_*, e} \mu_* = L_{\mu_*, e} \mu_*(S) \mu_*$ . But that total mass equals  $w(\mu_*, e)$ . Hence the equilibrium equation is exactly  $L_{\mu_*, e} \mu_* = \lambda \mu_*$ .

The difference from the exogenous case is substantial. Previously one solved an eigenproblem for one fixed positive operator. Now one solves the self-consistent nonlinear problem

$$L_{\mu} \mu \propto \mu.$$

## 7 Generalized Price identity

Let  $\varphi: S \rightarrow R$  be an observable. Define

$$(K_{\mu,e} \varphi)(s) = \int_S \varphi(t) K_{\mu,e}(s, dt).$$

### Theorem 7.1 Generalized Price equation

For one evolutionary step  $\mu' = R(\mu, e)$ ,

$$E_{\mu'}[\varphi] = E_{\mu}[\varphi] + \frac{\text{Cov}_{\mu}(w_{\mu,e}, \varphi)}{w(\mu, e)} + \frac{E_{\mu}[w_{\mu,e}(K_{\mu,e} \varphi - \varphi)]}{w(\mu, e)}.$$

**Proof.**

By definition,

$$E_{\mu'}[\varphi] = \frac{E_{\mu}[w_{\mu,e} K_{\mu,e} \varphi]}{w(\mu, e)}.$$

Add and subtract  $E_{\mu}[w_{\mu,e} \varphi]$  in the numerator:

$$E_{\mu'}[\varphi] = \frac{E_{\mu}[w \varphi]}{w} + \frac{E_{\mu}[w(K \varphi - \varphi)]}{w}.$$

The covariance identity

$$E_{\mu}[w \varphi] = \text{Cov}_{\mu}(w, \varphi) + w E_{\mu}[\varphi]$$

then yields the stated formula.

The two correction terms are selection covariance and transmission change. The decomposition remains valid even though fitness depends on the population.

## 8 Pairwise interaction games

A particularly important endogenous fitness model is induced by pairwise interaction. Let

$$a: S \times S \times E \longrightarrow R$$

be a payoff function. For a state  $s$ , population  $\mu$ , and environment  $e$ , define

$$f(s, \mu, e) = \int_S a(s, t, e) d\mu(t).$$

This is the expected payoff obtained by  $s$  when interacting with a random member of the population  $\mu$ .

## 9 From payoff to positive fitness

Discrete-time fitness should be positive. A canonical choice is

$$w(s, \mu, e) = \exp(\beta f(s, \mu, e)),$$

where  $\beta \geq 0$  is selection strength. For weak selection,

$$w = 1 + \beta f + O(\beta^2).$$

Thus additive payoff dynamics appears infinitesimally.

## 10 Finite evolutionary game

Let  $S = \{1, \dots, n\}$  and let  $A = (A_{ij})$  be a payoff matrix. For a population  $x = (x_1, \dots, x_n)$ , the payoff of type  $i$  is

$$f_i(x) = (Ax)_i = \sum_j A_{ij} x_j.$$

The population mean payoff is

$$\bar{f}(x) = \sum_i x_i f_i(x) = x^\top A x.$$

## 11 Frequency-dependent replicator equation

Assume no mutation. Under the continuous-time weak-selection limit,

$$\dot{x}_i = x_i(f_i(x) - \bar{f}(x)).$$

Hence

$$\dot{x}_i = x_i((Ax)_i - x^\top A x).$$

This is the classical frequency-dependent replicator equation.

## 12 Conservation theorem

### Theorem 12.1 Invariance of the simplex

The simplex

$$\Delta = \{x_i \geq 0, \sum_i x_i = 1\}$$

is invariant under replicator dynamics.

**Proof.**

$$\begin{aligned} \sum_i \dot{x}_i &= \sum_i x_i(f_i - \bar{f}) \\ &= \bar{f} - \bar{f} \\ &= 0. \end{aligned}$$

Thus total mass remains 1. Moreover, if  $x_i = 0$ , then  $\dot{x}_i = 0$ . Trajectories cannot cross a boundary face into negative coordinates.

### 13 Interior equilibria

#### Theorem 13.1 Equal-payoff characterization

Let  $x_*$  be an interior population:  $x_{*,i} > 0$  for every  $i$ . Then  $x_*$  is a fixed point of the pure replicator equation if and only if

$$f_i(x_*) = \bar{f}(x_*) \quad \text{for every } i.$$

**Proof.**

At equilibrium,

$$x_{*,i}(f_i(x_*) - \bar{f}(x_*)) = 0.$$

Because  $x_{*,i} > 0$ , we must have  $f_i(x_*) = \bar{f}(x_*)$ . Conversely, if every payoff equals the mean payoff, every derivative vanishes.

Thus every semantic type appearing with positive frequency at an interior equilibrium has equal reproductive payoff.

### 14 Relation to Nash equilibrium

For a symmetric population game, a population  $x_*$  is a symmetric Nash equilibrium if

$$f_i(x_*) \leq x_*^\top A x_*$$

for every pure type  $i$ , with equality for every type in the support of  $x_*$ . Every interior replicator equilibrium satisfies the Nash condition because all pure strategies have the same payoff. Boundary replicator equilibria need not be Nash equilibria, since an absent strategy may have strictly greater payoff without affecting the instantaneous derivative of its zero frequency.

### 15 Symmetric games and a Lyapunov function

Assume  $A = A^\top$ . Define  $V(x) = x^\top A x$ .

#### Theorem 15.1 Mean-payoff monotonicity

For symmetric  $A$ , along the replicator equation,

$$\frac{d}{dt}(x^\top A x) = 2\text{Var}_x(f) \geq 0.$$

**Proof.**

Since  $A = A^\top$ ,

$$\frac{d}{dt}(x^\top A x) = 2\dot{x}^\top A x.$$

With  $(Ax)_i = f_i$ ,

$$\begin{aligned}
\dot{x}^\top A x &= \sum_i x_i (f_i - \bar{f}) f_i \\
&= \sum_i x_i f_i^2 - \bar{f} \sum_i x_i f_i \\
&= \sum_i x_i f_i^2 - \bar{f}^2 \\
&= \text{Var}_x(f).
\end{aligned}$$

Multiplying by 2 gives the result.

Symmetric frequency-dependent selection performs ascent on mean interaction payoff, except at states where all supported strategies have equal payoff.

## 16 Shahshahani gradient formulation

On the interior of the simplex define the Shahshahani metric

$$g_{ij} = \frac{\delta_{ij}}{x_i}.$$

For the potential  $\Phi(x) = \frac{1}{2} x^\top A x$  with symmetric  $A$ , the replicator vector field is the constrained gradient of  $\Phi$  under this metric:

$$\dot{x} = \nabla_{\text{Sh}} \Phi.$$

The geometry of frequency-dependent selection is not ordinary Euclidean gradient flow but information-geometric gradient flow on the probability simplex.

## 17 Mutation–selection games

Let  $L(x)$  be a population-dependent Markov generator satisfying  $L_{ij}(x) \geq 0$  for  $i \neq j$ , and  $L(x)1 = 0$ . The continuous-time endogenous mutation–selection dynamics is

$$\dot{x}_j = x_j (f_j(x) - \bar{f}(x)) + \sum_i x_i L_{ij}(x).$$

### Proposition 17.1 Probability conservation

$$\sum_j \dot{x}_j = 0.$$

**Proof.**

The replicator contribution sums to zero by Theorem 12.1. For the mutation term,

$$\sum_j \sum_i x_i L_{ij}(x) = \sum_i x_i \sum_j L_{ij}(x) = 0,$$

because each row of  $L(x)$  sums to zero.

## 18 Nonlinear Markov processes

The state-dependent mutation law  $\mu \mapsto K_\mu$  turns the Markov process into a nonlinear Markov process. The transition law is no longer  $K_* \mu_t$  for one fixed  $K$ , but

$$(K_{\mu_t})_* \mu_t.$$

Adding selection gives

$$(K_{\mu_t})_* \left( \frac{w_{\mu_t} \mu_t}{\langle w_{\mu_t}, \mu_t \rangle} \right).$$

The population modifies its own transition operator.

## 19 Interacting populations

Let  $A$  be a finite set of population classes. For every  $a \in A$ , let  $S_a$  be its semantic state space. The joint population object is

$$P = \prod_{a \in A} P(S_a).$$

A joint state is  $(\mu_a)_{a \in A}$ .

## 20 Coupled fitness fields

### Definition 20.1 Coupled fitness and mutation

For each population  $a$ , let

$$w_a: S_a \times P \times E \rightarrow R_{>0}.$$

The fitness of a state in population  $a$  may depend on all populations simultaneously. Likewise let

$$K_a: S_a \times P \times E \rightarrow P(S_a).$$

## 21 Coupled evolutionary operator

Define

$$R_a(\mu, e) = (K_{a, \mu, e})_* \left( \frac{w_a(-, \mu, e) \mu_a}{w_a(\mu, e)} \right).$$

The full coupled operator is

$$R = \prod_{a \in A} R_a,$$

so  $R: P \times E \rightarrow P$ .

## 22 Pairwise multi-population games

Suppose population  $a$  has states  $i = 1, \dots, n_a$ . Let  $A_{ij}^{ab}$  be the payoff to type  $i$  in population  $a$  from interacting with type  $j$  in population  $b$ . Then

$$f_i^a(x) = \sum_{b \in A} \sum_j A_{ij}^{ab} x_j^b.$$

Mean payoff within population  $a$  is

$$\bar{f}^a = \sum_i x_i^a f_i^a.$$

## 23 Coupled replicator equations

The continuous dynamics becomes

$$\dot{x}_i^a = x_i^a (f_i^a(x) - \bar{f}^a(x)).$$

With mutation,

$$\dot{x}_i^a = x_i^a (f_i^a - \bar{f}^a) + \sum_j x_j^a L_{ji}^a(x).$$

### Theorem 23.1 Separate population normalization

For every  $a$ , the condition  $\sum_i x_i^a = 1$  is invariant.

**Proof.**

For each fixed  $a$ , the replicator contribution sums to  $\bar{f}^a - \bar{f}^a = 0$ . The mutation generator satisfies  $\sum_i L_{ji}^a = 0$ . Hence

$$\frac{d}{dt} \sum_i x_i^a = 0.$$

## 24 Environment as a dynamical object

So far  $e \in E$  has been treated parametrically. Let

$$G: E \times P(S) \longrightarrow E$$

be an environmental update. The closed coupled system is

$$F: P(S) \times E \longrightarrow P(S) \times E,$$

given by

$$F(\mu, e) = (R(\mu, e), G(e, \mu)).$$

A closed equilibrium is a pair  $(\mu_*, e_*)$  satisfying

$$\mu_* = R(\mu_*, e_*), \quad e_* = G(e_*, \mu_*).$$

This is a simultaneous population–environment fixed point.

## 25 Open evolutionary systems

Instead of closing the environment internally, one can expose it as an interface. Let  $(X, X')$  denote a system with forward state type  $X$  and backward feedback type  $X'$ .

## 26 Stochastic lenses

### Definition 26.1 Stochastic lens

A stochastic lens  $L: (X, X') \rightarrow (Y, Y')$  consists of a forward map

$$L^\rightarrow : X \rightarrow Y$$

and a stochastic backward update

$$L^\leftarrow : X \times Y' \rightarrow P(X').$$

The forward map exposes an observable output. The backward kernel propagates downstream feedback back into the upstream state or control variable.

### 27 Identity stochastic lens

The identity on  $(X, X')$  has forward part  $\text{id}_X$  and backward kernel  $\delta_{X'}$ .

## 28 Composition of stochastic lenses

Let  $L: (X, X') \rightarrow (Y, Y')$  and  $M: (Y, Y') \rightarrow (Z, Z')$ . The composite has forward part

$$(M \circ L)^\rightarrow = M^\rightarrow \circ L^\rightarrow.$$

For a backward feedback  $z'$ , first sample

$$y' \sim M^\leftarrow(L^\rightarrow(x), z'),$$

and then sample

$$x' \sim L^\leftarrow(x, y').$$

Hence the backward kernel is

$$(M \circ L)^\leftarrow(x, z') = \int_{Y'} L^\leftarrow(x, dy') M^\leftarrow(L^\rightarrow(x), z').$$

More invariantly, this is Kleisli composition using the strength of the commutative probability monad.

### Theorem 28.1 Associativity

Composition of stochastic lenses is associative up to the canonical coherence of the probability monad.

**Proof.**

The forward components compose associatively because morphism composition does. The backward components compose by iterated Kleisli integration. Associativity therefore follows from associativity of the monad multiplication  $P^3 \rightarrow P$ .

Stochastic lenses form a category, or an appropriate  $\infty$ -categorical enhancement when higher morphisms are retained.

## 29 Evolution as a stochastic lens

Take  $X = P(S)$ . Suppose a population exposes an observable  $O: P(S) \rightarrow Y$ . The environment returns a fitness or interaction signal  $r \in Y'$ . An evolutionary response kernel has the form

$$U: P(S) \times Y' \rightarrow P(P(S)).$$

Then

$$(O, U): (P(S), P(S)) \longrightarrow (Y, Y')$$

is a stochastic evolutionary lens. If the evolutionary update is deterministic, the backward kernel is  $\delta_{R_r(\mu)}$ .

## 30 Feedback closure

Suppose an environment lens converts the exposed population observable into a fitness signal. Composing the population and environment lenses closes the loop

Population, environment, fitness signal, and then the population update.

Endogenous fitness is a feedback phenomenon. The equation  $w = w_\mu$  is the algebraic shadow of this closed interaction.

## 31 Open evolutionary games

### Definition 31.1 Evolutionary open game

An evolutionary open game consists of:

- a semantic strategy object  $S$ ;
- a population object  $P(S)$ ;
- an environment interface  $E$ ;
- a payoff field  $u: S \times P(S) \times E \rightarrow R$ ;
- an evolutionary response operator  $R: P(S) \times E \rightarrow P(S)$ .

The payoff field need not itself prescribe the update. Different evolutionary rules may act on the same game.

### 32 Best-response correspondence

For fixed  $(\mu, e)$ , define

$$\text{BR}(\mu, e) = \underset{s \in S}{\operatorname{argmax}} u(s, \mu, e).$$

In a homotopical setting, BR should more properly be treated as a subobject or correspondence rather than an ordinary set.

#### Definition 32.1 Nash population

A population  $\mu_*$  is a Nash population at environment  $e$  if

$$\operatorname{supp}(\mu_*) \subseteq \text{BR}(\mu_*, e).$$

Every semantic strategy represented with nonzero population weight must be a best response to the current population.

### 33 Interior replicator equilibria and Nash equilibrium

#### Theorem 33.1 Interior equilibria are Nash

For a finite symmetric population game, every interior equilibrium of the pure replicator equation is a Nash equilibrium.

**Proof.**

By Theorem 13.1,  $f_i(x_*) = \bar{f}(x_*)$  for every  $i$ . Since  $x_*$  is interior, every pure strategy appears in its support. Hence every pure strategy obtains exactly the same payoff, every pure strategy is a best response, and  $\operatorname{supp}(x_*) \subseteq \text{BR}(x_*)$ .

### 34 Evolutionary dynamics versus rational optimization

The framework distinguishes two maps:

$$\text{BR}: P(S) \rightrightarrows S$$

and

$$R: P(S) \rightarrow P(S).$$

The first is an optimization correspondence. The second is a population dynamical law. Replicator dynamics need not jump to best responses. It continuously amplifies strategies according to relative payoff.

A game equilibrium is not an evolutionary update rule.

The two become linked through equilibrium theorems, not by definition.

### 35 Observer-relative games

Let  $q: M \rightarrow S$  be a capability-relative semantic quotient. Suppose microscopic interaction payoff is

$$u: M \times M \times E \rightarrow R.$$

A closed game exists on  $S$  only if the payoff descends: there is  $u: S \times S \times E \rightarrow R$  with

$$u(m, n, e) = u(q(m), q(n), e).$$

If this fails, two states indistinguishable to the observer may nevertheless have different interaction consequences. The coarse observer then does not possess a closed game description.

### 36 Game-theoretic lumpability

#### Definition 36.1 Strategic lumpability

A semantic quotient  $q: M \rightarrow S$  is strategically lumpable if payoff factors through  $q$ , mutation and transmission are  $q$ -lumpable, and environmental coupling factors through the coarse state. Under these conditions, the observer-relative population dynamics on  $S$  closes autonomously.

#### Theorem 36.2 Observer-relative evolutionary closure

If  $q$  is strategically lumpable, then there exists an induced evolutionary game on  $S$  whose population trajectory is exactly the pushforward of the microscopic trajectory:

$$q_* \circ R \simeq R_S \circ q_*.$$

**Proof.**

Payoff factorization implies that the induced fitness depends only on the coarse state. Kernel lumpability implies that stochastic transmission commutes with  $q_*$ . Environmental factorization ensures that feedback signals are also determined by the coarse population. The coarse-graining theorem for replicator–mutator dynamics therefore applies at every feedback step.

### 37 Failure of strategic lumpability

If strategic lumpability fails, coarse population state is not sufficient to predict its own future. One must augment the state with hidden variables. Formally, if

$$q_* \mu = q_* \nu$$

but

$$q_* R(\mu) \neq q_* R(\nu),$$

then no map  $F: P(S) \rightarrow P(S)$  can satisfy

$$F \circ q_* = q_* \circ R$$

for all microscopic states. The observer's coarse semantics is dynamically incomplete.

### 38 Memory from hidden semantic variables

One can repair non-Markovian coarse dynamics by enlarging the observed state. Suppose

$$h: M \rightarrow H$$

records additional hidden semantic information. Replace  $q: M \rightarrow S$  by

$$(q, h): M \rightarrow S \times H.$$

If the enlarged quotient is lumpable, then  $(S, H)$  is a sufficient state representation. This yields a formal criterion for semantic state augmentation:

Add enough hidden structure until the dynamics closes.

### 39 Higher-order frequency dependence

Pairwise interaction is not essential. Let

$$a_n: S^{n+1} \times E \rightarrow R.$$

Define

$$f_n(s, \mu, e) = \int_{S^n} a_n(s, s_1, \dots, s_n, e) d\mu(s_1) \cdots d\mu(s_n).$$

Total payoff may be

$$f(s, \mu, e) = \sum_{n \geq 0} \lambda_n f_n(s, \mu, e).$$

This is the stochastic-population analogue of the earlier operadic theory. Fitness can depend on genuinely multi-agent semantic interactions rather than only pairwise games.

### 40 Polynomial interaction laws

An even more categorical formulation uses a polynomial

$$I \xrightarrow{s} E \xrightarrow{p} B \xrightarrow{t} I.$$

Each operation  $b \in B$  specifies a finite interaction pattern. Given a population  $\mu$ , one samples inputs according to the corresponding fibers  $E_b$ , evaluates their interaction, and produces a fitness contribution. Polynomial functors specify the combinatorial shape of interactions, while the probability monad integrates over randomly sampled participants.

Polynomial interaction together with probability yields endogenous fitness.

### 41 Operadic game interaction

If  $O$  is an  $\infty$ -operad of interaction forms, one may assign to  $o \in O(n)$  a payoff map  $u_o: S^{n+1} \rightarrow R$ . The population-level expected payoff is obtained by integrating the other  $n$  arguments against  $\mu$ :

$$\int_{S^n} u_o(s, s_1, \dots, s_n) d\mu^{\otimes n}.$$

Operadic recombination and evolutionary game interaction can be encoded by the same arity structure.

#### 42 Endogenous semantic fidelity

Semantic fidelity relative to the class  $[S]_c$ , in the exogenous theory, was  $K(s, [S]_c)$ . With endogenous transmission,

$$F_c(s, \mu, e) = K_{\mu, e}(s, [S]_c).$$

Replication fidelity itself becomes frequency dependent. A meme may copy accurately when rare and inaccurately when common, or conversely.

#### 43 Effective endogenous reproduction number

Define

$$R_c(s, \mu, e) = w(s, \mu, e) F_c(s, \mu, e).$$

This quantity measures expected production of descendants that remain in the same observer-relative semantic class. Unlike a fixed reproduction number,  $R_c$  changes with the population itself.

#### 44 Frequency-dependent semantic selection

Suppose two semantic classes  $a$  and  $b$  satisfy

$$R_c(a, \mu) > R_c(b, \mu)$$

for one population state, but

$$R_c(a, \nu) < R_c(b, \nu)$$

for another. Then there is no absolute fitness ordering between  $a$  and  $b$ . Fitness is relational: a property of a state relative to a population and an environment. This is the categorical analogue of frequency-dependent selection.

#### 45 Nonlinear categorical dynamics

The exogenous theory had, for a fixed positive operator  $L$ ,

$$\mu_{t+1} = \frac{L\mu_t}{L\mu_t(S)}.$$

The endogenous theory has

$$\mu_{t+1} = \frac{L_{\mu_t} \mu_t}{L_{\mu_t} \mu_t(S)}.$$

The underlying geometry is not projective linear dynamics but a nonlinear self-map of the probability object. Equilibria solve

$$L_{\mu_*} \mu_* = \lambda \mu_*.$$

#### 46 Linearization around equilibrium

Let  $F(\mu) = R(\mu, e)$  for fixed  $e$ . At an equilibrium  $F(\mu_*) = \mu_*$ , local stability is controlled by the derivative  $DF_{\mu_*}$ . If  $T_{\mu_*} P(S)$  denotes the tangent space of normalized perturbations, then infinitesimal perturbations evolve by

$$\delta \mu_{t+1} = DF_{\mu_*} \delta \mu_t.$$

##### Definition 46.1 Local asymptotic stability

The equilibrium is linearly asymptotically stable if the spectral radius of

$$DF_{\mu_*} \mid_{T_{\mu_*} P(S)}$$

is less than 1 in discrete time. For continuous time, stability requires the relevant eigenvalues of the Jacobian to have negative real part. Second-order structure re-enters through the derivative of the population-dependent fitness field.

#### 47 Feedback derivatives

Since  $w = w(s, \mu, e)$ , the derivative of the evolutionary operator contains two contributions:

$$DF = D_{\text{direct}} + D_{\text{feedback}}.$$

The direct term changes population frequencies while holding the fitness field fixed. The feedback term arises because perturbing  $\mu$  changes  $w_\mu$  and potentially  $K_\mu$ . This term has no analogue in a fixed-fitness linear transfer system.

#### 48 Second-order interaction structure

For a finite game with payoff field  $f_i(x)$ , define

$$J_{ij} = \frac{\partial f_i}{\partial x_j}.$$

Then  $J$  measures how increasing population type  $j$  changes the reproductive payoff of type  $i$ . For the linear game  $f(x) = Ax$ , one has  $J = A$ . For nonlinear interaction games,  $D^2 f$  measures how those interaction effects themselves change with population composition. Higher derivatives classify increasingly high-order ecological or memetic feedback.

#### 49 Stochastic-lens interpretation of endogenous fitness

The complete feedback structure can be written schematically as

$$P(S) \xrightarrow{\mathcal{O}} Y \xrightarrow{\mathcal{E}^n Y'} \text{Fit}(S) \longrightarrow P(S).$$

Closing the interfaces gives

$$P(S) \longrightarrow P(S).$$

The endogenous nonlinear map is the closed semantics of an open stochastic system. Lens-like semantics is natural here because forward behavior and backward feedback are separate components before the system is closed.

## 50 Main theorem

### Theorem 50.1 Endogenous memetic evolution theorem

Let  $H = \text{Sh}_\infty(C, J)$  be equipped with a compatible probability monad  $P$ . Let  $S, E \in H$ , and suppose given

$$w: S \times P(S) \times E \rightarrow R_{>0}$$

and

$$K: S \times P(S) \times E \rightarrow P(S).$$

Then:

- **(i)** The formula

$$R(\mu, e) = (K_{\mu, e})_* \left( \frac{w_{\mu, e} \mu}{w(\mu, e)} \right)$$

defines a normalized nonlinear evolutionary operator. See Sections 4–5.

- **(ii)** Equilibria are precisely solutions of  $L_{\mu_*} \mu_* = \lambda \mu_*$ . See Theorem 6.2.
- **(iii)** Changes in any observable satisfy the generalized Price equation. See Theorem 7.1.
- **(iv)** Pairwise linear interaction generates

$$\dot{x}_i = x_i((Ax)_i - x^\top Ax).$$

See Sections 8–11.

- **(v)** For symmetric  $A$ , mean payoff is nondecreasing:

$$\frac{d}{dt}(x^\top Ax) = 2\text{Var}_x(Ax) \geq 0.$$

See Theorem 15.1.

- **(vi)** Multiple semantic populations admit coupled replicator systems. See Sections 19–23.
- **(vii)** Population–environment feedback is the closure of an open stochastic system. See Sections 24–30.
- **(viii)** Stochastic lenses compose such open systems associatively. See Theorem 28.1.

- **(ix)** A semantic quotient inherits an autonomous evolutionary game precisely when payoff, mutation, and environment interactions descend through that quotient. See Theorem 36.2.
- **(x)** Failure of such descent produces hidden-state dependence and generally non-Markovian observer-relative dynamics. See Sections 37–38.

**Proof.**

Statements (i)–(x) are established in Sections 4–6, Theorem 7.1, Sections 8–17, Sections 19–23, Sections 24–30, Theorem 28.1, and Sections 35–38.

## 51 Full specification

The endogenous theory is summarized by

$(C, J, H, P, M, Q, S, E, w, K, R).$

- $M$ : Fine memetic realizations.
- $Q: M \rightarrow S$ : Observer-relative semantics.
- $P(S)$ : Semantic population space.
- $E$ : Environment.
- $w(s, \mu, e)$ : Endogenous fitness.
- $K_{\mu, e}$ : Endogenous stochastic transmission.
- $R$ : Nonlinear population evolution.

The fundamental equation is

$$\mu_{t+1} = \frac{L_{\mu_t, e_t} \mu_t}{L_{\mu_t, e_t} \mu_t(S)}.$$

If the environment also evolves,

$$\begin{aligned} \mu_{t+1} &= R(\mu_t, e_t), \\ e_{t+1} &= G(e_t, \mu_t). \end{aligned}$$

## 52 Conceptual synthesis

The earlier fixed-fitness theory treated the memetic environment as external:

$$\mu \not\rightarrow \mu'.$$

The endogenous theory closes the feedback loop:

$$\mu \rightarrow w_\mu \rightarrow L_\mu \rightarrow \mu' \rightarrow w_{\mu'} \rightarrow \cdots.$$

The population constructs part of its own selective environment. For pairwise interaction,

$$f(s, \mu) = \int_S a(s, t) d\mu(t),$$

and consequently

$$\mu \mapsto w_\mu \mapsto R(\mu).$$

This turns memetic evolution from a linear transmission problem into a feedback game.

The stochastic-lens perspective separates the open interfaces: observation and feedback. The evolutionary-game perspective supplies interaction, then payoff, then relative reproduction. The  $\infty$ -topos layer determines which distinctions count as semantic distinctions for a given cognitive architecture. The probability monad determines stochastic inheritance. The game determines frequency-dependent fitness. The lens structure determines how open components are coupled.

Observer-relative stochastic game dynamics with feedback.

Equivalently, endogenous memetic evolution.

The theory no longer assigns a meme an absolute reproductive fitness. Fitness is a relational functional

$$w: S \times P(S) \times E \rightarrow R_{>0},$$

and replication occurs within the nonlinear closed loop generated by that functional.

## Part 6: Coevolution of Memes and Cognitive Architectures

### Abstract

We extend memetic replication to the case in which cognitive architectures themselves evolve under memetic influence. The semantic equivalence relation defining what counts as the same meme is therefore no longer fixed.

Let  $(C, J)$  be a cognitive  $\infty$ -site, and let  $M: C^{\text{op}} \rightarrow S$  be a memetic field. Its Grothendieck construction  $\int_C M$  has objects  $(c, m)$ , an architecture together with a memetic state as realized by that architecture. The joint population state space is the maximal  $\infty$ -groupoid

$$\Omega_M := (\int_C M)^{\approx}.$$

A population is  $\nu \in P(\Omega_M)$ , where  $P$  is a compatible probability monad. Fitness and transmission may depend on the complete joint population:

$$w: \Omega_M \times P(\Omega_M) \times E \rightarrow R_{>0},$$

$$K: \Omega_M \times P(\Omega_M) \times E \rightarrow P(\Omega_M).$$

The evolutionary equation is

$$R(\nu, e) = (K_{\nu, e})_* \left( \frac{w(-, \nu, e) \nu}{\int_{\Omega_M} w(x, \nu, e) d\nu(x)} \right).$$

Because the kernel may change the architecture coordinate, memes can modify the observer modality through which subsequent memetic identity is evaluated. If architecture  $c$  carries a semantic localization  $L_c$ , then  $m \simeq_c n$  need not imply  $m \simeq_d n$  after the architecture changes from  $c$  to  $d$ . Memetic evolution and semantic identity become mutually coupled.

Memes alter cognition, cognition alters memetic identity, and identity alters subsequent selection.

We formulate this coupled dynamics, derive its finite-state and continuous-time forms, characterize its equilibria, establish exact conditions under which architecture or meme marginals evolve autonomously, and prove an identity-transport theorem specifying when semantic identity is preserved under cognitive change. The fixed-architecture theory is Note V.

## Notation

- $\Omega_M$  — The joint state space: an architecture together with a meme as that architecture realizes it.
- $\simeq_c$  — Semantic equivalence as judged by architecture  $c$ . It can change when  $c$  changes.
- $\nu$  — A joint population, a probability state on architectures and memes together.
- $A$  — A cognitive plasticity kernel. Memes may change the next architecture.
- $\hat{q}$  — The total semantic quotient, from joint states to joint semantic classes.
- $T_{cd}$  — A translation of memes from architecture  $c$  to architecture  $d$ .

## 1 Cognitive architectures

Let  $C$  be a small  $\infty$ -category whose objects are cognitive architectures. A morphism  $u: c \rightarrow d$  represents an admissible semantic translation, abstraction, simulation, or architectural transformation. Equip  $C$  with a Grothendieck topology  $J$ , producing the cognitive  $\infty$ -site  $(C, J)$ . Its semantic  $\infty$ -topos is

$$H = \text{Sh}_\infty(C, J).$$

## 2 Memetic fields

### Definition 2.1 Memetic field

A memetic field is an  $\infty$ -sheaf

$$M: C^{\text{op}} \longrightarrow S.$$

For every architecture  $c$ ,  $M(c)$  is the space of memetic states representable at  $c$ . For every architecture morphism  $u: c \rightarrow d$ , there is a restriction or translation

$$u^*: M(d) \rightarrow M(c).$$

## 3 The joint architecture–meme state space

The appropriate state is no longer merely a meme  $m$ . It is a pair  $(c, m)$ , encoded by the Grothendieck construction.

**Definition 3.1 Memetic Grothendieck construction**

Let  $\int_C M$  denote the  $\infty$ -category obtained by unstraightening  $M: C^{\text{op}} \rightarrow S$ . There is a canonical right fibration

$$p: \int_C M \longrightarrow C.$$

Its objects are pairs  $(c, m)$  with  $m \in M(c)$ .

**Definition 3.2 Joint population space**

Define

$$\Omega_M := (\int_C M)^{\approx}$$

to be the maximal  $\infty$ -groupoid of the Grothendieck construction. The architecture projection

$$\pi: \Omega_M \rightarrow C^{\approx}$$

satisfies  $\pi(c, m) = c$ .

**4 Architecture-dependent semantics**

Suppose each architecture possesses its own semantic resolution. Let  $\text{Mod}(E)$  denote an appropriate  $\infty$ -category of left-exact accessible modalities on an ambient  $\infty$ -topos  $E$ .

**Definition 4.1 Cognitive modality assignment**

A cognitive modality assignment is a functor

$$L: C \longrightarrow \text{Mod}(E).$$

For  $c \in C$ , write  $L_c$  for the corresponding semantic localization.

**Definition 4.2 Architecture-relative semantic equivalence**

For ambient memetic realizations  $m, n$ , write  $m \simeq_c n$  when

$$\text{Id}_{L_c M}(\eta_c(m), \eta_c(n))$$

is inhabited. The architecture  $c$  participates directly in the definition of memetic identity.

**5 Dynamic semantic identity**

If the architecture changes, the equivalence relation itself may change.

### Definition 5.1 Semantic refinement

A transition  $c \rightsquigarrow d$  is semantically refining on  $M$  if  $m \simeq_d n$  implies  $m \simeq_c n$ . Equivalence at  $d$  is then at least as discriminating as equivalence at  $c$ .

### Definition 5.2 Semantic coarsening

A transition  $c \rightsquigarrow d$  is semantically coarsening if  $m \simeq_c n$  implies  $m \simeq_d n$ .

### Definition 5.3 Identity-changing transition

The transition changes semantic identity if there exist  $m, n$  such that either  $m \simeq_c n$  but  $m \not\simeq_d n$ , or  $m \not\simeq_c n$  but  $m \simeq_d n$ . Architecture evolution can split or merge semantic equivalence classes.

## 6 Identity transport along architecture morphisms

Suppose  $u: c \rightarrow d$  induces a semantic translation  $u^*: M(d) \rightarrow M(c)$ .

### Theorem 6.1 Identity transport

If  $x \simeq_d y$ , then  $u^* x \simeq_c u^* y$ .

**Proof.**

A witness of  $x \simeq_d y$  is a path  $p: x \simeq y$  in  $M(d)$ . Applying  $u^*$  gives a path  $u^* p: u^* x \simeq u^* y$ . Hence  $u^* x \simeq_c u^* y$ .

This concerns functorial translation. It does not assert that arbitrary architecture changes preserve the same equivalence relation on an underlying presentation space.

## 7 Identity-preserving cognitive change

### Definition 7.1 Semantically conservative transition

Suppose an architecture transition  $c \rightarrow d$  comes with a translation  $T_{cd}: M(c) \rightarrow M(d)$ . It is semantically conservative if

$$x \simeq_c y \iff T_{cd}(x) \simeq_d T_{cd}(y).$$

### Theorem 7.2 Identity invariance along conservative paths

Let  $c_0 \rightarrow c_1 \rightarrow \dots \rightarrow c_n$  be a sequence of semantically conservative architecture changes. Then for all  $x, y \in M(c_0)$ ,

$$x \simeq_{c_0} y$$

if and only if

$$T_{0n}(x) \simeq_{c_n} T_{0n}(y),$$

where

$$T_{0n} = T_{c_{n-1}c_n} \circ \dots \circ T_{c_0c_1}.$$

**Proof.**

Each transition preserves and reflects semantic equivalence. Apply this equivalence successively along the path.

**Corollary 7.3 Where identity can change**

Semantic identity can change along a cognitive trajectory only if at least one architecture transition fails to be conservative.

**8 Reflexive memetic influence**

A meme may influence the architecture through which it is interpreted.

**Definition 8.1 Architecture transition kernel**

A cognitive plasticity kernel is a Markov kernel

$$A: \Omega_M \longrightarrow P(C^\approx).$$

For  $(c, m) \in \Omega_M$ , the distribution  $A(c, m)$  describes possible successor cognitive architectures.

**Definition 8.2 Memetically induced cognitive plasticity**

The cognitive transition is memetically induced if  $A(c, m)$  depends nontrivially on  $m$ : there exist  $m, n \in M(c)$  with  $A(c, m) \neq A(c, n)$ . Different memes can alter the future cognitive architecture differently.

**9 Joint transition kernel**

Rather than factor architecture and meme transitions separately, the invariant formulation uses one kernel on the total state space.

**Definition 9.1 Joint evolutionary kernel**

A joint architecture–meme transition kernel is

$$K: \Omega_M \longrightarrow P(\Omega_M).$$

For a state  $x = (c, m)$ , the distribution  $K(x)$  describes the joint law of successor architectures and successor memetic realizations. The architecture kernel is recovered by pushforward:

$$A = P(\pi) \circ K.$$

**10 Conditional factorization**

Under a suitable disintegration, the joint kernel factors schematically as an architecture choice followed by a conditional memetic choice:

$$K((c, m), d(c', m')) = A((c, m), dc') B((c, m), c', dm').$$

Here  $A((c, m), dc')$  chooses the successor architecture, while  $B((c, m), c', dm')$  chooses the successor memetic realization conditional on that architecture. The factorization is useful conceptually but is not needed for the invariant theory.

## 11 Joint populations

Assume a compatible probability monad  $P$ .

### Definition 11.1 Joint cognitive–memetic population

A population is

$$\nu \in P(\Omega_M).$$

It describes the joint distribution of architectures and memetic states.

## 12 Architecture marginal

Define

$$a = \pi_* \nu \in P(C^\approx).$$

This is the distribution of cognitive architectures.

## 13 Conditional memetic populations

Where disintegration exists, write

$$d\nu(c, m) = a(dc) \mu_c(dm),$$

where  $\mu_c \in P(M(c))$  is the conditional memetic distribution within architecture  $c$ . A joint population is architecture frequencies together with architecture-conditioned meme distributions.

## 14 Endogenous joint fitness

Let  $E$  be an environment object.

### Definition 14.1 Joint fitness field

A joint fitness field is

$$w: \Omega_M \times P(\Omega_M) \times E \rightarrow R_{>0}.$$

Thus  $w((c, m), \nu, e)$  may depend on the architecture, the meme, the entire joint population, and the environment.

## 15 Joint selection

Define mean fitness

$$w(\nu, e) = \int_{\Omega_M} w(x, \nu, e) d\nu(x).$$

The selected population is

$$d\text{Sel}_w(\nu, e)(x) = \frac{w(x, \nu, e)}{w(\nu, e)} d\nu(x).$$

## 16 Coevolutionary operator

Allow the transition kernel itself to depend on  $(\nu, e)$ :

$$K_{\nu, e}: \Omega_M \rightarrow P(\Omega_M).$$

### Definition 16.1 Cognitive–memetic evolutionary operator

Define

$$R(\nu, e) = (K_{\nu, e})_* \text{Sel}_w(\nu, e).$$

Explicitly, for an event  $B$ ,

$$R(\nu, e)(B) = \frac{\int_{\Omega_M} w(x, \nu, e) K_{\nu, e}(x, B) d\nu(x)}{w(\nu, e)}.$$

This is the central evolution equation.

## 17 Population-dependent transfer operator

For fixed  $(\nu, e)$ , define  $L_{\nu, e}$  on finite positive measures by

$$(L_{\nu, e} \rho)(B) = \int_{\Omega_M} w(x, \nu, e) K_{\nu, e}(x, B) d\rho(x).$$

Then

$$R(\nu, e) = \frac{L_{\nu, e} \nu}{L_{\nu, e} \nu(\Omega_M)}.$$

## 18 Coevolutionary equilibrium

### Definition 18.1 Joint equilibrium

A population  $\nu_*$  is an equilibrium at environment  $e$  if  $R(\nu_*, e) = \nu_*$ .

### Theorem 18.2 Nonlinear eigenmeasure theorem

A joint population  $\nu_*$  is an equilibrium if and only if

$$L_{\nu_*, e} \nu_* = \lambda \nu_*,$$

where  $\lambda = w(\nu_*, e)$ .

**Proof.**

As in the fixed-architecture theory,  $R(\nu_*, e) = \nu_*$  if and only if  $L_{\nu_*, e} \nu_* = L_{\nu_*, e} \nu_*(\Omega_M) \nu_*$ . Since the kernel is normalized,

$$L_{\nu_*, e} \nu_*(\Omega_M) = \int_{\Omega_M} w(x, \nu_*, e) d\nu_*(x) = w(\nu_*, e).$$

## 19 Evolution of the architecture marginal

Let  $a' = \pi_* \nu'$ , and write  $A_{\nu, e} = P(\pi) \circ K_{\nu, e}$ .

### Theorem 19.1 Architecture marginal equation

The successor architecture distribution is

$$a'(D) = \frac{\int_{\Omega_M} w(x, \nu, e) A_{\nu, e}(x, D) d\nu(x)}{w(\nu, e)}.$$

**Proof.**

By definition  $a' = \pi_* R(\nu, e)$ , so  $a'(D) = R(\nu, e)(\pi^{-1}(D))$ . Substitute Definition 16.1:

$$a'(D) = \frac{\int w(x, \nu, e) K_{\nu, e}(x, \pi^{-1}(D)) d\nu(x)}{w}.$$

The inner kernel probability is  $A_{\nu, e}(x, D)$ .

## 20 Memes can select cognitive architectures

Even if cognitive architectures have no fitness term independent of memes, architecture frequencies may change because fitness is correlated with the memes carried within each architecture. Suppose architecture never mutates, so

$A((c, m), D) = \delta_c(D)$ . Then

$$a'(c) = \frac{\int_{M(c)} w((c, m), \nu, e) d\nu(c, m)}{w}.$$

Architectures correlated with highly reproducing memes increase in frequency.

## 21 Memes can directly transform architectures

If  $A((c, m), d)$  depends on  $m$ , then architecture evolution occurs even without differential survival between architectures. This is a stronger feedback mechanism:

$m \rightarrow c' \rightarrow L_{c'} \rightarrow \text{new semantic equivalence.}$

The meme changes the modality under which subsequent memes are classified.

## 22 Reflexive memes

### Definition 22.1 Reflexive meme

A memetic state  $m \in M(c)$  is reflexive if its transmission can alter the architecture in a way that changes semantic distinctions relevant to future memetic transmission. Formally, there exists an architecture  $d$  in the support of  $A(c, m)$  and states  $x, y$  such that  $x \approx_c y$  but  $T_{cd}(x) \not\approx_d T_{cd}(y)$ , or conversely. A reflexive meme modifies the equivalence relation used to classify later memes.

## 23 Semantic self-modification

The resulting causal structure is

$$m_t \longrightarrow c_{t+1} \longrightarrow \approx_{c_{t+1}} \longrightarrow \text{classification of } m_{t+1}.$$

This is stronger than ordinary frequency-dependent fitness. The state changes not merely its future reproduction probability but the semantic state space in which reproduction is evaluated.

## 24 Identity drift

### Definition 24.1 Identity drift

For two persistent presentations  $x, y$ , define their time-dependent semantic relation

$$I_t(x, y) = \text{Id}_{L_{c_t} M}(\eta_{c_t}(x), \eta_{c_t}(y)).$$

Identity drift occurs if the homotopy type of  $I_t(x, y)$  changes with  $t$ . Identity can therefore itself be a dynamical variable.

## 25 Identity-drift theorem

### Theorem 25.1 Conservative paths do not drift

Suppose every architecture transition along a trajectory is semantically conservative. Then no semantic identity drift occurs along that trajectory.

**Proof.**

By Theorem 7.2, every conservative transition preserves and reflects semantic equivalence. Therefore the truth and homotopy class of  $x \approx_{c_t} y$  is invariant along the trajectory, up to the coherent transport supplied by the transitions.

### Corollary 25.2 Drift requires a nonconservative step

Every identity-changing trajectory contains a nonconservative cognitive transition.

## 26 Dynamic coarse-graining

Suppose every architecture  $c$  has a semantic quotient  $q_c : M(c) \rightarrow S(c)$ . The semantic state at time  $t$  is not merely  $q(m_t)$ , but

$$q_{c_t}(m_t).$$

Both arguments evolve.

## 27 Total semantic field

Let  $S: C^{\text{op}} \rightarrow S$  be the semantic quotient field, with a natural transformation  $q: M \Rightarrow S$ . Taking Grothendieck constructions yields

$$\int_C M \longrightarrow \int_C S.$$

Passing to cores gives

$$\hat{q}: \Omega_M \rightarrow \Omega_S.$$

The observer-relative semantic state of a joint architecture–meme state is a point of the total semantic space  $\Omega_S$ .

## 28 Semantic population

The semantic population induced by  $\nu$  is

$$\sigma = \hat{q}_* \nu \in P(\Omega_S).$$

A central question is whether  $\sigma$  evolves autonomously.

## 29 Dynamic semantic lumpability

### Definition 29.1 Dynamic semantic lumpability

The joint evolutionary system is lumpable through  $\hat{q}: \Omega_M \rightarrow \Omega_S$  if there exists an induced operator  $R_S: P(\Omega_S) \rightarrow P(\Omega_S)$  such that

$$R_S \circ \hat{q}_* = \hat{q}_* \circ R.$$

## 30 Closure theorem

### Theorem 30.1 Exact semantic closure

Suppose fitness factors through the total semantic quotient,

$$w(x, \nu, e) = w_S(\hat{q}(x), \hat{q}_* \nu, e),$$

and the transition kernel is lumpable,

$$P(\hat{q}) \circ K_{\nu, e} \approx K_{\hat{q}_* \nu, e}^S \circ \hat{q}.$$

Then there exists a closed evolutionary operator  $R_S$  on semantic populations satisfying

$$\hat{q}_* R(\nu, e) = R_S(\hat{q}_* \nu, e).$$

**Proof.**

The first condition ensures that selection weights depend only on the coarse semantic state and the coarse population, so semantic pushforward commutes with selection. The second ensures that pushforward commutes with transition. Composing these two commutative operations gives the stated equation.

### 31 Failure of semantic closure

If two joint populations satisfy  $\hat{q}_* \nu = \hat{q}_* \rho$  but

$$\hat{q}_* R(\nu) \neq \hat{q}_* R(\rho),$$

then no autonomous first-order semantic dynamics exists on  $P(\Omega_S)$ . The observer has discarded information needed to predict semantic evolution.

### 32 Architecture as hidden state

A common source of nonclosure is failure to retain architecture. Suppose two agents currently assign the same semantic class to a meme,  $q_c(m) \approx q_d(n)$ . If  $c$  and  $d$  induce different future distinctions or transition laws, then the current semantic class alone is not sufficient to predict the next semantic state. Architecture functions as a hidden dynamical variable.

The same current semantic state need not have the same future semantic dynamics.

### 33 Minimal sufficient semantic state

One may seek an augmented quotient  $r: \Omega_M \rightarrow Z$  such that the intended semantic observable factors through  $r$  and the dynamics is lumpable through  $r$ . Then  $Z$  is a dynamically sufficient semantic state object:

$$P(\Omega_M) \longrightarrow P(Z) \longrightarrow P(Z)$$

closes. This generalizes the earlier principle: add hidden state until the dynamics becomes Markovian.

### 34 Finite architecture–meme model

Let cognitive architectures be indexed by  $\alpha = 1, \dots, r$  and memetic states by  $i = 1, \dots, n$ . Let  $x_{\alpha i}$  be the population fraction in the joint state  $(\alpha, i)$ . Then

$$x_{\alpha i} \geq 0, \quad \sum_{\alpha, i} x_{\alpha i} = 1.$$

### 35 Joint fitness matrix

Let  $w_{\alpha i}(x) > 0$  be endogenous fitness, and let  $Q_{\alpha i, \beta j}(x)$  be the probability that a reproducing state  $(\alpha, i)$  produces successor state  $(\beta, j)$ . Thus

$$Q_{\alpha i, \beta j} \geq 0, \quad \sum_{\beta, j} Q_{\alpha i, \beta j} = 1.$$

### 36 Finite coevolutionary replicator–mutator equation

#### Theorem 36.1 Finite joint update

The joint update is

$$x'_{\beta j} = \frac{\sum_{\alpha, i} x_{\alpha i} w_{\alpha i}(x) Q_{\alpha i, \beta j}(x)}{\sum_{\alpha, i} x_{\alpha i} w_{\alpha i}(x)}.$$

**Proof.**

This is the finite-state expression of Definition 16.1 on the state space  $\{1, \dots, r\} \times \{1, \dots, n\}$ .

### 37 Architecture marginal in the finite theory

Define  $a_\beta = \sum_j x_{\beta j}$ . Then

$$a'_\beta = \frac{\sum_{\alpha, i} x_{\alpha i} w_{\alpha i}(\sum_j Q_{\alpha i, \beta j})}{w}.$$

Architecture frequencies change by both differential fitness and transition into new architectures.

### 38 Meme marginal

If the same meme label  $i$  is meaningful across architectures, define  $m_j = \sum_\beta x_{\beta j}$ . Then

$$m'_j = \frac{\sum_{\alpha, i} x_{\alpha i} w_{\alpha i}(\sum_\beta Q_{\alpha i, \beta j})}{w}.$$

Neither marginal generally closes independently.

### 39 Marginal nonclosure theorem

#### Theorem 39.1 Marginals do not close

The architecture marginal  $a_t$  generally does not determine  $a_{t+1}$ , and the meme marginal  $m_t$  generally does not determine  $m_{t+1}$ .

**Proof.**

The formulas above depend on the full joint quantities  $x_{\alpha i}$ . Two joint distributions can have the same architecture marginal but different conditional meme distributions  $P(i \mid \alpha)$ . If fitness or transition depends on  $i$ , they produce different architecture updates. The dual argument applies to the meme marginal.

Architecture and meme must generally be modeled jointly.

### 40 Continuous-time coevolution

Let  $z = (\alpha, i)$  index joint states. Let  $f_z(x)$  be Malthusian fitness and  $L_{zz'}(x)$  a population-dependent Markov generator on joint states. Then

$$\dot{x}_{z'} = x_{z'}(f_{z'}(x) - \bar{f}(x)) + \sum_z x_z L_{zz'}(x).$$

Equivalently,

$$\dot{x}_{\beta j} = x_{\beta j}(f_{\beta j}(x) - \bar{f}(x)) + \sum_{\alpha, i} x_{\alpha i} L_{\alpha i, \beta j}(x).$$

#### 41 Conservation theorem

##### Theorem 41.1 Conservation of total probability

If  $\sum_{z'} L_{zz'}(x) = 0$ , then  $\sum_z x_z = 1$  is preserved.

**Proof.**

Summing the replicator term gives  $\bar{f} - \bar{f} = 0$ . Summing the transition term gives

$$\sum_z \sum_{z'} x_z L_{zz'}(x) = \sum_z x_z \sum_{z'} L_{zz'}(x) = 0.$$

Hence  $\frac{d}{dt} \sum_z x_z = 0$ .

#### 42 Decomposition of architecture change

Let  $\phi: \Omega_M \rightarrow R$  be any architecture observable depending only on  $c$ . The generalized Price equation gives

$$\Delta E[\phi] = \frac{\text{Cov}(w, \phi)}{\bar{w}} + \frac{E[w(K\phi - \phi)]}{\bar{w}}.$$

Architecture change decomposes into

Differential reproduction of architectures, plus within-lineage cognitive transformation.

#### 43 Decomposition of semantic change

For a semantic observable  $\psi: \Omega_S \rightarrow R$ , pull it back along  $\hat{q}$  and apply the same identity:

$$\Delta E[\psi \circ \hat{q}] = \frac{\text{Cov}(w, \psi \circ \hat{q})}{\bar{w}} + \frac{E[w(K(\psi \circ \hat{q}) - \psi \circ \hat{q})]}{\bar{w}}.$$

The second term now includes both ordinary memetic mutation and changes caused by altered cognitive architecture.

#### 44 Semantic mutation versus architectural reinterpretation

A change in semantic class may occur in at least two distinct ways.

**Memetic mutation.**  $(c, m) \rightarrow (c, m')$  with  $q_c(m) \neq q_c(m')$ .

**Cognitive reinterpretation.**  $(c, m) \rightarrow (d, T_{cd} m)$  with  $q_c(m) \neq q_d(T_{cd} m)$ , despite preservation of the underlying presentation. Semantic change need not require a change in the presentation itself.

## 45 Semantic conversion by architecture change

### Definition 45.1 Pure reinterpretation event

A transition is a pure reinterpretation event if the presentation is transported without substantive alteration but its semantic class changes:

$$m' \simeq T_{cd} m \quad \text{while} \quad [m]_c \neq [m']_d.$$

This is impossible in a theory with a fixed semantic quotient, but natural in the present framework.

## 46 Memetic niche construction

A population can alter the cognitive architectures that determine its future fitness landscape. This is a semantic analogue of niche construction. The loop is

$$\nu_t \rightarrow A_{\nu_t} \rightarrow a_{t+1} \rightarrow L_{a_{t+1}} \rightarrow Q_{t+1} \rightarrow w_{t+1} \rightarrow \nu_{t+1}.$$

Selection acts on a landscape partly constructed by earlier memetic evolution.

## 47 Semantic niche

### Definition 47.1 Semantic niche

The semantic niche of a joint population  $\nu$  is the architecture-weighted family of modalities

$$\{L_c : c \in \text{supp}(\pi_* \nu)\},$$

together with their frequencies and translation structure. A change in architecture distribution changes the semantic niche.

## 48 Reflexive selection

### Definition 48.1 Reflexive selection

Selection is reflexive when present memetic states influence future cognitive modalities in a manner that changes their own or their descendants' future fitness. Schematically,

$$m \rightarrow L' \rightarrow [m]_{L'} \rightarrow w'(m).$$

This creates a higher-order feedback loop:

State, observer, classification, fitness, and then state again.

## 49 Fixed semantic theory as a special case

Suppose  $A(c, m) = \delta_c$  for every  $(c, m)$ . Then architecture never changes. If the initial population is concentrated on one architecture  $c_0$ , then  $\nu_t$  remains supported on  $M(c_0)$ . The theory reduces to the endogenous memetic evolutionary system of Note V on the fixed semantic state space  $M(c_0)$ .

## 50 Architecture-only evolution as a special case

If every fiber  $M(c)$  is contractible, then no nontrivial memetic state remains. The total state space reduces to  $C^\infty$ , and the theory becomes ordinary evolutionary dynamics on cognitive architectures. The coupled theory contains both meme evolution and architecture evolution as limiting cases.

## 51 Joint stability

Let  $F(\nu) = R(\nu, e)$ . At equilibrium  $F(\nu_*) = \nu_*$ , perturbations satisfy

$$\delta \nu_{t+1} = DF_{\nu_*} \delta \nu_t.$$

Unlike fixed-architecture dynamics, the derivative contains architecture–meme cross terms. Schematically,

$$DF_{\nu_*} = \begin{pmatrix} D_{\text{meme} \rightarrow \text{meme}} & D_{\text{arch} \rightarrow \text{meme}} \\ D_{\text{meme} \rightarrow \text{arch}} & D_{\text{arch} \rightarrow \text{arch}} \end{pmatrix}.$$

The off-diagonal blocks encode the coevolutionary feedback.

## 52 Coevolutionary instability

Even if the meme subsystem and the architecture subsystem would each be stable when considered separately, the coupled system can be unstable because of the off-diagonal interaction terms. For a linearized block system

$$\begin{pmatrix} A & B \\ C & D \end{pmatrix} \begin{pmatrix} m \\ a \end{pmatrix},$$

stability depends on the spectrum of the full block operator, not merely the spectra of  $A$  and  $D$ . Feedback  $m \rightarrow a \rightarrow m$  can create new unstable modes.

## 53 Higher-order cognitive evolution

Architecture itself may be represented by a hierarchy

$$c^{(0)}, \quad c^{(1)}, \quad c^{(2)}, \quad \dots$$

where  $c^{(1)}$  controls transformations of  $c^{(0)}$ ,  $c^{(2)}$  controls transformations of  $c^{(1)}$ , and so forth. One may encode this by an iterated fibration  $C_{n+1} \rightarrow C_n$ . The total evolutionary state then lies in the iterated Grothendieck construction

$$\int_{C_n} \cdots \int_{C_1} M.$$

This is a formal route toward evolution of meta-cognitive architecture.

## 54 Self-modeling

Suppose some memetic states encode aspects of the architecture  $c$  itself. Then there exists a representation map

$$r_c: M(c) \rightarrow \text{Rep}(c),$$

where  $\text{Rep}(c)$  is an object of internal models of architecture  $c$ . A state may influence architecture through an internally represented model:

$$m \rightarrow r_c(m) \rightarrow c' \rightarrow M(c').$$

This is the natural categorical setting for self-model-mediated cognitive modification.

## 55 Self-referential semantic dynamics

The coupled structure may now take the form

$$M(c) \rightarrow \text{Rep}(c) \rightarrow C \rightarrow M(c') \rightarrow \text{Rep}(c') \rightarrow \cdots.$$

The state modifies the interpreter, and the modified interpreter changes the meaning assigned to later states. This should be distinguished from ordinary self-replication. It is semantic self-modification.

## 56 Main theorem

### Theorem 56.1 Cognitive–memetic coevolution theorem

Let  $(C, J)$  be a cognitive  $\infty$ -site,  $M: C^{\text{op}} \rightarrow S$  a memetic field,  $\Omega_M = (\int_C M)^\approx$ , and  $P$  a compatible probability monad. Suppose given

$$w: \Omega_M \times P(\Omega_M) \times E \rightarrow R_{>0}$$

and

$$K: \Omega_M \times P(\Omega_M) \times E \rightarrow P(\Omega_M).$$

Then:

- **(i)** The normalized operator of Definition 16.1 defines a joint evolutionary dynamics of memes and cognitive architectures.
- **(ii)** Its equilibria satisfy  $L_{\nu_*} \nu_* = \lambda \nu_*$ . See Theorem 18.2.
- **(iii)** The architecture marginal evolves by the pushforward of the joint selection–transition process. See Theorem 19.1.
- **(iv)** Neither architecture nor meme marginals generally form closed dynamical systems. See Theorem 39.1.
- **(v)** Memetically dependent architecture kernels permit memes to change future cognitive modalities. See Definition 8.2 and Definition 22.1.
- **(vi)** Nonconservative architecture changes permit semantic equivalence classes to split or merge. See Sections 5–7.
- **(vii)** Semantic identity is invariant along trajectories consisting entirely of conservative architecture transitions. See Theorem 25.1.

- **(viii)** The total semantic quotient  $\hat{q}: \Omega_M \rightarrow \Omega_S$  inherits an autonomous evolutionary dynamics exactly when the joint fitness and transition structure are lumpable through  $\hat{q}$ . See Theorem 30.1.
- **(ix)** Failure of semantic lumpability produces hidden-state dependence and generally non-Markovian coarse semantics. See Section 31.
- **(x)** The fixed-architecture theory is recovered when the architecture transition kernel is concentrated on the identity. See Section 49.

**Proof.**

Statements (i)–(x) follow from Sections 16–18, Theorem 19.1, Theorem 39.1, Definitions 8.2 and 22.1, Sections 5–7, Theorem 25.1, Theorem 30.1, Section 31, and Section 49.

## 57 Full formal specification

The coevolutionary theory is specified by

$(C, J, M, L, \Omega_M, P, E, w, K)$ .

- $C$ : Cognitive architecture  $\infty$ -category.
- $J$ : Cognitive topology.
- $M$ : Architecture-indexed memetic field.
- $L$ : Architecture-indexed semantic modalities.
- $\Omega_M$ : Joint architecture–meme state space.
- $P$ : Probability monad.
- $E$ : Environment.
- $w$ : Joint endogenous fitness.
- $K$ : Joint stochastic transition kernel.

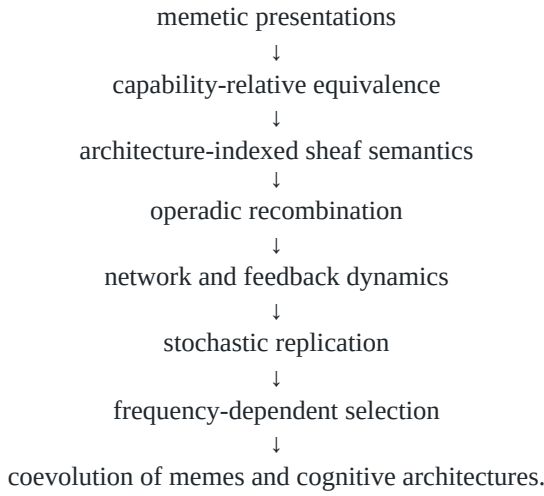
The population equation is

$$\nu_{t+1} = \frac{L_{\nu_t, e_t} \nu_t}{L_{\nu_t, e_t} \nu_t (\Omega_M)}.$$

The semantic state of an individual is  $q_{c_t}(m_t)$ . The architecture marginal is  $\pi_* \nu_t$ .

## 58 Conceptual hierarchy

The complete framework developed through the successive extensions is now



The final step changes the logical status of semantic identity itself. Previously  $\approx_c$  was fixed while states evolved. Now  $c = c_t$  also evolves, so  $\approx_{c_t}$  is time dependent.

## 59 Final principle

The strongest formulation is therefore not that memes evolve inside cognition, but that

Memes and the cognitive semantics that classify them coevolve.

A complete state is  $(c, m)$ , not merely  $m$ . Its semantic interpretation is  $q_c(m)$ . Its reproduction may produce  $(c', m')$ , and therefore change both  $m \mapsto m'$  and  $q_c \mapsto q_{c'}$ .

The full causal loop is

$$(c_t, m_t) \longrightarrow (c_{t+1}, m_{t+1}) \longrightarrow q_{c_{t+1}}(m_{t+1}) \longrightarrow w_{t+1} \longrightarrow (c_{t+2}, m_{t+2}).$$

Memetic replication is evaluated relative to a semantic modality that can itself be modified by the replicating population. The resulting theory is a dynamics not merely of states, but of states together with their changing equivalence relations: dynamics of objects, dynamics of observers, and dynamics of identity.

## Part 7: Evolution of the Semantic Universe

### Abstract

We extend capability-relative memetic replication beyond evolution inside a fixed cognitive site. The category of cognitive architectures, its translation morphisms, and its Grothendieck topology are now themselves dynamical.

At time  $t$ , let

$$\Sigma_t = (C_t, J_t)$$

be a cognitive  $\infty$ -site and let

$$E_t = \text{Sh}_\infty(C_t, J_t)$$

be its semantic  $\infty$ -topos. A transition of semantic universes is a left-exact colimit-preserving functor

$$F_{t,t'} : E_t \longrightarrow E_{t'},$$

or, more generally, a correspondence of  $\infty$ -topoi. Memetic objects are sections of an indexed family of semantic universes rather than objects of one fixed  $\infty$ -topos.

We distinguish four phenomena: change of site presentation without semantic change; refinement of the Grothendieck topology; creation or deletion of cognitive architectures and translations; and genuine change of the semantic  $\infty$ -topos. Left-exact transport preserves identity witnesses, conservative transport preserves and reflects semantic identity, refinement of a topology induces a left-exact localization, and equivalent site presentations yield equivalent semantic theories.

Stochastic evolution across changing semantic spaces uses a variable-space mutation kernel

$$K_t : S_t \longrightarrow P_{t+1}(S_{t+1}),$$

and the non-autonomous replicator–mutator equation

$$\nu_{t+1} = (K_t)_* \left( \frac{w_t \nu_t}{\int_{S_t} w_t d\nu_t} \right).$$

Finally, the semantic universe itself may be selected or transformed by its resident memes. A total state is a point of a Grothendieck construction over an  $\infty$ -category of admissible semantic universes. The resulting hierarchy is

State evolution, inside observer evolution, inside identity evolution, inside semantic-universe evolution.

The preceding fixed-universe theory is Note VI.

## Notation

- $(C_t, J_t)$  — The cognitive site at stage  $t$ . Objects, translations, and covers may all change.
- $E_t$  — The semantic  $\infty$ -topos of that site.
- $F_{t,t'}$  — Left-exact transport of semantic objects from one universe into the next.
- $K_t$  — A mutation kernel whose target lives in the next semantic space.
- $U$  — The  $\infty$ -category of admissible semantic universes.
- $(u, s)$  — A universe together with a semantic state inside it.

## 1 Why the site must itself become dynamical

In the preceding theory, one fixed cognitive site  $(C, J)$  determined one semantic universe  $\text{Sh}_\infty(C, J)$ . Cognitive architectures could evolve inside  $C$ , but three things were held fixed: the objects of  $C$ , the admissible translations  $\text{Map}_C(c, d)$ , and the Grothendieck topology  $J$ . We now allow all three to vary. At time  $t$ , write

$$\Sigma_t = (C_t, J_t).$$

Semantic evolution can alter objects, morphisms, covers, and sheaves, and therefore the ambient notion of semantic object itself.

## 2 Time as an indexing $\infty$ -category

Let  $T$  be an  $\infty$ -category of times or evolutionary stages. For discrete time one may take  $T = N$ , regarded as a category with a unique arrow  $n \rightarrow m$  whenever  $n \leq m$ . More general choices permit branching timelines, partially ordered stages, or higher homotopies between evolutionary histories.

## 3 Dynamical cognitive sites

### Definition 3.1 Time-indexed cognitive site

A time-indexed cognitive site is a family

$$(C_t, J_t), \quad t \in T.$$

The associated semantic universe is

$$E_t = \text{Sh}_\infty(C_t, J_t).$$

A raw family is not yet a dynamics. Transitions between semantic universes must still be specified.

## 4 Invariant semantic-universe transitions

Because a single  $\infty$ -topos can have many inequivalent-looking site presentations, the evolution of sites should be formulated at the level of  $\infty$ -topoi.

### Definition 4.1 Forward semantic transport

A forward semantic transport  $t \rightarrow t'$  is a colimit-preserving left-exact functor

$$F_{t,t'}: E_t \longrightarrow E_{t'}.$$

Left exactness means that  $F_{t,t'}$  preserves finite limits. Such a functor may be regarded as the inverse-image part of a geometric morphism, with the displayed direction following evolutionary time.

### Definition 4.2 Semantic-universe evolution

A semantic-universe evolution is a functorial assignment

$$E: T \longrightarrow \text{Pr}_{\text{lex}}^L$$

with  $E(t) = E_t$  an  $\infty$ -topos. For a composable pair  $t \rightarrow t' \rightarrow t''$ ,

$$F_{t',t''} \circ F_{t,t'} \simeq F_{t,t''}.$$

## 5 Dynamic semantic objects

A memetic object can no longer be one fixed  $M \in E$ . It must vary with the semantic universe.

### Definition 5.1 Dynamical semantic object

A dynamical semantic object consists of objects  $M_t \in E_t$  together with coherent transition morphisms

$$\theta_{t,t'}: F_{t,t'} M_t \longrightarrow M_{t'}.$$

If every  $\theta_{t,t'}$  is an equivalence, the family is transport-cartesian. The family  $t \mapsto E_t$  determines an indexed  $\infty$ -category

$$p: \int_T E \longrightarrow T.$$

A transport-cartesian semantic object is a section of this fibration.

## 6 Identity transport

Let  $x, y: 1 \rightarrow X$  be points of  $X \in E_t$ . Their identity object is the pullback

$$\text{Id}_X(x, y) \simeq 1 \times_{X \times X} X,$$

along the diagonal.

### Theorem 6.1 Left-exact transport preserves identity objects

Let  $F: E_t \rightarrow E_{t'}$  be left exact. Then

$$F \text{Id}_X(x, y) \simeq \text{Id}_{FX}(F x, F y).$$

**Proof.**

The identity object is a finite-limit construction from the terminal object, the product  $X \times X$ , and the diagonal. Since  $F$  preserves terminal objects, products, and pullbacks,

$$F(1 \times_{X \times X} X) \simeq 1 \times_{FX \times FX} F X,$$

which is  $\text{Id}_{FX}(F x, F y)$ .

### Corollary 6.2 Forward persistence of equivalence

If  $x \simeq y$  in  $E_t$ , then  $F x \simeq F y$  in  $E_{t'}$ . Legitimate semantic transport cannot destroy an already supplied identity witness. It may, however, identify distinctions that were previously visible.

## 7 Conservative evolution

### Definition 7.1 Conservative semantic transport

A functor  $F: E \rightarrow F$  is conservative if  $F(f)$  an equivalence implies that  $f$  is an equivalence.

### Theorem 7.2 Conservative transport preserves and reflects identity

If  $F$  is left exact and conservative, then

$$x \simeq y \iff Fx \simeq Fy.$$

**Proof.**

The forward implication is Corollary 6.2. Conversely, if  $Fx \simeq Fy$ , then the image of the identity object is inhabited. By Theorem 6.1,

$$F\text{Id}_X(x, y) \simeq \text{Id}_{FX}(Fx, Fy).$$

Conservativity reflects the corresponding path. Hence  $x \simeq y$ .

Failure of conservativity is the mechanism by which semantic distinctions disappear.

## 8 Presentation change versus genuine universe change

A different site presentation does not necessarily produce a different semantic universe.

### Definition 8.1 Presentational change

A transition  $(C, J) \rightsquigarrow (D, K)$  is merely presentational if

$$\text{Sh}_\infty(C, J) \simeq \text{Sh}_\infty(D, K).$$

### Definition 8.2 Genuine semantic-universe change

A site transition is genuine when the associated  $\infty$ -topoi are not equivalent.

### Theorem 8.3 Invariance under equivalent site presentation

Suppose  $\Phi$  is an equivalence of the two sheaf  $\infty$ -topoi. Then every semantic construction expressible internally using finite limits, colimits, mapping objects, identity types, and equivalence-invariant structure transports uniquely up to equivalence across  $\Phi$ .

**Proof.**

An equivalence of  $\infty$ -categories preserves and reflects limits, colimits, mapping spaces, and equivalences. Constructions made from these are invariant.

Changing coordinates on the semantic universe is not the same as changing the semantic universe.

## 9 Evolution of the Grothendieck topology

One way to change a semantic universe without changing the architecture category is to modify which families count as covers. Let  $J \subseteq J'$  be Grothendieck topologies on the same  $\infty$ -category  $C$ , with  $J'$  finer than  $J$ .

### Theorem 9.1 Topology refinement induces a left-exact localization

There is a reflective inclusion

$$\mathrm{Sh}_\infty(C, J') \hookrightarrow \mathrm{Sh}_\infty(C, J)$$

whose left adjoint is  $J'$ -sheafification

$$L_{J'} : \mathrm{Sh}_\infty(C, J) \rightarrow \mathrm{Sh}_\infty(C, J').$$

Moreover  $L_{J'}$  is left exact.

**Proof.**

Every  $J'$ -sheaf satisfies the descent conditions of  $J$ , since  $J \subseteq J'$ . The inclusion is fully faithful, and sheafification supplies the reflector. Sheafification of an  $\infty$ -topos is an accessible left-exact localization.

## 10 Interpretation of topology refinement

Adding covers imposes additional local-to-global compatibility equations. Thus

$$J \subseteq J'$$

produces

$$E_J \xrightarrow{L_{J'}} E_{J'}.$$

A semantic object that fails the new descent conditions is replaced by its closest  $J'$ -local approximation. Evolution of  $J$  can change which distributed representations count as legitimate global semantic objects.

## 11 Descent constraints as semantic laws

Suppose  $\{U_i \rightarrow U\}$  becomes a cover only after the transition  $J \rightsquigarrow J'$ . Then a formerly unconstrained semantic object  $X$  must after refinement satisfy

$$X(U) \simeq \mathrm{Tot} X(U.).$$

Evolution of the topology can create new equations among local cognitive perspectives.

New coverage imposes new semantic coherence laws.

## 12 Creation of new architectures

Changing  $C$  modifies the possible observer types. Suppose  $i: C_t \rightarrow C_{t+1}$  describes surviving architectures.

### Definition 12.1 Architectural birth

An object  $d \in C_{t+1}$  is new relative to  $i$  if it is not equivalent to any object in the essential image of  $i$ :

$$d \notin \mathrm{EssIm}(i).$$

New architectures can enlarge the space of possible semantic resolutions.

### 13 New translation morphisms

Even if the object set does not materially change, new morphisms may appear. Suppose  $c$  and  $d$  persist. If

$$\text{Map}_{C_t}(c, d) \rightarrow \text{Map}_{C_{t+1}}(c, d)$$

fails to be essentially surjective, then new translation procedures exist at time  $t + 1$ . Semantic-universe evolution can create new architectures, or merely new ways of translating among old ones. The latter matters because descent depends on morphisms, not only objects.

### 14 Why ordinary functors do not model deletion symmetrically

A forward functor  $C_t \rightarrow C_{t+1}$  models persistence and the creation of new target objects, but it does not symmetrically encode architectures that simply cease to participate. The more general formalism is a correspondence.

### 15 Site-transition correspondences

#### Definition 15.1 Architectural correspondence

A transition between architecture categories is a span

$$C_t \xleftarrow{p} R_t \xrightarrow{q} C_{t+1}.$$

The intermediate category  $R_t$  represents architecture continuations. An architecture  $c \in C_t$  is represented across the transition only if it lies in the essential image of  $p$ , and a later architecture has a predecessor only if it lies in the essential image of  $q$ .

#### Definition 15.2 Architectural death

An architecture  $c \in C_t$  dies relative to the correspondence when  $c \notin \text{EssIm}(p)$ .

#### Definition 15.3 Architectural birth, symmetrically

An architecture  $d \in C_{t+1}$  is born when  $d \notin \text{EssIm}(q)$ . Creation and deletion are then symmetric.

### 16 Semantic-universe correspondences

The same construction lifts from sites to their  $\infty$ -topoi. A general semantic transition may be

$$E_t \xleftarrow{p} X_t \xrightarrow{q} E_{t+1}.$$

This is an object of a correspondence  $(\infty, 2)$ -category of  $\infty$ -topoi. Such correspondences are useful when there is no preferred global functor  $E_t \rightarrow E_{t+1}$ . They represent partial translation, shared semantic subuniverses, or many-to-many reconstruction.

### 17 Composition of semantic correspondences

Given spans through  $E_1$ , their composite is the homotopy pullback

$$X_{02} = X_{01} \times_{E_1} X_{12}.$$

This produces a span from  $E_0$  to  $E_2$ . Associativity holds up to the coherent equivalence of homotopy pullbacks. Semantic-universe histories live in a higher category of correspondences.

## 18 Dynamic descent

Suppose  $M_t \in E_t$  satisfies descent for a cover  $U_{\cdot,t} \rightarrow U_t$ , and  $F_{t,t'}$  is left exact and colimit-preserving. If the image diagram remains a cover, descent transports.

### Theorem 18.1 Transport of descent

Assume

$$M_t(U_t) \simeq \text{Tot} M_t(U_{\cdot,t}).$$

Then

$$F_{t,t'} M_t(U_t) \simeq \text{Tot} F_{t,t'} M_t(U_{\cdot,t}),$$

provided  $F_{t,t'}$  preserves the relevant totalization.

**Proof.**

Apply  $F_{t,t'}$  to the descent equivalence. If it preserves the limit defining the totalization, it passes inside Tot.

For finite Čech descent, left exactness suffices for the finite-limit stages. Hyperdescent may require stronger limit preservation.

## 19 Semantic novelty

### Definition 19.1 Semantic novelty

Let  $F: E_t \rightarrow E_{t+1}$  be semantic transport. An object  $X \in E_{t+1}$  is novel relative to  $F$  if

$$X \notin \text{EssIm}(F).$$

This is stronger than ordinary memetic novelty. Ordinary novelty is a new point inside the same universe. Semantic-universe novelty is an object not equivalent to the transport of any earlier semantic object.

## 20 Novel distinctions

A later universe may possess observables unavailable earlier. Let  $X, Y \in E_t$ . Suppose their images are distinguished by a new observable

$$q: F(X) \sqcup F(Y) \rightarrow Z$$

that does not descend from any corresponding observable at time  $t$ . The later universe has acquired a new semantic distinction: a structural increase in expressive power, not merely a new data value.

## 21 Semantic collapse

### Definition 21.1 Semantic collapse

A semantic transport  $F: E_t \rightarrow E_{t+1}$  exhibits collapse if it is not conservative. Equivalently, there exists a non-equivalence  $f: X \rightarrow Y$  such that  $F(f)$  is an equivalence. The transition is a loss of semantic distinctions.

## 22 Semantic phase transition

### Definition 22.1 Semantic phase transition

A transition  $E_t \rightarrow E_{t+1}$  is a semantic phase transition when at least one of the following occurs:

- the two  $\infty$ -topoi cease to be equivalent;
- the transition is nonconservative;
- new semantic objects appear outside the essential image;
- the topology introduces genuinely new descent constraints;
- the architecture category acquires objects or translations that alter semantic reconstruction.

The terminology is structural. It denotes a change of semantic organization, not a thermodynamic claim.

## 23 Dynamic identity

Let  $M_t \in E_t$  be a dynamical memetic object and let  $x_t, y_t \in M_t$ . The identity space  $\text{Id}_{M_t}(x_t, y_t)$  now varies in a varying  $\infty$ -topos.

### Theorem 23.1 Identity persistence under conservative universe evolution

Suppose every transition  $F_{t,t+1}$  is left exact and conservative, and the memetic states are transported compatibly. Then  $x_t \simeq y_t$  if and only if  $x_{t+1} \simeq y_{t+1}$ .

**Proof.**

Left exactness preserves identity objects by Theorem 6.1. Conservativity reflects equivalence by Theorem 7.2. Compatibility identifies  $x_{t+1} \simeq F_{t,t+1}x_t$  and  $y_{t+1} \simeq F_{t,t+1}y_t$ .

### Corollary 23.2 When identity can change

A change in whether two persistent states count as identical requires a nonconservative semantic transition, a failure of compatible transport, or a change of the semantic objects being compared.

## 24 Variable semantic state spaces

Let  $S_t \in E_t$  be the observer-relative semantic state object at time  $t$ . A population is

$$\nu_t \in P_t(S_t),$$

where the probability structure may now vary with the semantic universe.

## 25 Compatible probability structure

### Definition 25.1 Dynamic probability monad

A dynamic probability structure consists of probability monads

$$P_t : E_t \rightarrow E_t$$

and coherent comparison transformations

$$F_{t,t'} P_t \Longrightarrow P_{t'} F_{t,t'}.$$

When these are equivalences, probability transport commutes with semantic transport. The hypothesis is not automatic. It is additional structure.

## 26 Mutation across different semantic universes

### Definition 26.1 Variable-space mutation kernel

A one-step stochastic semantic transition is a kernel

$$K_t : S_t \longrightarrow P_{t+1}(S_{t+1}).$$

For  $s \in S_t$ , the distribution  $K_t(s)$  lives on the next semantic universe. This is strictly more general than an endomorphism  $S \rightarrow P(S)$ .

## 27 Variable-space selection

Let  $w_t : S_t \rightarrow R_{>0}$  be the current fitness field. Define

$$\bar{w}_t = \int_{S_t} w_t(s) d\nu_t(s).$$

Selection yields

$$d\nu_t(s) = \frac{w_t(s)}{\bar{w}_t} d\nu_t(s).$$

## 28 Non-autonomous replicator–mutator equation

### Definition 28.1 Population update

The population update across changing semantic universes is

$$\nu_{t+1} = (K_t)_* \nu_t,$$

equivalently

$$\nu_{t+1} = (K_t)_* \left( \frac{w_t \nu_t}{\int_{S_t} w_t d\nu_t} \right).$$

### Theorem 28.2 Normalization

If  $\nu_t$  is a probability state and  $K_t(s)$  is normalized for every  $s$ , then  $\nu_{t+1}$  is normalized.

**Proof.**

Selection preserves total mass, so  $\nu_t(S_t) = 1$ . Then

$$\begin{aligned} \nu_{t+1}(S_{t+1}) &= \int_{S_t} K_t(s, S_{t+1}) d\nu_t(s) \\ &= \int_{S_t} 1 d\nu_t(s) \\ &= 1. \end{aligned}$$

## 29 Non-autonomous transfer operators

Define

$$(L_t \nu)(A) = \int_{S_t} w_t(s) K_t(s, A) d\nu(s).$$

Then  $L_t$  maps positive measures on  $S_t$  to positive measures on  $S_{t+1}$ . The normalized update is

$$\nu_{t+1} = \frac{L_t \nu_t}{L_t \nu_t(S_{t+1})}.$$

There is no single fixed transfer operator. One has a cocycle  $L_{t+n-1} \cdots L_t$ .

## 30 Iterated variable-universe evolution

### Proposition 30.1 Projective iteration

Ignoring intermediate normalization factors, the state after  $n$  transitions is

$$\nu_{t+n} = \frac{L_{t+n-1} \cdots L_t \nu_t}{(L_{t+n-1} \cdots L_t \nu_t)(S_{t+n})}.$$

**Proof.**

The scalar normalization factors cancel under projectivization, as in the fixed-space theory. Induct on  $n$ .

Evolutionary dynamics on changing semantic spaces is projective dynamics generated by a sequence of positive operators between different cones.

### 31 No fixed equilibrium in general

When  $S_t$  changes with time, the equation  $\nu_* = R(\nu_*)$  is not even type-correct unless the semantic universes repeat or are canonically identified. The appropriate invariant is not necessarily a fixed point.

#### Definition 31.1 Evolutionary coherent family

A family  $\{\nu_t\}$  is evolutionarily coherent if  $\nu_{t+1} = R_t(\nu_t)$  for every  $t$ . For periodic universe evolution one can seek periodic population sections. For stationary universe evolution, ordinary equilibria are recovered.

### 32 The space of semantic universes

Semantic-universe evolution can itself be endogenous. Let  $U$  be an  $\infty$ -category of admissible cognitive semantic universes. A point  $u \in U$  may encode  $(C_u, J_u, E_u)$  together with the probability, modality, and memetic structures required.

### 33 Indexed semantic universes

Suppose we have a functor

$$E: U \longrightarrow Pr_{\text{lex}}^L$$

with  $u \mapsto E_u$ . Choose in each universe a semantic state object  $S_u \in E_u$ . These form an indexed semantic family.

### 34 Total universe–semantic state space

Straightening determines a total  $\infty$ -category  $\int_U S$ . Define

$$\Omega = (\int_U S)^\approx.$$

A point is a pair  $(u, s)$ , a semantic universe together with a semantic state inside that universe.

### 35 Universe projection

There is a projection

$$\pi_U: \Omega \rightarrow U^\approx.$$

Every population on  $\Omega$  has a universe marginal  $\pi_{U*} \nu$ . The system can contain a distribution over different semantic universes.

### 36 Universe-changing kernels

#### Definition 36.1 Universe transition kernel

A stochastic universe transition is

$$A: \Omega \times P(\Omega) \times E \longrightarrow P(U^\approx).$$

The probability of moving to a new semantic universe can depend on the current universe, the current semantic state, the total population, and the external environment.

### 37 Full universe–semantic kernel

The invariant dynamics is represented by

$$K: \Omega \times P(\Omega) \times E \longrightarrow P(\Omega).$$

Its universe marginal is  $P(\pi_U) \circ K$ . One transition can simultaneously change  $u \mapsto u'$  and  $s \mapsto s' \in S_{u'}$ .

### 38 Universe-dependent fitness

Let

$$w: \Omega \times P(\Omega) \times E \rightarrow R_{>0}.$$

Then  $w((u, s), \nu, e)$  measures the reproductive value of state  $s$  as represented inside universe  $u$ .

### 39 Semantic-universe coevolution operator

Define

$$w(\nu, e) = \int_{\Omega} w(x, \nu, e) d\nu(x).$$

Then

$$R(\nu, e) = (K_{\nu, e})_* \left( \frac{w(-, \nu, e) \nu}{w(\nu, e)} \right).$$

This is formally the joint replicator–mutator operator, but the state coordinate now includes the semantic universe itself.

### 40 Universe selection

The marginal distribution on semantic universes evolves according to

$$a'(B) = \frac{\int_{\Omega} w(x, \nu, e) A_{\nu, e}(x, B) d\nu(x)}{w(\nu, e)}.$$

Semantic universes can spread, disappear, or transform under selection and transition.

### 41 Memes that alter their semantic universe

#### Definition 41.1 Universe-reflexive meme

A semantic state  $s \in S_u$  is universe-reflexive if the distribution of future universes depends nontrivially on  $s$ : there exist  $s, s' \in S_u$  with

$$A((u, s), -) \neq A((u, s'), -).$$

Such a state influences not merely an observer architecture inside one semantic universe, but the future semantic universe itself.

## 42 Strong semantic reflexivity

### Definition 42.1 Strongly universe-reflexive state

A state is strongly universe-reflexive if it can induce a transition  $u \rightarrow u'$  for which  $E_u \neq E_{u'}$ . Its propagation can change the ambient  $\infty$ -topos. The causal loop is

$$s \rightarrow u' \rightarrow E_{u'} \rightarrow \text{new semantic objects} \rightarrow \text{new descendants}.$$

## 43 Evolution of the coverage relation

A particularly important universe-reflexive transition changes only  $J_u$  while leaving the underlying architecture category fixed. Suppose  $J_u \subsetneq J_{u'}$ . Then  $E_{u'}$  is a left-exact localization of  $E_u$ . A resident semantic population can therefore induce new descent requirements.

Memetic activity can change what counts as sufficient agreement among observers.

## 44 Evolution of translation structure

Another transition changes  $\text{Map}_{C_u}(c, d)$ . A newly created translation may cause previously disconnected semantic regions to participate in common descent diagrams. New communication capability can change global semantic structure even when local observer objects remain unchanged.

## 45 Emergent global semantic objects

Suppose at time  $t$  the architectures  $c_1$  and  $c_2$  have no relevant common cover. At  $t + 1$ , new morphisms produce a cover

$$\{c_1 \rightarrow c, c_2 \rightarrow c\}.$$

Compatible local states  $x_1 \in M(c_1)$  and  $x_2 \in M(c_2)$  may now descend to a global object  $x \in M(c)$ . Evolution of the site can make a new distributed semantic object globally definable.

## 46 Loss of global semantic objects

Conversely, changes in topology or architecture may destroy the compatibility conditions required for a previous global representation. A formerly meaningful global section may cease to admit transport into the new semantic universe. Universe evolution can produce both semantic emergence and semantic extinction.

## 47 Semantic extinction

### Definition 47.1 Semantic extinction

Let  $F: E_t \rightarrow E_{t+1}$  be the available transport. A semantic object  $X \in E_t$  is extinct relative to the transition when no admissible successor in the specified dynamical family is equivalent to  $F(X)$ , or when the transition is represented only by a correspondence whose support contains no continuation of  $X$ . This is stronger than population extinction. The semantic type itself ceases to have a continuation.

## 48 Creation of semantic dimensions

A new architecture or observable may introduce a semantic coordinate unavailable at earlier times. Model this by a map  $S_{t+1} \rightarrow S_t$  whose homotopy fibers are noncontractible. Then multiple new semantic states project to one old state.

### Definition 48.1 Semantic dimensional refinement

A transition  $r: S_{t+1} \rightarrow S_t$  is a strict semantic refinement if some homotopy fiber of  $r$  contains more than one connected component. This is the universe-level analogue of capability refinement.

## 49 Creation of new homotopy structure

The refinement need not occur only at the level of connected components. A semantic state may acquire nontrivial  $\pi_1, \pi_2$ , and higher information. Universe evolution can alter the relevant homotopy  $n$ -type of semantic objects, gaining or losing higher coherence.

## 50 Dynamic Postnikov resolution

A simple example is a time-dependent truncation level  $n_t$ . Let  $E_t$  retain semantic information only through  $\tau_{\leq n_t}$ . If  $n_{t+1} > n_t$ , new higher identity structure becomes visible. If  $n_{t+1} < n_t$ , previously visible higher distinctions collapse. The capability hierarchy is a one-dimensional special case of evolving semantic universes.

## 51 Path dependence

When the universe itself changes, the final semantic state may depend on the path. Suppose one history is  $u_0 \rightarrow u_1 \rightarrow u_3$  and another is  $u_0 \rightarrow u_2 \rightarrow u_3$ . The composites

$$F_{13} \circ F_{01} \quad \text{and} \quad F_{23} \circ F_{02}$$

need not be equivalent unless the evolution diagram contains a specified coherent 2-cell between them. Semantic evolution is generally path dependent.

## 52 Coherent path independence

### Definition 52.1 Path coherence

A semantic-universe diagram is path coherent on a square when there is a specified equivalence

$$F_{13} \circ F_{01} \simeq F_{23} \circ F_{02}.$$

Higher-dimensional diagrams require higher coherence data. This is another reason the natural ambient structure is higher categorical.

### 53 Hysteresis

If different paths through semantic-universe space produce non-equivalent transports between the same endpoints, the system exhibits semantic hysteresis:

$$F_\gamma \neq F_{\gamma'}$$

for two paths  $\gamma, \gamma'$  from a common start to a common end  $\nu$ . Semantic history cannot be reconstructed from the endpoint universe alone.

### 54 Universe-level hidden state

If an observer sees only a coarse semantic object  $s_t$  but not the current semantic universe  $u_t$ , the observed dynamics can be non-Markovian even when  $(u_t, s_t)$  is Markovian. The semantic universe can itself function as latent state.

### 55 Universe lumpability

Let  $q: \Omega \rightarrow Z$  be a coarse observable that forgets some or all universe information.

#### Definition 55.1 Universe lumpability

The full dynamics is lumpable through  $q$  when there exists  $R_Z: P(Z) \rightarrow P(Z)$  such that

$$R_Z \circ q_* = q_* \circ R.$$

If this fails, the forgotten semantic-universe data affects future observable evolution.

### 56 Universe-level Price equation

Let  $\phi: \Omega \rightarrow R$  be an observable of the semantic universe, such as a numerical invariant of complexity or of the number of architecture types. Then

$$\Delta E[\phi] = \frac{\text{Cov}(w, \phi)}{\bar{w}} + \frac{E[w(K\phi - \phi)]}{\bar{w}}.$$

The first term measures selection among semantic universes. The second measures direct transformation of semantic universes within lineages.

### 57 Endogenous generation of semantic universes

The kernel  $K$  may assign positive probability to universe states  $u'$  not previously represented in the support of the population. Then

$$\text{supp}(a_{t+1}) \not\subseteq \text{supp}(a_t).$$

Genuinely new semantic universes can enter the evolutionary process.

## 58 Semantic innovation rate

Given the current support  $U_t = \text{supp}(a_t)$ , the probability of semantic-universe innovation is

$$a_{t+1}(U^\approx \setminus U_t).$$

In continuous settings this requires a measurable notion of previous support. The quantity measures population flow into previously unoccupied semantic universes.

## 59 Fixed-universe theory as a special case

Suppose the universe transition kernel is  $\delta_u$ . Then  $u_t = u_0$  almost surely, and the theory reduces to memetic and cognitive evolution inside the fixed  $\infty$ -topos  $E_{u_0}$ .

## 60 Fixed-site but evolving architecture theory

If the semantic universe  $u$  is fixed but the state object includes an architecture coordinate  $c \in C_u$ , we recover the cognitive–memetic coevolution theory of Note VI. The inclusions of generality are

$$\begin{array}{c} \text{fixed memetic state space} \\ \subset \\ \text{evolving cognitive architecture} \\ \subset \\ \text{evolving semantic universe.} \end{array}$$

## 61 Four distinct notions of change

The theory now distinguishes four categorically distinct phenomena.

- **State change.**  $m \rightarrow m'$  inside one semantic state space.
- **Observer change.**  $c \rightarrow c'$  inside one architecture category.
- **Semantic-rule change.**  $J \rightarrow J'$ , or a change of translation morphisms.
- **Universe change.**  $E \rightarrow E'$  with  $E \neq E'$ .

## 62 Main semantic-universe evolution theorem

### Theorem 62.1 Evolution of semantic universes

Let  $T$  be an evolutionary indexing  $\infty$ -category and let  $E: T \rightarrow Pr_{\text{lex}}^L$  be a diagram of semantic  $\infty$ -topoi. Then:

- **(i)** A dynamical semantic object is a section of the associated indexed  $\infty$ -category. See Section 5.
- **(ii)** Left-exact transport preserves identity objects. See Theorem 6.1.
- **(iii)** Conservative transport preserves and reflects semantic equivalence. See Theorem 7.2.
- **(iv)** Equivalent site presentations define equivalent semantic theories. See Theorem 8.3.
- **(v)** Refinement of a Grothendieck topology induces a left-exact localization. See Theorem 9.1.

- **(vi)** New objects or morphisms in the architecture category can create new local perspectives and new descent diagrams. See Sections 12–13.
- **(vii)** Site correspondences model architectural birth and death symmetrically. See Section 15.
- **(viii)** Nonconservative transport represents loss of semantic distinctions. See Section 21.
- **(ix)** Objects outside the essential image of transport represent semantic novelty. See Definition 19.1.
- **(x)** Nontrivial changes in topology, architecture, or topos can change the space of globally realizable semantic objects. See Sections 11, 18, and 45–46.

**Proof.**

Statements (i)–(x) are established in Sections 5–21, as cited.

### 63 Main evolutionary theorem

#### Theorem 63.1 Variable-universe replicator–mutator theorem

Let  $S_t \in E_t$  be semantic state objects with compatible probability structures,  $w_t : S_t \rightarrow R_{>0}$ , and  $K_t : S_t \rightarrow P_{t+1}(S_{t+1})$ . Then

$$\nu_{t+1} = (K_t)_* \left( \frac{w_t \nu_t}{\int_{S_t} w_t d\nu_t} \right)$$

defines a normalized population evolution between varying semantic spaces. If  $L_t$  is the corresponding positive transfer operator, then

$$\nu_{t+n} = \frac{L_{t+n-1} \cdots L_t \nu_t}{(L_{t+n-1} \cdots L_t \nu_t)(S_{t+n})}.$$

**Proof.**

Normalization is Theorem 28.2. The iterated expression is Proposition 30.1.

### 64 Main reflexivity theorem

#### Theorem 64.1 Universe-reflexive memetic dynamics

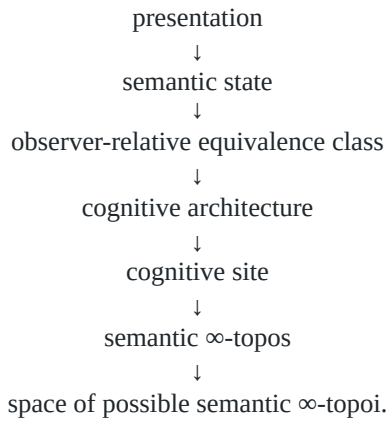
Let  $U$  be an  $\infty$ -category of semantic universes and  $\Omega = (\int_U S)^\approx$ . Suppose  $K$  and  $w$  are as in Sections 37–38. Then the normalized operator of Section 39 simultaneously evolves semantic universes, cognitive architectures, semantic states, and their population frequencies. If the universe marginal of  $K$  depends on the resident semantic state, then semantic states can modify the future semantic universes in which their descendants are represented.

**Proof.**

The total state space  $\Omega$  records both universe and semantic-state coordinates. The kernel outputs a distribution over the same total space, so its pushforward can change both coordinates. Dependence of  $P(\pi_U)K((u, s), -)$  on  $s$  is exactly the condition that semantic state influences semantic-universe transition.

## 65 Total hierarchy

The complete formal hierarchy is now



Evolution can occur at every level.

## 66 Final principle

In the fixed-universe theory the question was whether a transformation preserves a meme. In the observer-relative theory it became whether it preserves the meme relative to a cognitive modality. In the coevolutionary theory it became how the meme alters the modality that later judges its identity. The present theory asks how the evolving population alters the entire semantic universe in which such questions can be formulated.

How does the population alter the semantic universe in which identity can be asked?

The causal chain is semantic state, architecture change, site change, new descent structure, a new ∞-topos, new identity types, and then new replication dynamics.

Memetic evolution can modify the universe of semantic possibility itself: dynamics of states, of observers, of descent, and of the ambient ∞-topos. In compact form,

$$(E_t, M_t, \nu_t) \mapsto (E_{t+1}, M_{t+1}, \nu_{t+1}),$$

where even the type of the state being evolved is permitted to change.